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Edge plasma boundary layer generated by kink modes in tokamaks

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This paper describes the structure of the electric current generated by external kink modes at the plasma edge using the ideally conducting plasma model. It is found that the edge current layer is created by both wall touching and free boundary kink modes. Near marginal stability, the total edge current has a universal expression as a result of partial compensation of the δ -functional surface current by the bulk current at the edge. The resolution of an apparent paradox with the pressure balance across the plasma boundary in the presence of the surface currents is provided.

I. INTRODUCTION

Recently the important role of electric currents due to kink modes at the plasma edge of tokamaks was understood. When the plasma column, which initially is in magneto-hydrodynamic (MHD) equilibrium, is deformed it requires special external fields to maintain the equilibrium. Without them, the surface currents would be excited at the plasma edge in order to eliminate the normal component of magnetic field to the plasma boundary [1–3]. Fig.1 illustrates this statement for the case of a straight plasma column with the current I_{pl} in the longitudinal magnetic field B_ϕ , where plasma is deformed by the $m/n = 1/1$ kink mode (m, n are poloidal and toroidal wave numbers).

FIG. 1. (a) Toroidal magnetic field lines punch the plasma surface; (b) Surface currents eliminate B_{norm} : ■ opposite to I_{pl} , ■ in the direction of I_{pl} ; (c) Hiro currents in the wall are always opposite to I_{pl} . The eddy currents remain localized (red spot) inside the wall sectors.

Fig.1(a) shows the existence of the normal field component B_{norm} if there would be no surface currents. This situation is not consistent with the properties of the high temperature plasma, which instead always excites surface (not necessary δ -functional) currents. For the $1/1$ kink mode their direction is shown in Fig. 1(b) with blue indicating current in opposite to the plasma current I_{pl} direction and red in the same direction as I_{pl} .

The surface currents are always negative at the plasma side moving toward the wall. As a result, after touching the wall, the surface currents at the plasma boundary will be shared with the wall in the form of negative Hiro currents, as is illustrated in Fig. 1(c) regardless the gaps in the wall. This creates a new regime for MHD instabilities, called the wall touching kink mode (WTKM) [3].

In the case of the $m/n = 1/1$ WTKM the negative Hiro currents in the wall explain the asymmetry in the plasma current measurements in JET [4–6]. There is every reason to expect that the same Hiro currents can explain the positive current spike in internal plasma current measurements in conventional disruptions as well.

Because of Hiro currents, the physics of the plasma edge and plasma-wall interactions is of the great importance for plasma dynamics during disruptions. Recently, the first step in this direction was made in Ref. [7]. In particular, it was determined that for the free boundary kink modes (not WTKM) the surface currents remain δ -functional (as in the perturbed equilibrium theory) even taking into account the finite growth rate. They always flow along the field lines.

In this paper, the simplest model of the plasma edge, based on the ideal linear MHD approximation is presented as a reference case for future models. The circular cross-section plasma in the strong longitudinal magnetic field is considered as a simplest case revealing the properties of the plasma edge. In Sect. 2 the basic set of equations is specified. Sect. 3 describes the δ -functional surface currents for both WTKM and free boundary kink modes (FBKM).

Sect. 4 describes the structure of the boundary layer for both kinds of external kink modes. The analytical expressions for characteristics of the boundary layer are obtained. In particular, they correct one of the estimates of the δ -functional current given earlier [7]. Sect. 5 explains the mechanism, compensating the component of surface current perpendicular to the magnetic field and resolves the apparent inconsistency between direction of surface currents along the field lines and the periodicity constraint. The Appendix contains corrections to some technical mistakes in Ref. [7].

II. LINEARIZED MHD EQUATIONS

The linearized MHD equations are well-known [8–11]. For a stratified model of a tokamak plasma column in a strong magnetic field we for simplicity follow the recent Ref. [7].

