OAK RIDGE NATIONAL LABORATORY
Operated by
UNION CARBIDE NUCLEAR COMPANY
Division of Union Carbide Corporation

Post Office Box X
Oak Ridge, Tennessee

MASTER

EXTERNAL TRANSMITTAL
AUTHORIZED

ORNL CENTRAL FILES NUMBER 60-12-38
--

DATE: December 4, 1960
SUBJECT: A Statistical Model of Nuclear Level Spacings

TO: Distribution
FROM: L. Dresner and E. Inönd

COPY NO. 24

LEGAL NOTICE

This report was prepared as an account of Government sponsored work. Neither the United States, nor the Commission, nor any person acting on behalf of the Commission:
A. Makes any warranty or representation, expressed or implied, with respect to the accuracy, completeness, or usefulness of the information contained in this report, or that the use of any information, apparatus, method, or process disclosed in this report may not infringe privately owned rights; or
B. Assumes any liabilities with respect to the use of, or for damages resulting from the use of any information, apparatus, method, or process disclosed in this report.
As used in the above, "person acting on behalf of the Commission" includes any employee or contractor of the Commission, or employee of such contractor, to the extent that such employee or contractor of the Commission, or employee of such contractor prepares, disseminates, or provides access to, any information pursuant to his employment or contract with the Commission, or his employment with such contractor.

NOTICE

This document contains information of a preliminary nature and was prepared primarily for internal use at the Oak Ridge National Laboratory. It is subject to revision or correction and therefore does not represent a final report. The information is not to be abstracted, reprinted or otherwise given public dissemination without the approval of the ORNL patent branch, Legal and Information Control Department.

LEGAL NOTICE

This report was prepared as an account of Government sponsored work. Neither the United States, nor the Commission, nor any person acting on behalf of the Commission

A. Makes any warranty or representation, expressed or implied, with respect to the accuracy, completeness, or veridicality of the information contained in this report, or that the use of any information, apparatus, method, or process disclosed in this report may not infringe privately owned rights; or

B. Assumes any liabilities with respect to the use of, or for damages resulting from the use of any information, apparatus, method, or process disclosed in this report.

As used in the above, "person acting on behalf of the Commission" includes any employee or contractor of the Commission, or employee of such contractor to the extent that such employee or contractor of the Commission, or employee of such contractor prepares, disseminates, or provides access to, any information pursuant to his employment or contract with the Commission, or his employment with such contractor.

DISCLAIMER

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency Thereof, nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.

DISCLAIMER

Portions of this document may be illegible in electronic image products. Images are produced from the best available original document.

Abstract

A generalization of Wigner's simple model for the distribution of nuclear level spacings is studied. It is based on a stochastic process which reproduces the correct joint probability distribution of N energy levels for small spacings. The case $N = 3$, which includes the effect of the correlation between adjacent spacings, is discussed in detail. The resulting distribution and the correlation coefficient are compared with experimental data. No definite conclusion can be drawn except to say that the effect of the correlations on the spacing distribution is very small.

Introduction

Before 1955 the main emphasis in the study of the nuclear energy level distribution was on the energy dependence of the local mean level density, with practically no consideration given to the probability of a definite spacing for a given mean. Indeed, most texts omitted mention of the latter question or limited their remarks to the observation that if the energy levels were randomly placed and uncorrelated then the spacing distribution would be exponential. A notable exception to this practice was the book of Landau and Smorodinsky¹ where it was pointed out that the probability of very small spacings should be very small, a fact first noted in 1929 by Wigner and von Neumann.² Thus the spacing distribution should in fact be quite unlike an exponential one, and, in particular, should vanish at the origin rather than have a maximum there. This latter feature of the spacing distribution has recently been called a "level repulsion" effect.

