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UPPER HYBRID SOLITONS AND
OSCILLATING-TWO-STREAM INSTABILITIES

BY

MIKLOS PORKOLAB

AND

MARTIN V. GOLDMAN

PLASMA PHYSICS
LABORATORY

MASTER

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PRINCETON UNIVERSITY
PRINCETON, NEW JERSEY

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Upper-Hybrid Solitons and Oscillating-
Two-Stream Instabilities

by

Miklos Porkolab

Plasma Physics Laboratory, Princeton University,
Princeton, New Jersey 08540

and

Martin V. Goldman

Department of Astrogeophysics, University of Colorado
Boulder, Colorado 80302

ABSTRACT

A warm two-fluid theory of soliton formation near the upper-hybrid frequency is developed. Several forms of the nonlinear Schroedinger equation are obtained, depending on whether the electric field is completely perpendicular to the dc magnetic field or whether it has an additional small component parallel to the magnetic field. For the perpendicular case, the character of the soliton depends on its scale length, L , and on β . For low β , when $L < c/\omega_{pe}$, we find stationary envelope and hole solitons, whereas when $L > c/\omega_{pi}$ we obtain the super-Alfvénic solitons described by Kaufman and Stenflow by MHD theory. However, the case $E_{||} \neq 0$ may be of more interest, since it couples the pump to the excited waves more efficiently. In the limit of linearization about an infinite wavelength pump, the nonlinear Schroedinger equations yield purely growing (oscillating-two-stream) instabilities in both cases.

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I. Introduction

The formation of envelope solitons and associated density depletion in the absence of magnetic fields has been studied in considerable detail in recent years.¹⁻⁶ In particular, the understanding of soliton formation is of importance since it represents the nonlinear stage of the oscillating-two-stream instability (purely growing mode),² and thus it is expected to be present during intense microwave and laser heating of plasma.⁴

One of the very few papers dealing with such effects in a magnetically confined plasma is that of Kaufman and Stenflo,⁷ who examined super-Alfvénic soliton formation due to rf fields with frequencies near the upper-hybrid frequency and electric fields perpendicular to the magnetic field.⁷ These authors considered MHD-type perturbations. That is, they assumed that the ambient magnetic field and density are perturbed by the same amount (i.e., magnetic field lines are "frozen in").

We present here a more general study of upper-hybrid wave nonlinear behavior, relevant to a variety of low, intermediate, and high β plasmas. It is motivated in part by interest in heating near the upper-hybrid frequency by intense microwave sources.⁸⁻¹¹ In addition, there may be strong self-generated magnetic fields present in laser-pellet interactions,¹² providing the possibility of nonlinear upper-hybrid wave involvement in such systems. Heating in plasma focus devices by intense laser beams has also been proposed.¹³ Here the magnetic field may be on the order of a megagauss, and is provided

by the pinch current. Thus, it is desirable to understand the non-linear states of purely growing (oscillating-two-stream) instabilities which may be pumped by an intense wave at the upper-hybrid frequency^{8,14} in any of these plasmas. The pump waves may be excited by electromagnetic waves incident upon the inhomogeneous plasma from the high or low density regions.¹⁵ Good linear accessibility from the low density side to the upper-hybrid layer can be arranged even with an incident "ordinary" electromagnetic wave.^{15,16}

In this paper we take a more general look at upper-hybrid solitons, starting from warm two fluid equations and assuming that all scale lengths are much longer than the electron gyro-radius. We find that the perpendicular MHD solitons of Kaufman and Stenflo are a very special case, requiring a low β plasma, and a scale length, L , much larger than c/ω_{pi} , where ω_{pi} is the ion plasma frequency. In addition, they require $\omega_{pe} < \Omega_e$, where Ω_e is the electron cyclotron frequency. In different low β perpendicular limits, we find stationary solitons when $L \ll c/\omega_{pe}$ and $\omega_{pe} > \Omega_e$. However, the most interesting low β case may be that in which the disturbance has a small component of electric field along the magnetic field, for in this case the nonlinearity can be strongest. High β plasmas are also treated. For the various cases we discuss the undriven soliton solutions and the corresponding driven linear instabilities.

The plan of this paper is as follows: In Section II we consider strict perpendicular propagation, but allow for the low-frequency response to be electromagnetic in character. Nonlinear Schroedinger equations are derived in various limits, and undriven soliton solutions are obtained. Contact is made with the MHD theory of Kaufman and Stenflo.⁷

In Section III we obtain the nonlinear Schroedinger equation for the case where the pump has a small but finite electric field parallel to the background magnetic field. In Section IV the conclusions are given.

II. Upper-Hybrid Solitons with Spatial Variation Perpendicular to the Magnetic Field

We begin with the two-fluid equations for ions and electrons in the presence of an external dc magnetic field and a large amplitude electric field, together with smaller perturbations in both fields. The fields and configuration are shown in Figs. 1 and 2. The uniform background magnetic field, $\underline{B}_0$, is in the z direction. The total electric field is

$$\underline{E} = \underline{E}^H \cos \omega_0 t + \underline{E}^L \quad (1)$$

where $\underline{E}^H$ is the envelope of the total upper-hybrid field, and $\underline{E}^L$ is the low frequency field, obtained from an ambipolar potential ϕ :

$$\underline{E}^L = -\nabla \phi. \quad (2)$$

The frequency ω_0 will be assumed to be close to the upper-hybrid frequency. Initially, we allow the electric fields to vary in the x and z directions. We then specialize to the case $\underline{E}^H = \hat{x} E_x^H(x)$, in which only variation and upper-hybrid wave polarization in the $\hat{x}$ direction are allowed. This is the case illustrated in Fig. 1 (Fig. 2 pertains to Section III). In addition, we allow low frequency electromagnetic field components $E_y^L(x)$ and $B_z^L(x)$ to exist, as they are generally driven by currents in the y direction and are essential for the MHD limit of Kaufman and Stenflo.⁷ We now write down the fundamental fluid and Maxwell equations:

$$n_s \left(\frac{\partial}{\partial t} + \underline{v}_s \cdot \underline{\nabla} \right) \underline{v}_s + \frac{\gamma_s^T}{m_s} \underline{\nabla} n_s = \frac{q_s}{m_s} n_s \left(\underline{E} + \frac{\underline{v}_s}{c} \times \underline{B} \right), \quad (3)$$

$$\frac{\partial n_s}{\partial t} + \underline{\nabla} \cdot (\underline{v}_s n_s) = 0, \quad (4)$$

$$\underline{\nabla} \cdot \underline{E} = 4\pi \sum_s q_s n_s, \quad (5)$$

$$\underline{\nabla} \times \underline{B} = \frac{4\pi}{c} \sum_s q_s n_s \underline{v}_s + \frac{1}{c} \frac{\partial \underline{E}}{\partial t}, \quad (6)$$

$$\underline{\nabla} \times \underline{E} = - \frac{1}{c} \frac{\partial \underline{B}}{\partial t}, \quad (7)$$

$$\underline{\nabla} \cdot \underline{B} = 0. \quad (8)$$

s denotes ions ($q_i = e$) or electrons ($q_e = -e$). We solve Eq. (3) for the (zero-order) high frequency components of the velocity, and obtain,

$$v_{ox} = -PE_y \cos \omega_o t - QE_x \sin \omega_o t \quad (9a)$$

$$v_{oy} = PE_x \cos \omega_o t - QE_y \sin \omega_o t \quad (9b)$$

$$v_{oz} = RE_z \sin \omega_o t, \quad (9c)$$

where

$$P = \frac{e}{m} \frac{\Omega_e}{\omega_o^2 - \Omega_e^2} \quad (9d)$$

$$Q = \frac{e}{m} \frac{\omega_o}{\omega_o^2 - \Omega_e^2} \quad (9e)$$

$$R = - \frac{e}{m\omega_o} \quad (9f)$$

Here, e is the magnitude of the electronic charge, m is the electron mass, and $\Omega_e = |eB/mc|$ is the electron cyclotron frequency.

