THE RELATIVE COMPLEXITY OF VARIOUS CLASSIFICATION PROBLEMS AMONG COMPACT METRIC SPACES

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In this thesis, we discuss three main projects which are related to Polish groups and their actions on standard Borel spaces. In the first part, we show that the complexity of the classification problem of continua is Borel bireducible to a universal orbit equivalence relation induce by a Polish group on a standard Borel space. In the second part, we compare the relative complexity of various types of classification problems concerning subspaces of $[0,1]^n$ for all natural number n. We show that both of the homeomorphic relation of closed subsets of $[0,1]^n$ and the restricted homeomorphic relation of closed subsets of $[0,1]^n$ are Borel reducible to the homeomorphic relation of connected closed subsets of $[0,1]^{n+2}$; and the restricted homeomorphic relation of closed subsets of $[0,1]$ is Borel reducible to the restricted homeomorphic relation of connected closed subsets of $[0,1]^3$. In the last chapter, we give a topological characterization theorem for the class of locally compact two-sided invariant non-Archimedean Polish groups. Using this theorem, we show the non-existence of a universal group and the existence of a surjectively universal group in the class.

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CHAPTER 1

INTRODUCTION

Over the last two decades, descriptive set theory and its applications have been an active field of research, and there are a lot of development. The three main projects in this work are closely related to Polish groups and their actions on standard Borel spaces.

(1) The complexity of the classification problem of continua is Borel bireducible to a universal orbit equivalence relation.
(2) The Borel reducibility theory of the complexity of various classification problems of closed subsets of $[0,1]^n$.
(3) The non-existence of a universal locally compact TSI non-Archimedean Polish group and the existence of a surjectively universal one.

1.1. The Complexity of the Classification Problem of Continua

One of the main subjects of descriptive set theory is to study the complexity for the orbit equivalence relation induced by a Borel action of a Polish group on a standard Borel space.

Recall that the *Borel space* of a topological space (X,τ) is the σ-algebra $\sigma(\tau)$ generated by the open sets. A measurable space $(X,\mathcal{S})$ is a *standard Borel space* if it is isomorphic to $(Y,\sigma(\tau))$ for some Polish space (Y,τ).

Let G be a Polish group, X a standard Borel space, and $a : G\times X\to X$ an action of G on X. If a is a Borel function (i.e., the pre-images of Borel sets are Borel) then we say that X is a *Borel G-space*. And the *orbit equivalence relation*, denoted E_G^X, is given by

$$xE_G^X y \Longleftrightarrow \exists g\in G(a(g,x)=y).$$

Given two equivalence relations, we first need to define in what sense one is at most as complicated as the other. This is made precise by means of the concept of reducibility. For standard Borel spaces X,Y and equivalence relations E on X and F on Y, we say that

E is *Borel reducible* to F, denoted by $E \leq_B F$, if there is a Borel function $f : X \to Y$ such that, for any $x, y \in X$,

$$xEy \Longleftrightarrow f(x)Ff(y).$$

If $E \leq_B F$ and $F \leq_B E$, then we say that E is *Borel bireducible* with F and write $E \sim_B F$. The preorder $\leq_B$ imposes a hierarchy of complexity on equivalence relations.

By a theorem of Becker-Kechris (c.f. [2] Theorem 3.3.4), for any Polish group G, there is a universal equivalence relation in the class of all orbit equivalence relations induced by Borel actions of G. If follows from results of Mackey (c.f. [8] Theorem 3.5.3) and Uspenskij (c.f. [8] Theorem 2.5.2) that there is a universal equivalence relation E_G^X in the class of all orbit equivalence relations induced by Borel actions of all Polish group, i.e. for all Borel G'-space X', $E_{G'}^{X'} \leq_B E_G^X$. We simply refer to such an equivalence relation as a *universal orbit equivalence relation.*

Suppose we have a class of mathematical objects so that these objects can be viewed as forming a standard Borel space, one natural question would be determining when two objects are or are not isomorphic in an appropriate sense. The isomorphic relation becomes an equivalence relation on the space, so the complexity of the classification problem can be compared with other equivalence relations.

Very often, the complexity of classification problems can be shown to be Borel bireducible to an orbit equivalence relation induce by a Polish group action. For example, the isometric classification problem for all Polish metric spaces([4], [10]), the isometric classification problem of all separable Banach spaces([16]), the isometric problem of all separable (nuclear) C^*-algebras([19]) and the homeomorphism problem of all compact metric spaces([21]) are all Borel bireducible to some universal orbit equivalence relations, i.e. they are the most complex equivalence relations in the class of orbit equivalence relations induced by a Polish group on a standard Borel space.

In Chapter 3, we consider the complexity of the homeomorphic classification problem among continua and show that it is Borel bireducible to the most complex orbit equivalence relations induced by a Polish group action.

1.2. The Complexity of Classifying Various Closed Subsets of $[0,1]^n$

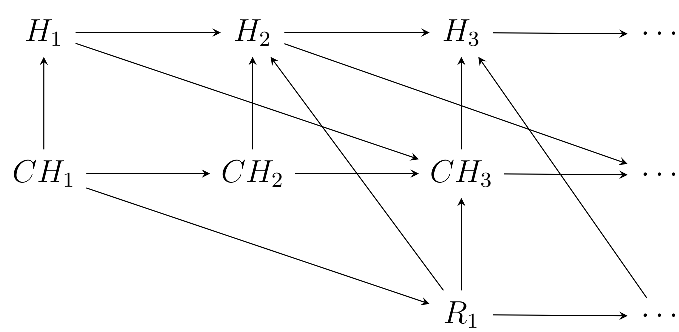
In Chapter 4, we will compare the complexity of various types of classification problems concerning subspaces of $[0,1]^n$ for $n \in \mathbb{N}$.

Let $\mathrm{Hom}([0,1]^n)$ denote the group of all homeomorphisms of the compact metric space $[0,1]^n$. For all $n \in \mathbb{N}$, define some equivalence relations:

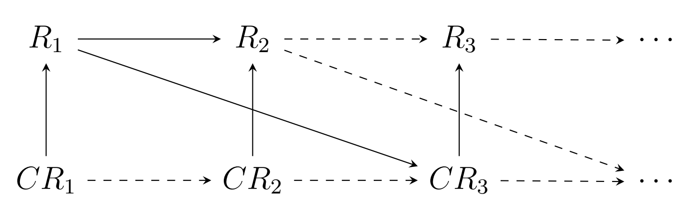
- H_n: homeomorphic relation of closed subsets of $[0,1]^n$;
- CH_n: homeomorphic relation of connected closed subsets of $[0,1]^n$;
- R_n: restricted homeomorphic relation of closed subsets of $[0,1]^n$;
- CR_n: restricted homeomorphic relation of connected closed subsets of $[0,1]^n$,

where two closed subsets (resp. connected closed subsets) $A, B \subseteq [0,1]^n$ are restricted homeomorphic equivalent, denoted $A \cong_{R_n} B$ (resp. $A \cong_{CR_n} B$), if there exists $f \in \mathrm{Hom}([0,1]^n)$ such that $f(A) = B$.

The goal of this chapter is to show the following Borel reducibility diagram. Note that some of the equivalence relations can be trivially reduced to some other equivalence relations. In particular, we show that $H_n \leq_B CH_{n+2}$ and $R_n \leq_B CH_{n+2}$, for all $n \in \mathbb{N}$.



Finally, we show the Borel reducibility between R_1 and CR_3. We will have the following diagram (dashed lines represent plausible results, but no proof provided).



1.3. Locally Compact TSI Non-Archimedean Polish Groups

A topological group is *Polish* if it has a Polish topology (i.e., separable and completely metrizable). A metric d on a Polish group G is *two-sided invariant* if for all $g_1, g_2, h, k \in G$,

$$d(hg_1k, hg_2k) = d(g_1, g_2).$$

If G admits a compatible two-sided invariant metric, we say that G is TSI. Abelian Polish groups and compact Polish groups are all TSI, but locally compact Polish groups need not be TSI, an example is $SL_2(\mathbb{R})$.

A Polish group is called *non-Archimedean* if it has a neighborhood base for the identity that consists of open subgroups. The group S_∞ of all permutations of $\mathbb{N}$ with the composition as the group operation and with the pointwise convergence topology is a non-Archimedean Polish group.

Recall that if $\mathcal{C}$ is a class of topological groups, then a *universal* element of $\mathcal{C}$ is a group $G \in \mathcal{C}$ such that any other group $H \in \mathcal{C}$ is topologically isomorphic to a closed subgroup of G. Similarly, a *surjectively universal element* of $\mathcal{C}$ is a group $G \in \mathcal{C}$ such that for any $H \in \mathcal{C}$, H is topologically isomorphic to a quotient group of G. It is of interest to ask whether there are universal or surjectively universal groups in this class.

In [2], Becker and Kechris showed that every non-Archimedean Polish group is topologically isomorphic to a closed subgroup of S_∞, thus S_∞ is a universal non-Archimedean Polish group; Gao([7]) showed that there is a surjectively universal group for all non-Archimedean Polish groups. But it is still unknown that whether there is a universal or surjectively universal group in the class of locally compact non-Archimedean Polish groups.

In Chapter 5, we consider the class of all locally compact TSI non-Archimedean Polish groups and give exact answers to these questions. In order to achieve this, we need to have a thorough understanding of these topological groups. In [11], Xuan and Gao gave a characterization of all TSI non-Archimedean Polish groups, which is a bigger class of topological groups containing the class of our interest. Inspired by that, we will show a characterization theorem of locally compact TSI non-Archimedean Polish groups.

In [15], the author showed that there are uncountably many non-isomorphic finitely generated simple groups, and each such a group falls into the class of our interest. Then by applying the characterization theorem of locally compact TSI non-Archimedean Polish groups, it turns out that there is no such a group in which uncountably many non-isomorphic countable simple groups can be embedded. So we show that the answer to the existence of a universal group in that class is negative.

At the end of the chapter, we give a very explicit construction and show the existence of a surjectively universal group in the class of our interest, by utilizing the characterization theorem again.

CHAPTER 2

PRELIMINARIES

The content of this chapter consists of a brief list of notations, some well-known definitions and facts from point-set topology, topological groups, and descriptive set theory. We assume the reader already has familiarity with the basics of sets, groups, and topological spaces.

2.1. Notations

Suppose X is a topological space, we write an element in $X \times \mathbb{I}$ as (x, λ) for some $x \in X$ and $\lambda \in \mathbb{I}$. If $A \subseteq X$ and $\lambda \in \mathbb{I}$, we sometimes write the set $\{(a, \lambda), a \in A\}$ simply as (A, λ) or $A \times \{\lambda\}$.

- $\cong_{(1,1)}$: Restricted homeomorphic relation
- $\overline{A}$: The closure of the set A
- E_G^X: The orbit equivalence relation induced by a Borel action of a Polish group G on a standard Borel space X
- $F(X)$: The space of closed subsets of X
- $\mathrm{Hom}(X, Y)$: The set of homeomorphisms from X to Y
- $\mathbb{I}$: The unit interval (or $[0, 1]$)
- id: The identity map
- $\mathrm{Int}\, A$: The interior points of the set A
- $\mathcal{K}(X)$: The space of compact subsets of X
- $\varprojlim G_i$: The inverse limit of the inverse system $\{G_i\}_i$
- $\mathcal{Q}$: The Hilbert cube (or $\mathbb{I}^{\mathbb{N}}$)
- S_∞: The group of the permutations of all natural numbers
- TSI: two-sided invariant

2.2. Definitions and Facts

A topological space X is *disconnected* if there are disjoint nonempty open sets H and K in X such that $X = H \cup K$. When no such disconnection exists, X is *connected*. A space X is *path connected* if for any two points x and y in X, there is a continuous function $f : \mathbb{I} \to X$ such that $f(0) = x$, $f(1) = y$. Such a function f (as well as its range $f(\mathbb{I})$, when confusion is not possible) is called a *path* from x to y.

Subspaces of connected spaces are not usually connected. In fact, the only subspace theorem available dealing with connectedness is just a useful way of rephrasing the definition so that it can be applied to a subspace without passing to the relative topology.

THEOREM 2.1 ([20]). *A subspace E of X is connected if and only if there are no nonempty disjoint sets H and K in X with $E = H \cup K$, such that*

$$H \cap \overline{K} = \overline{H} \cap K = \emptyset.$$

The next theorem says that the closure of a connected set is also connected.

LEMMA 2.2 ([20]). *If E is a connected subset of X and $E \subseteq A \subseteq \overline{E}$, then A is connected.*

Here is another useful theorem to show the connectedness of a set.

LEMMA 2.3 ([20]). *If $X = \bigcup X_\alpha$, where each X_α is connected and $\bigcap X_\alpha \neq \emptyset$, then X is connected.*

If $x \in X$, the largest connected subset C_x of X containing x is called the *component* of x. The components of X are closed sets, but they need not be open. The *path components* of a space X are the equivalence classes in X under the equivalence relation $x \sim y$ if there is a path joining x to y.

A space X is *Hausdorff* (or T_2) if two distinct points can be separated by disjoint open sets, i.e. whenever x and y are distinct points of X, there are disjoint open sets U and V in X with $x \in U$ and $y \in V$.

A space X is *compact* if each open cover of X has a finite subcover, i.e. for every arbitrary collection $\{U_\alpha\}_{\alpha \in A}$ of open subsets of X such that

$$X = \bigcup_{\alpha \in A} U_\alpha,$$

there is a finite subset B of A such that

$$X = \bigcup_{\alpha \in B} U_\alpha.$$

Every closed subset of a compact space is compact. Any products of compact spaces are compact. The continuous image of a compact space is compact.

A family $\mathcal{E}$ of subsets of X has the *finite intersection property* if the intersection of any finite subcollection from $\mathcal{E}$ is nonempty. For a compact space X, each family $\mathcal{E}$ of closed subsets of X with the finite intersection property has nonempty intersection.

THEOREM 2.4 ([20]). *The continuous image of a compact metric space in a Hausdorff space is metrizable.*

The next theorem provides a tool to simplify the process of showing a map is homeomorphism into showing the map is continuous bijective.

THEOREM 2.5 ([20]). *A one-one continuous map from a compact space X onto a Hausdorff space Y is a homeomorphism.*

Now we are ready to define some terms of the main topic of Chapter 3.

DEFINITION 2.6. A *continuum* is a compact, connected Hausdorff space.

Among the continua we find many familiar spaces. Some examples of continua are the unit interval $\mathbb{I}$, the circle $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$.

DEFINITION 2.7. Let X be a connected T_2-space. A *cut point* of X is a point $p \in X$ such that $X - \{p\}$ is not connected. If p is not a cut point of X, we call it a *non-cut point* of X.

The property of being a cut point, is preserved under homeomorphism; but continuous maps can destroy cut points. We will use extensively the cut (or non-cut) property in the proof.

LEMMA 2.8. *Suppose we have two homeomorphic continua, say* $X \xrightarrow{f} Y$*, and* $x, f(x)$ *are two cut points in their respective spaces, then* $f \restriction_{X-\{x\}}$ *maps each component in* $X - \{x\}$ *onto a component in* $Y - \{f(x)\}$.

PROOF. We can write $X - \{x\} = \bigcup C_i$ and $Y - \{f(x)\} = \bigcup D_i$, where C_i, D_i are components. Since f is a continuous map, then for each $i \in \mathbb{N}$, $f(C_i) \subseteq D_{j_i}$ for some $j_i \in \mathbb{N}$. Suppose there are two components C_{k_1}, C_{k_2} mapped into one single component D_k, then $f^{-1}(D_k)$ is disconnected, which is impossible. □

Let $(X, \mathcal{T})$ be a topological space. The class of *Borel sets* of X is the σ-algebra generated by the open sets of X. We denote it by $B(X, \mathcal{T})$. We call $(X, B(X))$ the *Borel space* of X.

DEFINITION 2.9. A measurable space $(X, \mathcal{S})$ is a *standard Borel space* if it is isomorphic to $(Y, B(Y))$ for some Polish space Y or equivalently, if there is a Polish topology $\mathcal{T}$ on X with $\mathcal{S} = B(\mathcal{T})$.

Given a topological space X we denote by $F(X)$ the set of closed subsets of X. We endow $F(X)$ with the σ-algebra generated by the sets

$$\{F \in F(X) : F \cap U \neq \emptyset\},$$

where U varies over open subsets of X. If X has a countable basis $\{U_n\}$, it is clearly enough to consider U in that basis. The space $F(X)$ with this σ-algebra is called the *Effros Borel space* of $F(X)$.

THEOREM 2.10 ([13]). *If* X *is Polish, the Effros Borel space of* $F(X)$ *is standard.*

Next we will explain how the space of continua can be viewed as a standard Borel space. For a Polish space X, let $\mathcal{K}(X) = \{A \subseteq X : A \text{ is compact}\}$ denote the hyperspace of compact subsets of X. $\mathcal{K}(X)$ is endowed with the *Vietoris topology*, which is the topology generated by sets of the form

$$\{K \in \mathcal{K}(X) : K \cap U \neq \emptyset\}$$

and those of the form

$$\{K \in \mathcal{K} : K \subseteq U\}$$

where $U \subseteq X$ is open. $\mathcal{K}(X)$ is a Polish space. If X is compact, then $\mathcal{K}(X)$ is also compact([13] 4.F).

Let $\mathcal{Q} = \mathbb{I}^{\mathbb{N}}$ denote the Hilbert cube. Since every separable metric space embeds into $\mathcal{Q}$, so in particular, every compact metric space can be embedded into a (compact) subspace of $\mathcal{Q}$. We view $\mathcal{K}(\mathcal{Q})$ as the class of compact metric spaces, so it is a standard Borel space. If we consider the subclass of connected compact metric spaces in $\mathcal{K}(\mathcal{Q})$, denoted $\mathcal{C}$. Then $\mathcal{C}$ is a closed subset of $\mathcal{K}(\mathcal{Q})$, since for any $K \in \mathcal{K}(\mathcal{Q})$,

$$K \in \mathcal{C} \iff \forall \text{ open } U, V(U \cap V = \emptyset \Rightarrow U \cap K = \emptyset \text{ or } V \cap K = \emptyset \text{ or } K \not\subseteq U \cup V).$$

Thus $\mathcal{C}$ is also a standard Borel space.

In Chapter 5, we will explore characterizations of a certain class of topological groups. A *topological group* is a group $(G, \cdot)$ together with a topology on G such that $(x, y) \mapsto x \cdot y^{-1}$ is continuous (from G^2 into G). A *Polish group* is a separable completely metrizable topological group. For example, all countable groups with the discrete topology are Polish groups, $(\mathbb{R}, +)$ with the usual metric is a Polish group.

There are some basic properties about Polish groups,

- If G is a Polish group and $H \leq_c G$ is a closed subgroup, then H is a Polish group with the subspace topology.
- If G is a Polish group and $H \trianglelefteq_c G$ is a closed normal subgroup, then the quotient group G/H is a Polish group with the quotient topology.
- If $\{G_n\}_{n \in \mathbb{N}}$ are Polish groups, then $\prod_n G_n$ is a Polish group with the product topology.

The group S_∞ of all permutations of $\mathbb{N}$ with the composition as the group operation and with the pointwise convergence topology is a Polish group. A compatible metric on S_∞

is

$$d(\mathbf{x},\mathbf{y}) = \begin{cases} 0 & \text{if } \mathbf{x} = \mathbf{y}, \\ 2^{-n} & \text{if } \mathbf{x} \neq \mathbf{y},\, n = \min\{k : \mathbf{x}(k) \neq \mathbf{y}(k)\}. \end{cases}$$

A basic open set of S_∞ has the form

$$[x_0, x_1, \ldots, x_n] := \{\mathbf{x} \in S_{\mathbb{N}} : \mathbf{x}(k) = x_k \text{ for all } k \leq n\}.$$

Moreover, S_∞ has an open neighborhood base at the identity consisting of open subgroups, say $\{[0, 1, \ldots, n]\}_{n\in\mathbb{N}}$.