In early 1955 Harvey³ called attention to experimental evidence of a nuclear "level repulsion" effect in the resonance levels of U^{238} , the spacings of which were more uniformly distributed than an exponential distribution would allow. The next year a paper by Gurevich and Pevsner⁴ appeared reinforcing Harvey's

-
1. L. Landau and Ya. Smorodinsky, Lectures on Nuclear Theory, State Technical Theoretical Literature Press, Moscow 1955. Trans: Consultants Bureau, Inc., New York, 1958.
 2. J. von Neumann and E. P. Wigner, Physik. Z. 30, 467 (1929).
 3. J. A. Harvey, Phys. Rev. 98, 1162 (1955). Interesting early discussions of the U^{238} data can be found in "Statistical Aspects of the Nucleus," BNL-331 (C-21), April, 1955, and in the Proceedings of the Gatlinburg Conference on Neutron Physics by Time-of-Flight, Nov. 1-2, 1956, published as ORNL-2309 (TID-4500).
 4. I. I. Gurevich and M. I. Pevsner, J. Exptl. Theoret. Phys. (U.S.S.R.) 31, 162 (1956) Trans.: Soviet Phys. JETP 4, 278 (1957). See also Nuclear Phys. 2, 575 (1957).

conclusion; and in 1958 further strong evidence in favor of a "level repulsion" effect was adduced by a very careful analysis of the then existing data on Th^{232} and $\text{U}^{234,6,8}$ by Harvey and Hughes.⁵

The earliest suggestions for theoretical treatment of the level spacing distribution were made by Wigner in 1956.⁶ He noted, for example, that when two levels are very close together the probability of finding a given spacing, s , between them is proportional to the spacing, s , itself.* Based on extending this proportionality to larger spacings he derived the distribution $\rho s \exp\left(-\frac{1}{2}\rho s^2\right)$. Whether this extended hypothesis had any merit remained of course to be decided by comparison with experiment; in any case the required linear behavior for small s provided the observed level repulsion effect. As it turned out, the analysis of Harvey and Hughes⁵ showed that Wigner's distribution fit the available experimental data rather well.

It was furthermore pointed out by Wigner that one model root structure on which the foregoing ideas could be tested is the root structure of a large random symmetric matrix. As a matter of fact, he pointed out the existence in the mathematical literature of one case in which the joint probability density of all

-
5. J. D. Harvey and D. J. Hughes, Phys. Rev. 109, 471 (1958). Some preliminary aspects of this work were presented by D. J. Hughes at the International Conference on the Neutron Interactions with the Nucleus held at Columbia University, Sept. 9-13, 1957 (proceedings published as TID-7547).
 6. E. P. Wigner, Gatlinburg Conference (see Ref. 3). Substantially the same remarks were made at the Columbia Conference (Ref. 5) whose proceedings may be more readily available.

*Wigner's observation is based on the (valid) use of first order perturbation theory as will be subsequently indicated. The same conclusion has been derived using a quite different approach by Dresner⁷ who showed that the nuclear resonance levels can be considered as the roots of a Gaussian random function. For such functions it can be proved that the root- (level-) spacing distribution is linear in the spacing for small spacings.

7. L. Dresner, Phys. Rev. 113, 633 (1959).

the eigenvalues could be explicitly written down.⁶ This case is that in which all the independent matrix elements are normally distributed about zero, and the dispersion of the above-diagonal elements is half that of the diagonal elements. The joint probability density of the eigenvalues is then

$$p(\lambda_1, \lambda_2, \dots, \lambda_N) = C \cdot \prod_{i < j=2}^N |\lambda_j - \lambda_i| \cdot \exp \left[- \sum_{i=1}^N \lambda_i^2 / 4\sigma^2 \right] \quad (1)$$

where C is a normalizing constant, and σ^2 is the dispersion of the above-diagonal elements. It is the product of differences which appears in Eq. (1) that produces the level repulsion effect.

It has not yet been possible to integrate this joint probability appropriately so as to obtain from it the distribution of spacings between adjacent eigenvalues. However, in lieu of an analytic approximation numerical experiments have been performed by Blumberg and Porter,⁸ Rosenzweig,⁹ and Porter and Rosenzweig.¹⁰ In all cases the authors found that for random symmetric matrices of large order the distribution of spacings between adjacent eigenvalues agreed very closely with Wigner's earlier surmise. Quite significant is the fact that this result occurred even though several different distributions were used for the individual matrix elements.

A limitation of Wigner's original argument is that it does not allow the computation of the correlation between adjacent level spacings. Such a generalization is possible by considering the probability of finding not two but any number, N, of levels very close together, and is explained in detail below. In the case N = 3, which allows correlation only between adjacent spacing intervals, the distribution of the individual spacings has been calculated as well as the correlation coefficient between adjacent spacings. These results have been compared with the

8. S. Blumberg and C. E. Porter, Phys. Rev. 110, 786 (1958).

9. N. Rosenzweig, Phys. Rev. Lett. 1, 101 (1958).

10. C. E. Porter and N. Rosenzweig (to be published).

results of the $N = 2$ (Wigner) case as well as with experiment, and are discussed in detail below.

