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1-23-76
LA-6051

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Reporting Date: August 1975

Issued: December 1975

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by

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LOS ALAMOS, NEW MEXICO 87545

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UNITED STATES
ENERGY RESEARCH AND DEVELOPMENT ADMINISTRATION
CONTRACT W-7405-ENG. 36

Printed in the United States of America. Available from
National Technical Information Service
U.S. Department of Commerce
5285 Port Royal Road
Springfield, VA 22151
Price: Printed Copy \$4.50 Microfiche \$2.25

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DIRECT EVALUATION OF TRANSIENT SURFACE TEMPERATURES AND HEAT FLUXES

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ABSTRACT

Evaluations of transient surface temperatures resulting from the absorption of radiation are required in laser fusion reactor systems studies. A general method for the direct evaluation of transient surface temperatures and heat fluxes on the boundaries of bounded media is developed by constructing fundamental solutions of the scalar Helmholtz equation and performing certain elementary integrations.

INTRODUCTION

The analysis of first-wall erosion phenomena in fusion reactors caused by evaporation requires a knowledge of the transient temperature on the inner surface. This problem is discussed in a review paper by Behrisch.¹ An expression for the increase of the surface temperature of a semi-infinite solid that results from introducing total energy E per area A over the time interval τ is given in Eq. (5) of Ref. 1. If the time interval τ is allowed to go to zero, then Eq. (5) of Ref. 1 predicts an infinite increase in the surface temperature and very large temperature increases when the time interval is very small. Because these predictions seem insignificant when finite amounts of the total energy E are involved, the problem of analyzing directly transient surface temperatures is examined in this study.

for $0 \leq x < \infty$ and $t > 0$. We seek to determine directly the transient surface temperature $T(0,t)$ for an arbitrary thermal source $S(x,t)$ and arbitrary, time-dependent heat flux on the surface. This direct determination can be accomplished by first calculating some fundamental solutions of the scalar Helmholtz equation and by carrying out some elementary integrations.

The Laplace transform of the temperature field $T(x,p)$ satisfies

$$\frac{d^2}{dx^2} T(x,p) - q^2 T(x,p) = -\frac{T(x,0)}{\kappa} - \frac{S(x,p)}{k}, \quad (2)$$

where $q^2 = p/\kappa$ and $T(x,0)$ is the initial temperature distribution. Let K_i for $i = 1, 2$ be two linearly independent solutions of

$$\frac{d^2 K_i}{dx^2} - q^2 K_i = 0. \quad (3)$$

It follows from Eqs. (2) and (3) that

$$K_1 T''(x,p) - T(x,p) K_1'' = -\frac{T(x,0)}{\kappa} K_1 - \frac{K_1}{k} S(x,p). \quad (4)$$

Integrating Eq. (4) over the half space produces

TRANSIENT SURFACE TEMPERATURES FOR A SEMI-INFINITE MEDIUM

General Result for Surface Temperature Laplace Transform

When the thermal properties are independent of temperature, the transient temperature field in a half space satisfies

$$\frac{1}{\kappa} \frac{\partial T}{\partial t}(x,t) = \frac{\partial^2 T}{\partial x^2}(x,t) + \frac{S(x,t)}{k} \quad (1)$$

$$-K_1'(0)T(0,p) = -K_1(0)T'(0,p) + \int_0^\infty dx \frac{K_1}{\kappa} T(x,0) + \int_0^\infty dx \frac{K_1}{\kappa} S(x,p). \quad (5)$$

Computing the transient surface temperature of a semi-infinite medium requires only one fundamental solution from Eq. (3). We take

$$K_1(x) = e^{-qx} \quad (6)$$

to avoid exponential growth at infinity. It follows from Eqs. (5) and (6) that the Laplace transform of the half-space surface temperature is given by

$$T(0,p) = -\frac{T'(0,p)}{q} + \frac{1}{\kappa q} \int_0^\infty dx e^{-qx} T(x,0) + \frac{1}{\kappa q} \int_0^\infty dx e^{-qx} S(x,p). \quad (7)$$

Let the surface heat flux be

$$-k \frac{\partial T}{\partial x}(0,t) = F_0 f(t), \quad (8)$$

where $f(t)$ is an arbitrary function of time. Then the surface temperature Laplace transform is

$$T(0,p) = \frac{F_0 f(p)}{\kappa q} + \frac{1}{\kappa q} \int_0^\infty dx e^{-qx} T(x,0) + \frac{1}{\kappa q} \int_0^\infty dx e^{-qx} S(x,p). \quad (9)$$

In this last expression, the initial temperature distribution, the thermal source, and the time dependence of the surface heat flux are all regarded as specified functions. Further simplification of Eq. (9) requires selecting explicit forms for these specified functions.