We now wish to solve for the low frequency (background) density variations including the time averaged values of the $\underline{v} \cdot \underline{\nabla} \underline{v}$ term, which gives the ponderomotive force. In order to simplify the problem we shall first neglect E_z , and consider the simple case of upper-hybrid one-dimensional perpendicular variation so that

$$\underline{E}^H = \hat{x} E_x^H(x) \quad (10)$$

We break up the densities and velocities into high and low frequency parts:

$$\underline{v} = \underline{v}_0 + \underline{u} \quad (11)$$

$$n = n_o(1 + \eta) \quad , \quad \eta \equiv \delta n/n_o \quad (12)$$

Taking the time-averaged $\hat{x}$ and $\hat{y}$ components of (3) and (4) we obtain

for the low frequency fluctuations the following equations:

$$\frac{\partial}{\partial t} m_s u_x^s = \frac{\partial}{\partial x} (-T_s \eta_s - q_s \phi - \psi_s) + \frac{q_s}{c} u_y^s B_o \quad (13)$$

$$\frac{\partial}{\partial t} m_s u_y^s = q_s E_y^L - \frac{q_s}{c} u_x^s B_o \quad (14)$$

$$\frac{\partial}{\partial t} \eta_s + \frac{\partial}{\partial x} u_x^s = 0 \quad (15)$$

Note that we have ignored $B_z^L(x)$ relative to B_o . Here, ϕ is the ambipolar potential given by (2), and ψ_s is the ponderomotive potential energy:

$$\psi_e \equiv \frac{m_e}{2} \left[\frac{\omega_o}{2\pi} \int_0^{2\pi/\omega_o} dt (u_x^e)^2 \right] = \frac{m_e Q^2}{4} |E_x^H|^2, \quad (16)$$

ψ_i is smaller by the electron to ion mass ratio, and will be ignored here. The ions see essentially only the ambipolar potential.

Note that for pump frequency at the upper-hybrid frequency, i.e.,

$$\omega_o^2 = \omega_{pe}^2 + \Omega_e^2 = \omega_{uh}^2, \quad (17)$$

the ponderomotive potential reduces to

$$\psi = \frac{e^2}{4m} \frac{\omega_o^2}{\omega_{pe}^2} |E_x^H|^2, \quad (18)$$

which differs from (1) in Ref. 7 by the factor $\omega_o^2/4\omega_{pe}^2$.¹⁷

In addition to the fluid equations (13)-(15), we require the linearized versions of Maxwell's equations (5)-(8). Combining (6) and (7) to eliminate E_y^L we obtain a wave equation for $B^L = B_z^L(x)$:

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) B^L = \frac{4\pi en_o}{c} \frac{\partial}{\partial x} (u_y^e - u_y^i). \quad (19)$$

To generate equations for u_y^s we take a time derivative of (14) and insert into (13). This yields,

$$\frac{m_s}{q_s} \frac{\partial^2 u_y^s}{\partial t^2} = \frac{\partial E_y}{\partial t} + \frac{B_o}{m_s c} \left[\frac{\partial}{\partial x} (\psi_s + q_s \phi + T_s \eta) - \frac{q_s}{c} u_y^s B_o \right], \quad (20)$$

We now assume that the slow time scale, T , is much greater than the inverse cyclotron frequencies:

$$T \gg \Omega_s^{-1} = \frac{m_s}{eB_o}, \quad s = e, i. \quad (21)$$

This allows us to ignore the inertial term on the left side of (20) and we then have an immediate solution for u_y^s . We require the x-derivative of u_y^s to insert into (19). Using Maxwell's equation (7) and the approximation based on (21), (19) becomes

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{v_A^2} \frac{\partial^2}{\partial t^2} \right) \left(\frac{B^L}{B_o} \right) = -\beta \frac{\partial^2}{\partial x^2} \left(\frac{\psi_e}{T_e + T_i} + \eta \right), \quad (22)$$

where, β is the effective particle pressure over magnetic pressure,

$$\beta \equiv \frac{4\pi n_o}{B_o^2} (T_e + \bar{T}_i), \quad \eta = \eta_e, \quad \bar{T}_i \equiv T_i (\eta_i / \eta_e) \quad (23)$$

and v_A^2 is the Alfvén velocity:

$$v_A^2 = \frac{B_o^2}{4\pi n_o M}.$$

We have used the following approximations in deriving equation (22):

m_e has been ignored relative to M , and $1/c^2$ has been ignored relative to $1/v_A^2$. Finally, in the linearized limit we can show:¹⁷

$$\eta_i / \eta_e \approx \omega_{uh}^2 / \omega_{pe}^2,$$

and thus

$$\bar{T}_i = T_i (1 + \Omega_e^2 / \omega_{pe}^2) \quad (24)$$

Note that quasineutrality is recovered in the limit $\Omega_e^2 \ll \omega_{pe}^2$.

Equation (22) is a wave equation for B^L in terms of ψ_e and η . We now need an equation for η in terms of B^L . To this end, we first differentiate the y-momentum equation (14), with respect to x and use Maxwell's equation (7) and the continuity equation (15), to obtain a time integral:

$$m_s \frac{\partial}{\partial x} u_y^s = \frac{-q_s}{c} B^L + \frac{q_s}{c} B_o \eta^s. \quad (25)$$

If we now take an x -derivative of the x -momentum equation (13) and once more use the continuity equation we obtain,

$$-m_s \frac{\partial^2}{\partial t^2} \eta_s = \frac{\partial^2}{\partial x^2} (-T_s \eta_s - q_s \phi - \psi_s) + \frac{q_s}{c} B_o \frac{\partial u_y^s}{\partial x}. \quad (26)$$

Next, we combine (25) and (26) to eliminate $\partial u_y^s / \partial x$, and then take the electron and ion equations, together with the quasineutrality condition (24) to eliminate the ambipolar potential, ϕ . The result, to lowest order in m_e/M , is

$$\frac{\partial^2}{\partial x^2} \left(\frac{\psi_e}{T_e + T_i} + \eta \right) = R_g^{-2} \left(\eta - \frac{B^L}{B_o} \right), \quad (27)$$

where

$$R_g^{-2} = \frac{m_e \Omega_e^2}{T_e + T_i} = \frac{\omega_{pe}^2}{c^2 \beta}. \quad (28)$$

is an effective electron gyro-radius. Equations (22) and (27) now determine η as a function of ψ_e , which is the basic relationship needed to generate a nonlinear Schroedinger equation for the upper-hybrid wave.

An approximation intrinsic to the fluid approach is that the scale length L must be much greater than the effective gyro-radius R_g , so that

$$\frac{\partial^2}{\partial x^2} \eta \ll R_g^{-2} \eta. \quad (29)$$

Thus, (27) and (22) may be rewritten as

$$\frac{\partial^2}{\partial x^2} \left(\frac{\psi}{m \Omega_e^2} \right) = \frac{\delta n}{n_o} - \frac{B^L}{B_o}, \quad (30)$$

$$\left[(1 + \beta) \frac{\partial^2}{\partial x^2} - v_A^{-2} \frac{\partial^2}{\partial t^2} \right] \frac{B^L}{B_o} = - \frac{\omega_{pe}^2}{c^2} \frac{\partial^2}{\partial x^2} \left[\frac{\psi}{m \Omega_e^2} \right]. \quad (31)$$

Several different cases may now be distinguished.