Wigner's Distribution

Let us begin by discussing Wigner's idea, which can be expressed as follows: Suppose out of the ensemble of all possible spacing sequences we consider the sub-ensemble of spacing sequences for which $\lambda_2 - \lambda_1 \ll \bar{s}$, where $\bar{s}$ is the local mean level spacing. We can construct such a sub-ensemble by considering the sub-ensemble of spacing sequences for which $\lambda_2 = \lambda_1$, and applying to it a suitable real, random perturbation, V . In case we restrict ourselves to first-order perturbation theory, which should be applicable for sufficiently small spacings, the eigenvalues λ_2 and λ_1 are determined solely by the perturbation matrix elements V_{11} , $V_{12} = V_{21}$, V_{22} . Furthermore since we are interested in the level spacing only, no generality is lost by taking the perturbation matrix to be traceless, i.e., by taking $V_{11} = -V_{22}$,

whence the level separation is $s = 2 \sqrt{V_{11}^2 + V_{12}^2}$. It is noteworthy that under the foregoing assumptions s does not depend on the positions of the other levels of the sub-ensemble.

If we transform the joint probability density of V_{11} and V_{12} from these variables to s and an angular variable and integrate over the latter, the resulting distribution, that of s , will be proportional to s , since the element of area in plane polar coordinates is proportional to the radius vector. The remaining factor depends on the behavior of the joint probability of V_{11} and V_{12} near $V_{11} = V_{12} = 0$, since when s is very small so must V_{11} and V_{12} be. If this joint probability does not vanish at the origin, then for sufficiently small s the probability of a given spacing is proportional to the spacing itself.

Next the argument proceeds as follows: In general the probability of finding a spacing between s and $s + ds$ is the product of the probability of finding no

level λ_2 in the interval $(\lambda_1, \lambda_1 + s)$ and the probability of finding λ_2 in the interval $\lambda_1 + s \leq \lambda_2 \leq \lambda_1 + s + ds$. When s is small, the first factor is unity; hence, for small s the probability of finding λ_2 in ds around s is proportional to s itself. If we now make the ad hoc assumption that this proportionality holds for large s as well, the level spacing distribution can easily be calculated. For, let $P(s)$ be the probability of finding no level in the interval $(\lambda_1, \lambda_1 + s)$. Then:

$$P(s + ds) = P(s)(1 - \rho s ds) \quad (2)$$

since the second factor on the right hand side is the probability of not finding λ_2 in the interval $(\lambda_1 + s, \lambda_1 + s + ds)$. (ρ is a constant.) Solving the differential equation equivalent to (2) gives $P(s)$ as

$$P(s) = e^{-\frac{1}{2} \rho s^2} \quad (3)$$

from which it follows that spacing distribution $p(s) = -(dP/ds)$ is

$$p(s) = \rho s e^{-\frac{1}{2} \rho s^2} \quad (4)$$

Generalization of Wigner's Distribution

As noted in the introduction, the foregoing argument is deficient in the respect that it supplies no information about the correlation of successive level spacings. However, a generalization which includes this feature is possible without introducing any new assumptions beyond those mentioned in the previous section. For this purpose, let us consider the sub-ensemble of level sequences for which $\lambda_N - \lambda_1 \ll N\bar{s}$, i.e., for which there is nearly an N -fold degenerate level. Again let us study the spacings between these levels by first imagining them degenerate and then applying first-order perturbation theory to a real, random perturbation, V . As before the λ 's are determined in first order by diagonalizing the real, symmetric, (traceless) random $N \times N$ matrix

$V_{ik}(i, k = 1, \dots, N).$

Suppose we now transform the independent variables of the joint probability of the perturbation matrix elements from the independent matrix elements to the eigenvalues and the independent eigenvector components. It has been shown by Wilks¹¹ that the Jacobian of this transformation contains as its eigenvalue-

dependent part the product of differences $\prod_{i < j=2}^N (\lambda_j - \lambda_i).$ Again if the multi-

variate matrix element distribution does not vanish at the origin, then for small spacings the joint probability of the eigenvalues is given by the foregoing product of differences.