Explicit Results for Special Cases

If there is no volumetric thermal source in the half space assumed to be at a spatially uniform

initial temperature T_0 , then Eq. (9) simplifies to

$$T(0,p) = \frac{T_0}{p} + \frac{F_0 f(p)}{\kappa q}. \quad (10)$$

By applying the convolution theorem, the inverse Laplace transform of Eq. (10) is

$$T(0,t) - T_0 = \frac{F_0}{\kappa} \sqrt{\frac{\kappa}{\pi}} \int_0^t d\xi \frac{f(t-\xi)}{\sqrt{\xi}}. \quad (11)$$

This result gives the rise in the half-space surface temperature due to an arbitrary, time-dependent heat flux at the surface.

Suppose that an amount of energy E per area A enters the half space over the time interval τ at a constant rate. In this case, we can write

$$F_0 = \frac{E}{A\tau} \quad (12)$$

and

$$f(t) = \begin{cases} 1, & \text{for } 0 \leq t \leq \tau \\ 0, & \text{for } t > \tau. \end{cases} \quad (13)$$

If we assume that $t < \tau$, then Eq. (11) simplifies to

$$T(0,t) - T_0 = \frac{2E}{A\tau} \sqrt{\frac{t}{\kappa \rho c \pi}} \quad (14)$$

when combined with Eqs. (12) and (13). Evaluating Eq. (14) at $t = \tau$ yields

$$T(0,\tau) - T_0 = \frac{2E}{A\sqrt{\tau \kappa \rho c \pi}}, \quad (15)$$

which is Eq. (5) of Ref. 1 used by Behrisch in evaporation studies.

Because Eq. (15) is singular for finite values of E when τ equals zero, we consider an alternative way to model energy absorption and the transient surface temperature for evaporation studies. Suppose that the energy entering a half space is absorbed in a thin region with thickness δ adjacent to the surface. Assume that no energy flows out of the half space and that energy is introduced at a constant

rate over the time interval $0 \leq t \leq \tau$ in the amount of E per area A . Then we have

$$S(x,t) = \begin{cases} \frac{E}{A\delta\tau}, & \text{for } 0 \leq t \leq \tau, 0 \leq x \leq \delta \\ 0, & \text{for } t > \tau \end{cases} \quad (16)$$

and

$$f(t) = 0, \quad 0 \leq t < \infty. \quad (17)$$

Consequently, the surface temperature Laplace transform for this case is

$$T(0,p) = \frac{T_0}{p} + \frac{\kappa}{k} \frac{E}{A\delta\tau} \frac{1}{p^2} (1 - e^{-q\delta})(1 - e^{-p\tau}), \quad (18)$$

as computed from Eq. (9) with a spatially uniform initial temperature T_0 . If we assume that the time $t \leq \tau$, then the inverse Laplace transform of Eq. (18) is found to be

$$T(0,t) = T_0 + \frac{\kappa}{k} \frac{E}{A\delta\tau} \left[t - \left(t + \frac{\delta^2}{2\kappa} \right) \operatorname{erfc} \left(\frac{\delta}{2\sqrt{\kappa t}} \right) + \delta \sqrt{\frac{t}{\pi\kappa}} e^{-\delta^2/4\kappa t} \right], \quad (t \leq \tau). \quad (19)$$

This is the transient surface temperature on the time interval $0 \leq t \leq \tau$ caused by a volumetric source over a thin region adjacent to the surface of a half space. To determine the value of the temperature rise at the end of the time interval, when the source is switched off, Eq. (19) is evaluated at the time τ with the result that

$$T(0,\tau) - T_0 = \frac{E}{\rho c_p A \delta \tau} \left[\tau - \left(\tau + \frac{\delta^2}{2\kappa} \right) \operatorname{erfc} \left(\frac{\delta}{2\sqrt{\kappa\tau}} \right) + \delta \sqrt{\frac{\tau}{\pi\kappa}} e^{-\delta^2/4\kappa\tau} \right]. \quad (20)$$