THEOREM 2.11 (Becker, Kechris, [2]). *Let G be a Polish group. Then the following are equivalent:*

(1) G is isomorphic to a closed subgroup of S_∞.

(2) G admits a countable neighborhood base of 1_G consisting of open subgroups.

Isomorphisms in this work are all between topological groups, i.e., they are both group isomorphisms and homeomorphisms. Because of the characterization in the previous theorem, we can now define non-Archimedean Polish groups.

DEFINITION 2.12. A Polish group G is *non-Archimedean* if it has a basis at the identity that consists of open subgroups.

A metric d on a topological group G is *left-invariant* if $d(gh, gk) = d(h, h)$ for all $g, h, k \in G$. Birkhoff-Kakutani ([8], Theorem 2.1.1) showed that if G is a metrizable topological group, then G admits a compatible left-invariant metric. A metric d on a group G is *two-sided invariant* (TSI) if $d(g_1hg_2, g_1kg_2) = d(h, k)$ for all $g_1, g_2, h, k \in G$. Klee ([14]) showed a characterization of TSI groups.

THEOREM 2.13 ([14]). *A metrizable group G is TSI if and only if there is a countable open neighborhood base $\{V_n\}$ about 1_G such that $gV_ng^{-1} = V_n$ for all $n \in \mathbb{N}$ and $g \in G$.*

An *inverse system* of topological groups consists of a sequence of $\{G_i\}_{i\in\mathbb{N}}$ topological groups, and a collection of continuous homomorphisms $\pi_{j,i} : G_j \longrightarrow G_i$ for all $i \leq j$, so that

(1) $\pi_{i,i} : G_i \longrightarrow G_i$ is the identity, and

(2) whenever $i \leq j \leq k$, we have $\pi_{k,i} = \pi_{j,i} \circ \pi_{k,j}$.

Given an inverse system $\{G_i\}_{i\in\mathbb{N}}$ of topological groups, its direct product is the topological group $\prod G_i$, endowed with the product topology, and the group operation is defined coordinatewise.

The *inverse limit* $\varprojlim G_i$ is a subgroup of $\prod G_i$ defined as

$$\left\{(g_i)_{i\in\mathbb{N}} \in \prod G_i : \pi_{j,i}(g_i) = g_j \text{ for all } i \leq j\right\}.$$

If each G_i is a Hausdorff space for all $i \in \mathbb{N}$, then $\varprojlim G_i$ is a closed subspace of $\prod G_i$.

Lemma 2.14. *If $\{G_i\}_{i\in\mathbb{N}}$ is an inverse system of Hausdorff topological groups, then $\varprojlim G_i$ is a closed subspace of $\prod G_i$.*

Proof. Let $(g_i) \in \prod G_i - \varprojlim G_i$. Then there exists m, n with $n \geq m$ and $\pi_{n,m}(g_n) \neq g_m$. Choose open disjoint neighborhoods U and V or $\pi_{n,m}(g_n)$ and g_m in G_m, respectively. Let U' be an open neighborhood of g_n in G_n, such that $\pi_{m,n}(U') \subseteq U$. Consider the basic open subset $W = \prod_{i\in\mathbb{N}} V_i$ of $\prod G_i$ where $V_n = U'$, $V_m = V$ and $V_i = G_i$ for $i \neq m, n$. Then W is a open neighborhood of (g_i), disjoint from $\varprojlim G_i$. This shows that $\varprojlim G_i$ is closed. □

Actually, we can define an inverse system in a more general context. Let $(I, \leq)$ be a *directed partially ordered set*, that is, I is a set with a binary relation $\leq$ satisfying the following conditions:

(1) $i \leq i$, for $i \in I$;

(2) $i \leq j$ and $j \leq k$ imply $i \leq k$, for $i, j, k \in I$;

(3) $i \leq j$ and $j \leq i$ imply $i = j$, for $i, j \in I$; and

(4) if $i, j \in I$, there exists some $k \in I$ such that $i, j \leq k$.

An inverse system of topological groups over I consists of an indexed collection $\{G_i\}_{i\in I}$ of topological groups, and a collection of continuous homomorphisms $\pi_{j,i} : G_j \to G_i$ for all $i \leq j$, so that

(1) $\pi_{i,i} : G_i \longrightarrow G_i$ is the identity, and

(2) whenever $i \leq j \leq k$, we have $\pi_{k,i} = \pi_{j,i} \circ \pi_{k,j}$.

Let I be a directed partial order and $J \subseteq I$. We say that J is *cofinal* in I if for any $i \in I$ there is $j \in J$ such that $i \leq j$. If J is cofinal in I then $(J, \leq |J)$ is also a directed partial order. If $\{G_i\}_{i \in I}$ is an inverse system of topological groups and $J \subseteq I$ is cofinal, then $\{G_j\}_{j \in J}$ is also an inverse system, moreover, $\varprojlim_{i \in I} G_i$ and $\varprojlim_{j \in J} G_j$ are isomorphic.

CHAPTER 3

THE COMPLEXITY OF THE CLASSIFICATION PROBLEM OF CONTINUA

3.1. The $\tilde{I}(X, A)$ Construction

Suppose X is a (path) connected perfect compact metric space and $A \subseteq X$ is a closed subset. We will construct a (path) connected compact metric space $\tilde{I}(X, A)$.

Since our desired set $\tilde{I}(X, A)$ is built on from the set $I(X, A)$, which is defined in [21], so we need to first review the construction of $I(X, A)$. Here we have to point out that neither the I-construction nor the $\tilde{I}$-construction depends on the (path) connectivity or the perfect property of X. As long as X is a compact metric space and $A \subseteq X$ is a closed subset of X, the $I(X, A)$ and $\tilde{I}(X, A)$ can be constructed in the following ways.

Pick a countable dense subset of A and enumerate it in such a way that each element occurs infinitely many times, denote the enumeration by $\{a_1, a_2, \dots\}$. Then the set $I(X, A) \subseteq X \times \mathbb{I}$ is defined by

$$I(X, A) := X \times \{0\} \cup \{\mathbf{a}_1, \mathbf{a}_2, \dots\},$$

where $\mathbf{a}_i = (a_i, \frac{1}{i+1})$ for all $i \in \mathbb{N}$.

Note that $I(X, A)$ consists of two parts: one part is the copy of the space X as $X \times \{0\}$; the other part is the countable set of isolated points $D_A := \{\mathbf{a}_1, \mathbf{a}_2, \dots\}$ in $X \times (0, 1]$ (Figure 3.1).

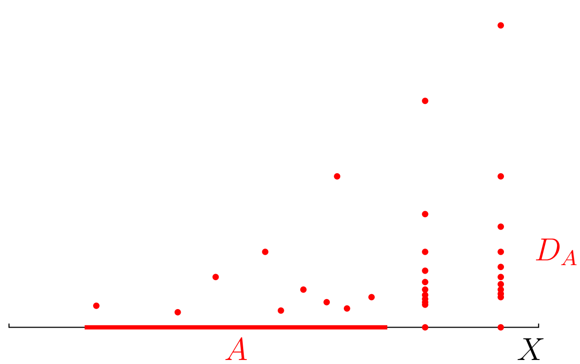


FIGURE 3.1. $I(X, A)$

Generally speaking, the construction described above codes a pair of sets (X, A) into one set $I(X, A)$ and in the meantime we can still keep track of the original pair. This information preserving construction is made precisely by the following theorem.

THEOREM 3.1 ([21]). *Let X, Y be compact metric spaces, with A, B closed subsets so that A (resp. B) contains all isolated points of X (resp. Y). Let $I(X, A)$ and $I(Y, B)$ be constructed as above. Then every homeomorphism $g : X \to Y$ with $g(A) = B$ extends to a homeomorphism $I(X, A) \to I(Y, B)$. Conversely, if $f : I(X, A) \to I(Y, B)$ is a homeomorphism, then $f(X) = Y$ and $f(A) = B$.*

The author mentioned in [21] that D_A is unique up to homeomorphism on the choices of countable dense subsets of A and the ways of enumerations of those sets. We will show next that D_A is unique up to homeomorphism in a broader sense, as long as D_A satisfies two conditions stated in the following theorem.

PROPOSITION 3.2. *Let (X, ρ) be a perfect compact metric space, $A \subseteq X$ be a closed subset, and $I(X, A) = X \times \{0\} \cup D_A$ be constructed as above. Suppose $D'_A \subseteq X \times \mathbb{I}$ is a countable set of isolated points that satisfies*

(C1) $D'_A \subseteq X \times (0, 1]$,

(C2) $\overline{D'_A} - D'_A = A \times \{0\}$,

then $X \times \{0\} \cup D'_A \cong I(X, A)$.

PROOF. Clearly, the set D_A specifically constructed above satisfies both (C1), (C2). Instead of proving the statement directly, we will show a more general argument.

Suppose D_A, D'_A are arbitrary countable sets of isolated points satisfying (C1) and (C2). We will find a homeomorphism $f : X \times \{0\} \cup D_A \to X \times \{0\} \cup D'_A$ (Figure 3.2) such that $f \restriction_{X \times \{0\}} = \text{id}$. Since the spaces under consideration are compact metric spaces, it is enough to find a continuous bijection f by Theorem 2.5.

Enumerate D_A, D'_A

$$D_A = \{\mathbf{d}_1, \mathbf{d}_2, \ldots\}$$

$$D'_A = \{\mathbf{d}'_1, \mathbf{d}'_2, \ldots\}$$

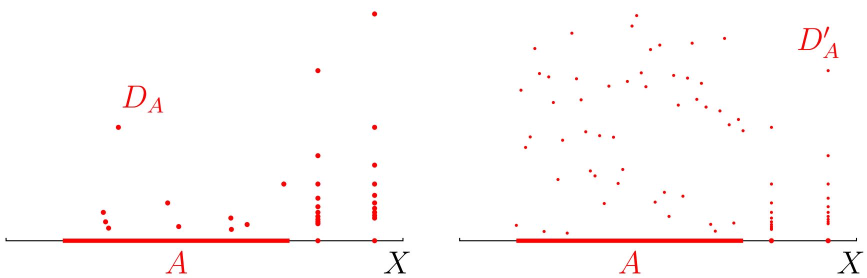


FIGURE 3.2. $X \times \{0\} \cup D_A$ and $X \times \{0\} \cup D'_A$

in such a way that the vertical distance of $\mathbf{d}$ to $X \times \{0\}$ is decreasing, i.e. $\pi_2(\mathbf{d}_{i+1}) \leq \pi_2(\mathbf{d}_i)$ for $i \in \mathbb{N}$, where π_2 is the projection map $\pi_2 : X \times \mathbb{I} \to \mathbb{I}$. Notice that for all $n \in \mathbb{N}$, there are only finitely many $\mathbf{d} \in D_A$ and $\mathbf{d}' \in D'_A$ such that $\mathbf{d}, \mathbf{d}' \in X \times [\frac{1}{n+1}, \frac{1}{n})$. Otherwise, suppose there are infinitely many $\mathbf{d} \in D_A$ or $\mathbf{d}' \in D'_A$ in $X \times [\frac{1}{n+1}, \frac{1}{n})$ for some n, then there will be a limit point in D_A or D'_A, respectively, since both $X \times \{0\} \cup D_A, X \times \{0\} \cup D'_A$ are compact spaces. But this contradicts with that D_A, D'_A consist of isolated points.

From (C2), we have

$$A \subseteq \overline{\pi_1(D_A)} \text{ and } A \subseteq \overline{\pi_1(D'_A)}, \tag{1}$$

where π_1 is the projection map $\pi_1 : X \times \mathbb{I} \to X$. Suppose $\{\mathbf{d}_{n_1}, \mathbf{d}_{n_2}, \ldots, \mathbf{d}_{n_k}\} \subseteq D_A$ and $\{\mathbf{d}'_{m_1}, \mathbf{d}'_{m_2}, \ldots, \mathbf{d}'_{m_k}\} \subseteq D'_A$, we have

$$A \subseteq \overline{\pi_1(D_A - \{\mathbf{d}_{n_1}, \mathbf{d}_{n_2}, \ldots, \mathbf{d}_{n_k}\})} \text{ and } A \subseteq \overline{\pi_1(D_A - \{\mathbf{d}'_{m_1}, \mathbf{d}'_{m_2}, \ldots, \mathbf{d}'_{m_k}\})}. \tag{2}$$

It is possible that for some $\mathbf{d} \in D_A$ or $\mathbf{d}' \in D'_A$, $\pi_1(\mathbf{d}) \notin A$ or $\pi_1(\mathbf{d}') \notin A$. Next we will define some points a, a' in A that are the closest to $\pi_1(\mathbf{d}), \pi_1(\mathbf{d}')$ respectively.

CLAIM 3.3. *For all $\mathbf{d}_i \in D_A$, there exists $a_i \in A$ such that the distance between $\pi_1(\mathbf{d}_i)$ and A achieves its minimum at a_i, i.e.*

$$\rho(\pi_1(\mathbf{d}_i), a_i) = \rho(\pi_1(\mathbf{d}_i), A),$$

where ρ is a compatible metric on X. Similarly, for all $\mathbf{d}'_i \in D'_A$, there exists $a'_i \in A$ with $\rho(\pi_1(\mathbf{d}'_i), a'_i) = \rho(\pi_1(\mathbf{d}'_i), A)$.

Proof of Claim: Since X is a compact space, so any family of closed subsets of X with the finite intersection property has nonempty intersection. Fix an arbitrary $\mathbf{d} \in D_A$ and denote $\rho(\pi_1(\mathbf{d}), A)$ by r. For all $n \in \mathbb{N}$, the closed ball $B_n := \overline{B_\rho(\pi_1(\mathbf{d}), r + 1/n)}$ intersects with A. Now the family of closed subsets $\{B_n\}_n \cup A$ has the finite intersection property, so $\bigcap_n B_n \cap A \neq \emptyset$. Since $\bigcap_n B_n = \overline{B_\rho(\pi_1(\mathbf{d}), r)}$, so there exists $a \in A$ with $\rho(\pi_1(\mathbf{d}), a) = r$. $\square$

Fix the corresponding sequences $\{a_i\}_i, \{a'_i\}_i \subseteq A$ for $\{\mathbf{d}_i\}_i$ and $\{\mathbf{d}'_i\}_i$, respectively. We will show that $\{a_i\}_i, \{a'_i\}_i$ and $\{\pi_1(\mathbf{d}_i)\}_i, \{\pi_1(\mathbf{d}'_i)\}_i$ behave closely.

Claim 3.4. *If $\{\mathbf{d}_{n_k}\}_k \subseteq D_A$ converges to some $(a_0, 0) \in X \times \mathbb{I}$, then $\{a_{n_k}\}_k$ converges to a_0. Similarly, if $\mathbf{d}'_{n_k} \xrightarrow{k} (a'_0, 0)$, then $a'_{n_k} \xrightarrow{k} a'_0$.*

Proof of Claim: We show $\{a_{n_k}\}_k$ is Cauchy. By the way we defined a_{n_k}, we have

$$\rho(\pi_1(\mathbf{d}_{n_k}), a_{n_k}) \leq \rho(\pi_1(\mathbf{d}_{n_k}), a_0).$$

Then by triangle inequality,

$$\begin{aligned} \rho(a_{n_k}, a_{n_{k'}}) &\leq \rho(\pi_1(\mathbf{d}_{n_k}), a_{n_k}) + \rho(\pi_1(\mathbf{d}_{n_k}), \pi_1(\mathbf{d}_{n_{k'}})) + \rho(\pi_1(\mathbf{d}_{n_{k'}}), a_{n_{k'}}) \\ &\leq \rho(\pi_1(\mathbf{d}_{n_k}), a_0) + \rho(\pi_1(\mathbf{d}_{n_k}), \pi_1(\mathbf{d}_{n_{k'}})) + \rho(\pi_1(\mathbf{d}_{n_{k'}}), a_0). \end{aligned}$$

Since the sequence $\{\pi_1(\mathbf{d}_{n_k})\}_k$ is Cauchy and converging to a_0 by assumption, so $\{a_{n_k}\}_k$ is also Cauchy. Hence $a_{n_k} \xrightarrow{k} a_0$, by the fact that $\pi_1(\mathbf{d}_{n_k}) \xrightarrow{k} a_0$ again. $\square$

We are now ready to define the map f. The idea is that f acts as the identity function on the set $X \times \{0\}$ and f gives a one-one correspondence between points in D_A and D'_A. The goal of the correspondence $\mathbf{d} \xrightarrow{f} \mathbf{d}'$ is that as $\mathbf{d} \in D_A$ approaches some point in $A \times \{0\}$, we want to have $\mathbf{d}' \in D'_A$ approach $\mathbf{d}$ and thus approach to the same point in $A \times \{0\}$ as well.

Stage 1:

Suppose D_A or D'_A has some points in $X \times [\frac{1}{2}, 1)$, otherwise, we move on to the next stage and set $m'_1 = 0$ and $i_{m'_1} = 0$. List all $\mathbf{d}_i \in D_A$ such that $\pi_2(\mathbf{d}_i) \in [\frac{1}{2}, 1)$, say $\mathbf{d}_1, \ldots, \mathbf{d}_{n_1}$, and list all $\mathbf{d}'_j \in D'_A$ so that $\pi_2(\mathbf{d}'_j) \in [\frac{1}{2}, 1)$, say $\mathbf{d}'_1, \ldots, \mathbf{d}'_{m_1}$. Because of (1), we know that $B_\rho(a_1, 1) \cap \pi_1(D'_A) \neq \emptyset$. So for $\mathbf{d}_1$, take the smallest j_1 so that $\pi_1(\mathbf{d}'_{j_1}) \in B_\rho(a_1, 1) \cap \pi_1(D'_A)$. Suppose we have defined $\mathbf{d}_{i-1} \xrightarrow{f} \mathbf{d}'_{j_{i-1}}$. By (2), we know that $B_\rho(a_i, 1) \cap \pi_1(D'_A - \{\mathbf{d}'_1, \ldots, \mathbf{d}'_{j_{i-1}}\}) \neq \emptyset$. So for $\mathbf{d}_i$, let j_i be the smallest number such that $\mathbf{d}'_{j_i}$ has not been used and $\pi_1(\mathbf{d}'_{j_i}) \in B_\rho(a_i, 1) \cap \pi_1(D'_A)$.

Let $m'_1 := \max\{j_1, \ldots, j_{n_1}, m_1\}$ and for each $\mathbf{d}'_j, j \leq m'_1$ that has not been used, we will find $\mathbf{d}_{i_j}$ like in the previous paragraph. This completes the first stage, and let $i_{m'_1}$ record the largest index of $\mathbf{d}_i \in D_A$ that has been used.

Stage 2

Suppose D_A or D'_A has some points in $X \times [\frac{1}{3}, \frac{1}{2})$ that have not been used, otherwise, we move on to the next stage and set $m'_2 = 0$ and $i_{m'_2} = 0$. We list all $\mathbf{d}_i \in D_A$ such that $\pi_2(\mathbf{d}_i) \in [\frac{1}{3}, \frac{1}{2})$, say $\mathbf{d}_{n_1+1}, \ldots, \mathbf{d}_{n_2}$, and list all the $\mathbf{d}'_j \in D'_A$ such that $\pi_2(\mathbf{d}'_j) \in [\frac{1}{3}, \frac{1}{2})$, say $\mathbf{d}'_{m_1+1} \ldots, \mathbf{d}'_{m_2}$. For each such $\mathbf{d}_i \in D_A$ that has not occurred, let j_i be the smallest number so that $\mathbf{d}'_{j_i}$ has not been used and $\pi_1(\mathbf{d}'_{j_i}) \in B_\rho(a_i, \frac{1}{2}) \cap \pi_1(D'_A)$.

Let m'_2 be the maximum of all the indexes j_i defined so far and m_2. We will assign each $\mathbf{d}'_j, j \leq m'_2$ that has not been assigned some $\mathbf{d}_{i_j}$ like before. This completes the second stage and let $i_{m'_2}$ record the largest index of $\mathbf{d}_i \in D_A$ that has been used.