The next step in generalizing Wigner's distribution will be to define a stochastic process by which N levels λ_i can be marked on the real axis successively, and such that for very small spacings the joint probability of the N levels is just given by the foregoing product of differences. This process is defined by the statement that the conditional probability of finding the k^{th} level at $d\lambda_k$ around λ_k is proportional to the product of its distances to all previous levels, viz.:

$\prod_{j=1}^{k-1} (\lambda_k - \lambda_j).$ For, let $P(s | \lambda_{k-1}, \dots, \lambda_1)$ be the conditional probability that

$\lambda_k - \lambda_{k-1} > s$, given the positions of $\lambda_1, \dots, \lambda_{k-1}$. Then:

$$P(s + ds | \lambda_{k-1}, \dots, \lambda_1) = P(s | \lambda_{k-1}, \dots, \lambda_1) \left[1 - \rho_k \prod_{j=1}^{k-1} (\lambda_k - \lambda_j) \right] \quad (5)$$

The ρ_k are constants and are related by the requirement that the expected value of the last spacing be the same at every step. From (5) it follows that

11. S. S. Wilks, Mathematical Statistics, Princeton University Press, Princeton, New Jersey.

$$-\frac{\partial P(s|\lambda_{k-1}, \dots, \lambda_1)}{\partial s} = P(s|\lambda_{k-1}, \dots, \lambda_1) \rho_k \prod_{j=1}^{k-1} (\lambda_k - \lambda_j) \quad (6)$$

If $p(\lambda_k, \dots, \lambda_1)$ is the joint probability of $\lambda_k, \dots, \lambda_1$ it follows from (6) that

$$p(\lambda_k, \dots, \lambda_1) = P(s|\lambda_{k-1}, \dots, \lambda_1) p(\lambda_{k-1}, \dots, \lambda_1) \rho_k \prod_{j=1}^{k-1} (\lambda_k - \lambda_j) \quad (7)$$

since the left hand side of (6) is just the conditional probability of a spacing s given the positions of the first $(k - 1)$ levels. It can easily be shown from (7) that

$$p(\lambda_k, \dots, \lambda_1) = \prod_{i < j=2}^k (\lambda_j - \lambda_i) \cdot \prod_{j=2}^k \rho_j P(\lambda_j - \lambda_{j-1} | \lambda_{j-1}, \dots, \lambda_1) p(\lambda_1) \quad (8)$$

When all the level spacings are small the conditional probabilities in the second factor are close to unity, and the joint probability is proportional to the characteristic product of root differences, as desired.

When $2 = N \geq k > 1$, Eq. (6) becomes

$$-\frac{\partial P(s)}{\partial s} = \rho_2 s P(s) \quad (9)$$

the solution of which is just given by Eq. (3).

When $N = 3$, Eq. (6) yields Eq. (9) in the case $k = 2$, and

$$-\frac{\partial P(s/t)}{\partial s} = \rho_3 s(s + t) P(s/t) \quad (10)$$

where $s = \lambda_3 - \lambda_2$ and $t = \lambda_2 - \lambda_1$ when $k = 3$. Equation (10) applied successively to each pair of spacing intervals will then be taken to describe the level structure.

Since $P(s/t)$ is a distribution function, $P(0/t) = 1$; this boundary condition and Eq. (10) serve to determine $P(s/t)$ as

$$P(s/t) = e^{-\rho \left(\frac{1}{2} s^2 t + \frac{1}{3} s^3 \right)} \quad (11)$$

where the subscript 3 has been dropped. $p(s)$ can then be determined from the integral equation

$$p(s) = \int_0^{\infty} \left[- \frac{\partial P(s/t)}{\partial s} \right] p(t) dt \quad (12a)$$

$$= \rho s e^{-\frac{1}{3} \rho s^3} \int_0^{\infty} (s+t) e^{-\frac{1}{2} \rho s^2 t} p(t) dt \quad (12b)$$