A. Low β , $v > v_A$ MHD Limit

First we treat low β . Equation (31) then reduces to equation (4) of Kaufman and Stenflo⁷ except that our result shows that the ponderomotive term is $(1/4)(1 + \Omega_e^2/\omega_{pe}^2)(E_o^2/B_o^2)$ rather than just E_o^2/B_o^2 .¹⁷ They were concerned with super-Alfvén solitons, in which $B^L = B^L(x - Vt)$ has the form of an undamped propagating disturbance, with $V > v_A$. They also assumed frozen-in field lines, or $\delta n/n_o \approx B^L/B_o$. This corresponds to our Eq. (30), provided

$$\frac{1}{L^2} \left(\frac{\psi}{m\Omega_e^2} \right) \ll \frac{B^L}{B_o}$$

But, this is consistent with (31) only provided that

$$\left| \frac{v^2}{v_A^2} - 1 \right| \ll \frac{\omega_{pe}^2}{c^2} L^2,$$

$$\beta \ll 1.$$

(32)

This is not hard to satisfy when $V \approx v_A$. However, we have ignored the inertial term in (20) by assuming the inequality (21), which here takes the form $L^2/V^2 \gg \Omega_i^{-2}$, or, equivalently,

$$1 \ll \frac{\omega_{pi}^2 L^2}{c^2} \frac{v_A^2}{V^2}.$$

(33)

This is a much more stringent condition on L when $v_A \gtrsim V$ than (32) and limits the applicability of the Kaufman-Stenflo theory to relatively broad solitons.

B. Low β , $V \ll v_A$, $B^L \rightarrow 0$ Limit

There is a different low β limit, consistent with larger scale lengths L , and not treated by Kaufman and Stenflo. Suppose we consider a sufficiently slow time scale T that

$$\frac{1}{v_A^2} \frac{\partial^2}{\partial t^2} B^L \ll \frac{\partial^2}{\partial x^2} B^L$$

(e.g., subsonic solitons, with $V \ll v_A$). Then, an integral of (31) is

$$\frac{B^L}{B_0} + \frac{\omega_{pe}^2}{c^2} \left(\frac{\psi_e}{m\Omega_e^2} \right) = \text{Const.} \quad (34)$$

which represents a pressure balance between ponderomotive and perturbed magnetic pressures. Taking the constant equal to zero, and comparing with (30), we see that, provided,

$$1 \gg \frac{L^2 \omega_{pe}^2}{c^2}, \quad (35)$$

a solution is

$$\frac{\delta n}{n_0} = \frac{\partial^2}{\partial x^2} \left(\frac{\psi}{m\Omega_e^2} \right) \gg \frac{B^L}{B_0} \quad (36)$$

This is a complementary limit to case A (compare with Eqs. (32) and (33)), in which the induced magnetic field is negligible. The drift, $v_y(x) \propto -(\nabla\psi \times \underline{B}_0)_y$, has an x-gradient, which produces the density perturbation δn (via continuity and y-momentum conservation). This limit has the interesting feature that the density is proportional to the second derivative of the ponderomotive potential. We shall treat this case in more detail later.

C. High β , $V \approx c_s$ Limit

In this limit, (31) may be written as

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{c_s^2} \frac{\partial^2}{\partial t^2} \right) \frac{B}{B_0} = - \frac{1}{T_e + T_i} \frac{\partial^2}{\partial x^2} \psi, \quad (37)$$

where $c_s^2 = (T_e + T_i)/M$ is sound speed. If we assume $V > c_s$, then the condition for frozen-in field lines ($\delta n/n_0 \approx B/B_0$) becomes,

$$\left| \frac{v^2}{c_s^2} - 1 \right| \ll \frac{L^2}{R_g^2}, \quad (38)$$

which is not hard to satisfy. However, the condition (21) for neglect of the inertial terms can now be written as

$$1 \ll \frac{\omega_{pi}^2 L^2}{c^2} \frac{c_s^2}{v^2} \beta^{-1}. \quad (39)$$

Since $\beta \gg 1$, this is even harder for L to satisfy when $v^2 \approx c_s^2$ than (33). Accordingly, we shall not treat this case further.

D. High β , $V \ll c_s$ Limit

Once more assuming $\beta \gg 1$, we now assume $V \ll c_s$, and

$$\frac{1}{c_s^2} \frac{\partial^2}{\partial t^2} B \ll \frac{\partial^2}{\partial x^2} B.$$

Now an integral of (37) is

$$\frac{B^L}{B_0} + \frac{\psi}{T_e + T_i} = \text{Const.} \quad (40)$$

Inserting this into (30) and using $L^2 \gg R_g^2$, we obtain

$$\frac{\delta n}{n_0} = \frac{-\psi + C}{T_e + T_i} \quad (41)$$

$$\frac{B^L}{B_0} = \frac{\delta n}{n_0} = \frac{-\psi + C}{T_e + T_i} \quad (42)$$

Equation (41) yields the usual low frequency density response to the ponderomotive force obtained in the limit $B_0 \rightarrow 0$. Equation (40) with (42) therefore represents a balance of perturbed particle and ponderomotive pressures. The perturbed magnetic pressure is negligible because of the high β condition. However, the field lines B^L are "frozen-in" according to (42), and will provide an additional term in the non-linear Schroedinger equation. The constant C is usually chosen as the spatial average of ψ , in order to conserve particles.

We can now obtain from linear theory the equation for the high frequency density variations (upper-hybrid wave), and simply add the additional density and magnetic perturbations given here. We assume only variations in the x direction. Proceeding along the lines of previous calculations, it is easy to show from (3)-(8) that the following equation governs the high frequency component of density (electric field):

$$\frac{\partial^2 E_x}{\partial t^2} + \left(\frac{4\pi e^2}{m} n_L^{\text{TOT}} + (\Omega_e^{\text{TOT}})^2 \right) E_x - \omega_0^2 \frac{\partial^2 E_x}{\partial x^2} + 2v_e \frac{\partial E_x}{\partial t} = 0 \quad (43)$$

where n_L^{TOT} is the total low frequency density, i.e.,

$$n_L^{\text{TOT}} = n_0 + \delta n,$$

and Ω_e^{TOT} is the cyclotron frequency corresponding to the total magnetic field, $B^{\text{TOT}} = B_o + B^L$. ν_e is the linear damping rate, and D describes the dispersion of the upper-hybrid wave. In particular, we distinguish two regimes:

$$(i) \text{ For } \Omega_e < \omega_{uh} \ll 2\Omega_e, \quad \omega_o D = -(T_e/m) (\omega_{pe}^2 / \Omega_e^2) \quad (44a)$$

$$(ii) \text{ For } 2\Omega_e \ll \omega_{uh}, \quad \omega_o D \approx 3T_e/m. \quad (44b)$$

Thus (44a) describes backward waves (negative dispersion)⁸ and (44b) describes forward waves (positive dispersion).

The nonlinear wave equation (43) can be reduced further by the usual WKB techniques, i.e., we assume

$$E^H = E^H(t) e^{-i\omega_o t},$$

where $E^H(t)$ is a slowly varying "envelope" of the electric field, and we obtain a nonlinear Schroedinger equation:

$$\begin{aligned} -2i\omega_o \frac{\partial E_x^H}{\partial t} - \omega_o D \frac{\partial^2 E_x^H}{\partial x^2} - 2\nu_e i\omega_o E_x^H + (\omega_{uh}^2 - \omega_o^2) E_x^H \\ + \left(\omega_{pe}^2 \frac{\delta n_o}{n_o} + 2\omega_{ce}^2 \frac{B^L}{B_o} \right) E_x^H = 0. \end{aligned} \quad (45)$$

Next, we introduce the dimensionless time variable, $\tau = t\omega_o$, and define

$$p \equiv 1 - \omega_{uh}^2 / \omega_o^2. \quad (46)$$

Then the nonlinear Schroedinger equation becomes,

$$\begin{aligned} \frac{i\partial E_x^H}{\partial \tau} + \frac{D}{2\omega_o} \frac{\partial^2}{\partial x^2} E_x^H + \frac{iv_e}{\omega_o} E_x^H + \frac{P}{2} E_x^H \\ - \frac{1}{2} \left[\frac{\omega_{pe}^2}{\omega_o^2} \frac{\delta n}{n_o} + \frac{2\omega_{ce}^2}{\omega_o^2} \frac{B^L}{B_o} \right] E_x^H = 0, \end{aligned} \quad (47)$$

where the nonlinear expressions for δn and B^L are the solutions to Eqs. (30) and (31) in the various limits discussed.