$\vdots$

We can continue this process. It is clear that the correspondence between points in D_A and D'_A is one-one. To see that the correspondence defined this way is indeed a bijection, note that as $n \to \infty$, the set $X \times [\frac{1}{n+1}, \frac{1}{n})$ will eventually contain every point of D_A and D'_A. For all **Stage n**, we have either assigned $f(\mathbf{d})$ or $f^{-1}(\mathbf{d}')$, for $\mathbf{d} \in D_A \cap X \times [\frac{1}{n+1}, \frac{1}{n})$ and $\mathbf{d}' \in D'_A \cap X \times [\frac{1}{n+1}, \frac{1}{n})$, respectively.

The last thing we want to show is the continuity. f is continuous on D_A, since both D_A and D'_A consist of isolated points and f is a one-one correspondence between them. f is also continuous on $(X - A) \times \{0\}$, as $f \restriction_{(X-A)\times\{0\}} = \text{id}$ and $(X - A) \times \{0\}$ is open in $I(X, A)$. So we just need to check f is continuous at $(a, 0)$, for all $a \in A$.

Since $I(X, A)$ is a metric space with a compatible metric r:

$$r[(x_1, a_1), (x_2, a_2)] = \rho(x_1, x_2) + |a_1 - a_2|,$$

where ρ is a compatible metric on X, so we really need to check that for all $\{\mathbf{a}_n\}_n \subseteq I(X, A)$ with $\mathbf{a}_n \xrightarrow{n} (a, 0)$, $f(\mathbf{a}_n) \xrightarrow{n} (a, 0)$. Suppose there exists N_0 with $\mathbf{a}_n \notin D_A$ for all $n > N_0$, then $f(\mathbf{a}_n) = \mathbf{a}_n$ for all $n > N_0$, thus $f(\mathbf{a}_n) \xrightarrow{n} (a, 0)$. So without loss of generality, we can assume that there are infinitely many $n \in \mathbb{N}$ such that $\mathbf{a}_n \in D_A$, say $\{\mathbf{a}_{n_k}\}_k \subseteq D_A$. It is enough to show that $f(\mathbf{a}_{n_k}) \xrightarrow{k} (a, 0)$. Equivalently, we need to show that for all $\{\mathbf{d}_{i_k}\}_k \subseteq D_A$ with $\mathbf{d}_{i_k} \xrightarrow{k} (a, 0)$, $f(\mathbf{d}_{i_k}) \xrightarrow{k} (a, 0)$.

By the way we enumerate the set D_A, we have $i_k \xrightarrow{k} \infty$. Then,

$$
\begin{aligned}
r[f(\mathbf{d}_{i_k}), (a, 0)] &= \rho(\pi_1(f(\mathbf{d}_{i_k})), a) + \pi_2(f(\mathbf{d}_{i_k})) \\
&\leq \rho[\pi_1(\mathbf{d}_{i_k}), \pi_1(f(\mathbf{d}_{i_k}))] + \rho[\pi_1(\mathbf{d}_{i_k}), a] + \pi_2(f(\mathbf{d}_{i_k})). \qquad (3)
\end{aligned}
$$

Recall the two sequences of numbers associated with each stage of construction, $\{m'_1, m'_2, \ldots\}$ and $\{i_{m'_1}, i_{m'_2}, \ldots\}$. By (C2), there exists a strictly increasing subsequence $\{m'_{k_1}, m'_{k_2}, \ldots\}$ with $m'_{k_n} \neq 0$ for all $n \in \mathbb{N}$, and the corresponding non-decreasing subsequence $\{i_{m'_{k_1}}, i_{m'_{k_2}}, \ldots\}$ with $i_{m'_{k_n}} \neq 0$ for all $n \in \mathbb{N}$. We claim the following:

Claim 3.5. *For all $i > i_{m'_{k_n}}$, $\pi_2(f(\mathbf{d}_i)) < \frac{1}{k_n}$, and $\rho[\pi_1(\mathbf{d}_i), \pi_1(f(\mathbf{d}_i))] < \frac{1}{k_n}$.*

Proof of Claim: Note that at each stage k_n, we have assigned all $\mathbf{d}'_j \in D'_A$ such that $\pi_2(\mathbf{d}'_j) \geq \frac{1}{k_n}$ with some $\mathbf{d}_{i_j} \in D_A$ and $i_j \leq i_{m'_{k_n}}$. So $\pi_2(f(\mathbf{d}_i)) < \frac{1}{k_n}$ for all $i > i_{m'_{k_n}}$.

On the other hand, for all $i > i_{m'_{k_n}}$, $\mathbf{d}_i \in D_A$ must have not been used in the stage k_n or earlier stages. Thus, $\rho[\pi_1(\mathbf{d}_i), \pi_1(f(\mathbf{d}_i))] < \frac{1}{k_n}$. □

Now combine the inequality (3) with Claim 3.4, 3.5, we have proved that $f(\mathbf{d}_{i_k}) \xrightarrow{k} (a, 0)$. □

Theorem 3.1 and Proposition 3.2 together will immediately imply the following result, we will only state the theorem without providing a proof.

Corollary 3.6. *Let X, Y be perfect compact metric spaces and $A \subseteq X$, $B \subseteq Y$ be closed subsets. Suppose $D_A \subseteq X \times \mathbb{I}$ and $D_B \subseteq Y \times \mathbb{I}$ are countable sets of isolated points that satisfy:*

(C1) $D_A \subseteq X \times (0,1]$, $D_B \subseteq Y \times (0,1]$;

(C2) $\overline{D_A} - D_A = A \times \{0\}$, $\overline{D_B} - D_B = B \times \{0\}$.

Then $X \times \{0\} \cup D_A \cong Y \times \{0\} \cup D_B$ *if and only if there exists a homeomorphism* $f : X \to Y$ *such that* $f(A) = B$ *(Figure 3.3).*

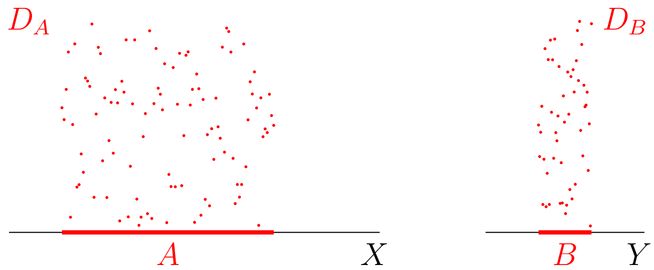


FIGURE 3.3. $I(X, A)$ and $I(Y, B)$

REMARK 3.7. In Corollary 3.6 and Proposition 3.2, we can make weaker assumptions about the pairs $(X, A), (Y, B)$, and the statements are still true. Similar to the assumptions made in Theorem 3.1, we can assume X, Y are compact metric spaces, with A, B closed subsets so that A (resp. B) contains all isolated points of X (resp. Y).

Continue our construction of $\tilde{I}(X, A)$. Let $X_A := \overline{D_A}$ and embed $I(X, A)$ into $I(X, A) \times \{0\} \cup X_A \times \mathbb{I}$ as $I(X, A) \times \{0\}$ (Figure 3.4).

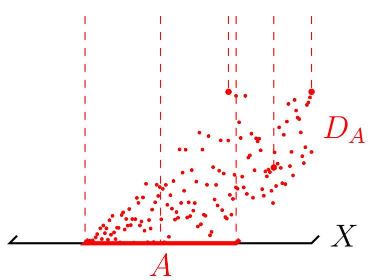


FIGURE 3.4. $I(X, A) \times \{0\} \cup (X_A \times \mathbb{I})$

Finally, we put all points $X_A \times \{1\}$ into one equivalent class, denoted $\mathbf{a}^*$, and all the other points into distinct classes of their own, call the resulting quotient space $\tilde{I}(X, A)$ (Figure 3.5).

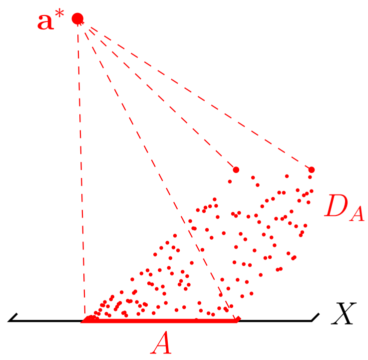


FIGURE 3.5. $\tilde{I}(X, A)$

Since the continuous image of a compact space is still compact, so $\tilde{I}(X, A)$ is clearly a compact space. By Theorem 2.4, $\tilde{I}(X, A)$ is also metrizable. Next, we will define a compatible metric τ on $\tilde{I}(X, A)$.

LEMMA 3.8. *Suppose ρ is a compatible metric for $I(X, A)$ with $\rho < 1$, then $\tilde{I}(X, A)$ has a compatible metric*

$$\tau[(\mathbf{x}, r), (\mathbf{y}, s)] = 2|r - s| + (1 - \max\{r, s\})\rho(\mathbf{x}, \mathbf{y}),$$

for $(\mathbf{x}, r), (\mathbf{y}, s) \in \tilde{I}(X, A)$.

PROOF. First we check τ is a metric. Actually, we just need to check the triangle inequality. Given $(\mathbf{x}, r), (\mathbf{y}, s)$, and $(\mathbf{z}, t)$ in $\tilde{I}(X, A)$, we look at the following cases:

- $s < r = t$:

$$\tau[(\mathbf{x}, r), (\mathbf{y}, s)] + \tau[(\mathbf{y}, s), (\mathbf{z}, t)] \geq 2|r - t| + (1 - r)\rho(\mathbf{x}, \mathbf{z})$$

- $r = t < s$:

$$\tau[(\mathbf{x}, r), (\mathbf{y}, s)] + \tau[(\mathbf{y}, s), (\mathbf{z}, t)] \geq 4|s - r| + (1 - s)\rho(\mathbf{x}, \mathbf{z})$$

$$= 4|s - r| + (1 - r)\rho(\mathbf{x}, \mathbf{z}) - (s - r)\rho(\mathbf{x}, \mathbf{z}) > (1 - r)\rho(\mathbf{x}, \mathbf{z})$$

- $s < r < t$:

$$\tau[(\mathbf{x}, r), (\mathbf{y}, s)] + \tau[(\mathbf{y}, s), (\mathbf{z}, t)] \geq 2|r - t| + (1 - t)\rho(\mathbf{x}, \mathbf{z})$$

- $r < s < t$:

$$\tau[(\mathbf{x}, r), (\mathbf{y}, s)] + \tau[(\mathbf{y}, s), (\mathbf{z}, t)] \geq 2|r - t| + (1 - t)\rho(\mathbf{x}, \mathbf{z})$$

- $r < t < s$:

$$\tau[(\mathbf{x}, r), (\mathbf{y}, s)] + \tau[(\mathbf{y}, s), (\mathbf{z}, t)] \geq 2|s - r| + 2|s - t| + (1 - s)\rho(\mathbf{x}, \mathbf{z})$$

$$\geq 2|s - r| + (1 - t)\rho(\mathbf{x}, \mathbf{z}) \geq 2|t - r| + (1 - t)\rho(\mathbf{x}, \mathbf{z})$$

Next, we check τ is compatible with the quotient topology. There are two types of points in $\tilde{I}(X, A)$, say $\mathbf{a}^*$ as one type and all other points as the other type.

An open neighborhood of $\mathbf{a}^*$ in the quotient topology has the form $X_A \times (1 - \epsilon, 1]$ for some $\epsilon > 0$, and it is equal to the open ball $B_\tau(\mathbf{a}^*, \epsilon)$.

Fix some $\mathbf{x}' := (\mathbf{x}, r) \in \tilde{I}(X, A)$, where $r < 1$. For one direction, consider an open ball $B_\tau(\mathbf{x}', \epsilon)$ for some $\epsilon > 0$. We can find an open set in the quotient topology, say $B_\rho(\mathbf{x}, \epsilon/4) \times (r - \epsilon/4, r + \epsilon/4)$, such that it is contained in the open ball. On the other hand, suppose there is an open neighborhood $V \subseteq \tilde{I}(X, A)$ of $\mathbf{x}'$ in the quotient topology, we will find an open ball at $\mathbf{x}'$ with respect to the metric τ contained in V.

Without loss of generality, we can assume V does not contain $\mathbf{a}^*$. Then the quotient map is 1-1 on the pre-image of V in the product space $X \times \{0\} \cup X_A \times \mathbb{I}$. Hence there exists $\epsilon > 0$ such that $B_\rho(\mathbf{x}, \epsilon) \times (r - \epsilon, r + \epsilon) \subseteq V$ in the quotient topology.

Finally, let $\epsilon' > 0$ such that $\epsilon' < \epsilon$ and $\epsilon' < \frac{1-r}{\frac{1}{2} + \frac{1}{\epsilon}}$, we can check that $B_\tau(\mathbf{x}', \epsilon') \subseteq B_\rho(\mathbf{x}, \epsilon) \times (r - \epsilon, r + \epsilon)$. For all $(\mathbf{y}, s) \in B_\tau(\mathbf{x}', \epsilon')$,

$$2|r - s| \leq \tau[(\mathbf{x}, r), (\mathbf{y}, s)] < \epsilon' < \epsilon, \tag{4}$$

so $|r - s| < \epsilon/2$. Moreover,

$$(1 - r - \epsilon'/2)\rho(\mathbf{x}, \mathbf{y}) \leq (1 - \max\{r, s\})\rho(\mathbf{x}, \mathbf{y}) \leq \tau[(\mathbf{x}, r), (\mathbf{y}, s)] < \epsilon',$$

where the leftmost inequality is because of (4). So

$$\rho(\mathbf{x}, \mathbf{y}) < \epsilon'/(1 - r - \epsilon'/2) < \epsilon.$$

Hence $(\mathbf{y}, s) \in B_\rho(\mathbf{x}, \epsilon) \times (r - \epsilon, r + \epsilon)$. □

In the space $\tilde{I}(X, A)$, there is a very special point, namely $\mathbf{a}^*$. Intuitively, $\mathbf{a}^*$ is path connected to all the other points (assuming the space X is path connected), this property will turn out to be very important in uncovering the original pair (X, A). We will make this topological property precisely in the following lemma.

LEMMA 3.9. *If X is a (path) connected perfect compact metric space, then the space $\tilde{I}(X, A)$ defined above is (path) connected. Moreover, $\mathbf{a}^*$ is a cut point and $\tilde{I}(X, A) - \{\mathbf{a}^*\}$ has infinitely many components.*

PROOF. The subset $X_A \times \mathbb{I}$ is clearly path connected, as all the points in it are connected to $\mathbf{a}^*$. If X is (path) connected, then $\tilde{I}(X, A)$ is also (path) connected, since $X_A \times \mathbb{I} \cap X \times \{0\} \neq \emptyset$.

For the second part, we will write out all the components of $\tilde{I}(X, A) - \{\mathbf{a}^*\}$:

$$C_{\mathbf{a}} := \{(\mathbf{a}, \lambda), \lambda \in [0, 1)\}, \text{ for all } \mathbf{a} \in D_A,$$

and

$$C_X^\infty := X \times \{0\} \times \{0\} \cup A \times \{0\} \times [0, 1).$$

Actually, all the components $C_{\mathbf{a}}$ for $\mathbf{a} \in D_A$ are clopen sets in $\tilde{I}(X, A) - \{\mathbf{a}^*\}$. To check C_X^∞ is (path) connected, notice that each $X \times \{0\}^2 \cup \{a\} \times \{0\} \times [0, 1)$ is (path) connected for all $a \in A$, so $C_X^\infty = \bigcup_{a \in A} X \times \{0\}^2 \cup \{a\} \times \{0\} \times [0, 1)$ is (path) connected, as $\bigcap_{a \in A} X \times \{0\}^2 \cup \{a\} \times \{0\} \times [0, 1) = X \times \{0\}^2 \neq \emptyset$. □

3.2. The Borel Reduction Map

Next, we will show the construction from the pair (X, A) with $A \subseteq X$ into the space $\tilde{I}(X, A)$ is a Borel map. Since this is a pure computational problem, we will divide it up into some small lemmas and then combine all of them in the end.

Zielinski mentioned in [21] that the map I from the space of pairs $\{(X, A) \in \mathcal{K}(\mathcal{Q}) \times \mathcal{K}(\mathcal{Q}) : A \subseteq X\}$ into $\mathcal{K}(\mathcal{Q} \times [0, 1])$ is Borel. For the completeness, we briefly explain it here. We will use the Kuratowski-Ryll Nardzewski selector functions stated in the following theorem.

THEOREM 3.10 ([13]). *Let X be Polish. There is a sequence of Borel functions $d_n : F(X) \to X$, such that for nonempty $F \in F(X)$, $\{d_n(F)\}$ is dense in F.*

Fix $h : \mathbb{N} \to \mathbb{N}$, an enumeration with inifinite repetition, and let $\{d_n\}_{n\in\mathbb{N}}$ be the functions in Theorem 3.10. Define $I(X, A) = X \times \{0\} \cup D_A$ as before, where $D_A = \{(d_{h(1)}(A), 1/2), (d_{h(2)}(A), 1/3), \dots\}$. Then for any open U in $\mathcal{Q} \times [0, 1]$, $I(X, A) \cap U \neq \emptyset$ if and only if

$$\exists n, (d_n(X), 0) \in U \text{ or } \exists n, (d_{h(n)}(A), 1/(n + 1)) \in U,$$

a Borel condition by the measurability of the selector functions. Thus $(X, A) \xrightarrow{I} I(X, A)$ is a Borel map.

Suppose X is a topological space, we embed it into the cylinder $X \times \mathbb{I}$ as $X \times \{0\}$, and obtain the cone ΛX over X as the quotient space of $X \times \mathbb{I}$, by identifying all the points $(x, 1)$ in $X \times \mathbb{I}$ as a single point. Note that if $X \in \mathcal{K}(\mathcal{Q})$, then ΛX belongs to $\mathcal{K}(\Lambda\mathcal{Q})$.

LEMMA 3.11. *Suppose $X \subseteq \mathcal{Q}$ is a closed subset, then the map $f : \mathcal{K}(\mathcal{Q}) \to \mathcal{K}(\Lambda\mathcal{Q})$ by $f(X) = \Lambda X$ is Borel.*

PROOF. We can write f as a composition of two functions

$$X \xrightarrow{f_1} X \times \mathbb{I} \xrightarrow{f_2} \Lambda X.$$

f_1 is Borel, since for all basic open set $U \times V$ of $\mathcal{Q} \times \mathbb{I}$, $(X \times \mathbb{I}) \cap (U \times V) \neq \emptyset$ if and only if $X \cap U \neq \emptyset$.

Suppose U is an open set in $\Lambda\mathcal{Q}$, let $\{(u_n, \lambda_n)\}_n$ be a countable dense subset of U. Then $\Lambda X \cap U \neq \emptyset$ if and only if

$$\bigcup_n \bigcup_{\{i: B_\rho(u_n, \frac{1}{i}) \times (\lambda_n - \frac{1}{i}, \lambda_n + \frac{1}{i}) \subseteq U\}} \{X \cap B_\rho(u_n, \frac{1}{i}) \neq \emptyset\},$$

where ρ is a compatible metric on X. Thus f_2 is Borel as well. □

The construction of $\tilde{I}(X, A)$ is more or less like constructing a partial cone with the base $X_A = \overline{D_A}$, the closure of all isolated points in $I(X, A)$. Because of the previous lemma, we know the map from X_A to ΛX_A is Borel. We will show next that the map $I(X, A) \to X_A$ is also Borel.

Lemma 3.12. *Suppose (X, ρ) is a compact metric space. For all $A \in \mathcal{K}(X)$, let D_A denote the set of isolated points in A. Then the map $f : \mathcal{K}(X) \to \mathcal{K}(X)$ by $f(A) = \overline{D_A}$ is Borel.*

Proof. For all $A \in \mathcal{K}(X)$, let $A' = A - D_A$, i.e., A' is the set of limit points of A. We first check that the map $A \mapsto A'$ is Borel. Suppose $U \subseteq X$ is an open set and $\{u_n\}_n$ is a countable dense subset of U, then $A' \cap U \neq \emptyset$ if and only if

$$\bigcup_n \bigcap_i \{A \cap B_\rho(u_n, \frac{1}{i}) \neq \emptyset\},$$

where B_ρ is an open ball. Thus the map is Borel.