We consider the straight cylinder case as a basic approximation for a toroidal plasma and accordingly use $r, \omega, R\phi$ ($0 \leq r \leq a$) as cylindrical coordinates with a, R being minor and major radii of toroidal plasma and perturbations of the form

$$r = r_0 + \xi(r)e^{\gamma t} \cos(m\omega - n\phi), \quad (1)$$

where r_0 is the cylindrical radius of the unperturbed magnetic surface. The perturbation of the magnetic field $\tilde{\mathbf{B}}$ can be specified as [12]

$$\tilde{\mathbf{B}} = \tilde{B}_r \mathbf{e}_r + \tilde{B}_\omega \mathbf{e}_\omega + \tilde{B}_\phi \mathbf{e}_\phi = \nabla\psi \times \mathbf{e}_\phi + \tilde{B}_\phi \mathbf{e}_\phi, \quad \tilde{B}_r = \frac{1}{r}\psi'_\omega, \quad \tilde{B}_\omega = -\psi'_r. \quad (2)$$

The perturbation of vector potential ψ and plasma displacement ξ are related through

$$\psi = B_\omega^* \xi, \quad B_\omega^* \equiv B_\omega - \frac{nr}{mR} B_\phi = \frac{B_\phi r}{R} \frac{f}{m}, \quad (3)$$

where we introduced notations

$$\mu(r) \equiv \frac{1}{q(r)} = \frac{RB_\omega}{rB_\phi}, \quad f(r) \equiv m\mu - n, \quad (4)$$

The function $f(r)/m$ is the same as Δ in Eq. (32) of Ref. [7].

In terms of normalized growth rate $\bar{\gamma}_m$ and variable η

$$\bar{\gamma}_m^2 \equiv \frac{\mu_0 \rho \gamma^2 R^2}{B_\phi^2}, \quad \eta \equiv r\xi, \quad (5)$$

where ρ is the plasma density (assumed to be uniform), and μ_0 is the vacuum magnetic permeability, the linearized equation determining η and ψ has the form [7]

$$[r(\bar{\gamma}^2 + f^2)\eta']' = \frac{m^2}{r}(\bar{\gamma}^2 + f^2)\eta + (f^2)'\eta, \quad \psi = \frac{B_\phi}{R} \frac{f}{m} \eta. \quad (6)$$

In the vacuum outside the plasma

$$\psi^v = \frac{\frac{a^m}{r^m} - \lambda^m \frac{r^m}{a^m}}{1 - \lambda^m} \frac{B_\phi}{R} \frac{f}{m} \eta_a, \quad \frac{a\psi'^v(a)}{\psi^v} = -m - \frac{2m\lambda^m}{1 - \lambda^m}, \quad \lambda \equiv \frac{a^2}{b^2}, \quad (7)$$

where λ takes into account the presence of the wall with radius b .

The growth rate is determined by the matching condition obtained by integrating Eq. (6) across the plasma boundary

$$(\bar{\gamma}^2 + f^2) \frac{a\eta'}{\eta} \Big|_a = f_a^2 \frac{a\eta'^v}{\eta} = f_a^2 \frac{a\psi'^v}{\psi^v} - m f_a a \mu'^v = m(2\mu_a - f_a) f_a - \frac{2m\lambda^m}{1 - \lambda^m} f_a^2. \quad (8)$$

The perturbation $\tilde{B}_\xi$ of the poloidal component of magnetic field at the perturbed magnetic surface is determined by

$$\tilde{B}_\xi = \tilde{B}_\omega + B'_\omega(r)\xi = -\psi' + B'_\omega(r)\xi = \frac{B_\phi}{R} \left(\mu - \frac{f}{m} \frac{r\eta'}{\eta} \right) \xi. \quad (9)$$

The difference between $\tilde{B}_\xi^v$ at the vacuum side of the plasma surface and $\tilde{B}_\xi^c$ inside the plasma core represents the edge current $I^{edge}(r)$ in the layer $a - r$

$$I^{edge}(r) \equiv a\tilde{B}_\xi^v(a) - r\tilde{B}_\xi^c(r). \quad (10)$$

In terms of vector potential ψ it can be expressed as

$$I^{edge}(r) = \left(\frac{r\psi'}{m\psi} f - \frac{jR}{B_\phi} + \mu \right) \frac{m\psi}{f} + \left(f_a + \frac{2\lambda^m f_a}{1 - \lambda^m} - \mu_a \right) \frac{m\psi_a}{f_a} \quad (11)$$

or in terms of solution $\eta(r)$ to Eq. (6) as

$$I^{edge}(r) = \frac{B_\phi}{R} \left[\left(f \frac{r\eta'}{m\eta} - \mu \right) \eta(r) + \left(f_a + \frac{2\lambda^m f_a}{1 - \lambda^m} - \mu_a \right) \eta_a \right]. \quad (12)$$