The result for the temperature rise of the surface of a semi-infinite medium given in Eq. (20) is not singular in the limit as $\tau \rightarrow 0$ in contrast to that of Eq. (15). This can be demonstrated by using the asymptotic expansion of the complementary error function, namely,

$$\operatorname{erfc} \left(\frac{\delta}{2\sqrt{\kappa\tau}} \right) \xrightarrow{\tau \rightarrow 0} \frac{e^{-\delta^2/4\kappa\tau}}{\sqrt{\pi}} \times \left[\frac{2\sqrt{\kappa\tau}}{\delta} - \frac{8(\kappa\tau)^{3/2}}{\delta^3} + \dots \right]. \quad (21)$$

Combining Eqs. (20) and (21) yields

$$T(0,\tau) - T_0 = \frac{E}{\rho c_p A \delta} + \frac{E e^{-\delta^2/4\kappa\tau}}{\rho c_p A \delta \sqrt{\pi}} \left[\frac{2\sqrt{\kappa\tau}}{\delta} + \left(\frac{2\sqrt{\kappa\tau}}{\delta} \right)^3 + \dots \right], \quad (22)$$

which, in the limit as $\tau \rightarrow 0$, becomes

$$\lim_{\tau \rightarrow 0} [T(0,\tau) - T_0] = \frac{E}{\rho c_p A \delta}. \quad (23)$$

Accordingly, to a first approximation, the temperature rise of the surface of a semi-infinite medium is independent of the time interval over which the energy is introduced provided that the time interval is short and provided that the energy is introduced as a volumetric source over a thin region adjacent to the surface rather than as a surface heat flux source. The exact result for an arbitrary time of energy deposition at a constant rate and arbitrary thickness of the volumetric source region is contained in Eq. (20).

TRANSIENT SURFACE TEMPERATURES FOR OTHER GEOMETRIES

Slabs

The transient temperature field in a slab satisfies Eq. (1) for $0 \leq x \leq a$ and $t > 0$, where a is the slab thickness. Integrating Eq. (4) over the slab thickness gives

$$T(a,p)K_1'(a) - T(0,p)K_1'(0) - K_1(a)T'(a,p) + K_1(0)T'(0,p) = \int_0^a dx K_1 \left[\frac{S(x,p)}{k} + \frac{T(x,0)}{\kappa} \right], \quad (24)$$

where the K_i for $i = 1, 2$ are two linearly independent solutions of Eq. (3). Written out in full, Eq. (24) is

$$T(a,p)K_1'(a) - T(0,p)K_1'(0) - K_1(a)T'(a,p)$$

$$+ K_1(0)T'(0,p) = \int_0^a dx K_1 \left[\frac{S(x,p)}{k} + \frac{T(x,0)}{\kappa} \right] \quad (25)$$

and

$$T(a,p)K_2'(a) - T(0,p)K_2'(0) - K_2(a)T'(a,p)$$

$$+ K_2(0)T'(0,p) = \int_0^a dx K_2 \left[\frac{S(x,p)}{k} + \frac{T(x,0)}{\kappa} \right]. \quad (26)$$

These last two equations are interpreted as follows. Either the temperature or the heat flux is specified at $x = 0$. The same is true at $x = a$, except that heat fluxes would not be specified at both surfaces of the slab in such a way as to introduce energy at both surfaces if a bounded temperature at infinite time is to be obtained. Once two of the four quantities $[T(a,p), T(0,p), T'(a,p) \text{ and } T'(0,p)]$ are specified, we regard Eqs. (25) and (26) as a set of two algebraic equations for the remaining two quantities. For example, if the heat flux at $x = 0$ and the temperature at $x = a$ are both specified, then Eqs. (25) and (26) are solved to obtain the Laplace transforms of the temperature at $x = 0$ and the heat flux at $x = a$.