Now, $\overline{D_A} \cap U \neq \emptyset$ if and only if

$$\bigcup_n \bigcup_i \{A \cap B_\rho(u_n, \frac{1}{i}) \neq \emptyset\} \cap \{A' \subseteq X - \overline{B_\rho}(u_n, \frac{1}{i})\},$$

where $\overline{B_\rho}$ is a closed ball. Hence, we have that $A \mapsto \overline{D_A}$ is Borel. □

Given a perfect compact metric space X and a closed subset $A \subseteq X$, let $\tilde{I}(X, A)$ be constructed as before, then $\tilde{I}(X, A)$ is a subspace of the cone $\Lambda I(X, A)$. Actually, recall that $I(X, A) = X \times \{0\} \cup D_A$, then $\tilde{I}(X, A)$ is the union of the cone $\Lambda \overline{D_A} \subseteq \Lambda I(X, A)$ and $I(X, A) \times \{0\} \subseteq \Lambda I(X, A)$. So we will check next the union operation is Borel.

Lemma 3.13. *Suppose X is a compact metric space. The map $(K, L) \mapsto K \cup L$ from $\mathcal{K}(X) \times \mathcal{K}(X)$ into $\mathcal{K}(X)$ is Borel.*

Proof. Let $U \subseteq X$ be an open set, then $(K \cup L) \cap U \neq \emptyset$ if and only if

$$K \cap U \neq \emptyset \text{ or } L \cap U \neq \emptyset.$$

This is clearly Borel. □

We are ready to compute the Borel measurability of the map $(X, A) \xrightarrow{\tilde{I}} \tilde{I}(X, A)$.

Proposition 3.14. *The map $\tilde{I}$ from $\{(X,A) \in \mathcal{K}(\mathcal{Q}) \times \mathcal{K}(\mathcal{Q}) : A \subseteq X\}$ into $\tilde{I}(X,A) \in \mathcal{K}(\Lambda(\mathcal{Q} \times \mathbb{I}))$ is Borel.*

Proof. Consider the following maps

$$(X,A) \xrightarrow{f_0} I(X,A) \xrightarrow{f_1} \overline{D_A} \xrightarrow{f_2} \Lambda\overline{D_A},$$

$$(X,A) \xrightarrow{f_0} I(X,A) \xrightarrow{f_3} I(X,A) \times \{0\},$$

$$(\Lambda\overline{D_A}, I(X,A) \times \{0\}) \xrightarrow{f_4} \Lambda\overline{D_A} \cup I(X,A) \times \{0\}.$$

Notice that the map f_3 from $I(X,A)$ to $I(X,A) \times \{0\}$ is actually a composition of two maps: Firstly, $I(X,A) \subseteq \mathcal{K}(\mathcal{Q} \times \mathbb{I})$ is mapped to $I(X,A) \times \{0\} \subseteq \mathcal{K}(\mathcal{Q} \times \mathbb{I} \times \mathbb{I})$, then $I(X,A) \times \{0\}$ is mapped to $I(X,A) \times \{0\} \subseteq \mathcal{K}(\Lambda(\mathcal{Q} \times \mathbb{I}))$ by the quotient map.

Since all the maps f_0, f_1, f_2, f_3 and f_4 are Borel by previous lemmas, so $\tilde{I}(X,A) = f_4(f_2 \circ f_1 \circ f_0(X,A), f_3 \circ f_0(X,A))$ is Borel. □

3.3. Code One Closed Set

Now let us consider a class $\mathcal{P}$ of perfect compact metric spaces that has a certain property,

$$\mathcal{P} = \{X \text{ is perfect compact metric space } : \forall x \in X, X - \{x\} \text{ is path connected}\}.$$

Example 3.15.

- For all $n \geq 2$, $\mathbb{I}^n \in \mathcal{P}$,
- The Hilbert cube $\mathcal{Q}$ is in $\mathcal{P}$.

The Hilbert cube $\mathcal{Q}$ is path connected. For example, let $\mathbf{x} = (x_1, x_2, \ldots), \mathbf{y} = (y_1, y_2, \ldots)$ be two points in $\mathcal{Q}$. Then for all $n \in \mathbb{N}$, there is a continuous map $f_n : \mathbb{I} \to \mathbb{I}$ such that $f_n(0) = x_n$ and $f_n(1) = y_n$, say $f_n(\lambda) = (1-\lambda)x_n + \lambda y_n$. Hence, $f : \mathbb{I} \to \mathcal{Q}$ given by $f(\lambda) = (f_1(\lambda), f_2(\lambda), \ldots)$ is a continuous path from $\mathbf{x}$ to $\mathbf{y}$.

If we remove an arbitrary point $\mathbf{x}^* = (x_1^*, x_2^*, \ldots) \in \mathcal{Q}$, the remaining space $\mathcal{Q} - \{\mathbf{x}^*\}$ is still path connected.

Again, consider two points $\mathbf{x}, \mathbf{y} \in \mathcal{Q} - \{\mathbf{x}^*\}$. In a simpler case, suppose there exist i, j such that $x_i \neq \mathbf{x}_i^*$ and $y_j \neq \mathbf{x}_j^*$. Let $f_n, n \in \mathbb{N}$ be the same maps as in the previous case, then

$$(f_1(\lambda), f_2(\lambda), \dots, f_{i-1}(\lambda), x_i, f_{i+1}(\lambda), \dots)$$

is a continuous path that sends $\mathbf{x}$ to $(y_1, \dots, y_{i-1}, x_i, y_{i+1}, \dots)$. Note that $\mathbf{x}^*$ is not on the path since the ith coordinate is a constant x_i which is not equal to $\mathbf{x}_i^*$. Then we just move $(y_1, \dots, y_{i-1}, x_i, y_{i+1}, \dots)$ to $\mathbf{y}$ by a path

$$(y_1, \dots, y_{i-1}, f_i(\lambda), y_{i+1}, \dots).$$

Once again, $\mathbf{x}^*$ is not on the path since $y_j \neq \mathbf{x}_j^*$.

On the other hand, suppose the only coordinate that $\mathbf{x}, \mathbf{y}$ are different from $\mathbf{x}^*$ is the ith. Then we can move $\mathbf{x}$ along some jth coordinate ($j \neq i$) to a different $x'_j \neq x_j$, now that $\mathbf{x}', \mathbf{y}$ have more than one coordinates combined that are different from $\mathbf{x}^*$, we can apply the previous case to find a path from $\mathbf{x}$ to $\mathbf{y}$ without $\mathbf{x}^*$ on it.

Next, we will show that the $\tilde{I}$ construction will preserve the information about the original pair in the sense that made precisely by the following theorem.

THEOREM 3.16. *Suppose $X, Y \in \mathcal{P}$, and $A \subseteq X, B \subseteq Y$ are closed subsets with at least two points, i.e. $|A|, |B| \geq 2$. Then $\tilde{I}(X, A) \cong \tilde{I}(Y, B)$ if and only if there exists a homeomorphism $f : X \to Y$ such that $f(A) = B$.*

PROOF. Suppose $f' : X \to Y$ is a homeomorphism with $f'(A) = B$, then by Theorem 3.1, we can extend f' to a homeomorphism $f : I(X, A) \to I(Y, B)$ with $f(X_A) = Y_B$ and $f(X \times \{0\}) = Y \times \{0\}$. And we want to extend f further into a homeomorphism $\tilde{f}$ between $\tilde{I}(X, A)$ and $\tilde{I}(Y, B)$.

We assign $\tilde{f}(\mathbf{a}^*) = \mathbf{b}^*$. For all $\mathbf{x} \in X_A$ and $\lambda \in \mathbb{I}$, let $\tilde{f}((\mathbf{x}, \lambda)) := (f(\mathbf{x}), \lambda)$. Now $\tilde{f}$ is a bijection. We need to show $\tilde{f}$ is continuous.

- For all point in $(X-A) \times \{0\} \times \{0\}$, $\tilde{f}$ behaves exactly like f, since $(X-A) \times \{0\} \times \{0\}$ is open in $\tilde{I}(X, A)$, so $\tilde{f}$ is continuous.

- For all point $\mathbf{x} \in X_A \times [0,1) \cup \{\mathbf{a}^*\}$, a basic open neighborhood for $\tilde{f}(\mathbf{x})$ has the form $U \times V$, where U is open in $I(Y,B)$ and V is open in $\mathbb{I}$, then $\tilde{f}^{-1}(U \times V) = f^{-1}(U) \times V$, which is open in $\tilde{I}(X,A)$.

So $\tilde{f}$ is a bijective continuous map between two compact metric spaces, and hence $\tilde{f}$ is a homeomorphism.

For the other direction, assume $\tilde{f} : \tilde{I}(X,A) \to \tilde{I}(Y,B)$ is a homeomorphism, we will show that $\tilde{f}(I(X,A) \times \{0\}) = I(Y,B) \times \{0\}$, which implies $I(X,A) \cong I(Y,B)$. Then by Theorem 3.1 again, we will have a homeomorphism $f : X \to Y$ such that $f(A) = B$.

We first check the cut (or non-cut) property of all the points in $\tilde{I}(X,A)$. Recall that $\tilde{I}(X,A) = I(X,A) \times \{0\} \cup X_A \times (0,1) \cup \{\mathbf{a}^*\}$ (Figure 3.5), where $I(X,A) = X \times \{0\} \cup D_A$ and $X_A = \overline{D_A}$.

- All the points in $D_A \times \{0\}$ are non-cut points. Actually, remove an arbitrary point $\mathbf{x} \in D_A \times \{0\}$ from $\tilde{I}(X,A)$, the remaining space $\tilde{I}(X,A) - \{\mathbf{x}\}$ is still path connected.
- All the points in $X \times \{0\}^2$ are non-cut points. Suppose $(x,0,0)$ has been removed for some $x \in X - A$, then by the assumption $X \in \mathcal{P}$, $X - \{x\}$ is still path connected, so is $X \times \{0\}^2 - \{(x,0,0)\}$. Hence $\tilde{I}(X,A) - \{(x,0,0)\}$ being the union of two path connected sets with nonempty intersection $X_A \times \{0\}$ is path connected.

  Suppose we have removed an arbitrary point $(a,0,0)$ for some $a \in A$, we show the remaining points are still path connected to $\mathbf{a}^*$. By assumption, A has at least two points, so at least one of them is still path connected to $\mathbf{a}^*$, thus all the points in $X \times \{0\}^2$ are path connected to $\mathbf{a}^*$, by using the property $X \in \mathcal{P}$ again. The rest of the points in $\tilde{I}(X,A) - \{(a,0,0)\}$ are clearly path connect to $\mathbf{a}^*$.
- All the points in $A \times \{0\} \times (0,1)$ are non-cut points. The argument is similar to the second part of the previous case. Suppose $(a,0,\lambda)$ has been removed for some $a \in A$ and $0 < \lambda < 1$. Since $|A| \geq 2$ and $X \in \mathcal{P}$, so all the points in $X \times \{0\}^2$ are still path connected to $\mathbf{a}^*$. The rest of the points in $\tilde{I}(X,A) - \{(a,0,\lambda)\}$ can be path connected to $\mathbf{a}^*$ either directly or through the points in $X \times \{0\}^2$.
- All the points in $D_A \times (0,1)$ are cut points. For all $\mathbf{x} \in D_A \times (0,1)$, $\tilde{I}(X,A) - \{\mathbf{x}\}$

has two components, and they are both clopen sets.

- Finally, $\mathbf{a}^*$ is a cut point. By Lemma 3.9, $\tilde{I}(X,A) - \{\mathbf{a}^*\}$ has infinitely many components.

The exact same analysis applies for the corresponding points in $\tilde{I}(Y,B)$.

Since cut (or non-cut) property is a topological property, and by the previous analysis, $\mathbf{a}^*$ is the only cut point in $\tilde{I}(X,A)$ such that removing it will result in infinitely many components, so $\tilde{f}(\mathbf{a}^*) = \mathbf{b}^*$. Now $\tilde{f}$ must send each component in $\tilde{I}(X,A) - \{\mathbf{a}^*\}$ onto some component in $\tilde{I}(Y,B) - \{\mathbf{b}^*\}$.

Recall the two types of components in $\tilde{I}(X,A) - \{\mathbf{a}^*\}$ (Lemma 3.9):

$$C_{\mathbf{a}} := \{(\mathbf{a},\lambda), \lambda \in [0,1)\}, \mathbf{a} \in D_A,$$

$$C_X^\infty := (X - A) \times \{0\} \times \{0\} \cup A \times \{0\} \times [0,1).$$

Each component $C_{\mathbf{a}}, \mathbf{a} \in D_A$ has exactly one non-cut point, say $(\mathbf{a}, 0)$, whereas, the component C_X^∞ has infinitely many non-cut points. Thus we must have

$$\tilde{f}(C_X^\infty) = C_Y^\infty, \tag{5}$$

and $\tilde{f}$ sends $C_{\mathbf{a}}, \mathbf{a} \in D_A$ onto some component $C_{\mathbf{b}}, \mathbf{b} \in D_B$. In particular, $\tilde{f}$ will send the exact non-cut point in each $C_{\mathbf{a}}$ to the non-cut point in some $C_{\mathbf{b}}$, i.e. $\tilde{f}(D_A \times \{0\}) = D_B \times \{0\}$. Hence

$$\tilde{f}(X_A \times \{0\}) = Y_B \times \{0\}. \tag{6}$$

We still need to show $\tilde{f}((X - A) \times \{0\} \times \{0\}) = (Y - B) \times \{0\} \times \{0\}$. But because of equation (5), it is enough to show that $\tilde{f}(X_A \times [0,1)) = X_B \times [0,1)$. Consider the spaces $\tilde{I}(X,A) - X_A \times \{0\}$ and $\tilde{I}(Y,B) - Y_B \times \{0\}$, $\tilde{f}$ is a homeomorphism between them because of equation (6). So $\tilde{f}$ sends each component in one space onto a component in the other. Note that one of the components in their respective spaces is $X_A \times (0,1) \cup \{\mathbf{a}^*\}$ and $Y_B \times (0,1) \cup \{\mathbf{b}^*\}$. Since $\tilde{f}(\mathbf{a}^*) = \mathbf{b}^*$, then we must have $\tilde{f}(X_A \times (0,1)) = X_B \times (0,1)$. This completes the proof. □

REMARK 3.17. We assume in the statement of the theorem that both A and B have at least two elements for the purpose of convenience in proof. The only place in the proof that we used that assumption is when analyzing the cut (or non-cut) property of all the points in $\tilde{I}(X, A), \tilde{I}(X, B)$, more precisely, we try to single out the points $\mathbf{a}^*, \mathbf{b}^*$ from all the other points in their respective spaces. Without the assumption of $|A|, |B| \geq 2$, we are still able to achieve this, so we can eliminate that assumption in the theorem.

There is a slightly different version of Theorem 3.16. We will loosen the condition on X a little. In Theorem 3.16, we require that removing an arbitrary point $x \in X$, the remaining space $X - \{x\}$ is still path connected. In the next theorem, we only require that X is path connected and X has no cut point. The only difference is that when we remove a point $\mathbf{x}$ from $\tilde{I}(X, A)$, some of the components in the remaining space $\tilde{I}(X, A) - \{\mathbf{x}\}$ may not be path connected anymore under the new assumption; whereas in Theorem 3.16, all the components of the remaining space are path connected. We can use a similar argument as before to show the revised theorem, but here we will give a different proof.

THEOREM 3.18. *Suppose X, Y are perfect compact metric spaces such that*

(1) X, Y are path connected,

(2) X, Y have no cut point,

and $A \subseteq X, B \subseteq Y$ are closed subsets with at least two elements. Then $\tilde{I}(X, A) \cong \tilde{I}(Y, B)$ if and only if $I(X, A) \cong I(Y, B)$.

PROOF. The proof of the direction $I(X, A) \cong I(Y, B) \Rightarrow \tilde{I}(X, A) \cong \tilde{I}(Y, B)$ is identical to the previous theorem. We will just check the direction $\tilde{I}(X, A) \cong \tilde{I}(Y, B) \Rightarrow I(X, A) \cong I(Y, B)$.

Suppose $\tilde{f} : \tilde{I}(X, A) \to \tilde{I}(Y, B)$ is a homeomorphism, we will show that $\tilde{f}(X \times \{0\} \times \{0\}) = Y \times \{0\} \times \{0\}$ and $\tilde{f}(A \times \{0\} \times \{0\}) = B \times \{0\} \times \{0\}$.

By assumption, all points of $I(X, A) \times \{0\}$ and $A \times \{0\} \times (0, 1)$ are non-cut points. For all $(\mathbf{a}, \lambda)$, where $\mathbf{a} \in D_A$ and $\lambda \in (0, 1)$, $\tilde{I}(X, A) - \{(\mathbf{a}, \lambda)\}$ has exactly two components, thus it is a cut-point. Similar properties hold for the corresponding points in $\tilde{I}(Y, B)$. And

$\mathbf{a}^*, \mathbf{b}^*$ are two special cut-points in their spaces, saying that removing them will result in infinitely many components. Thus $\tilde{f}(\mathbf{a}^*) = \mathbf{b}^*$.

Remove $\mathbf{a}^*$ and $\mathbf{b}^*$ from their respective spaces, then $\tilde{f}$ sends one component in $\tilde{I}(X, A) - \{\mathbf{a}^*\}$ onto some component in $\tilde{I}(Y, B) - \{\mathbf{b}^*\}$.

There are two types of components in $\tilde{I}(X, A) - \{\mathbf{a}^*\}$:

$$C_{\mathbf{a}} := \{(\mathbf{a}, \lambda), \lambda \in [0, 1)\}, \mathbf{a} \in D_A$$

$$C_X^\infty := X \times \{0\} \times \{0\} \cup A \times \{0\} \times (0, 1).$$

For all point in the component $C_{\mathbf{a}}, \mathbf{a} \in D_A$, there is an open neighborhood that is contained in the component. But for all $(a, 0, 0) \in C_X^\infty$, every open neighborhood is not entirely contained inside C_X^∞. The components $C_{\mathbf{b}}, \mathbf{b} \in D_B$ and C_Y^∞ in $\tilde{I}(Y, B) - \{\mathbf{b}^*\}$ have similar properties. Thus $\tilde{f}$ must send some component $C_{\mathbf{a}}$ to some $C_{\mathbf{b}}$. In particular,

$$\tilde{f}(D_A \times \{0\}) = D_B \times \{0\},$$

and hence $\tilde{f}(\overline{D_A} \times \{0\}) = \overline{D_B} \times \{0\}$.

Next, let us remove all points $\overline{D_A} \times \{0\}, \overline{D_B} \times \{0\}$ from the original spaces. Clearly, all points $\overline{D_A} \times (0, 1) \cup \{\mathbf{a}^*\}$ are in one component in the remaining space, same for $\overline{D_B} \times (0, 1) \cup \{\mathbf{b}^*\}$. Since we already know $\tilde{f}(\mathbf{a}^*) = \mathbf{b}^*$, then we must have

$$\tilde{f}(\overline{D_A} \times (0, 1) \cup \{\mathbf{a}^*\}) = \overline{D_B} \times (0, 1) \cup \{\mathbf{b}^*\}.$$

This combined with the result that $\tilde{f}(C_X^\infty) = C_Y^\infty$ tells us $\tilde{f}(X \times \{0\} \times \{0\}) = Y \times \{0\} \times \{0\} \times \{0\}$. □

3.4. The $\tilde{I}_2(X, B, A)$ Construction

Next, we will show that we can code two closed subsets of a (path) connected compact metric space into a (path) connected compact metric space.