The surface value $I^{edge}(a)$ of edge current represents the δ -functional part of it, i.e., the surface current $\mu_0 i^{surf}$ excited by the kink mode

$$\mu_0 i^{surf} \equiv \frac{I^{edge}(a)}{a} = \frac{B_\phi}{R} \left[-\frac{jR}{B_\phi} + \left(\frac{r\psi'}{m\psi} + 1 + \frac{2\lambda^m}{1 - \lambda^m} \right) f_a \right] \xi_a, \quad (13)$$

or

$$\mu_0 i^{surf} = \frac{B_\phi}{R} \left(f \frac{a\eta'}{m\eta} + f - 2\mu + \frac{2f\lambda^m}{1 - \lambda^m} \right)_a \xi_a. \quad (14)$$

The Ref. [7], Eq. (29) has suggested also an expression in terms of the growth rate when η' is substituted by eigen-value $\bar{\gamma}^2$ from the boundary condition Eq. (8)

$$\mu_0 i^{surf} = \frac{B_\phi}{R} \frac{\bar{\gamma}^2}{\bar{\gamma}^2 + f^2} \left(f - 2\mu + \frac{2f\lambda^m}{1 - \lambda^m} \right)_a \xi_a. \quad (15)$$

The important property of surface currents i^{surf} at the plasma boundary is that they do not contain the resonant factor $f = m\mu - n$ in their amplitude. They vanish only at another marginal stability point $nq_a = nq_L < m$, which is determined by the current distribution and geometry of the stabilizing wall. In contrast, the eddy currents i^{eddy} in the wall, which are generated by the perturbation of magnetic field outside the plasma $\tilde{B}_r = \mathbf{B} \cdot \nabla \xi$ ($\mathbf{B}$ is the equilibrium magnetic field), do have the resonant factor f in their amplitude. Using Eq. (7) the expression for i^{eddy} can be obtained as

$$\mu_0 i^{eddy} = -\tilde{B}_\omega(b) = -f \frac{B_\phi}{R} \frac{2}{1 - \lambda^m} \frac{a^m}{b^m} \frac{\eta_a}{b}. \quad (16)$$

Eddy currents are always much smaller than the surface current at the plasma as well as the Hiro currents in the WTKM.

III. SURFACE CURRENTS OF THE WALL TOUCHING AND FREE BOUNDARY KINK MODES

In the wall touching kink mode regime the plasma core remains in MHD equilibrium. The force acting on the surface currents is applied to the wall where they are converted into the Hiro currents. The core equilibrium can be described by Eq. (6) with $\bar{\gamma}^2 = 0$ or, alternatively, by the perturbed equilibrium equation for vector potential ψ

$$\Delta \Psi^* = -\bar{j}(\Psi^*), \quad \Psi^* = \Psi_0^*(r) + \psi, \quad B_\omega^* = -\Psi_0'^*, \quad (r\psi')' = \frac{m^2}{r} \psi + \frac{m\bar{j}'}{f} \psi, \quad \bar{j} \equiv \frac{\mu_0 j R}{B_\phi}. \quad (17)$$

The individual kink modes m/n do not represent the WTKM (see, Ref. [3]) in the real situation. At the same time, they serve as the elementary components of WTKM and their properties are important. For conventional free boundary kink modes, FBKM, the surface currents are in equilibrium but the core is not in equilibrium and $\bar{\gamma}^2$ is determined as an eigen-value by Eqs. (6, 8).

The kink mode $m = 1$ is a special case, specific for cylindrical and toroidal geometry. The solution to both Eq. (6) and Eq. (17) is simply

$$\eta = \frac{r}{a}\eta_a, \quad \psi = \frac{B_\phi}{R}\frac{r}{a}f\eta_a \quad (18)$$

and the surface current Eq. (14) for both WTKM and FBKM is given by

$$\mu_0 i_{m=1}^{surf} = -2\frac{B_\phi}{R} \left[n - \frac{\lambda}{1-\lambda}(\mu_a - n) \right] \xi_a, \quad (19)$$

where $\lambda = a^2/b^2$ takes into account the effect of eddy currents in the wall. At $\mu_a \simeq n$ eddy currents are not excited and their effect is absent.