This process can be simplified by choosing the appropriate fundamental solutions, $K_1(x)$ and $K_2(x)$. If we take $K_1(x)$ and $K_2(x)$ as solutions of Eq. (26) such that

$$K_1(a) = 0 \quad (27)$$

and

$$K_2'(0) = 0, \quad (28)$$

then Eqs. (25) and (26) become

$$\begin{aligned} -K_1'(0)T(0,p) &= -K_1'(a)T(a,p) - K_1(0)T'(0,p) \\ &+ \int_0^a dx K_1 \left[\frac{S(x,p)}{k} + \frac{T(x,0)}{\kappa} \right] \end{aligned} \quad (29)$$

and

$$\begin{aligned} -K_2(a)T'(a,p) &= -K_2'(a)T(a,p) - K_2(0)T'(0,p) \\ &+ \int_0^a dx K_2 \left[\frac{S(x,p)}{k} + \frac{T(x,0)}{\kappa} \right]. \end{aligned} \quad (30)$$

When the heat flux at $x = 0$ and the temperature at $x = a$ are regarded as specified boundary conditions, Eq. (29) gives the Laplace transform of the temperature on the surface at $x = 0$, and Eq. (30) gives the Laplace transform of the heat flux on the surface at $x = a$. Other combinations of boundary conditions can be treated in the same way.

In view of Eq. (27), we have

$$K_1(x) = \sinh q(a-x), \quad (31)$$

and in view of Eq. (28),

$$K_2(x) = \cosh qx. \quad (32)$$

Consequently, Eqs. (29) and (30) reduce to

$$\begin{aligned} q \cosh qa T(0,p) &= q T(a,p) - \sinh qa T'(0,p) \\ &+ \int_0^a dx \sinh q(a-x) \left[\frac{S(x,p)}{k} + \frac{T(x,0)}{\kappa} \right] \end{aligned} \quad (33)$$

and

$$\begin{aligned} -\cosh qa T'(a,p) &= -q \sinh qa T(a,p) - T'(0,p) \\ &+ \int_0^a dx \cosh qx \left[\frac{S(x,p)}{k} + \frac{T(x,0)}{\kappa} \right]. \end{aligned} \quad (34)$$

If the surface at $x = a$ is maintained at the initial temperature $[T(a,t) = T_0 = T(x,0)]$, Eqs. (33) and (34) become

$$\begin{aligned} T(0,p) &= \frac{T_0}{p \cosh qa} + \frac{\tanh qa}{q} \frac{F_0}{k} f(p) \\ &+ \frac{1}{q \cosh qa} \int_0^a dx \sinh q(a-x) \left[\frac{S(x,p)}{k} + \frac{T_0}{\kappa} \right] \end{aligned} \quad (35)$$

and

$$-T'(a, p) = -\frac{T_0 q \tanh qa}{p} + \frac{F_0 f(p)}{k \cosh qa} + \frac{1}{\cosh qa} \int_0^a dx \cosh qx \left[\frac{S(x, p)}{k} + \frac{T_0}{\kappa} \right], \quad (36)$$

when the heat flux at $x = 0$ is specified as in Eq. (9). These last two equations simplify to

$$T(0, p) = \frac{T_0}{p} + \frac{\tanh qa}{qa} \frac{aF_0}{k} f(p) + \frac{1}{q \cosh qa} \int_0^a dx \sinh q(a-x) \frac{S(x, p)}{k} \quad (37)$$

and

$$-T'(a, p) = \frac{F_0 f(p)}{k \cosh qa} + \frac{1}{\cosh qa} \int_0^a dx \cosh qx \frac{S(x, p)}{k}. \quad (38)$$

Accordingly, if the slab is initially at the uniform temperature T_0 , and the surface at $x = a$ is kept at this temperature, then Eq. (37) gives the Laplace transform of the transient temperature on the surface $x = 0$ for an arbitrary time-dependent heat flux on this surface and an arbitrary volumetric source term.

When there is no volumetric heat source in the slab, the transient surface temperature at $x = 0$ is

$$T(0, t) = T_0 + \frac{F_0}{k} \int_0^t d\xi f(t - \xi) \sqrt{\frac{\kappa}{\pi\xi}} \left[1 - 2(e^{-a^2/\kappa\xi} - e^{-4a^2/\kappa\xi} + e^{-9a^2/\kappa\xi} \dots) \right]. \quad (39)$$

The first term in the integral gives the half-space solution in agreement with Eq. (11). The remaining terms in the integral in Eq. (39) consider the finite thickness of the slab.