Suppose $(X, B, A) \in \mathcal{K}(\mathcal{Q})^3$ with $A \subseteq B \subseteq X$, we first construct $I(X, A) \subseteq X \times \mathbb{I}$ as before, i.e.

$$I(X, A) = (X \times \{0\}) \cup D_A,$$

where D_A is a countable set of isolated point such that $\overline{D_A} - D_A = A \times \{0\}$. Note that $(B \times \{0\}) \cup D_A$ is a closed subset of $I(X, A)$. Then we define $I_2(X, B, A) \subseteq X \times \mathbb{I}^2$ as

$$I_2(X, B, A) := I(I(X, A), (B \times \{0\}) \cup D_A).$$

Let D_B^2 denote the countable set of isolated points of $I_2(X, B, A)$ and $X_B^2 := \overline{D_B^2}$ (Figure 3.6).

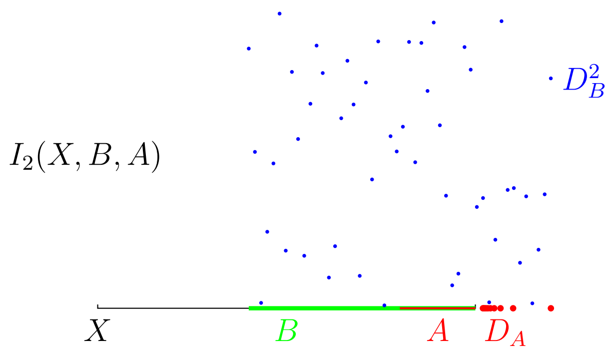


FIGURE 3.6. $I_2(X, B, A)$

Observe that D_A is the set of isolated points in the space $I(X, A)$, but all the points in $D_A \times \{0\}$ are not isolated in the space $I_2(X, B, A)$ anymore, as by the I-construction, we have

$$X_B^2 - D_B^2 = (B \times \{0\} \times \{0\}) \cup D_A \times \{0\}.$$

$I_2(X, B, A)$ is clearly disconnected, actually, it will always have infinitely many isolated points.

Embed $I_2(X, B, A)$ into $(I_2(X, B, A) \times \{0\}) \cup X_B^2 \times \mathbb{I}$ as $I_2(X, B, A) \times \{0\}$, then identify all the points in $X_B^2 \times \{1\}$ as an equivalent class, denoted $\mathbf{x}^{**}$, and call the resulting quotient space $\tilde{I}_2(X, B, A)$ (Figure 3.7). The $\tilde{I}_2$-construction is Borel, actually, $\tilde{I}_2(X, B, A) = \tilde{I}(I(X, A), (B \times \{0\}) \cup D_A)$, and as pointed out earlier that both $\tilde{I}$ and I are Borel. It is not hard to check the connectedness property of $\tilde{I}_2(X, B, A)$.

LEMMA 3.19. *Suppose $X \in \mathcal{P}$ and $A \subseteq B \subseteq X$ are closed subsets. Then $\tilde{I}_2(X, B, A)$ is path connected.*

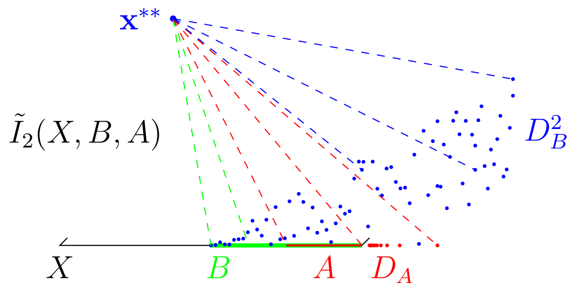


FIGURE 3.7. $\tilde{I}_2(X,B,A)$

PROOF. Note that $\tilde{I}_2(X,B,A) = I(X,A) \times \{0\} \times \{0\} \cup X_B^2 \times [0,1) \cup \{\mathbf{x}^{**}\}$, where $\mathbf{x}^{**} = \{(\mathbf{x},1) : \mathbf{x} \in X_B^2\}$. It is enough to show that every point of $\tilde{I}_2(X,B,A)$ is path connected with $\mathbf{x}^{**}$.

It is clear that all points in $X_B^2 \times [0,1)$ are path connected to the point $\mathbf{x}^{**}$. Recall that $I(X,A) = X \times \{0\} \cup D_A$. Since $X_B^2 \supseteq D_A \times \{0\}$, and $X_B^2 \supseteq B \times \{0\} \times \{0\}$, then by the assumption of path connectedness of X, we know that all points of $I(X,A) \times \{0\} \times \{0\}$ are path connected to $\mathbf{x}^{**}$. $\square$

3.5. Code Two Nested Closed Subsets

Let $(X,B,A), (Y,D,C) \in \mathcal{K}(\mathcal{Q})^3$ with $A, B \subseteq X$ and $C, D \subseteq Y$, we say (X,B,A) and (Y,D,C) are restricted homeomorphic equivalent, denoted

$$(X,B,A) \cong_{(1,1)} (Y,D,C),$$

if there exists $f \in \mathrm{Hom}(X,Y)$ such that $f(B) = D$ and $f(A) = C$. We will show the $\tilde{I}_2$-construction preserves the restricted homeomorphic equivalent relation.

THEOREM 3.20. *Consider triples $(X,B,A), (Y,D,C) \in \mathcal{K}(\mathcal{Q})^3$, where $X, Y \in \mathcal{P}$ and $A \subseteq B \subseteq X$, $C \subseteq D \subseteq Y$ are infinite closed subsets. We have $(X,B,A) \cong_{(1,1)} (Y,D,C)$ if and only if $\tilde{I}_2(X,B,A) \cong \tilde{I}_2(Y,D,C)$.*

PROOF. For the forward direction, suppose there exists a homeomorphism $f : X \to Y$ with $f(A) = C$ and $f(B) = D$. We can apply Theorem 3.1 repeatedly to first extend f into a homeomorphism $f' : I(X,A) \to I(Y,C)$ with $f'(D_A) = D_C$, and then extend

f' into a homeomorphism $f'' : I_2(X, B, A) \to I_2(Y, D, C)$ with $f''(D_B^2) = D_D^2$. Now f'' induces a homeomorphism between $I_2(X, B, A) \times \{0\} \subseteq \tilde{I}_2(X, B, A)$ and $I_2(Y, D, C) \times \{0\} \subseteq \tilde{I}_2(Y, D, C)$. As we did in Theorem 3.16, f'' can be extended into a homeomorphism $\tilde{f} : \tilde{I}_2(X, B, A) \to \tilde{I}_2(Y, D, C)$ by defining

$$\tilde{f}((\mathbf{x}, \lambda)) := (f''(\mathbf{x}), \lambda) \text{ for all } \mathbf{x} \in \overline{D_B^2} \text{ and } \lambda \in \mathbb{I}.$$

For the backward direction, suppose $\tilde{f} : \tilde{I}_2(X, B, A) \to \tilde{I}_2(Y, D, C)$ is a homeomorphism. We will show $(X, B, A) \cong_{(1,1)} (Y, D, C)$ in the following order

$$\tilde{f}(A \times \{0\} \times \{0\} \times \{0\}) = C \times \{0\} \times \{0\} \times \{0\};$$

$$\tilde{f}(B \times \{0\} \times \{0\} \times \{0\}) = D \times \{0\} \times \{0\} \times \{0\};$$

$$\tilde{f}((X - B) \times \{0\} \times \{0\} \times \{0\}) = (Y - D) \times \{0\} \times \{0\} \times \{0\}.$$

First, let us discuss the non-cut (or cut) property of the points in $\tilde{I}_2(X, B, A)$. We will divide $\tilde{I}_2(X, B, A)$ up as the union $I_2(X, B, A) \times \{0\} \cup X_B^2 \times (0, 1) \cup \{\mathbf{x}^{**}\}$.

- Recall that $X_B^2 = (B \times \{0\} \times \{0\} \cup D_A \times \{0\}) \cup D_B^2$
  - All the points in $B \times \{0\} \times \{0\} \times (0, 1)$ are non-cut points. For all $(b, 0, 0, \lambda_0)$, where $b \in B$ and $\lambda_0 \in (0, 1)$, $\tilde{I}_2(X, B, A) - \{(b, 0, 0, \lambda_0)\}$ is still path connected (all points are path connected to $\mathbf{x}^{**}$). We just need to find a path from $(b, 0, 0, \lambda)$ for some $\lambda < \lambda_0$ to $\mathbf{x}^{**}$. By assumption that B is inifite, there exists $b' \in B, b' \neq b$. Then a natural path would be from $(b, 0, 0, \lambda)$ to $(b, 0, 0, 0)$, then to $(b', 0, 0, 0)$ (since X is path connected), and finally to $\mathbf{x}^{**}$.
  - All the points in $D_A \times \{0\} \times (0, 1)$ are non-cut points. For all $(\mathbf{a}_0, 0, \lambda_0) \in D_A \times \{0\} \times (0, 1)$, the set $\tilde{I}_2(X, B, A) - \{(\mathbf{a}_0, 0, \lambda) : \lambda \leq \lambda_0\}$ is connected (all points are path connected to $\mathbf{x}^{**}$). Since the closure of $\tilde{I}_2(X, B, A) - \{(\mathbf{a}_0, 0, \lambda) : \lambda \leq \lambda_0\}$ is the whole space $\tilde{I}_2(X, B, A)$, so $\tilde{I}_2(X, B, A) - \{(\mathbf{a}_0, 0, \lambda_0)\}$ is connected by Lemma 2.2.

  - All the points in $D_B^2 \times (0,1)$ are cut points. For all $(\mathbf{b}_0, \lambda_0)$, where $\mathbf{b}_0 \in D_B^2$ and $\lambda_0 \in (0,1)$, $\tilde{I}_2(X,B,A) - \{(\mathbf{b}_0, \lambda_0)\}$ has exactly two components. One of them is the clopen set $\{(\mathbf{b}_0, \lambda) : \lambda < \lambda_0\}$ and the other is $\tilde{I}_2(X,B,A) - \{(\mathbf{b}_0, \lambda) : \lambda \leq \lambda_0\}$.

- Recall that $I_2(X,B,A) = I(X,A) \times \{0\} \cup X_B^2$.
  - All the points of $X_B^2 \times \{0\}$ are non-cut points. Note that because of the assumption that B has more than one element, $\tilde{I}_2(X,B,A) - \{(\mathbf{b}, 0)\}$ is path connected for all $\mathbf{b} \in X_B^2$.
  - All the points in $I(X,A) \times \{0\} \times \{0\}$ are non-cut points. Recall that $I(X,A) = X \times \{0\} \cup D_A$. Since $D_A \times \{0\} \times \{0\} \subseteq X_B^2 \times \{0\}$, so they are non-cut points by the previous case. Using the assumption that $X \in \mathcal{P}$ and $|B| > 2$ again, it is clear $X \times \{0\} \times \{0\} \times \{0\}$ are all non-cut points.

- Finally, $\mathbf{x}^{**}$ is a cut point. Removing it will result in infinitely many components in the remaining space.

Similar conclusions can be made for the corresponding points in $\tilde{I}_2(Y,D,C)$. Based on the property of $\mathbf{x}^{**}$ as indicated in the last case of previous analysis, we have

$$\tilde{f}(\mathbf{x}^{**}) = \mathbf{y}^{**}. \tag{7}$$

Next, remove $\{\mathbf{x}^{**}\}$ from $\tilde{I}_2(X,B,A)$, let us analysis the components in the remaining space.

Claim 3.21. *In the space $\tilde{I}_2(X,B,A) - \{\mathbf{x}^{**}\}$, there are three types of components:*

i) $\{(\mathbf{b}, \lambda), \lambda \in [0,1)\}$, *for* $\mathbf{b} \in D_B^2$;

ii) $\{(\mathbf{a}, 0, \lambda), \lambda \in [0,1)\}$, *for* $\mathbf{a} \in D_A$;

iii) $(X - B) \times \{0\} \times \{0\} \times \{0\} \cup B \times \{0\} \times \{0\} \times [0,1)$.

Proof of Claim. It is clear that all sets of type i) are path connected. Moreover, they are clopen sets in $\tilde{I}_2(X,B,A) - \{\mathbf{x}^{**}\}$, since D_B^2 consists of isolated points in $I_2(X,B,A)$. So no component in $\tilde{I}_2(X,B,A) - \{\mathbf{x}^{**}\}$ would properly contain any component of type i).

In other words, the rest of components in $\tilde{I}_2(X,B,A) - \{\mathbf{x}^{**}\}$ are properly contained in $\tilde{I}_2(X,B,A) - \{\mathbf{x}^{**}\} - \bigcup_{\mathbf{b}\in D_B^2}\{(\mathbf{b},\lambda), \lambda \in [0,1)\}$.

All the sets of type ii) are clopen sets in $\tilde{I}_2(X,B,A) - \{\mathbf{x}^{**}\} - \bigcup_{\mathbf{b}\in D_B^2}\{(\mathbf{b},\lambda), \lambda \in [0,1)\}$, as the set D_A consists of isolated points in $I(X,A)$ and all the points of D_B^2 have been removed. Thus the sets of type ii) are components in $\tilde{I}_2(X,B,A) - \{\mathbf{x}^{**}\}$.

Finally, the set of type iii) is path connected, thus a component. □

Now we will define a topological property that will differentiate one type from the other two:

(8) P_x: there is an open neighborhood U of x such that U is path connected and all points but x are cut points in U.

Claim 3.22. *In the components of type i), only points of $D_B^2 \times \{0\}$ satisfy property (8). None of the point in the components of type ii) or iii) satisfies property (8).*

Proof of Claim. First of all, for all $(\mathbf{b}_0, 0)$ in $D_B^2 \times \{0\}$, the clopen sets $\{(\mathbf{b}_0,\lambda) : \lambda \in [0,1)\}$ is an open connected neighborhood and $(\mathbf{b}_0, 0)$ is the only non-cut point. So all of $D_B^2 \times \{0\}$ satisfy property (8). On the other hand, for all $(\mathbf{b}_0, \lambda_0)$ in a component of type i) with $\mathbf{b}_0 \in D_B^2$ and $\lambda_0 > 0$, a connected open neighborhood U of $(\mathbf{b}_0, \lambda_0)$ is homeomorphic to an open subinterval of $\mathbb{R}$, thus all of U are cut points.

For all $(\mathbf{a}_0, 0, \lambda_0)$ in a component of type ii), there exists a sequence of points in $D_B^2 \times \{0\}$ converging to it. This implies that all open neighborhoods of $(\mathbf{a}_0, 0, \lambda_0)$ are disconnected.

Similarly, none of the point in $B \times \{0\} \times \{0\} \times [0,1)$ satisfies property (8). Since all of the points in $(X - B) \times \{0\} \times \{0\} \times \{0\}$ are non-cut points within the component iii), so they do not satisfy property (8) either. □

All of the arguments in the previous two claims can be carried out to the corresponding points in the space $\tilde{I}_2(Y,D,C) - \{\mathbf{y}^{**}\}$. Thus, we must have

$$\tilde{f}(D_B^2 \times \{0\}) = D_D^2 \times \{0\} \text{ and } \tilde{f}(X_B^2 \times \{0\}) = X_D^2 \times \{0\}. \tag{9}$$

For all component $\{(\mathbf{a}, 0, \lambda) : \lambda \in [0, 1)\}$ of type ii), there is only one non-cut point, which is $(\mathbf{a}, 0, 0)$, but the component iii) has infinitely many non-cut points by the assumption $X \in \mathcal{P}$. Thus, we have

$$\tilde{f}(D_A \times \{0\} \times \{0\}) = D_C \times \{0\} \times \{0\}, \tag{10}$$

and hence $\tilde{f}(A \times \{0\} \times \{0\} \times \{0\}) = C \times \{0\} \times \{0\} \times \{0\}$. Combine equation (9) and (10), we conclude that $\tilde{f}(B \times \{0\} \times \{0\} \times \{0\}) = D \times \{0\} \times \{0\} \times \{0\}$.

The last thing we need to show is $\tilde{f}$ sends $(X - B) \times \{0\} \times \{0\} \times \{0\}$ onto $(Y - D) \times \{0\} \times \{0\} \times \{0\}$. Actually, it is enough to show $\tilde{f}(X_B^2 \times (0, 1)) = X_D^2 \times (0, 1)$, since we already know $\tilde{f}$ sends the type iii) component in $\tilde{I}_2(X, B, A) - \{\mathbf{x}^{**}\}$ to the same type of component in $\tilde{I}_2(Y, D, C) - \{\mathbf{y}^{**}\}$. Because of equation (9), we can consider the spaces $\tilde{I}_2(X, B, A) - X_B^2 \times \{0\}$ and $\tilde{I}_2(Y, D, C) - X_D^2 \times \{0\}$, all the points of $X_B^2 \times (0, 1) \cup \{\mathbf{x}^{**}\}$ and $X_D^2 \times (0, 1) \cup \{\mathbf{y}^{**}\}$ are in one clopen component, respectively. Since $\tilde{f}(\mathbf{x}^{**}) = \mathbf{y}^{**}$, then $\tilde{f}(X_B^2 \times (0, 1)) = X_D^2 \times (0, 1)$. □

In the previous theorem, we try to code two nested closed subsets of a (path) connected compact metric space into a (path) connected compact metric space. Next we will give a general construction of coding two arbitrary closed subsets of a compact metric space.

Given a triple $(X, B, A) \in \mathcal{K}(\mathcal{Q})^3$ such that $A, B \subseteq X$ and they both contain the isolated points of X, we first embed X into $X \times \mathbb{I}$ as $X \times \{0\}$ and add a countable sequence of isolated points $D_{A\cap B} \subseteq X \times (0, 1]$ that converges to $(A \cap B) \times \{0\}$, the resulting space is denoted $I(X, A \cap B) = X \times \{0\} \cup D_{A\cap B}$.

Secondly, we embed $I(X, A\cap B)$ into $I(X, A\cap B)\times \mathbb{I}$ as $I(X, A\cap B)\times \{0\}$ and add a sequence of isolated points $D_A^2 \subseteq I(X, A\cap B)\times (0, 1]$ that converges to $A\times \{0\}\cup D_{A\cap B}\times \{0\}$, the resulting space is $I_2(X, A, A \cap B) = I(I(X, A \cap B), A \times \{0\} \cup D_{A\cap B} \times \{0\}) = I(X, A \cap B) \times \{0\} \cup D_A^2$.

Thirdly, we embed $I_2(X, A, A\cap B)$ into $I_2(X, A, A\cap B)\times \mathbb{I}$ as $I_2(X, A, A\cap B)\times \{0\}$ and add a sequence of isolated points $D_{A\cup B}^3$ that converges to $(A\cup B)\times \{0\}\times \{0\}\cup D_A^2\times \{0\}$, the resulting space is $I_3(X, A\cup B, A, A\cap B) = I(I_2(X, A, A\cap B), (A\cup B)\times\{0\}\times\{0\}\cup D_A^2\times\{0\}) = I_2(X, A, A\cap B)\times\{0\}\cup D_{A\cup B}^3$.

Proposition 3.23. *Suppose we have two triples $(X, B, A), (Y, D, C) \in \mathcal{K}(\mathcal{Q})^3$ with $A, B \subseteq X$ and $C, D \subseteq Y$, moreover, assume A, B and C, D contain all isolated points of X and Y, respectively. Let $I_3(X, A\cup B, A, A\cap B)$ and $I_3(Y, C\cup D, C, C\cap D)$ be constructed as above. Then $(X, B, A) \cong_{(1,1)} (Y, D, C)$ if and only if $I_3(X, A\cup B, A, A\cap B) \cong I_3(Y, C\cup D, C, C\cap D)$.*

Proof. For the forward direction, suppose $f : X \to Y$ is a homeomorphism with $f(A) = C, f(B) = D$. Then we can extend f into $f_1 : I(X, A\cap B) \to I(Y, C\cap D)$ such that $f_1(D_{A\cap B}\cup A\times\{0\}) = D_{C\cap D}\cup C\times\{0\}$. Hence f_1 can be extended to $f_2 : I_2(X, A, A\cap B) \to I_2(Y, C, C\cap D)$ such that $f_2(D_A^2\times\{0\}\cup(A\cup B)\times\{0\}\times\{0\}) = D_C^2\times\{0\}\cup(C\cup D)\times\{0\}\times\{0\}$. Therefore, we can extend f_2 into $f_3 : I_3(A\cup B, A, X, A\cap B,) \to I_3(Y, C\cup D, C, C\cap D)$.