In particular, the nonlinear Schroedinger equation of Kaufman and Stenflo (case A, above) corresponds to negative dispersion, low β , and long scale lengths. The negative dispersion, according to (44a), implies that $\omega_{uh} \ll 2\Omega_e$, or $\omega_{pe} \ll \Omega_e$, so the dominant nonlinearity in (47) is the B^L/B_o term. As discussed by Kaufman and Stenflo, this equation has a super-Alfvén solitary wave solution of the hyperbolic secant form.

We next treat the nonlinear Schroedinger equation for case B, above, assuming positive dispersion, so that $\omega_{pe} > \Omega_e$. The dominant nonlinearity, from (36) and (18), is

$$- \frac{1}{2} \frac{\omega_{pe}^2}{\omega_o^2} \left(\frac{n_L}{n_o} \right) = - \frac{1}{8} \frac{c^2}{B_o^2 \omega_{pe}^2} \frac{\partial^2}{\partial x^2} |E_x^H|^2. \quad (48)$$

The nonlinear Schroedinger equation can now be written as

$$\begin{aligned} i \frac{\partial}{\partial \tau} E_x^H + \frac{1}{2} \frac{\partial^2}{\partial x^2} E_x^H + \frac{iv_e}{\omega_o} E_x^H + \frac{P}{2} E_x^H - \tilde{N} \left(\frac{\partial^2}{\partial x^2} |E_x^H|^2 \right) E_x^H \\ = 0, \end{aligned} \quad (49)$$

where

$$\bar{x} = (\omega_o/D)^{1/2} x, \quad (50)$$

and

$$\tilde{N} = (c^2/\omega_{pe}^2)(\omega_o/D)(1/8B_o^2) > 0, \quad (51)$$

In Appendix A we summarize the predictions of the parametric theory of the purely growing mode near $\omega_o \approx \omega_{uh}$.⁸ The growth rate found for perpendicular propagation at the upper-hybrid frequency is given by the following expression:

$$\gamma_{\perp} = -\nu + \frac{\omega_o}{2} \left[4\tilde{N}|E_o|^2 \bar{k}^2 (\bar{k}^2 - p) - (\bar{k}^2 - p)^2 \right]^{1/2}, \quad (52)$$

where,

$$\bar{k}^2 \equiv Dk^2/\omega_o^2 \approx 3k_e^2/\omega_o^2.$$

The propagation conditions for an extraordinary electromagnetic wave pump in an inhomogeneous plasma allow ω_o to be equal to or greater than ω_{uh} . Also, (52) requires $(\bar{k}^2 - p) > 0$, so we have the following condition on p (defined in (46))

$$\bar{k}^2 > p \geq 0.$$

The maximum growth rate with respect to $\bar{k}^2$ occurs at

$$\bar{k}_m^2 = p \frac{(1-2\tilde{N}|E_o|^2)}{1-4\tilde{N}|E_o|^2} . \quad (53)$$

Thus, we must have $4\tilde{N}|E_o|^2 < 1$ for the validity of the maximum growth rate formula. The simplest case is $4\tilde{N}|E_o|^2 \ll 1$. Thus we have approximately,

$$\bar{k}_m^2 \approx p(1 + 2\tilde{N}|E_o|^2) . \quad (54)$$

Inserting into (52) yields for the maximum growth rate,

$$\gamma_{\perp} \Big|_{\max} = -\nu + \omega_o p \tilde{N} |E_o|^2 , \quad (52a)$$

$$4\tilde{N}|E_o|^2 \ll 1 , \quad p \ll D/R_g^2 \omega_o^2 = (\Omega_e^2 / \omega_o^2) (3T_e / (T_e + T_i))$$

Another limit is $4\tilde{N}|E_o|^2 > 1$. Here, the growth rate has no maximum, but we can take $p = 0$ and obtain from (52),

$$\gamma_{\perp} = -\nu + \frac{\omega_o}{2} \bar{k}^2 (4\tilde{N}|E_o|^2 - 1)^{1/2} \quad (52b)$$

$$4\tilde{N}|E_o|^2 > 1 , \quad \bar{k}^2 \ll D/R_g^2 \omega_o^2 = (\Omega_e^2 / \omega_o^2) (3T_e / (T_e + T_i)) .$$

This growth rate increases with $\bar{k}^2$ until the fluid theory breaks down because $\bar{k}^2$ becomes of order R_g^{-2} . Here a kinetic theory is required.

We now show how to obtain the growth rate of (52) by linearizing the nonlinear Schroedinger equation (49) about an infinite wavelength perpendicular upper-hybrid pump E_o . That is, we assume

$$E_x^H(x) = (E_o + E_k^- e^{i\bar{k}x} e^{\gamma_1 \tau}) ,$$

where all frequencies are in units of ω_o and γ is taken to be real.

Substituting into (49), we note that

$$E_x^H \frac{\partial^2}{\partial x^2} |E_x^H|^2 = -\bar{k}^2 |E_o|^2 E_k - \bar{k}^2 E_o^2 E_k^* ,$$

and so (49) reduces to the following coupled linear equations for E_k and E_k^* :

$$(+i\gamma_{\perp} - \bar{k}^2/2 + \nu_e/\omega_o + p/2 + \tilde{N}\bar{k}^2 |E_o|^2) E_k = -\tilde{N}\bar{k}^2 E_o^2 E_k^* ,$$

$$(-i\gamma_{\perp} - \bar{k}^2/2 - \nu_e/\omega_o + p/2 + \tilde{N}\bar{k}^2 |E_o|^2) E_k^* = -\tilde{N}\bar{k}^2 E_o^2 E_k .$$

Setting the determinant equal to zero yields (52).

In the nonlinear undriven limit without damping ($\nu_e = 0$), (49) can be solved by the usual techniques described in Appendix B. The result is a cusped soliton with $|E^H|^2 < \rho_o$, or a cusped hole-soliton with $|E^H|^2 > \rho_o$, as illustrated in Fig. 2a and 2b.

Here $\rho_o \equiv (4\tilde{N})^{-1}$. The analytic expressions for these solitons are given by (B-17) and (B-18), with x interpreted as $\bar{x}$ in (50). We note that the cusp in $|E|^2$ as $x = 0$ is not physical, since the equation is only self-consistent when the scale length is greater than the effective gyro-radius R_g (Eq. (29)). This condition is here,

$$\left| \frac{1}{|E|^2} \frac{\partial |E|^2}{\partial \bar{x}} \right| \ll \frac{(D/\omega_o)^{1/2}}{R_g}.$$

The infinite slope of $|E|^2$ with respect to $\bar{x}$ in the vicinity of the cusp (near $\bar{x} = 0$) violates this condition. It is interesting to note that the character of the soliton depends crucially on its intensity relative to ρ_o . The solitons are either entirely above or entirely below ρ_o , but cannot cross ρ_o , which appears as a pseudopotential barrier in the formalism in Appendix B. The stability of these solutions is beyond the scope of this paper, but deserves future attention.