If the heat flux on the surface $x = 0$ vanishes, and if the thin region volumetric energy source

defined in Eq. (16) is entered into Eq. (37), the corresponding Laplace transform of the transient temperature on the surface $x = 0$ is found to be

$$T(0, p) = \frac{T_0}{p} + \left(\frac{1 - e^{-p\tau}}{p} \right) \frac{E}{A\delta\tau\kappa q^2} \left[1 - \frac{\cosh q(a - \delta)}{\cosh qa} \right]. \quad (40)$$

By using the exponential representations of the hyperbolic cosine functions, we find that Eq. (40) can be written in the form

$$T(0, p) = \frac{T_0}{p} + \frac{\kappa E(1 - e^{-p\tau})}{kA\delta\tau p^2} \left[1 - e^{-q\delta} + e^{-q(2a + \delta)} - e^{-q(2a - \delta)} - e^{-q(4a + \delta)} + e^{-q(4a - \delta)} \dots \right]. \quad (41)$$

The transient surface temperature on the surface $x = 0$ for small times such that $t \leq \tau$ is found by computing the inverse Laplace transform of Eq. (41). The result is

$$T(0, t) = T_0 + \frac{\kappa E}{kA\delta\tau} \left\{ t - \left(t + \frac{\delta^2}{2\kappa} \right) \operatorname{erfc} \left(\frac{\delta}{2\sqrt{\kappa t}} \right) + \delta \sqrt{\frac{t}{\pi\kappa}} e^{-\delta^2/4\kappa t} + \left[t + \frac{(2a + \delta)^2}{2\kappa} \right] \operatorname{erfc} \left(\frac{2a + \delta}{2\sqrt{\kappa t}} \right) - (2a + \delta) \sqrt{\frac{t}{\pi\kappa}} e^{-(2a + \delta)^2/4\kappa t} - \left[t + \frac{(2a - \delta)^2}{2\kappa} \right] \operatorname{erfc} \left(\frac{2a - \delta}{2\sqrt{\kappa t}} \right) + (2a - \delta) \sqrt{\frac{t}{\pi\kappa}} e^{-(2a - \delta)^2/4\kappa t} \dots \right\}, \quad (42)$$

provided that $t \leq \tau$. Evaluating Eq. (42) at $t = \tau$ produces

$$T(0, \tau) = T_0 + \frac{E}{\rho c_p A \delta} \left\{ 1 - \left(1 + \frac{\delta^2}{2\kappa\tau} \right) \operatorname{erfc} \left(\frac{\delta}{2\sqrt{\kappa\tau}} \right) \right\}$$

$$\begin{aligned}
& + \frac{\delta}{\sqrt{\pi\kappa\tau}} e^{-\delta^2/4\kappa\tau} + \left[1 + \frac{(2a + \delta)^2}{2\kappa\tau}\right] \operatorname{erfc}\left(\frac{2a + \delta}{2\sqrt{\kappa\tau}}\right) \\
& - \frac{(2a + \delta)}{\sqrt{\pi\kappa\tau}} e^{-(2a + \delta)^2/4\kappa\tau} - \left[1 + \frac{(2a - \delta)^2}{2\kappa\tau}\right] \operatorname{erfc}\left(\frac{2a - \delta}{2\sqrt{\kappa\tau}}\right) \\
& + \frac{(2a - \delta)}{\sqrt{\pi\kappa\tau}} e^{-(2a - \delta)^2/4\kappa\tau} \dots \quad (43)
\end{aligned}$$

The first three terms in the braces of Eq. (43) give the same solution as that obtained for a semi-infinite medium in Eq. (20). The remaining terms in Eq. (43) are due to the finite thickness of the slab and vanish as the slab thickness approaches infinity.