For the backward direction, suppose $\tilde{f} : I_3(X, A\cup B, A, A\cap B) \to I_3(Y, C\cup D, C, C\cap D)$ is a homeomorphism. Then $\tilde{f}$ must send isolated points $D_{A\cup B}^3$ to isolated points $D_{C\cup D}^3$, hence

$$\tilde{f}\left((A\cup B, 0, 0, 0)\cup(D_A^2, 0, 0)\right) = (C\cup D, 0, 0, 0)\cup(D_C^2, 0, 0). \tag{11}$$

$\tilde{f}$ must send non-isolated points $I_2(X, A, A\cap B)\times\{0\}$ to non-isolated points $I_2(Y, C, C\cap D)\times\{0\}$. Now we restrict $\tilde{f}$ to these two subspaces. Since $D_A^2\times\{0\}, D_C^2\times\{0\}$ are the set of isolated points in their respective spaces, so $\tilde{f}$ must send $D_A^2\times\{0\}$ to $D_C^2\times\{0\}$, hence

$$\tilde{f}((A, 0, 0, 0)\cup(D_{A\cap B}, 0, 0)) = (C, 0, 0, 0)\cup(D_{C\cap D}, 0, 0). \tag{12}$$

And $\tilde{f}$ sends non-isolated points $I(X, A\cap B)\times\{0\}\times\{0\}$ to non-isolated points $I(X, C\cap D)\times\{0\}\times\{0\}$. Now we restrict $\tilde{f}$ to these two subspaces. Since $D_{A\cap B}\times\{0\}\times\{0\}, D_{C\cap D}\times\{0\}\times\{0\}$ are the set of isolated points, so $\tilde{f}(D_{A\cap B}\times\{0\}\times\{0\}) = D_{C\cap D}\times\{0\}\times\{0\}$, and hence

$$\tilde{f}((A\cap B, 0, 0, 0)) = (C\cap D, 0, 0, 0). \tag{13}$$

Now combine equations (11), (12), and (13), we have that $\tilde{f}$ induces a homeomorphism that sends X to Y, $A\cup B$ to $C\cup D$, A to C, and $A\cap B$ to $C\cap D$. This shows that

$(X, B, A) \cong_{(1,1)} (Y, D, C)$. $\square$

Suppose $X \in \mathcal{P}$, then we can perform the exact same construction as before to obtain $\tilde{I}_3(X, A \cup B, A, A \cap B)$. For example,

$$\tilde{I}_3(X, A \cup B, A, A \cap B) = I_3(X, A \cup B, A, A \cap B) \times \{0\} \cup X^3_{A\cup B} \times (0,1) \cup \{\mathbf{x}^{***}\},$$

where $X^3_{A\cup B} = \overline{D^3_{A\cup B}}$ and $\mathbf{x}^{***}$ is the equivalence class $X^3_{A\cup B} \times \{1\}$.

Corollary 3.24. *Consider two triples $(X, B, A), (Y, D, C) \in \mathcal{K}(\mathcal{Q})^3$, where $X, Y \in \mathcal{P}$, $A, B \subseteq X$ and $C, D \subseteq Y$. Then $(X, B, A) \cong_{(1,1)} (Y, D, C)$ if and only if $\tilde{I}_3(X, A \cup B, A, A \cap B) \cong \tilde{I}_3(Y, C \cup D, C, C \cap D)$.*

Remark 3.25. Actually, we can code an arbitrarily finitely many number of closed subsets of a compact metric space such that the coding preserves restricted homeomorphic relations, moreover, the finite sequence of closed subsets are not necessary nested. Nonetheless, it is enough to get the main result with Theorem 3.20.

3.6. The Main Theorem

Theorem 3.26 ([21]). *The equivalence relation $\cong_{(1,1)}$ restricted on the space of triples (X, B, A), where $X \in \mathcal{P}$ and $A \subseteq B \subseteq X$, is Borel bireducible with a universal orbit equivalence relation.*

In fact, Zielinski ([21]) showed the much stronger result in which the spaces X in the triples are all the same:

$$X = \{(x, y) \in \mathcal{Q} \times \mathcal{Q} : \forall m \neq n \ (y_m = 0 \text{ or } y_n = 0)\}.$$

We will give a brief check that $X \in \mathcal{P}$.

First let us describe X as the disjoint union of points on the 'plane', say $\mathcal{Q} \times \{\mathbf{0}\}$, and points outside the 'plane'. Clearly, removing any point on the 'plane' will not affect the path connectivity of the remaining space (Example 3.15).

Now suppose we remove a point $(\mathbf{x}^*, \mathbf{y}^*) \in X$ such that $\mathbf{y}_n^* \neq 0$ for some $n \in \mathbb{N}$. The goal is to show that there is a connected path from any point outside the 'plane' to the 'plane', since the 'plane' is a path connected piece.

Without loss of generality, consider an arbitrary point above the removed one, $(\mathbf{x}^*, \mathbf{y}) \in X$ with $\mathbf{y}_n > \mathbf{y}_n^*$. We can first find a path from $(\mathbf{x}^*, \mathbf{y})$ to some point $(\mathbf{x}, \mathbf{y})$, where $\mathbf{x} \neq \mathbf{x}^*$, say

$$f(\lambda) := (\lambda \mathbf{x} + (1-\lambda)\mathbf{x}^*, \mathbf{y}), \lambda \in [0,1].$$

Then we can find path from $(\mathbf{x}, \mathbf{y})$ to $(\mathbf{x}, \mathbf{0})$, say

$$g(\lambda) := (\mathbf{x}, (1-\lambda)\mathbf{y}), \lambda \in [0,1].$$

Because of the last theorem, our effort in proving the main theorem is left to finding a Borel map that can code each triple in the class mentioned above to a single (path) connected compact metric space, and this has been achieved in Theorem 3.20.

THEOREM 3.27. *The homeomorphic equivalence relation of the class of connected compact metric spaces is complete.*

CHAPTER 4

THE COMPLEXITY OF CLASSIFYING VARIOUS CLOSED SUBSETS OF $[0,1]^N$

4.1. Comparing H_n and CH_n Relations

In this section, we will compare the complexity of classifying closed subsets of $[0,1]^n$ and connected closed subsets of $[0,1]^{n+2}$. We will first construct a map from the space of closed sets to the space of connected closed sets. The construction is slightly different from the I-construction in the previous chapter.

Suppose $A \subseteq \mathbb{I}^n$ is a closed subset and we will construct $\tilde{A}$ in the following way: First embed A into $\mathbb{I}^{n+1}$ as $A \times \{0\}$ and add a countable set of isolated points $D_A \subseteq \mathrm{Int}(\mathbb{I}^{n+1})$ such that their limit points are exactly $A \times \{0\}$; then embed $\overline{D_A}$ into $\mathbb{I}^{n+2}$ as $\overline{D_A} \times \{0\}$ and form the cylinder set $\overline{D_A} \times \mathbb{I}$; finally, glue all the points $\overline{D_A} \times \{1\}$ as one point to form the quotient space $\tilde{A} := (\overline{D_A} \times \mathbb{I})/\{(\mathbf{x}, 1) : \mathbf{x} \in \overline{D_A}\}$. We denote the equivalent class $\{(\mathbf{x}, 1) : \mathbf{x} \in \overline{D_A}\}$ by $\mathbf{a}^*$.

Remark 4.1. The metric introduced in Lemma 3.8 is still valid for the space $\tilde{A}$: For all $(\mathbf{x}, r), (\mathbf{y}, s) \in \tilde{A}$,

$$\tau[(\mathbf{x}, r), (\mathbf{y}, s)] = 2|r - s| + (1 - \max\{r, s\})\rho(\mathbf{x}, \mathbf{y}),$$

where ρ is a compatible metric on $\mathbb{I}^{n+1}$ with $\rho < 1$.

There is another way to construct $\tilde{A}$: first we embed A into $\mathbb{I}^{n+1}$ as $A \times \{0\}$ and add a countable set of isolated points $D_A \subseteq \mathrm{Int}(\mathbb{I}^{n+1})$ so that $\overline{D_A} - D_A = A \times \{0\}$; then we embed $A \times \{0\} \cup D_A$ into $\mathbb{I}^{n+2}$ as $A \times \{0\}^2 \cup D_A \times \{0\}$, called the 'floor' points; add an arbitrary point $\mathbf{a}^* \in \mathrm{Int}(\mathbb{I}^{n+2})$, and connect all the 'floor' points to $\mathbf{a}^*$. So $\tilde{A}$ constructed this way is a connected closed subset of $\mathbb{I}^{n+2}$ and consists of the points:

$$\{\lambda\mathbf{x} + (1 - \lambda)\mathbf{a}^* : \mathbf{x} \in A \times \{0\}^2 \cup D_A \times \{0\}, \lambda \in [0, 1]\}.$$

The $\tilde{A}$ constructed in these two ways are homeomorphic to each other, so without any confusion, we can use the same notation.

Now we will state a topological property that will separate points in $D_A \times \{0\}$ from the other points in $\tilde{A}$:

$$P_{\mathbf{x}}: \ \mathbf{x} \text{ is a non-cut point. For all open neighborhood } V \text{ of } \mathbf{x}, \text{ there exists an open subset } U \subseteq V \text{ such that } \mathbf{x} \in U \text{ and } U \text{ is path connected.} \tag{14}$$

LEMMA 4.2. *Suppose $A \subseteq \mathbb{I}^n$ is a closed subset, and $\tilde{A}$ is constructed as before. Then exactly, the points of $D_A \times \{0\}$ in $\tilde{A}$ satisfy Property (14).*

PROOF. Note that all the points in $D_A \times \{0\}$ are non-cut points, actually, for all $\mathbf{x} \in D_A \times \{0\}$, $\tilde{A} - \{\mathbf{x}\}$ is still path connected, since all points in $\tilde{A} - \{\mathbf{x}\}$ are path connected to $\mathbf{a}^*$. For all $\mathbf{x} \in D_A$ and for all open neighborhood V of $(\mathbf{x}, 0)$, there exists some $\epsilon > 0$ such that $U := \{(\mathbf{x}, r) : r < \epsilon\} \subseteq V$. Clearly, U is path connected.

Note that all the points in $\{(\mathbf{x}, r) : \mathbf{x} \in D_A, 0 < r \leq 1\}$ are cut points, so they don't satisfy the Property (14).

Finally, for $(\mathbf{x}, r) \in \tilde{A}$, where $\mathbf{x} \in A \times \{0\}, r \in \mathbb{I}$, there is a sequence of points from $D_A \times \{0\}$ converging to $\mathbf{x}$, then every open neighborhood V of $\mathbf{x}$ with $\mathbf{a}^* \notin V$ is not connected. □

THEOREM 4.3. *The homeomorphic equivalence relation of closed subsets of $\mathbb{I}^n$ is Borel reducible to the homeomorphic equivalence relation of connected closed subsets of $\mathbb{I}^{n+2}$, i.e. $H_n \leq_B CH_{n+2}$ for all $n \in \mathbb{N}$.*

PROOF. Suppose A, B are closed subsets of $\mathbb{I}$, and $\tilde{A}, \tilde{B}$ are constructed as in the beginning of this section. Moreover, assume that $\tilde{f} : \tilde{A} \to \tilde{B}$ is a homeomorphism. By Lemma 4.2, we have

$$\tilde{f}(D_A \times \{0\}) = D_B \times \{0\},$$

hence $\tilde{f}(A \times \{0\}^2) = B \times \{0\}^2$. Therefore, A, B are homeomorphic to each other.

On the other hand, suppose $f : A \to B$ is a homeomorphism. With the same argument as in the proof of Proposition 3.2, we can extend f into a homeomorphism $f' : \overline{D_A} \to \overline{D_B}$ such that $f' \restriction_{A \times \{0\}} = f$. Then we can extend f' further to $\tilde{f}$ by sending $\mathbf{a}^*$ to $\mathbf{b}^*$, and $(\mathbf{a}, \lambda) \in \overline{D_A} \times [0,1)$ to $(f'(\mathbf{a}), \lambda) \in \overline{D_B} \times [0,1)$. $\tilde{f} : \tilde{A} \to \tilde{B}$ is clearly one-to-one, onto and continuous. Since both $\tilde{A}$ and $\tilde{B}$ are compact metric spaces, then the continuity of $\tilde{f}$ implies homeomorphism. $\square$

Since the notion of being a connected closed subset is stronger than just being a closed subset, so trivially, we have $CH_n \leq_B H_n$ for all $n \in \mathbb{N}$. Thus we have the following diagram (Figure 4.1).

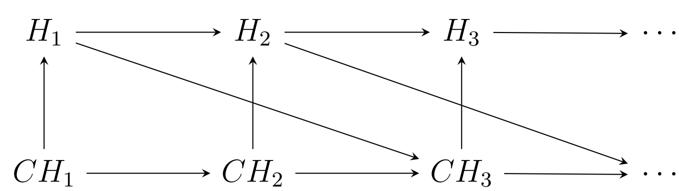


FIGURE 4.1. Reductions between H_n and CH_n

4.2. Comparing R_n and CH_n Relations

As we have mentioned before, $\mathbb{I}^n \in \mathcal{P}$ for all $n \geq 2$, so a direct application of Theorem 3.16 and Remark 3.17 implies that for all $n \geq 2$ and $A, B \subseteq \mathbb{I}^n$ closed, we have

$$A \cong_{R_n} B \Longleftrightarrow \tilde{I}(\mathbb{I}^n, A) \cong \tilde{I}(\mathbb{I}^n, B).$$

Here we have to point out again that the connected spaces $\tilde{I}(\mathbb{I}^n, A), \tilde{I}(\mathbb{I}^n, B)$ can be treated as subsets of $\mathbb{I}^{n+2}$, so we have $R_n \leq_B CH_{n+2}$ for all $n \geq 2$. Now the only case left is when $n = 1$.

THEOREM 4.4. *The restricted homeomorphic relation on $\mathbb{I}$ is Borel reducible to the connected homeomorphic relation on $\mathbb{I}^3$, i.e. $R_1 \leq_B CH_3$.*

PROOF. Let $A, B \subseteq \mathbb{I}$ be closed subsets and $\tilde{I}(\mathbb{I}, A), \tilde{I}(\mathbb{I}, B)$ be constructed. The direction from $A \cong_{R_1} B \Rightarrow \tilde{I}(\mathbb{I}, A) \cong \tilde{I}(\mathbb{I}, B)$ is identical to the proof in Theorem 3.16, so we will only look at the other direction.

Suppose $\tilde{f} : \tilde{I}(\mathbb{I}, A) \to \tilde{I}(\mathbb{I}, B)$ is a homeomorphism, we need to check that

$$\tilde{f}(\mathbb{I} \times \{0\}^2) = \mathbb{I} \times \{0\}^2,$$

and

$$\tilde{f}(A \times \{0\}^2) = B \times \{0\}^2.$$

First, we show that $\mathbf{a}^*$ has a unique topological property that no other point has.

Claim 4.5. *In the quotient space $\tilde{I}(\mathbb{I}, A)$, $\mathbf{a}^*$ is the unique cut point such that $\tilde{I}(\mathbb{I}, A) - \{\mathbf{a}^*\}$ has infinitely many components.*

Proof of Claim: It's clear that $\mathbf{a}^*$ is a cut point such that $\tilde{I}(\mathbb{I}, A) - \{\mathbf{a}^*\}$ has infinitely many components. For the remaining points, we can divide them up into the following cases:

- For all $(\mathbf{x}, \lambda) \in \tilde{I}(\mathbb{I}, A)$, where $\mathbf{x} \in D_A$ and $\lambda \in (0, 1)$, $\tilde{I}(\mathbb{I}, A) - \{(\mathbf{x}, \lambda)\}$ has exactly two components.
- For all $(\mathbf{x}, 0) \in \tilde{I}(\mathbb{I}, A)$, where $\mathbf{x} \in D_A$, it's a non-cut point.
- For all $(a, 0, 0) \in \tilde{I}(\mathbb{I}, A)$, where $a \geq \max\{A\}$ or $a \leq \min\{A\}$, $\tilde{I}(\mathbb{I}, A) - \{(a, 0, 0)\}$ has exactly two components.
- For all $(a, 0, \lambda) \in \tilde{I}(\mathbb{I}, A)$, where $\min A < a < \max A$ and $\lambda < 1$, $(a, 0, \lambda)$ is a non-cut point.

□

Thus $\tilde{f}$ sends $\mathbf{a}^*$ to $\mathbf{b}^*$. If we remove $\mathbf{a}^*, \mathbf{b}^*$ from their respective spaces, then $\tilde{f}$ sends each component in the domain to some component in the codomain.

Claim 4.6. *In the space $\tilde{I}(\mathbb{I}, A) - \{\mathbf{a}^*\}$, there are two types of components:*

(1) First type

$$\{(\mathbf{a}, \lambda) : 0 \leq \lambda < 1\}, \mathbf{a} \in D_A$$

(2) Second type

$$\mathbb{I} \times \{0\}^2 \cup \{(a, 0, \lambda) : a \in A, \lambda < 1\}$$

Proof. Note that for all $\mathbf{a} \in D_A$, the set $\{(\mathbf{a}, \lambda) : 0 \leq \lambda < 1\}$ is clopen and path connected, thus it's a component. It's clear that the only non-cut point is $(\mathbf{a}, 0)$. Similarly, the second type of component is also clopen and path connected. Depending on if A contains 0 or 1, the second type of component could have up to two non-cut points or all cut points. □

In order to differentiate these two types of components, we will state the following property in the space $\tilde{I}(\mathbb{I}, A) - \{\mathbf{a}^*\}$:

(15) $P_{\mathbf{x}}$: For all open neighborhood V of $\mathbf{x}$, V is disconnected as a subspace of $\tilde{I}(\mathbb{I}, A) - \{\mathbf{a}^*\}$.

Claim 4.7. *None of the point in the first type of component satisfies Property (15). On the other hand, all the points in* $\{(a, 0, \lambda) : a \in A, \lambda < 1\}$ *satisfy Property (15).*

Proof. The first sentence is obvious, since the component itself is a path connected clopen set. For all $(a, 0, \lambda)$ where $a \in A, \lambda < 1$, there is a sequence $\{(\mathbf{a}_n, \lambda) : \mathbf{a}_n \in D_A\}_n$ converging to $(a, 0, \lambda)$. Suppose V is an open neighborhood of $(a, 0, \lambda)$, then V must contain some $(\mathbf{a}_n, \lambda)$, thus it is disconnected. □

Therefore, $\tilde{f}$ sends each component $\{(\mathbf{a}, \lambda) : \lambda < 1\}, \mathbf{a} \in D_A$ to some component $\{(\mathbf{b}, \lambda) : \lambda < 1\}, \mathbf{b} \in D_B$, and sends the component $\mathbb{I} \times \{0\}^2 \cup \{(a, 0, \lambda) : a \in A, \lambda < 1\}$ to $\mathbb{I} \times \{0\}^2 \cup \{(b, 0, \lambda) : b \in B, \lambda < 1\}$. Since $(\mathbf{a}, 0)$ is the only non-cut point in the component $\{(\mathbf{a}, \lambda) : \lambda < 1\}$ for all $\mathbf{a} \in D_A$, similarly, $(\mathbf{b}, 0)$ is the only non-cut point in the component $\{(\mathbf{b}, \lambda) : \lambda < 1\}$ for all $\mathbf{b} \in D_B$, so we have

$$\tilde{f}(D_A \times \{0\}) = D_B \times \{0\}.$$

Hence, we also have $\tilde{f}(\overline{D_A} \times \{0\}) = \tilde{f}(\overline{D_B} \times \{0\})$, which implies that $\tilde{f}(A \times \{0\}^2) = B \times \{0\}^2$.