Equation (12) can be solved readily by iteration. Plotted in Fig. 1 are the distributions of level spacings obtained from it as well as from Eq. (4), both normalized to unit mean. The difference is small which is gratifying, since the Wigner surmise ($N = 2$) is known⁵ to be in reasonably good agreement with experiment. This small difference indicates a relatively small influence of t on s , a fact which can be better appreciated by examination of Fig. 2 which is a plot of the conditional probability $p(s/t)$ given by the left hand side of Eq. (10). There it is shown that the curves labelled with reasonably probable values of t are quite similar. Furthermore the correlation coefficient C of adjacent level spacings, which is defined as

$$C = \frac{\overline{st} - \bar{s} \bar{t}}{\overline{s^2} - \bar{s}^2} = \frac{\overline{(s - \bar{s})(t - \bar{t})}}{\overline{(s - \bar{s})^2}} \quad (13)$$

UNCLASSIFIED
ORNL-LR-DWG 46171

Fig. 1.

UNCLASSIFIED
ORNL-LR-DWG 46172

Fig. 2.

and is unity when s and t are identical and zero when they are uncorrelated, is calculated to be -12%.

There is an unsatisfactory feature about the Eq. (10). The joint probability distribution $p(s,t)$ of two adjacent spacings, obtained from the defining relation

$$p(s,t) = - p(t) \frac{\partial P(s/t)}{\partial s}$$

turns out to be not symmetric in s and t . * This shows that the Eq. (10) is too simple a way to account for the correlation correctly. However, since the asymmetry is very slight (e.g., one obtains $\overline{st^2} = 2.50$ as compared with $\overline{s^2t} = 2.52$), our simple approach should be useful in estimating the effect of the correlation on Wigner's distribution.

Comparison with Experimental Data

As the spacing distribution given by Eq. (12) is quite similar to the Wigner distribution, very precise data are needed to distinguish between them. Unfortunately, such data are not available at present. We do not yet have large enough samples of spacings which surely belong to a unique distribution. We shall, therefore, restrict ourselves to a few remarks on the analysis of existing data.

The "cleanest" samples are obtained from the determination of slow-neutron resonances of even-even nuclei. Assuming they are formed by s-wave neutrons, all such levels have the same spin and parity and clearly belong to the same distribution. There still remains one uncertainty introduced by the possibility of missing some small levels which are very close to large levels. This effect

*We would like to thank Professor E. P. Wigner for drawing our attention to this point.

produces a slight bias in the samples against small spacings. In their analysis of 35 spacings obtained from Th^{232} , U^{234} , U^{236} , and U^{238} , Harvey and Hughes⁵ made an approximate correction for this effect. For each of the spacing distributions to be tested they estimated separately the number of levels which would remain undetected; for the Wigner distribution, this correction increased the number of spacings from 35 to 37. Their analysis showed a definite preference for the Wigner distribution over some other qualitatively different distributions. However, the statistical dispersion in these data is still too large to distinguish between the Wigner distribution and a similar distribution like the one given by Eq. (12). There are many more levels known in even-odd nuclei, but these levels may have at least two different spins and the uncertainty in the population ratio of the two different distributions makes it impossible to use these data for such a precise comparison as we have in mind.

Recently J. L. Rosen¹² at Columbia University made a very careful study of the slow-neutron resonances in U^{238} between 100 ev and 1000 ev, using the Nevis velocity selector. He determined the energies and widths of many new levels, as well as increasing the accuracy in the parameters of the previously known levels. Adding his results to the previous work below 100 ev, we now have 55 adjacent levels, or 54 spacings for U^{238} between 0 and 1000 ev. A comparison of the experimental reduced neutron width distribution with the theoretical Porter-Thomas distribution and of the experimental spacing distribution with the Wigner distribution shows rather good agreement and indicates that very few, if any, levels are missed. Hence, as also pointed out by Rosen, one may assume the effect of the missed levels to be insignificant here. However, there is now another uncertainty connected with the possible existence of p-wave levels. The levels which have very small reduced widths have an appreciable probability of