Next, we consider the moderate or high β case (D), in which we found (Eq. (42)),

$$\frac{\delta n}{n_o} = \frac{B^L}{B_o} = \frac{-\psi}{T_e + T_i}.$$

Inserting this nonlinearity into the Schroedinger equation (47) yields the nonlinear term

$$+ \frac{1}{2} \left(1 + \frac{\Omega_e^2}{\omega_{uh}^2} \right) \frac{\psi}{T_e + T_i} E_x^H.$$

This is the form of nonlinearity familiar from non-magnetic soliton theory,¹⁻⁴ and thus, for positive dispersion, or $\omega_{pe} \gg \Omega_e$, the customary Langmuir envelope solitons are retrieved.

III. Upper-Hybrid Solitons with a Small Component Along the Magnetic Field

Let us not consider the case when the upper-hybrid electric field $\underline{E}^H$ has a component E_z^H along the external dc magnetic field, but $|E_z^H| \ll |E_x^H|$, and $E_y^H = 0$. We also allow z-variation in both E_z^H and E_x^H . The low frequency electrostatic field $\underline{E}^L$ is also allowed to have both z and x components and variation. For simplicity, we do not consider any low-frequency electromagnetic field. The configuration is shown in Fig. 3. Taking the time averages to obtain the ponderomotive force contributions, we find,

$$\overline{\underline{v}_0 \cdot \underline{\nabla} v_{0x}} = \begin{cases} \frac{\partial}{\partial x} \psi_{xx}^e \\ \frac{\partial}{\partial z} \psi_{zz}^e \end{cases}, \quad (55)$$

where,

$$\psi_{xx}^e \equiv \frac{1}{4} \left[Q^2 |E_x^H|^2 + Q|R| |E_z^H|^2 \right], \quad (56a)$$

and

$$\psi_{zz}^e \equiv \frac{1}{4} \left[R^2 |E_z^H|^2 + Q|R| |E_x^H|^2 \right], \quad (56b)$$

and we have made use of $\underline{\nabla} \times \underline{E}^H = 0$. The momentum and continuity equation for the low frequency variables then become:

$$\frac{\partial}{\partial t} m_s u_x^s = \frac{\partial}{\partial x} P_x^s + \frac{q_s B_0}{c} u_y^s \quad (57a)$$

$$\frac{\partial}{\partial t} m_s u_y^s = - \frac{q_s B_0}{c} u_x^s \quad (57b)$$

$$\frac{\partial}{\partial t} m_s u_z^s = \frac{\partial}{\partial z} p_z^s \quad (57c)$$

$$\frac{\partial \eta_s}{\partial t} + \frac{\partial u_x^s}{\partial x} + \frac{\partial u_z^s}{\partial z} = 0, \quad (57d)$$

where,

$$p_{xz}^s = -\gamma_s T_s \eta_s - q_s \phi - \psi_{xx}^s. \quad (58)$$

Here, as before, $\eta_s \equiv \delta n_s^L / n_0$, and ψ^i is negligible. If we take the x-derivatives of the x and y momentum equations, and the z-derivative of the z-momentum equation, then the continuity equation becomes,

$$\frac{\partial^2}{\partial t^2} \eta_s + \frac{\partial^2}{\partial x^2} \frac{p_x^s}{m_s} + \Omega_s \frac{\partial u_y^s}{\partial x} + \frac{\partial^2}{\partial z^2} \frac{p_z^s}{m_s} = 0, \quad (59)$$

and we have the following equation for $\partial u_y^s / \partial x$:

$$\Omega_s \left[1 + \Omega_s^{-2} \frac{\partial^2}{\partial t^2} \right] \frac{\partial u_y^s}{\partial x} = - \frac{\partial^2}{\partial x^2} p_x^s / m, \quad (60)$$

where $\Omega_s = q_s B_0 / m_s c$. As in (21), we assume a time scale slow compared to Ω_i^{-1} , so we may solve (60) by iteration to obtain,

$$\Omega_s \frac{\partial u_y^s}{\partial x} \approx \left(-1 + \Omega_s^{-2} \frac{\partial^2}{\partial t^2} \right) \frac{\partial^2}{\partial x^2} \frac{p_x^s}{m_s} \quad (61)$$

Thus, we see that the second and third terms in (59) are negligible compared to the fourth, provided that

$$1 \ll \frac{L_z}{L_x} \ll T\Omega_i, \quad (62)$$

where, L_z is the scale length in the z -direction, L_x is the scale length in the x -direction, and T is the usual slow time scale. We also have assumed a much sharper gradient in the x -direction than in the z -direction ($L_z \gg L_x$). Finally, we can neglect the first term compared to the fourth term in (59) if we assume,

$$L_z \ll Tv_i, \quad (63)$$

where $v_i = (T_i/M)^{1/2}$ is the ion thermal speed. The continuity equation then simply gives $\partial u_z / \partial z = 0$, or

$$p_z^s = -\langle \psi_{zz} \rangle_{sp}, \quad (64)$$

where we have chosen the integration constant equal to the spatial average, $\langle \psi_{zz} \rangle_{sp}$, of ψ_{zz} so as to conserve particles. Writing out (64) for electrons and for ions, and using the quasineutrality condition $\eta_e = \eta_i$, we obtain a solution for the depleted density:

$$\frac{\delta n_L}{n_0} = \frac{-e^2 (|E_x^H|^2 - \langle |E_x^H|^2 \rangle_{sp})}{4(T_e + T_i)m(\omega_0^2 - \Omega_e^2)} , \quad (65)$$

when we have dropped $|E_z^H|^2$ relative to $|E_x^H|^2$, since we have $Q|R|$ on the order of R^2 . Note, this result does not depend explicitly on the E_z^H field or the scale length L_z . However, the scale length L_z cannot be allowed to go to infinity, but must satisfy the inequality (62).

We further note that our result is equivalent to the approximation $\partial P_z^S / \partial z = 0$ using only the z-momentum equation (57c) with no interial term. Since this does not use the continuity equation, we may interpret η_s more generally, as $\eta_s = \ln n_L^S / n_0$. The solution of equations (64), using the quasineutrality condition $\eta_e = \eta_i$, then yield

$$n_L = n_0 \exp \left[-m_e (\psi_{zz} - \langle \psi_{zz} \rangle_{sp}) / (T_e + T_i) \right] . \quad (66)$$

Since,

$$mQ|R| = e^2 / m(\omega_0^2 - \Omega_e^2) , \quad \text{and} \quad mK^2 = e^2 / m\omega_0^2 ,$$

equation (66) agrees with the results obtained earlier by Gurevich and Pitaevskii.¹⁸ However, the present derivation holds only if a finite E_z exists, the inequalities (63) and (64) are satisfied, and particle motion (i.e., pressure) is allowed along the magnetic field (which was not pointed out in Reference 18). The expansion of (66) yields (65).