We still need to show that $\tilde{f}(\mathbb{I} \times \{0\}^2) = \mathbb{I} \times \{0\}^2$. Consider the spaces $\tilde{I}(\mathbb{I}, A) - \overline{D_A} \times \{0\}$ and $\tilde{I}(\mathbb{I}, B) - \overline{D_B} \times \{0\}$. $\tilde{f}$ must send the component containing $\mathbf{a}^*$ to the component containing $\mathbf{b}^*$, i.e.

$$\tilde{f}(\{(\mathbf{a}, \lambda) : \mathbf{a} \in \overline{D_A}, \lambda > 0\}) = \{(\mathbf{b}, \lambda) : \mathbf{b} \in \overline{D_B}, \lambda > 0\}.$$

Thus, we have showed $\tilde{f}(\mathbb{I} \times \{0\}^2) = \mathbb{I} \times \{0\}^2$. □

In general, the reduction from CH_n to R_m for some $n, m \in \mathbb{N}$ is unknown at the moment, but we can briefly discuss about the reduction when both n and m are equal to 1.

Define the reduction map $f : \mathbb{I} \to \mathbb{I}$ by $f(\lambda) = (\lambda + 1)/3$. Suppose A, B are closed interval of $\mathbb{I}$, and $g : A \to B$ is a homeomorphism. Without loss of generality, we can assume g is increasing, then g can be extended to g' on $\mathbb{I}$ with $g' \restriction_A = g$, for example,

$$g'(\lambda) = \begin{cases} \frac{\min B}{\min A}\lambda & \text{if } \lambda < \min A \\ g(\lambda) & \text{if } \lambda \in A \\ 1 - (1-\lambda)\frac{1-\max B}{1-\max A} & \text{if } \lambda > \max A \end{cases}.$$

The other direction is trivial. Thus $CH_1 \leq_B R_1$, and we have the following diagram (Figure 4.2).

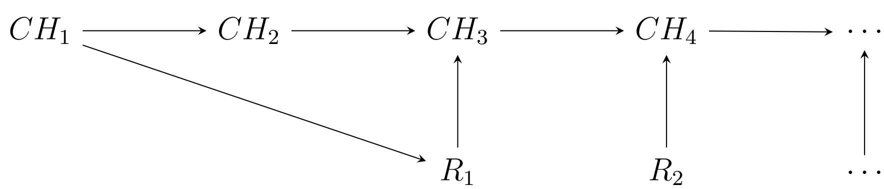


FIGURE 4.2. Reductions between CH_n and R_n

4.3. Comparing R_1 and CR_3 Relations

Suppose $A \subseteq \mathbb{I}$ is a closed set, let $I(\mathbb{I}, A) = \mathbb{I} \times \{0\} \cup D_A$ be constructed as in the Chapter 3. Recall that we have defined D_A in a specific way, say $D_A = \{(a_1, \frac{1}{2}), (a_2, \frac{1}{3}), \dots, (a_n, \frac{1}{n+1}), \dots\}$, where $\{a_1, a_2, \dots\}$ is an enumeration with infinite repetition of a countable dense subset of A. Next, we explain that if we move each point in D_A a little bit upwards and call the new set D'_A, say

$$D'_A := \{(a_1, \frac{1}{2 - 1/2}), (a_2, \frac{1}{3 - 1/3}), \dots, (a_n, \frac{1}{n + 1 - 1/(n+1)}), \dots\}.$$

Then $I(\mathbb{I}, A)$ and $\mathbb{I} \times \{0\} \cup D'_A$ are restricted homeomorphic to each other, i.e. $I(X, A) \cong_{R_2} \mathbb{I} \times \{0\} \cup D'_A$.

For the purpose of notation simplification, we will use a different way representing the sets D_A, D'_A. Suppose $\{a_n\}_n \subseteq A$ is a countable increasing dense subset. Let $D_A = \bigcup_n \{(a_n, \lambda_i^n)\}_i$, where $\lambda_{i+1}^n > \lambda_i^n$ for all $n, i \in \mathbb{N}$, and for each $n \in \mathbb{N}$, $\lambda_i^n \xrightarrow{i} 0$. Similarly, let

$D'_A = \bigcup_n \{(a_n, \mu_i^n)\}_i$, where $\mu_{i+1}^n > \mu_i^n$ for all $n, i \in \mathbb{N}$, and for each n, $\mu_i^n \xrightarrow{i} 0$. Assume $\mu_i^n > \lambda_i^n$ for all $n, i \in \mathbb{N}$ (Figure 4.3).

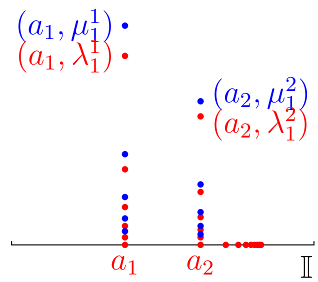


FIGURE 4.3

We claim there is a homeomorphism $\tilde{f} \in \text{Hom}(\mathbb{I}^2)$ such that $\tilde{f} \restriction_{\mathbb{I}\times\{0\}} = \text{id}$ and $\tilde{f}((a_n, \lambda_i^n) = (a_n, \mu_i^n)$ for all $n, i \in \mathbb{N}$.

For each $n \in \mathbb{N}$, we can find a homeomorphism $f_n \in \text{Hom}(\mathbb{I})$ such that $f_n(\lambda_i^n) = \mu_i^n$ for $i \in \mathbb{N}$. Let $\ell_{(n,i)}$ denote the linear function connecting the pair (a_n, λ_i^n) and $(a_{n+1}, \lambda_i^{n+1})$, so for all $a_n \leq x \leq a_{n+1}$, the point $(x, \ell_{(n,i)}(x))$ is on the line $\ell_{(n,i)}$. Similarly, let $\ell'_{(n,i)}$ be the linear function connecting the pair $(a_n, \mu_i^n), (a_{n+1}, \mu_i^{n+1})$.

The sequence $0 \leq a_1 < a_2 < \cdots$ split the unit interval into countable many subintervals. For all $x \in \mathbb{I}$, there exists n with $a_n \leq x \leq a_{n+1}$, let $f_x^2 \in \text{Hom}(\mathbb{I})$ be the map that sends points in$\{\ell_{(n,i)}(x)\}_i$ to the corresponding points in $\{\ell'_{(n,i)}(x)\}_i$.

Define $\tilde{f}$ on $\mathbb{I}^2$ by

$$\tilde{f}(x, y) = (x, f_x^2(y)),$$

then $\tilde{f}$ is clearly a one-to-one, onto, and continuos map, thus $\tilde{f} \in \text{Hom}(\mathbb{I}^2)$.

REMARK 4.8. In some special cases when $A \subseteq \text{Int}(\mathbb{I})$, with the same construction as above, we are able to find a homeomorphism $\tilde{f} \in \text{Hom}(\mathbb{I}^2)$ such that its restriction on the boundary points of $\mathbb{I}^2$ is the identity map and it maps (a_n, λ_i^n) to (a_n, μ_i^n) for all $n, i \in \mathbb{N}$.

In particular, if there are two distinct interior points $(a, \lambda_a), (b, \lambda_b) \in \text{Int}(\mathbb{I}^2)$, we can always find a map $\tilde{f} \in \text{Hom}(\mathbb{I}^2)$ such that $\tilde{f}((a, \lambda_a)) = (b, \lambda_b)$ and the restriction of $\tilde{f}$ on the boundary of $\mathbb{I}^2$ is the identity map.

Theorem 4.9. *The restricted homeomorphic relation on closed subsets of* $\mathbb{I}$ *is Borel reducible to the one on the closed subsets of* $\mathbb{I}^2$, *i.e.* $R_1 \leq_B R_2$.

Proof. The direction from $I(\mathbb{I}, A) \cong_{R_2} I(\mathbb{I}, B) \Rightarrow A \cong_{R_1} B$ is an application of Theorem 3.16. So we will only show the other direction.

Suppose $A, B \subseteq \mathbb{I}$ are closed subsets and $f \in \mathrm{Hom}(\mathbb{I})$ is a homeomorphism with $f(A) = B$, we need to extend f into a homeomorphism $F \in \mathrm{Hom}(\mathbb{I}^2)$ such that $F(I(\mathbb{I}, A)) = I(\mathbb{I}, B)$.

Note that the map $f \times \mathrm{id} \in \mathrm{Hom}(\mathbb{I}^2)$ sends $I(\mathbb{I}, A)$ onto $\mathbb{I} \times \{0\} \cup D'_B$, where $D'_B := (f \times \mathrm{id})(D_A)$ is the countable set of isolated points with $\overline{D'_B} - D'_B = B \times \{0\}$. Then Proposition 3.2 implies that there exists a homeomorphism $\tilde{f}$ between $\mathbb{I} \times \{0\} \cup D'_B$ and $I(\mathbb{I}, B)$ such that $\tilde{f} \restriction_{\mathbb{I}\times\{0\}} = \mathrm{id}$. Thus, we have the following maps

$$I(\mathbb{I}, A) \xrightarrow{f\times \mathrm{id}} \mathbb{I} \times \{0\} \cup D'_B \xrightarrow{\tilde{f}} I(\mathbb{I}, B).$$

So it is enough to extend the map $\tilde{f}$ into some $F \in \mathrm{Hom}(\mathbb{I}^2)$ such that $F \restriction_{\mathbb{I}\times\{0\}} = \mathrm{id}$ and $F(D'_B) = D_B$.

By the way we constructed the sets D_A, D_B, it is possible that $D'_B \cap D_B \neq \emptyset$. We will construct a set $D''_B \subseteq \mathbb{I} \times (0, 1)$, with $D''_B \cap D'_B = \emptyset$, so that $\mathbb{I} \times \{0\} \cup D_B \cong_{R_2} \mathbb{I} \times \{0\} \cup D''_B$. This is made possible by the construction at the beginning of this section. Hence, without loss of generality, we assume $D'_B \cap D_B = \emptyset$.

Suppose we have assigned a one-one correspondence $\tilde{f}$ between D'_B and D_B, now we need to extend $\tilde{f}$ to F. For each pair $\mathbf{b} \in D'_B$ and $\tilde{f}(\mathbf{b}) \in D_B$, we can find a union of rectangles of the form $L_{\mathbf{b}} := \bigcup_{i=1}^{k} [x_1^i, x_2^i] \times [y_1^i, y_2^i]$ for some $k \in \mathbb{N}$, where $L_{\mathbf{b}}$ is simply connected (Figure 4.4) such that

(1) $\{\mathbf{b}, \tilde{f}(\mathbf{b})\} \subseteq L_{\mathbf{b}}$ are interior points,

(2) $L_{\mathbf{b}}$ intersects with D'_B, D_B only at $\mathbf{b}, \tilde{f}(\mathbf{b})$, respectively,

(3) and for all $\mathbf{b}' \in D'_B, \mathbf{b}' \neq \mathbf{b}$, we have $L_{\mathbf{b}} \cap L_{\mathbf{b}'} = \emptyset$.

Note that each $L_{\mathbf{b}}$ is homeomorphic to the unit square $\mathbb{I}^2$.

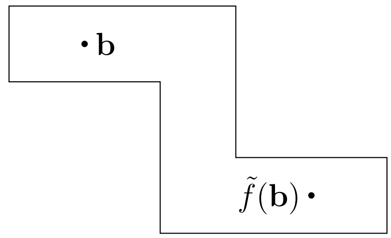


FIGURE 4.4

As stated in Remark 4.8, we could define a homeomorphism $f_{\mathbf{b}}$ on $L_{\mathbf{b}}$ with $f_{\mathbf{b}}(\mathbf{b}) = \tilde{f}(\mathbf{b})$ and such that $f_{\mathbf{b}} = \text{id}$ on the boundary points of $L_{\mathbf{b}}$. We can glue these maps together to form $\bigcup_b f_b : \bigcup_b L_b \to \bigcup_b L_b$. It is one-to-one, onto, and continuous. Finally, define F as the union of the map $\bigcup_{\mathbf{b}} f_{\mathbf{b}}$ and the identity map on the rest of $\mathbb{I}^2$. F is clearly a bijective and continuous map, thus a homeomorphism on $\mathbb{I}^2$. This completes the proof. □

Now that we are able to extend a restricted homeomorphic relation $A \cong_{R_1} B$ into $I(\mathbb{I}, A) \cong_{R_2} I(\mathbb{I}, B)$, the next question would naturally be whether we could extend $I(\mathbb{I}, A) \cong_{R_2} I(\mathbb{I}, B)$ into a restricted homeomorphic relation on the connected subsets of $\mathbb{I}^3$, i.e. is $\tilde{I}(\mathbb{I}, A) \cong_{CR_3} \tilde{I}(\mathbb{I}, B)$?

For one direction, suppose we have constructed the sets $\tilde{I}(\mathbb{I}, A), \tilde{I}(\mathbb{I}, B)$ for some given closed subsets $A, B \subseteq \mathbb{I}$, and assume $\tilde{I}(\mathbb{I}, A) \cong_{CR_3} \tilde{I}(\mathbb{I}, B)$, then Theorem 3.16 implies $A \cong_{R_1} B$.

On the other hand, given two closed subsets A, B with $A \cong_{R_1} B$, we can extend it into $I(\mathbb{I}, A) \cong_{R_2} I(\mathbb{I}, B)$. Let $\mathbf{x}_0 \in \text{Int}(\mathbb{I}^3)$ be an interior point, and we use it as the vertex and the six sides of the unit cube $\mathbb{I}^3$ as the bases to divide the unit cube into six cones.

One of the cones is homeomorphic to $\tilde{I}(\mathbb{I}, A)$, it consists of the points

$$\Lambda_{\tilde{I}} := \{\lambda\mathbf{x} + (1-\lambda)\mathbf{x}_0 : \mathbf{x} \in \mathbb{I} \times \mathbb{I} \times \{0\}, \lambda \in \mathbb{I}\}.$$

Suppose $F \in \text{Hom}(\mathbb{I}^2)$ such that $F(I(\mathbb{I}, A)) = I(\mathbb{I}, B)$, we can extend F into a homeomorphism $F_{\tilde{I}}$ on the cone $\Lambda_{\tilde{I}}$ by

$$F_{\tilde{I}}(\lambda\mathbf{x} + (1-\lambda)\mathbf{x_0}) = \lambda F(\mathbf{x}) + (1-\lambda)\mathbf{x}_0 \text{ for all } \lambda \in \mathbb{I} \text{ and } \mathbf{x} \in \mathbb{I}^2 \times \{0\}.$$

Similarly, we can define a homeomorphism for each of the other five cones so that they agree on the intersections. Hence, the union of these homeomorphism forms a homeomorphism on the unit cube $\mathbb{I}^3$.

Theorem 4.10. *The restricted homeomorphic relation on closed subsets of* $\mathbb{I}$ *is Borel reducible to the one on the connected closed subsets of* $\mathbb{I}^3$, *i.e.* $R_1 \leq_B CR_3$.

CHAPTER 5

LOCALLY COMPACT TSI NON-ARCHIMEDEAN POLISH GROUPS

5.1. The Characterization Theorem

Recall that if $\mathcal{C}$ is a class of topological groups, then a *universal* element of $\mathcal{C}$ is a group $G \in \mathcal{C}$ such that any other group $H \in \mathcal{C}$ is topologically isomorphic to a closed subgroup of G. Similarly, a *surjectively universal element* of $\mathcal{C}$ is a group $G \in \mathcal{C}$ such that for any $H \in \mathcal{C}$ there is a continuous homomorphism from G onto H.

In this chapter, we consider the class of locally compact TSI non-Archimedean Polish groups, and we try to answer the questions about the existence (or non-existence) of a universal or surjectively universal element in that class.

First, we will prove a characterization theorem of all the groups in the class.

Theorem 5.1. *Let G be a topological group. Then the following are equivalent:*

(1) G is a locally compact TSI non-Archimedean Polish group.

(2) G has a neighborhood base for the identity that consists of compact open normal subgroups.

(3) There is an inverse system $\{\Gamma_i\}_{i\in\mathbb{N}}$, where each Γ_i is a discrete countable group and for all $i \leq j, \ker \pi_{j,i}$ is finite, such that G is isomorphic to $\varprojlim \Gamma_i$.

Proof. (1) $\Rightarrow$ (2): By the assumption that G is TSI and Klee's Theorem 2.13, there exists a countable open neighborhood $\{U_n\}$ with $gU_ng^{-1} = U_n$ for all $n \in \mathbb{N}$, then the set of groups generated by these open neighborhood $\{\langle U_n\rangle\}$ forms a countable open normal subgroups.

Since G is a non-Archimedean Polish group, there exists a countable open neighborhood base $\{V_n\}$ of 1_G consisting of all open subgroups. We claim that $\{\langle U_n\rangle\}$ is a base for 1_G.

Let $W \subseteq G$ be any open neighborhood of 1_G, we need to find some $n \in \mathbb{N}$ with $\langle U_n\rangle \subseteq W$. For this let m be such that $V_m \subseteq W$ and n be such that $U_n \subseteq V_m$. Since V_m is a subgroup, we have $\langle U_n\rangle \subseteq V_m \subseteq W$, as required.

Since G is locally compact, then there exists a open subset V of 1_G whose closure is compact. Without loss of generality, we can assume $\langle U_n \rangle \subseteq V$ for all $n \in \mathbb{N}$, so all the groups $\langle U_n \rangle$ are compact normal open subgroups.

(2) $\Rightarrow$ (1): This follows from the definition of a locally compact TSI non-Archimedean Polish group.

(2) $\Rightarrow$ (3) : Suppose $\{U_i\}_{i \in \mathbb{N}}$ is such a base. Without loss of generality, we can assume $U_0 \supseteq U_1 \supseteq \cdots \supseteq U_i \supseteq \cdots$.Then $\{G/U_i\}_{i \in \mathbb{N}}$ is an inverse system of countable discrete groups. For all $i \leq j$, let $\pi_{j,i} : G/U_j \longrightarrow G/U_i$ be the natural homomorphism. Then $\ker \pi_{j,i} = U_i/U_j$ is finite, since U_i is compact.

To see that G is isomorphic to $\varprojlim G/U_i$, define a map $\varphi : G \to \prod G/U_i$ by $\varphi(g) = (gU_i)$. First, φ is injective, since for all $g \neq g'$, there exists U_n for some $n \in \mathbb{N}$ such that $g' \notin gU_n$, so $g'U_n \neq gU_n$, i.e. $\varphi(g) \neq \varphi(g')$. Secondly, φ maps G onto $\varprojlim G/U_i$. For all $(g_iU_i) \in \varprojlim G/U_i$, we have $\bigcap g_iU_i \neq \emptyset$, as each g_iU_i is compact for all i and $\{g_iU_i\}$ satisfies finite intersection property. Since $\bigcap U_i = \{1_G\}$, so $\bigcap g_iU_i$ is a unique element in G with $\varphi(\bigcap g_iU_i) = (g_iU_i)$.

φ is a clearly a continuous map. Finally, we check φ^{-1} is also continuous. It is enough to check φ^{-1} is continuous at $(1_GU_i) \in \prod G/U_i$. Suppose $U \subseteq G$ is an open neighborhood of 1_G, then there exists $U_n \subseteq U$ for some $n \in \mathbb{N}$, and $\prod_{i=0}^{n-1} G/U_i \times U_n \times \prod_{i=n+1}^{\infty} G/U_i$ is a basic open set in $\prod G/U_i$ such that the φ^{-1} image of it is contained in U. This shows φ is an isomorphism between G and $\varprojlim G/U_i$.

(3) $\Rightarrow$ (2) : Let $\{\Gamma_i\}_{i \in \mathbb{N}}$ be such an inverse system. Then the countable set $\{\prod_{i=0}^{n}\{1_{\Gamma_i}\} \times \prod_{i=n+1}^{\infty} \Gamma_i\}_n$, where 1_{Γ_i} is the identity element in Γ_i, is an open neighborhood of $1_{\prod \Gamma_i}$ consisting of open normal subgroups. Thus $\prod \Gamma_i$ is a TSI non-Archimedean Polish group. G being isomorphic to a closed subgroup $\varprojlim \Gamma_i$ of $\prod \Gamma_i$ is then a TSI non-Archimedean Polish group as well.