Spherical Shells

If the thermal properties are constant, the transient temperature field in a spherical shell satisfies

$$\begin{aligned}
\frac{1}{\kappa} \frac{\partial T}{\partial t} (r, t) &= \frac{1}{r^2} \frac{\partial}{\partial r} \left[r^2 \frac{\partial T}{\partial r} (r, t) \right] + \frac{S(r, t)}{k}, \\
a \leq r \leq b \quad (44)
\end{aligned}$$

in the case of spherical symmetry. The Laplace transform of Eq. (44) is

$$\begin{aligned}
\frac{1}{r^2} \frac{\partial}{\partial r} \left[r^2 \frac{\partial T}{\partial r} (r, p) \right] - q^2 T(r, p) \\
+ \frac{S(r, p)}{k} + \frac{T(r, 0)}{\kappa} = 0. \quad (45)
\end{aligned}$$

Let K_i for $i = 1, 2$ be two linearly independent solutions of

$$\frac{1}{r^2} \frac{d}{dr} \left[r^2 \frac{dK_i}{dr} \right] - q^2 K_i = 0. \quad (46)$$

By following the same procedure leading to Eqs. (25) and (26), we obtain

$$K_1(b)b^2T'(b, p) - K_1(a)a^2T'(a, p) - T(b, p)b^2K_1'(b)$$

$$+ T(a, p)a^2K_1'(a) = (-1) \int_a^b dr r^2 K_1 \left[\frac{S(r, p)}{k} + \frac{T(r, 0)}{\kappa} \right] \quad (47)$$

and

$$\begin{aligned}
K_2(b)b^2T'(b, p) - K_2(a)a^2T'(a, p) - T(b, p)b^2K_2'(b) \\
+ T(a, p)a^2K_2'(a) = (-1) \int_a^b dr r^2 K_2 \left[\frac{S(r, p)}{k} + \frac{T(r, 0)}{\kappa} \right]. \quad (48)
\end{aligned}$$

Any two of the four quantities $\{T(a, p), T'(a, p), T(b, p)$ and $T'(b, p)\}$ can be regarded as the unknowns to be computed from Eqs. (47) and (48). If we seek the transient temperature on the inner surface, $r = a$, of the shell and the transient heat flux on the outer surface, then $T(a, p)$ and $T'(b, p)$ are taken as the unknowns in Eqs. (47) and (48). In this case the two linearly independent solutions of Eq. (46) may be subjected to the boundary conditions

$$K_1(b) = 0 \quad (49)$$

and

$$K_2'(a) = 0. \quad (50)$$

Consequently, it follows from Eq. (47) that

$$\begin{aligned}
a^2 K_1'(a) T(a, p) &= a^2 K_1(a) T'(a, p) + b^2 K_1'(b) T(b, p) \\
- \int_a^b dr r^2 K_1 \left[\frac{S(r, p)}{k} + \frac{T(r, 0)}{\kappa} \right], \quad (51)
\end{aligned}$$

whereas Eq. (48) becomes

$$\begin{aligned}
b^2 K_2(b) T'(b, p) &= a^2 K_2(a) T'(a, p) + b^2 K_2'(b) T(b, p) \\
- \int_a^b dr r^2 K_2 \left[\frac{S(r, p)}{k} + \frac{T(r, 0)}{\kappa} \right]. \quad (52)
\end{aligned}$$

These last two equations give the Laplace transforms of the inner surface temperature and the outer surface heat flux.

The fundamental solution, $K_1(r)$, that satisfies the boundary condition of Eq. (49) is

$$K_1(r) = r^{-1} \sinh q(b-r). \quad (53)$$

With Eq. (53) we find that Eq. (51) reduces to

$$\begin{aligned} (\sinh q\Delta + aq \cosh q\Delta)T(a,p) = & -a \sinh q\Delta T'(a,p) \\ & + bqT(b,p) + \int_a^b dr r \sinh q(b-r) \left[\frac{S(r,p)}{k} + \frac{T(r,0)}{\kappa} \right], \end{aligned} \quad (54)$$

where $\Delta = b - a$ is the shell thickness. Equation (54) is a general result for the Laplace transform of the temperature on the inner surface of the shell and can be simplified for various special cases. Suppose that $S(r,t) = 0$, $T(r,0) = T_0$, and $T(b,t) = T_0$ and that

$$-k \frac{\partial T}{\partial r}(a,t) = F_0 f(t). \quad (55)$$

Then Eq. (54) becomes

$$\begin{aligned} (\sinh q\Delta + aq \cosh q\Delta)T(a,p) = & \frac{aF_0}{k} f(p) \sinh q\Delta \\ & + bq \frac{T_0}{p} + \frac{T_0}{\kappa} \int_a^b dr r \sinh q(b-r), \end{aligned} \quad (56)$$

which simplifies to

$$T(a,p) = \frac{T_0}{p} + \frac{\sinh q\Delta aF_0 f(p)}{k(\sinh q\Delta + aq \cosh q\Delta)}. \quad (57)$$