Using similar calculations as before, but retaining a finite parallel component of the high frequency density perturbation, in the limit $\omega_{pe} \gg \Omega_e$ we can easily obtain the following differential equation:

$$\frac{\partial^2 E}{\partial t^2} + \left(\frac{4\pi e^2}{m} n_L + \Omega_e^2 \right) E - \omega_o \left(D_{\perp} \frac{\partial^2}{\partial x^2} + D_{\parallel} \frac{\partial^2}{\partial z^2} \right) E + 2v_c \frac{\partial E}{\partial t} = 0, \quad (67)$$

where $\omega_o D_{\parallel} = \omega_o D_{\perp} = 3T_e/m$. For small E_z components we can ignore variations along the z direction, and substituting (66) for n_L , we obtain the conventional form of the nonlinear Schroedinger equation:

$$i \frac{\partial E^H}{\partial t} + \frac{D}{2} \frac{\partial^2 E^H}{\partial x^2} + M(|E^H|^2 - \langle |E^H|^2 \rangle) E^H + i v_e E^H + \frac{P}{2} E^H = 0, \quad (68)$$

where

$$M = \frac{e^2 \omega_{pe}^2}{8\omega_o m (T_e + T_i) (\omega_o^2 - \Omega_e^2)} = \frac{e^2}{8m\omega_{uh} (T_e + T_i)}, \quad (69)$$

If we consider perturbation solutions about a real pump field E_o , namely

$$E^H = E_o + E_k e^{ikx} e^{\gamma t},$$

then,

$$(|E^H|^2 + \langle |E^H|^2 \rangle_{sp}) E^H \approx E_o^2 (E_k + E_k^*)$$

and by similar techniques as those which led to (53), we obtain a purely growing mode with growth rate,

$$\gamma = -\nu_e + \omega_o \left[ME_o^2 (\bar{k}^2 - p) - (\bar{k}^2 - p)^2 \right]^{1/2}. \quad (70)$$

Thus, the wave must have positive dispersion ($D = 3Te/m > 0$), and so $\omega_o = \omega_{uh} \approx \omega_{pe} \gg \Omega_e$. This equation is identical to the result from parametric instability theory given in Appendix A. The growth rate (70) is a maximum when $(\bar{k}^2 = p + ME_o^2/2 \ll D/\omega_o R_g^2)$, so we obtain from (70) a result which may be written as

$$\gamma^{\max} = -\nu_e + \omega_o E_o^2 / 32\pi n(T_e + T_i). \quad (71)$$

This is essentially the growth rate of the purely growing mode in the absence of a dc magnetic field. Although it would appear from (62) that this growth rate must be much less than the ion cyclotron frequency, it can be shown that it is valid when $\Omega_i \ll \gamma^{\max} \ll \Omega_c$, since the ions only see the ambipolar potential. Let us now compare this with the growth rate, γ , obtained in (52) for the case of strict perpendicular upper-hybrid polarization.

When $2\tilde{N}E_o^2 \ll 1$, (52a) applies, and we find the ratio of $\gamma_{\perp \max}$ to $\gamma^{\max}$ of (71) is just $pR_g^2 = k_m^2 R_g^2 \ll 1$. Thus, in this limit the maximum growth rate for the purely growing upper-hybrid mode generally does not occur in the perpendicular direction under conditions for which the fluid approximation is valid.

With $v_e = 0$ and $D > 0$, (68) is of the standard form treated in Appendix B, case (i). Comparing with (B-11), the undriven soliton solution is the usual hyperbolic secant:

$$|E|^2 = \frac{2\bar{\Omega}}{M} \operatorname{sech}^2 \left((2M\bar{\Omega}D/\omega_o)^{1/2} x \right) \quad (72)$$

$$\bar{\Omega} = \Omega - p/2$$

IV. Conclusion

We have derived nonlinear Schroedinger equations and found soliton solutions for electrostatic disturbances near the upper-hybrid frequency, having spatial variation perpendicular and almost perpendicular to the magnetic field. When the equations are linearized about an infinite wavelength (electromagnetic) pump wave, near or at the upper-hybrid frequency, the solutions are familiar⁸ purely growing instabilities. Within the fluid approximation, waves with wave vectors slightly off perpendicular to the magnetic field have the fastest growth rate, leading us to expect that the corresponding solitons may represent the nonlinear saturated limit of this instability, as in

non-magnetic cases.¹⁻⁵ The principal limitation of the theory is that it is based on warm two-fluid equations, and so cannot adequately treat nonlinear solutions with scale lengths on the order of, or less than, the electron gyro-radius. In the linear limit, when the pump field is sufficiently intense, waves with these short wavelengths have the fastest growth rates.⁸ We should therefore like to point out the need for derivation and solution of nonlinear equations based on kinetic equations for upper-hybrid waves. In general, wave particle interactions should be taken into account to study the phenomena of damping and heating. Simulations may offer the best route to these effects, but we believe they should be at least two-dimensional to include the fastest growing waves.

Although we have assumed the high frequency unstable waves and solitons are electrostatic, the same assumption has not, and cannot, in general, be made for the low frequency waves.

In addition to the ambipolar potential created by charge separation due to electron response to the ponderomotive force, a low-frequency electromagnetic response is induced by the $\mathbf{F} \times \mathbf{B}$ current, where $\mathbf{F}$ consists of the ponderomotive, ambipolar, and pressure forces. The low-frequency magnetic field is essential to make contact with the MHD limit of Kaufman and Stenflo.⁷ They treat perpendicular upper-hybrid solitons with velocities greater than the Alfvén speed. We can derive their result, together with the frozen-in field line approximation, only provided that the scale length $L \gg c/\omega_{pi}$, where ω_{pi} is the ion plasma frequency. This is a very restrictive condition, also noted by Kaufman and Stenflo. For more reasonable scale

lengths $L \ll c/\omega_{pe}$, we again find perpendicular solitons and hole-solitons for which the low-frequency magnetic field is negligible, although the density perturbation is still related to the $\mathbf{E} \times \mathbf{B}$ drift through the continuity and momentum equations. The relevance of low-frequency magnetic fields to other electrostatic purely growing instabilities in a magnetic field, such as the lower-hybrid instabilities, is an interesting question which to our knowledge, has not yet been fully explored.

Finally, we wish to emphasize the possibility of good accessibility to the upper-hybrid layer in an inhomogeneous plasma, even if the density is decreasing outward from the interior, when the incident electromagnetic wave is at an angle to the magnetic field. This fact may result in possible application of the effects discussed in this paper for the purposes of microwave and laser heating of plasmas.

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APPENDIX A

The Purely Growing Mode Obtained from Linear Parametric Theory

The dispersion relationship for parametric instabilities in a magnetic field may be written in the following form:^{8,14}

$$\epsilon(\omega) + \frac{\mu^2}{4} \chi_i(\omega) (1 + \chi_e(\omega)) \left[\frac{1}{\epsilon(\omega - \omega_0)} + \frac{1}{\epsilon(\omega + \omega_0)} \right] = 0, \quad (\text{A-1})$$

where for $k_x \gg k_z$, $k_y = 0$, $E_0 = E_{ox}$, $\omega \approx \omega_{uh}$,

$$\mu^2 = \frac{e^2}{m^2} \frac{|E_{ox}|^2 k_x^2}{(\omega_0^2 - \Omega_e^2)^2}, \quad (\text{A-2})$$

and

where the magnetic field is in the $\hat{z}$ direction, and we assumed $\mu^2 \ll 1$.

Here $\epsilon(\omega \pm \omega_0)$, $\epsilon(\omega)$ are the dielectric functions, ω_0 is the pump frequency, $\omega \rightarrow i\gamma$ is the growth rate of the purely growing mode, $\chi_i(\omega)$ is the low frequency ion susceptibility, and $\chi_e(\omega)$ is the low frequency electron susceptibility.