12. J. L. Rosen, "Neutron Resonances in U^{238} ," Thesis, Columbia University, CU-185, 1959.

being due to p-wave neutrons and hence of belonging to a different distribution. Consider the relation $\Gamma_n = 2 \gamma_n^2 s_\ell$ which relates the neutron width to γ_n^2 , the square of the surface overlap integral which is responsible for the process and s_ℓ , the penetrability of the nucleus for the ℓ -wave neutrons. The ratio of the penetrabilities for s- and p-wave neutrons is $(s_1/s_0) = k^2 R^2$ which gives about $2 \times 10^{-3} E_{\text{ev}}^{1/2}$ for U^{238} , with $R = 9 \times 10^{-13}$ cm. Assuming that the average of γ_n^2 has approximately the same value for both s- and p-levels and using the value found by Rosen for the average reduced neutron width, i.e., $\overline{\Gamma}_n^0 = 1.76 \times 10^{-3}$ ev, we see that $(\overline{\Gamma}_n^0)_p \sim 0.7 \times 10^{-8} E_{\text{ev}}$. Thus a level with a ratio $(\overline{\Gamma}_n^0/E)$ approaching this value will have an appreciable probability of being due to a p-wave neutron and rather conservatively one may use with Rosen the condition $(\overline{\Gamma}_n^0/E_{\text{ev}} < 10^{-7})$ as an indication of a possible p-level. There are 14 such levels in the U^{238} data. If one assumes that some (about 5) of these levels are really p-levels and discards them, the resulting width and spacing distributions agree slightly better with the corresponding Porter-Thomas and Wigner distributions. If, on the other hand, all these levels are left out the reduced widths do not follow the Porter-Thomas distribution anymore, since there are not enough small widths left. These statistical considerations make it very likely that some of the small levels are due to p-wave neutrons, but it is not possible to single them out uniquely by such considerations. Figures 3 and 4 compare the experimental histograms for these two extreme cases with the theoretical distributions. It is clear that one cannot decide which distribution is favored by the data.

It is tempting to try to evaluate from the data the correlation coefficient between adjacent spacings. However, here again one is faced with the same difficulties. Consider a two-dimensional distribution $p(s,t)$. Let us denote by c the population value of the correlation coefficient between s and t and by c' the calculated value of the correlation coefficient for the experimental sample

UNCLASSIFIED
2-01-059-607

Fig. 3. Comparison of the Theoretical Distributions with the Experimental Histogram when All 54 Spacings are Included ($\bar{s} = 18.3$). The Wigner distribution is $p_W(s)$; $p(s)$ is given by Eq. 12.

UNCLASSIFIED
2-01-059-608

Fig. 4. Comparison of the Theoretical Distributions with the Experimental Histogram when Only 42 Spacings are Included ($\bar{s} = 23.6$). The Wigner distribution is $p_W(s)$; $p(s)$ is given by Eq. 12.

under consideration. If the values of s and t are independent of each other, c vanishes; but the sampling distribution of the correlation coefficient will have a dispersion which is for large samples approximately equal to $(1/\sqrt{n})$, n being the number of pairs (s,t) in the sample.¹³ If $c \neq 0$, the value of this dispersion will change, but its order of magnitude will remain the same for small values of c . It may be used, therefore, to estimate the statistical fluctuations in the determination of c . For the 53 pairs of spacings of U^{238} the dispersion in c will be of the order of ± 0.15 . Thus, even if we were sure of the identity of all these 55 levels, we would not be able to choose between $c = 0$ and $c = -0.12$ on the basis of the U^{238} data alone. Actually, the uncertainty introduced by the possible p -levels makes the situation much worse. If we assume all the 55 levels to be s -levels we find for the correlation coefficient $c = 0.12 \pm 0.05$, while if we assume that all the small levels with $(\Gamma_n^0/E_{ev}) < 10^{-7}$ should be left out of the sample, and consider only the spacings between the remaining levels, we obtain $c = -0.40 \pm 0.05$; the indicated errors are approximate experimental errors coming from the uncertainties in the positions of the levels. As the truth must be somewhere between these extremes, one feels that a small negative value for c is quite probable; but strictly speaking, any value of c between a small positive value (~ 0.10) and a large negative value (~ -0.40) is consistent with our incomplete knowledge of U^{238} levels. Finally, we should note that there are about 40 more levels known in the neighboring even-even nuclei (Th^{232} , U^{234} , U^{236} , Pu^{240}), but because of all the mentioned uncertainties their addition does not increase the statistical accuracy appreciably. It appears, therefore, that the precise experimental determination of the correlation coefficient between adjacent spacings must wait for the observation of many more levels in even-even nuclei.

13. Cf. e.g., H. Cramer, "Mathematical Methods of Statistics," Chapt. 27, Princeton University Press, 1946.