Let $H(t)$ be the inverse Laplace transform of $\sinh q\Delta/(\sinh q\Delta + aq \cosh q\Delta)$. Then applying the convolution theorem to find the inverse Laplace transform of Eq. (57) produces

$$T(a,t) = T_0 + \frac{aF_0}{k} \int_0^t d\xi H(\xi) f(t-\xi), \quad (58)$$

which gives the transient temperature on the inner surface of the spherical shell when an arbitrary

time-dependent heat flux is specified on this surface. If

$$F_0 = \frac{E}{4\pi a^2 \tau} \quad (59)$$

and

$$f(t) = \begin{cases} 1, & \text{for } 0 \leq t \leq \tau \\ 0, & \text{for } t > \tau, \end{cases} \quad (60)$$

we find that Eq. (58) gives

$$T(a,\tau) = T_0 + \frac{2E}{4\pi a^2 \sqrt{\tau k \rho c \pi}} \quad (61)$$

as the first approximation to the rise in the inner surface temperature that occurs over the time interval $0 \leq t \leq \tau$.

Because the result in Eq. (61) for this temperature rise is singular as $\tau \rightarrow 0$, we introduce the energy as a volumetric source spread over a thin spherical shell region lying next to the inner surface of the shell. That is, let $T'(a,p) = 0$, $T(r,0) = T_0$, $T(b,t) = T_0$, and

$$S(r,t) = \begin{cases} \frac{3E}{4\pi(c^3 - a^3)\tau} s(t), & \text{for } a \leq r \leq c = a + \delta \\ 0, & \text{for } c \leq r \leq b \end{cases} \quad (62)$$

and

$$s(t) = \begin{cases} 1, & \text{for } 0 \leq t \leq \tau \\ 0, & \text{for } t > \tau. \end{cases} \quad (63)$$

In this case the Laplace transform of the inner surface temperature as given in Eq. (54) becomes

$$\begin{aligned} (\sinh q\Delta + aq \cosh q\Delta)T(a,p) = & \frac{bqT_0}{p} \\ & + \int_a^c dr r \sinh q(b-r) \frac{3E s(p)}{4\pi(c^3 - a^3)\tau k} \end{aligned}$$

$$+ \frac{T_0}{\kappa} \int_a^b dr r \sinh q(b-r), \quad (64)$$

which simplifies to

$$T(a,p) = \frac{T_0}{p} + \frac{3E s(p)}{4\pi(c^3 - a^3)\tau k q^2} - \frac{3E s(p)}{4\pi(c^3 - a^3)\tau k q^2} \left[\frac{\sinh q(b-c) + cq \cosh q(b-c)}{\sinh q\Delta + aq \cosh q\Delta} \right]. \quad (65)$$

Inverting the first two terms on the right-hand side of Eq. (65) and evaluating the result at $t = \tau$ yields the following result for the temperature rise of the inner surface over the time τ as a first approximation,

$$T(a,\tau) = T_0 + \frac{3E}{\rho c_p 4\pi(c^3 - a^3)}. \quad (66)$$

Because

$$c^3 = (a + \delta)^3 = a^3 + 3a^2\delta + 3a\delta^2 + \delta^3,$$

Eq. (66) can be written as

$$T(a,\tau) = T_0 + \frac{E}{\rho c_p 4\pi a^2 \delta [1 + (\frac{\delta}{a}) + 1/3(\frac{\delta}{a})^2]}. \quad (67)$$

Correction terms to this expression for the temperature rise due to the finite thickness of the shell can be obtained by inverting the third term on the right-hand side of Eq. (65). These terms are small when $\delta \ll a$ and $\delta \ll b - a$. We also note that Eq. (67) and Eq. (22) obtained in the analysis of the semi-infinite medium problem give the same results for the surface temperature rise in the limit as $\tau \rightarrow 0$ when $\delta \ll a$ and when we identify A in Eq. (22) with $4\pi a^2$ in Eq. (67).

REFERENCES

1. R. Behrisch, "First-Wall Erosion in Fusion Reactors," Nucl. Fusion 12, 695 (1972).