1) Perpendicular Propagation

Taking the limit $k_{||} = 0$, $|\omega| \ll \Omega_i$, we have the following relations:^{8,14}

$$\chi_i(\omega) \approx \omega_{pi}^2 / \Omega_i^2; \quad \chi_e(\omega) \approx \omega_{pe}^2 / \Omega_e^2;$$

$$\epsilon(\omega \pm \omega_0) \approx \epsilon_{Re}(\omega_0) \pm i\epsilon_{Im}(\omega_0).$$

Assuming $k_x v_e / \Omega_e = k_x r_{ce} \ll 1$, near the upper-hybrid frequency (ω_{uh}) we have

$$\epsilon_{Re}(\omega_o) \approx 1 - \frac{(\omega_{pe}^2 + \omega_o^2 D k_x^2)}{\omega_o^2 - \Omega_e^2}, \quad (A-3)$$

$$\epsilon_{Im}(\omega_o) \approx (\gamma + \nu_e) \frac{\partial \epsilon}{\partial \omega_o},$$

and $\partial \epsilon / \partial \omega_o \approx 2\omega_{uh} / \omega_{pe}^2$, where D is the dispersion. In particular, we may distinguish two limits in which the dispersion assumes a simple form:

$$(a) \quad \Omega_e < \omega_o \ll 2\Omega_e, \quad \omega_o D \approx -r_{ce}^2 \omega_{pe}^2;$$

$$(b) \quad 2\Omega_e \ll \omega_o, \quad \omega_o D \approx 3\nu_e^2.$$

Substituting these relations into (A-1) we obtain

$$\frac{\mu^2}{2} \frac{\epsilon_{Re}(\omega_o)}{\epsilon_{Re}^2(\omega_o) + \epsilon_{Im}^2(\omega_o)} = - \frac{\Omega_e^2}{\omega_{uh}^2}. \quad (A-4)$$

It is clear that we need $\epsilon_{Re}(\omega_o) < 0$ for instability, which for $\omega_o \gtrsim \omega_{uh}$ can be satisfied only for $D > 0$, i.e., positive dispersion, Assuming

$$1 + \omega_{pe}^2 / \Omega_e^2 \ll \omega_{pi}^2 / \Omega_i^2,$$

and substituting the previous relations into (A-4), we obtain the following result for the growth rate:

$$\gamma = -\nu_e + \frac{\omega_o}{2} \left[4\tilde{N} E_{ox}^2 \bar{k}^2 (\bar{k}^2 - p) - (\bar{k}^2 - p)^2 \right]^{1/2}, \quad (A-5)$$

where

$$\bar{k}^2 \equiv Dk^2/\omega_o^2 = 3k_e^2/\omega_o^2,$$

$$p \equiv 1 - \omega_{uh}^2/\omega_o^2,$$

$$\tilde{N} \equiv (c^2/\omega_{pe}^2)(\omega_o/D)(1/8B_o^2).$$

We note that as long as $|\chi_i(\omega)| \gg 1 + |\chi_e(\omega)|$, (A-5) holds even for $\gamma > \Omega_i$.

We wish to point out here that if we ignore the k_x dependence of the coupling coefficient in (A-5), the minimum threshold (or maximum growth rate) in (A-4) is obtained for $\epsilon_{Re}(\omega_o) = -\epsilon_{Im}(\omega_o)$, so that (A-3) reduces to

$$\epsilon_{Im} = (\mu^2/4)(\Omega_e^2/\omega_{uh}^2), \quad (A-6)$$

or

$$\gamma = -\nu_e + \omega_o \bar{k}^2 \tilde{N} |E_o|^2,$$

which is the maximum growth rate obtained in Eq. (52a) for $4\tilde{N}|E_o|^2 \ll 1$

(since in this limit $\bar{k}^2 \approx p$). The threshold given by (A-6) has also been obtained previously by one of the present authors.⁸ The reason for the agreement is that at small pump fields the minimum threshold depends mainly on the mismatch between ω_o and ω_{ek} (the normal mode), and the dependence of μ^2 upon k^2 enters only in next order in $|E_o|^2$ (See Eq. (54)).

ii) Propagation at a Finite Angle to the Magnetic Field

In this case we assume a small but finite k_z , and take $k_z \ll k_x$, $\gamma/k_z \ll v_e$, $\gamma/k_z \ll v_i$, $\omega_o/k_z \gg v_e$ (where v_e and v_i are the electron and ion thermal velocities, respectively). The low frequency electron and ion susceptibilities are given by

$$\chi_e(\omega) \approx \frac{1}{k^2 \lambda_{De}^2} ; \quad \chi_i(\omega) \approx \frac{1}{k^2 \lambda_{Di}^2} ;$$

where we assume $k^2 \lambda_{De,i}^2 \ll 1$. For $2\Omega_e \ll \omega_o \approx \omega_{uh}$ we have

$$\epsilon_{Re}(\omega_o) \approx 1 - \frac{\omega_{pe}^2 + \omega_o D(k_z^2 + k_x^2)}{\omega_o^2 - \Omega_e^2} , \quad (A-7)$$

where $\omega_o D = 3v_e^2$, and where we assumed $k_y = 0$. Substituting these relations into (A-1), we obtain

$$\frac{\mu^2}{2k^2 \lambda_D^2} \frac{\epsilon_{Re}(\omega_o)}{\epsilon_{Re}^2(\omega_o) + \epsilon_{Im}^2(\omega_o)} = - \left(1 + \frac{T_i}{T_e} \right) , \quad (A-8)$$

which for the foregoing dielectric constants reduces to

$$\gamma = -\nu_e + \omega_o \left[M |E_o|^2 (\bar{k}^2 - p) - (\bar{k}^2 - p)^2 \right]^{1/2}, \quad (A-9)$$

where

$$M = \frac{e^2 \omega_{pe}^2}{8 \omega_o m (T_e + T_i) (\omega_o^2 - \Omega_e^2)} \approx \frac{e^2}{8 \omega_o m (T_e + T_i)}$$

Note that (A-8) shows $\epsilon_{Re}(\omega_o) < 0$ for instability, which can be satisfied for positive dispersion. The maximum growth rate (or minimum threshold) is obtained from (A-8) or (A-9) for $\epsilon_{Re}(\omega_o) = -\epsilon_{Im}(\omega_o)$ (since μ^2/k^2 is independent of k to order k_z^2/k_x^2), and the result is

$$\gamma = -\nu + \frac{e^2 |E_{ox}|^2}{8 m (T_e + T_i) \omega_o}, \quad (A-10)$$

which agrees with Eq. (71) for $\omega_o \approx \omega_{pe} \gg \Omega_e$. We note that had we assumed cold ions so that $\gamma/k_z \gg \nu_i$, $T_i \ll T_e$, but $\gamma \ll \Omega_i$, we would have obtained the same result as (A-10). (See Ref. 14.)

APPENDIX B

Pseudopotential Method for Solving Nonlinear Schroedinger Equations

In this appendix we generalize a well-known method for solving or characterizing the solutions to nonlinear wave equations. First, we essentially follow Hasegawa¹⁹ to obtain a constant of the motion, and then by interpreting this as an "energy," and defining a pseudopotential we characterize the final solutions. The latter procedure is often applied to low frequency collisionless shocks and solitons.²⁰ The object is to solve

$$\frac{i\partial E}{\partial \tau} + \frac{1}{2} \frac{\partial^2}{\partial x^2} E + f(|E|^2)E = 0, \quad (B-1)$$

where f is an arbitrary nonlinear function of $|E|^2$, and suitable dimensionless variables have been chosen. Make the substitution,

$$E = \rho^{1/2} e^{i\sigma}, \quad (B-2)$$

where ρ and σ are real. Separate real and imaginary parts to obtain the two equations,

$$\frac{\partial}{\partial \tau} \rho + \frac{\partial}{\partial x} \left(\frac{\rho \partial \sigma}{\partial x} \right) = 0 \quad (B-3)$$

$$\frac{1}{8} \frac{\partial}{\partial \rho} \left[\frac{(\rho')^2}{\rho} \right] + f(\rho) = \dot{\sigma} + \frac{1}{2} \left(\frac{\partial \sigma}{\partial x} \right)^2, \quad (B-4)$$

where $\rho' = \partial \rho / \partial x$ and $\dot{\sigma} \equiv \partial \sigma / \partial \tau$.