We just need to show there is an open compact subgroup in G. Or equivalently, we need to show there is an open compact subgroup in $\varprojlim \Gamma_i$. Consider the projection $\pi_0 : \varprojlim \Gamma_i \longrightarrow \Gamma_0$, $\pi_0^{-1}(1_{\Gamma_0})$ is an open subgroup of $\varprojlim \Gamma_i$. Let's check it is also compact.

Note that $\{\ker \pi_{i,0}\}_{i\in\mathbb{N}}$ is an inverse system, and each $\ker \pi_{i,0}$ is finite by the assumption. Since $\pi_0^{-1}(1_{\Gamma_0}) = \varprojlim \ker \pi_{i,0}$ is a closed subset of $\prod \ker \pi_{i,0}$, and $\prod \ker \pi_{i,0}$ is a compact space, we have that $\pi_0^{-1}(1_{\Gamma_0})$ is compact. □

Remark 5.2. If G is a TSI non-Archimedean Polish group (without the condition of locally compact), we are still able to express G as the inverse limit of an inverse system $\{\Gamma_i\}_{i\in\mathbb{N}}$, where each Γ_i is a discrete countable group.

5.2. Universal Groups

We will now show that there does not exist a universal element in the class of locally compact TSI non-Archimedean Polish groups. In order to achieve this, we will need the following fact.

Theorem 5.3 ([15]). *There are uncountably many non-isomorphic finitely generated simple groups.*

Note that each finitely generated simple group falls into the class of our interest. Then by the characterization theorem of locally compact TSI non-Archimedean Polish groups, it turns out that there is no such a group in which uncountably many non-isomorphic countable simple groups can be embedded.

Definition 5.4. Let DCG denote the class of all discrete countable groups; and let TSI non-Arch denote all TSI non-Archimedean Polish groups.

DCG contains all finitely generated simple groups. And all the groups in TSI non-Arch have a very nice form as stated in Remark 5.2. So the next theorem shows that any class of groups containing DCG and contained in TSI non-Arch does not have a universal element.

Theorem 5.5. *Let $\mathcal{L}$ be a class of topological groups, and assume*

$$DCG \subseteq \mathcal{L} \subseteq TSI\ non\text{-}Arch.$$

Then $\mathcal{L}$ does not have a universal group.

Proof. By way of contradiction, suppose there is a universal group in $\mathcal{L}$, then it isomorphic to a close subgroup of $\prod \Gamma_i$, where each Γ_i is a countable discrete group. In particular, every countable group can be embedded into $\prod \Gamma_i$. Since $\mathcal{L} \supseteq$ DCG, so every finitely generated simple group is isomorphic to a closed subgroup of $\prod \Gamma_i$.

Let $G_\lambda, \lambda \in \Lambda$, be an uncountable family of non-isomorphic finitely generated simple groups. For each $\lambda \in \Lambda$ let e_λ be an embedding from G_λ into $\prod \Gamma_n$. For each $i \in \mathbb{N}$, let $\pi_i : \prod \Gamma_i \to \Gamma_i$ be the projection map. Then each $\pi_i \circ e_\lambda : G_\lambda \to \Gamma_i$ is a group homomorphism. Since G_λ is simple, $\ker(\pi_i \circ e_\lambda)$ is either $\{1_{G_\lambda}\}$ or G_λ itself. For each λ, there exists some $i \in \mathbb{N}$ such that $\ker(\pi_i \circ e_\lambda) = \{1_{G_\lambda}\}$, because e_λ is a group isomorphism (and therefore a nontrivial homomorphism). For such an i, $\pi_i \circ e_\lambda$ is an embedding from G_λ into Γ_i.

We have concluded that for each $\lambda \in \Lambda$ there is $i \in \mathbb{N}$ such that G_λ is isomorphic to a subgroup of Γ_n. However, the collection

$$\{G : G \leq \Gamma_i \text{ for some } i \in \mathbb{N}\}$$

is countable, whereas $\{G_\lambda : \lambda \in \Lambda\}$ is uncountable. A contradiction. □

Corollary 5.6. *There does not exists a universal group for all locally compact TSI non-Archimedean Polish gorups.*

5.3. Surjectively Universal Groups

Suppose we have two inverse systems $\{\Gamma_n\}_n$ and $\{\Gamma'_n\}_n$,

$$\Gamma_0 \xleftarrow{\pi_{1,0}} \Gamma_1 \xleftarrow{\pi_{2,1}} \cdots$$

$$\Gamma'_0 \xleftarrow{\pi'_{1,0}} \Gamma'_1 \xleftarrow{\pi'_{2,1}} \cdots$$

if there are surjective homomorphisms $\phi_n : \Gamma_n \to \Gamma'_n$ such that the following diagram commutes

$$\begin{array}{ccccc} \Gamma_0 & \xleftarrow{\pi_{1,0}} & \Gamma_1 & \xleftarrow{\pi_{2,1}} & \cdots \\ \downarrow{\scriptstyle \phi_0} & & \downarrow{\scriptstyle \phi_1} & & \\ \Gamma'_0 & \xleftarrow{\pi'_{1,0}} & \Gamma'_1 & \xleftarrow{\pi'_{2,1}} & \cdots \end{array}$$

Then $(\phi_n)_n$ is a homomorphism from $\varprojlim \Gamma_n$ into $\varprojlim \Gamma'_n$.

Let $\mathbb{F}_\omega$ be the free group generated by some countable set X. We have a natural inverse system:

$$\mathbb{F}_\omega \overset{\mathrm{id}}{\leftarrow} \mathbb{F}_\omega \overset{\mathrm{id}}{\leftarrow} \cdots$$

Then for an arbitrary inverse system $\{\Gamma_n\}_{n\in\mathbb{N}}$ of countable discrete groups, it is easy to find surjective homomorphisms $\phi_n : \mathbb{F}_\omega \to \Gamma_n$ such that the following diagram commutes

$$\begin{array}{ccccc} \mathbb{F}_\omega & \xleftarrow{\mathrm{id}} & \mathbb{F}_\omega & \xleftarrow{\mathrm{id}} & \cdots \\ \downarrow{\scriptstyle \phi_0} & & \downarrow{\scriptstyle \phi_1} & & \\ \Gamma_0 & \xleftarrow{\pi_{1,0}} & \Gamma_1 & \xleftarrow{\pi_{2,1}} & \cdots \end{array}$$

However, the map $(\phi_n)_n$ is in general not surjective from $\varprojlim \mathbb{F}_\omega$ into $\varprojlim \Gamma_n$.

In the next theorem, we will construct an inverse system consisting of a certain class of quotient groups of $\mathbb{F}_\omega$ and show its inverse limit is a surjectively universal group.

THEOREM 5.7. *There exists a surjectively universal group for all locally compact TSI non-Archimedean Polish groups.*

PROOF. Let X be a countable free generating set of $\mathbb{F}_\omega$. Write $X = \bigcup_{n\in\mathbb{N}} X_n$ as a disjoint union, where each X_n is infinite. Define M_0 to be the kernel of the homomorphism $\varphi_0 : \mathbb{F}_\omega \longrightarrow \langle X_0 \rangle$, where φ_0 is generated by the map:

$$\varphi_0(g) = \begin{cases} g, & \text{if } g \in X_0; \\ 1, & \text{if } g \in \bigcup_{i\geq 1} X_i. \end{cases}$$

Consider the set

$$\mathcal{N} = \{N \trianglelefteq \mathbb{F}_\omega : N \leq M_0, M_0/N \text{ is finite}, \{n \geq 1 : X_n - N \neq \emptyset\} \text{ is finite}\}.$$

Claim 5.8. *$\mathcal{N}$ is countable and is closed under intersection.*

Proof of Claim. $\mathcal{N}$ is clearly closed under intersection, so we will just show it is countable. Note that

$$\begin{aligned}\mathcal{N} &\subseteq \{N \trianglelefteq M_0 : M_0/N \text{ is finite}, \{n \geq 1 : X_n - N \neq \emptyset\} \text{ is finite}\} \\ &\subseteq \{N \trianglelefteq M_0 : M_0/N \text{ is finite}\}.\end{aligned}$$

Then it is enough to show that $\{N \trianglelefteq M_0 : M_0/N \text{ is finite}\}$ is countable.

Since $M_0 \trianglelefteq \mathbb{F}_\omega$ is a free group, let us enumerate a free generating set of M_0, say $\{m_0, m_1, \dots\}$. Let $K_i = \langle m_i, \dots \rangle^{\mathrm{N}}$ be the normal subgroup in M_0 generated by $m_i, m_{i+1}, \dots$, or $K_i = \ker f_i$, where f_i is the homomorphism $f_i : M_0 \to \langle m_0, m_1, \dots, m_{i-1} \rangle$ generated by

$$f_i(g) = \begin{cases} g, & \text{if } g \in \{m_0, m_1, \dots, m_{i-1}\}; \\ 1, & \text{if } g \in \{m_i, m_{i+1}, \dots\}. \end{cases}$$

Then $M_0/K_i \cong \langle m_0, \dots, m_{i-1} \rangle$, i.e. M_0/K_i is finitely generated free group.

Note that

$$\{N \trianglelefteq M_0 : M_0/N \text{ is finite}\} = \bigcup_{i \in \mathbb{N}} \{N \trianglelefteq M_0 : N \geq K_i, M_0/N \text{ is finite}\}.$$

Since suppose N belongs to the left hand side of the equation above, then there exists a finite subset of the generating set, say $\{m_{i_0}, \dots, m_{i_j}\}$ for some $j \in \mathbb{N}$, such that $M_0 = \left\langle m_{i_0}, \dots, m_{i_j}, N \right\rangle$. Let $i_{-1} = \max\{i_0, \dots, i_j\} + 1$, then $\{m_{i_{-1}}, m_{i_{-1}+1}, \dots\} \subseteq N$ and hence $K_{i_{-1}} \leq N$. So it is enough to show that for all $i \in \mathbb{N}$,

$$\{N \trianglelefteq M_0 : N \geq K_i, M_0/N \text{ is finite}\} \text{ is countable.}$$

Fix i, for all $N \in \{N \trianglelefteq M_0 : N \geq K_i, M_0/N \text{ is finite}\}$, we have that

$$M_0/N \cong (M_0/K_i)/(N/K_i)$$

is finite. Since in addition, M_0/K_i is finitely generated, so N/K_i is finitely generated.

Then there are finitely many elements from M_0, say $n_0, n_1, \ldots, n_{i_k}$, for some $i_k \in \mathbb{N}$, such that $N/K_i = \langle n_0K_i, n_1K_i, \ldots, n_{i_k}K_i \rangle$.

Let $\mathcal{L}$ be the set of all cosets of K_i in M_0, i.e. $\mathcal{L} = \{gK_i : g \in M_0\}$, then $\mathcal{L}$ is countable. And there is a one-to-one map from $\{N \trianglelefteq M_0 : N \geq K_i, M_0/N \text{ is finite}\}$ into a finite subset of $\mathcal{L}$. Therefore $\{N \trianglelefteq M_0 : N \geq K_i, M_0/N \text{ is finite}\}$ is countable, this completes the proof of the claim.

□

The inverse inclusion is a directed partial order on $\mathcal{N}$. This gives an inverse system $\{\mathbb{F}_\omega/N\}_{N\in\mathcal{N}}$. We claim that $\varprojlim_{N\in\mathcal{N}} \mathbb{F}_\omega/N$ is surjectively universal for all locally compact TSI non-Archimedean Polish groups. Without loss of generality, we can assume $N_0 \leq N_1 \leq \cdots \leq N_i \leq \ldots$ is a cofinal sequence in $\mathcal{N}$. So $\varprojlim_{N\in\mathcal{N}} \mathbb{F}_\omega/N = \varprojlim_{i\in\mathbb{N}} \mathbb{F}_\omega/N_i$.

First we need to check $\varprojlim_{i\in\mathbb{N}} \mathbb{F}_\omega/N_i$ is a locally compact TSI non-Archimedean Polish group. For all $i \leq j$, the kernel of the map $\pi_{j,i} : \mathbb{F}_\omega/N_j \longrightarrow \mathbb{F}_\omega/N_i$ is N_i/N_j. And $N_i/N_j \subseteq M_0/N_j$ is finite. Thus by Theorem 5.1, $\varprojlim_{i\in\mathbb{N}} \mathbb{F}_\omega/N_i$ is such a group.

Next we need to show $\varprojlim_{i\in\mathbb{N}} \mathbb{F}_\omega/N_i$ is surjectively universal. Let $\{\Gamma_n\}_{n\in\mathbb{N}}$ be an inverse system, where Γ_n are countable discrete groups, and for all $i \leq j$, the homomorphism $\gamma_{j,i} : \Gamma_j \longrightarrow \Gamma_i$ has a finite kernel. Let D_0 be a subset of $\varprojlim \Gamma_i$ such that $\pi_0(D_0) = \Gamma_0$. And for each $i \geq 1$, let D_i be a finite subset of $\varprojlim \Gamma_i$ such that $\pi_i(D_i) = \ker \gamma_{i,0}$. Fix a surjective map $k_i : X_i \longrightarrow D_i$ for each $i \in \mathbb{N}$. Let $D = \bigcup_{i\in\mathbb{N}} D_i$ and $k = \bigcup_{i\in\mathbb{N}} k_i$, then define ϕ_i to be the group homomorphism generated by the map $\pi_i \circ k$, i.e.

$$\mathbb{F}_\omega \xrightarrow{k} \varprojlim \Gamma_i \xrightarrow{\pi_i} \Gamma_i.$$

Clearly, $\phi_i(\mathbb{F}_\omega) = \Gamma_i$, and $\mathbb{F}_\omega/\ker\phi_i \cong \Gamma_i$. Then we define $M_i' = M_0 \cap \ker\phi_i$ for $i \geq 1$.

Claim 5.9. *$M_i' \in \mathcal{N}$ for all $i \in \mathbb{N}$.*

Proof of Claim. Note that $M_0 \subseteq \ker\phi_0$. Actually, if $g \in M_0$ then g must be generated by elements from $\bigcup_{i\geq 1} X_i$, so $\phi_0(g) = 1_{\Gamma_0}$.

Fix $i \in \mathbb{N}$. Suppose $gM_i' \neq g'M_i'$ in M_0/M_i', then $gg'^{-1} \notin \ker \phi_i \cap M_0$. Since $gg'^{-1} \in M_0$, so $gg'^{-1} \notin \ker \phi_i$, i.e. $g \ker \phi_i \neq g' \ker \phi_i$. By assumption, $\ker \phi_0 / \ker \phi_i$ is finite, so there are at most finitely many cosets in M_0/M_i'. This completes the proof of the claim. $\square$

Note that $M_0, M_1', \ldots,$ may not be cofinal in $\mathcal{N}$, so we need to define a new sequence that is cofinal in $\mathcal{N}$. Let $M_i = M_i' \cap N_i$ for $i \geq 1$, then $\varprojlim_{i\in\mathbb{N}} \mathbb{F}_\omega/N_i \cong \varprojlim_{i\in\mathbb{N}} \mathbb{F}_\omega/M_i$. Now consider the following diagram:

$$
\begin{array}{ccccccc}
\mathbb{F}_\omega/M_0 & \longleftarrow & \mathbb{F}_\omega/M_1 & \longleftarrow & \mathbb{F}_\omega/M_2 & \longleftarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathbb{F}_\omega/\ker \phi_0 & \longleftarrow & \mathbb{F}_\omega/\ker \phi_1 & \longleftarrow & \mathbb{F}_\omega/\ker \phi_2 & \longleftarrow & \cdots \\
\phi_0 \downarrow & & \phi_1 \downarrow & & \phi_2 \downarrow & & \\
\Gamma_0 & \xleftarrow{\gamma_{1,0}} & \Gamma_1 & \xleftarrow{\gamma_{2,1}} & \Gamma_2 & \xleftarrow{\gamma_{3,2}} & \cdots
\end{array}
$$

All the unidentified maps are natural homomorphism π. It is clear that

$$\varprojlim \mathbb{F}_\omega/\ker \phi_i \cong \varprojlim \Gamma_i.$$

So it's enough to show that there is a map from $\varprojlim \mathbb{F}_\omega/M_i$ onto $\varprojlim \mathbb{F}_\omega/\ker \phi_i$. We first prove a lemma.

Lemma 5.10. *For each $i \in \mathbb{N}$, $gM_0 \cap r \ker \phi_{i+1} \neq \emptyset$ for all $g, r \in \ker \phi_0$.*

Proof of Lemma. It's enough to show $gM_0 \cap \ker \phi_{i+1} \neq \emptyset$ for all $g \in \ker \phi_0$. By taking ϕ_{i+1} on both sides, it's enough to show that for all $g \in \ker \phi_0$, there exists $g' \in M_0$ such that $\phi_{i+1}(gg') = 1$.

Fix $g \in \ker \phi_0$, i.e. $\phi_0(g) = 1$, then $\phi_{i+1}(g) \in \ker \gamma_{i+1,0}$. Hence there exists $g'' \in X_{i+1}$ such that $\phi_{i+1}(g'') = \phi_{i+1}(g)$. Note that $X_{i+1} \subseteq M_0$ for all $i \in \mathbb{N}$, then let $g' = g''^{-1} \in M_0$. And this completes the proof. $\square$

Claim 5.11. *The map $(\pi)_i : \varprojlim \mathbb{F}_\omega/M_i \to \varprojlim \mathbb{F}_\omega/\ker \phi_i$ is surjective.*

PROOF. Pick an arbitrary $(r_i \ker \phi_i)_i \in \varprojlim \mathbb{F}_\omega / \ker \phi_i$. Fix $g_0 M_0 \in \pi^{-1}(r_0 \ker \phi_0) \subseteq \mathbb{F}_\omega / M_0$, then $\pi^{-1}(g_0 M_0) \subseteq \mathbb{F}_\omega / M_i$ is a finite subset, where $\pi : \mathbb{F}_\omega / M_i \to \mathbb{F}_\omega / M_0$ is the natural homomorphism.

For all $i \geq 1$, since $r_i \ker \phi_i \subseteq r_0 \ker \phi_0$ and $g_0 M_0 \subseteq r_0 \ker \phi_0$, so both $r_0^{-1} r_i$ and $r_0^{-1} g_0$ belong to $\ker \phi_0$. By Lemma 5.10, $r_0^{-1} g_0 M_0 \cap r_0^{-1} r_i \ker \phi_i \neq \emptyset$, i.e. $g_0 M_0 \cap r_i \ker \phi_i \neq \emptyset$, for all $i \geq 1$.

Hence there exists some element in $\pi^{-1}(g_0 M_0) \subseteq \mathbb{F}_\omega / M_i$ that can be mapped to $r_i \ker \phi_i$, say $g_0 M_0 \cap r_i \ker \phi_i$ (see the following diagram).

$$\begin{array}{ccccc} (g_0 M_0) & \mathbb{F}_\omega / M_0 & \xleftarrow{\ \pi\ } & \mathbb{F}_\omega / M_i & (g_0 M_0 \cap r_i \ker \phi_i) \\ & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} & \\ (r_0 \ker \phi_0) & \mathbb{F}_\omega / \ker \phi_0 & \xleftarrow{\ \pi\ } & \mathbb{F}_\omega / \ker \phi_i & (r_i \ker \phi_i) \end{array}$$

Now we get a tree with root $g_0 M_0$ and finitely many nodes $\pi^{-1}(g_0 M_0) \subseteq \mathbb{F}_\omega / M_i$ of length i, and $\pi^{-1}(g_0 M_0) \subseteq \mathbb{F}_\omega / M_i$ is non-empty, for all $i \in \mathbb{N}$, so the tree is infinite. By König's Lemma (c.f. [13], p.20), there exists an infinite branch $(g_0 M_0, g_1 M_1, \dots)$. Clearly, $(g_0 M_0, g_1 M_1, \dots) \in \varprojlim_{i \in \mathbb{N}} \mathbb{F}_\omega / M_i$. □

□

REMARK 5.12. With the same method, we can show that there exists a surjectively universal group for the class of locally compact abelian Polish groups. But it is still unknown whether there is a universal one.

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