We now seek stationary solutions, $\dot{\rho} = 0$. Then, it can be shown¹⁹ that the solution to (B-3) is

$$\sigma = C_1 \int \frac{dx}{\rho} + \Omega t + C, \quad (B-5)$$

where C_1 , C , and Ω are arbitrary constants, independent of space and time. Inserting (B-5) into (B-4) yields

$$\frac{1}{8} \frac{\partial}{\partial \rho} \left[\frac{(\rho')^2}{\rho} \right] + f(\rho) = \Omega + \frac{C_1^2}{2\rho^2}, \quad (B-6)$$

which can be integrated to give,

$$\left(\frac{\partial \rho}{\partial x} \right)^2 + V(\rho) = 0, \quad (B-7)$$

where, the pseudopotential, V is defined by

$$V(\rho) \equiv 8\rho \left[\int f(\rho) d\rho - C_2 - \Omega\rho \right] + 4C_1^2. \quad (B-8)$$

Equation (B-7) has the form of an energy integral of a mechanical system, with ρ playing the role of displacement, x playing the role of time, V of potential, and $E = 0$. Thus, for any nonlinearity, $f(\rho)$, potential curve arguments may be used to characterize the solution for ρ . We now proceed to treat the various $f(\rho)$:

$$\text{Case i)} \quad f(\rho) = N\rho, \quad N = \text{const} . \quad (\text{B-9})$$

Now,

$$V(\rho) = 4N\rho^3 - 8C_2\rho - 8\Omega\rho^2 + 4C_1^2 . \quad (\text{B-10})$$

This is a cubic equation, and the configuration which gives a solitary solution is shown in Fig. 4. This configuration corresponds to $C_1 = C_2 = 0$, $\Omega/N > 0$. Under these conditions (B-6) can be integrated, and the solution is the usual

$$\rho = (2\Omega/N) \text{sech}^2 \left((2N\Omega)^{1/2} x \right) \quad (\text{B-11})$$

$$\text{Case ii)} \quad f(\rho) = -\tilde{N} \frac{\partial^2}{\partial x^2} \rho + p/2, \quad \tilde{N} > 0 . \quad (\text{B-12})$$

This is the form of nonlinearity in (49), for perpendicular low β solitons with $L \ll c/\omega_{pe}$, with the linear dispersive term added in. The above general treatment must be modified slightly to apply to this case. Since

$$-N\rho'' = -\frac{N}{2} \frac{\partial}{\partial \rho} (\rho')^2 ,$$

the integral of (B-6) is again of the form (B-7) but now with V defined as,

$$V(\rho) = \frac{-8\rho C_2 - 8\tilde{\Omega}\rho^2 + 4C_1^2}{1 - 4\tilde{N}\rho} , \quad (\text{B-13})$$

where $\tilde{\Omega} \equiv \Omega - p/2$.

The forms of this potential corresponding to localized solutions for ρ are illustrated in Fig. 5. In Fig. 5a, $\tilde{\Omega} = -C_2^2/2C_1^2$, and

$\rho_1 \equiv C_1^2/C_2$ is greater than ρ_0 , where,

$$\rho_0 \equiv (4\tilde{N})^{-1} . \quad (B-14)$$

Thus,

$$V(\rho) = \frac{(2|\tilde{\Omega}|/\tilde{N})(\rho_1 - \rho)^2}{\rho_0 - \rho} . \quad (B-15)$$

This gives rise to a hole soliton with a cusp, as shown in Fig. 2a .

Another possibility (Fig. 5b) is $C_1 = C_2 = 0$, and $\Omega > 0$, so that

$$V(\rho) = \frac{(2|\tilde{\Omega}|/\tilde{N})\rho^2}{\rho - \rho_0} . \quad (B-16)$$

This yields a regular soliton with a cusp, as shown in Fig. 2b.

Both cases have analytic solutions of the form,

$$\tanh^{-1}u - u = (|\tilde{\Omega}|/2\tilde{N})^{1/2}|x|/u_1 \quad (B-17)$$

where the values of u and u_1 for solitons, and hole solitons are:

a) cusped hole soliton:

$$u \equiv [(\rho - \rho_0)/(\rho_1 - \rho_0)]^{1/2} , \quad \rho_0 \leq \rho \leq \rho_1 ,$$

$$u_1 \equiv (\rho_1 - \rho_0)^{1/2} \quad (B-18a)$$

b) cusped soliton:

$$u \equiv (1 - \rho/\rho_0)^{1/2} , \quad 0 \leq \rho \leq \rho_0 , \quad u_1 \equiv \rho_0^{1/2} . \quad (B-18b)$$

Rewriting (B-17), we get the following simple results for case b):

$$\rho = \rho_0 \operatorname{sech}^2 \left[(2|\bar{\Omega}|)^{1/2} x + (1 - \rho/\rho_0)^{1/2} \right] . \quad (\text{B-18c})$$

In each case, the cusp does not prevent (B-17) from being a solution, since at $x = 0$ the first derivative is discontinuous, but the second derivative is continuous (and infinite). The scale lengths, L , for these solitons approach zero near $x = 0$, but the approximations leading to (B-12) require $L > R_g$ and thus break down near $x = 0$, as discussed in the text. The unphysical cusp would therefore not be present in the real soliton. The full width at half the maximum intensity, ρ , corresponds in both cases to $u = 1/2^{1/2}$, and is given by

$$\Delta x \approx u_1 (|\bar{\Omega}|/2\tilde{N})^{1/2}/3 . \quad (\text{B-19})$$

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17. Note that this differs from the results of Kaufman and Stenflo by the factor $\omega_{uh}^2/4\Omega_{pe}^2$. This discrepancy can be resolved by noting $\tilde{n}_i \approx \tilde{n}_e (\omega_{uh}^2/\omega_{pe}^2)$ (as given later in the manuscript), so that the ponderomotive potential given by Kaufman and Stenflo is that acting on the ion density component, which in the MHD description cannot be distinguished from electrons. However, strictly speaking the ponderomotive force acts on electrons only, and is transmitted to ions by the ambipolar potential. Also, these authors seem to be missing a factor of (1/4) which comes from integration and time averaging.
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Fig. 1. Geometry for perpendicular propagation. $E_x^H(x)$ is the (high-frequency) electrostatic upper-hybrid field. $E_x^L(x)$ is the low-frequency field. It can be obtained from the ambipolar potential, ϕ , which arises because of the charge separation caused by the ponderomotive force associated with $E_x^H(x)$. $E_y^L(x)$ and $B_z^L(x)$ represent the low-frequency electromagnetic field which can be driven by currents in the y direction.

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Fig. 2. Cusped soliton solutions to the nonlinear Schroedinger equation for strict orthogonality to the magnetic field, low β , and $L < c/\omega_{pe}$. The coordinate $\tilde{x} = (\omega_0/D)^{1/2} x$.

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Fig. 3. Geometry for oblique solitons. $\underline{E}^H$ is the (high-frequency) electrostatic upper-hybrid field with components and variation in the x and z directions. $\underline{E}^L$ is the low-frequency electrostatic field, also having components and variation in the x and z directions. $\underline{E}^L$ is derived from the ambipolar potential, ϕ . We have ignored the low-frequency electromagnetic field for simplicity.

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Fig. 4. Form of pseudopotential giving rise to a hyperbolic secant envelope soliton for a cubic nonlinearity in the nonlinear Schroedinger equation (See Eq. (B-7) and (B-10) in the Appendix B.

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Fig. 5. Pseudopotential forms giving rise to cusped solitons for a dispersive nonlinearity (Eq. (B-12)) in the nonlinear Schroedinger equation: Figure 5a shows the pseudopotential for a hold soliton, bounded between $\rho_0 = (4N)^{-1}$ and ρ_1 ; Figure 5b shows the pseudopotential for a soliton with intensity bounded above by ρ_0 .