For kink modes $m > 1$, a special case with an analytical solution to both Eq. (6) and Eq. (17) is represented by the uniform plasma current distribution $j_\phi = \text{const}$:

$$\eta = \frac{r^m}{a^m}\eta_a, \quad \psi = \frac{f}{m}\frac{B_\phi}{R}\frac{r^m}{a^m}\eta_a, \quad \frac{r\eta'}{m\eta} = \frac{r\psi'}{m\psi} = 1. \quad (20)$$

Then, Eq. (14) gives the value of the surface current

$$\mu_0 i_{j=\text{const}}^{surf} = 2\frac{B_\phi}{R} \left[(m-1)\mu_a - n + \frac{\lambda^m}{1-\lambda^m}(m\mu_a - n) \right] \xi_a. \quad (21)$$

In calculations of i^{surf} (14) for non-uniform plasma current profiles $a\eta'/\eta$ or $a\psi'/\psi$ should be obtained from a numerical solution of Eq. (17). In examples below the following current distribution $j(r)$ is used

$$\bar{j}(r) = \frac{4\mu_a}{1+j_1} \left[1 - (1-j_1)\frac{r^2}{a^2} \right], \quad \mu(r) = \frac{4\mu_a}{1+j_1} \left[\frac{1}{2} - (1-j_1)\frac{r^2}{4a^2} \right], \quad (22)$$

where $\mu_a = \mu(a) = 1/q_a$ and $j_1 = j(a)/j(0)$. The case $j_1 = 1$ corresponds to the uniform plasma current distribution. The results are presented in terms of a dimensionless form $\hat{i}^{surf}$ of the surface current and $\hat{\gamma}$, normalized to the poloidal Alfvén transit time, as defined by the relationships

$$\hat{i}^{surf} = \frac{q_a R}{\xi_a B_\phi} \cdot \mu_0 i^{surf}, \quad \hat{\gamma}^2 \equiv \bar{\gamma}^2 q_a^2, \quad (23)$$

with $\hat{i}^{surf}$ equal to $\hat{\sigma}$ of Ref. [7].

Fig. 2 shows the growth rate and amplitudes of $\hat{i}_{WTKM}^{surf}$ and $\hat{i}_{FBKM}^{surf}$ for WTKM and free-boundary kink modes, calculated for $m = 2, 3, 4$ and different values of the j_1 parameter, which controls the edge value of the current density. The eddy current effect in the wall is neglected, $\lambda = 0$. The Cbcyl code, based on `lsode()` routine [13], was created for these calculations.

FIG. 2. (a) Growth rate $\hat{\gamma}^2$ of free boundary kink modes; (b) Normalized surface current $-\hat{i}_{WTKM}^{surf}$ (with the opposite sign) for WTKM; (c) Normalized surface currents and $-\hat{i}_{FBKM}^{surf}$ for free boundary kink modes. Stabilizing wall is absent.

For a parabolic current distribution ($j_1 = 0$) the kink mode $m = 4$ is practically marginally stable (Fig. 2). Its maximum growth rate is $\hat{\gamma}^2 \simeq 0.0006$.

The surface currents are equal to zero at the left boundary $nq_a = nq_L$ of the instability zone on the nq_a axis for each separate mode (Fig. 2a), but are finite at the resonant value $nq_a = m$.

Note, that the value of $\hat{i}_{FBKM}^{surf}$ in Fig. 2c at the resonant point $nq_a = m$ is not equal to a “universal” number 2 of Ref. [7], as could be expected based on assumption that $\hat{\gamma}^2$ is dominant over f^2 in Eq. (15). The behavior of edge currents near the right boundary of the instability zone $nq \rightarrow m$ is considered in the next section in detail.

IV. STRUCTURE OF THE EDGE PLASMA CURRENTS

The previous section has presented the δ -functional part of the edge current, referred to as the surface current. Here, the contribution of the bulk current (12), excited by the kink modes near the plasma edge is analyzed as well. The most important case corresponds to the marginal situation near the resonant value of the safety factor $nq_a \simeq m$.

Fig. 3 shows the profiles of the normalized edge plasma current $\hat{I}^{edge}(r)$

$$\hat{I}^{edge}(r) \equiv \frac{Rq_a}{B_\phi \xi_a} I^{edge}(r) = \frac{aB_\xi^v(a) - rB_\xi^c(r)}{B_\omega(a)\xi_a} \quad (24)$$

for the current profile (22) with $j_1 = 1/7$ corresponding $q_q \bar{j}_a = 0.5$ and different values of $nq_a \rightarrow 2$ for the $m = 2$ mode.

FIG. 3. Normalized edge current $\hat{I}^{edge}(r)$ for $q_a \bar{j}_a = 0.5$, $m = 2$ for WTKM (blue curves) and FBKM (red curves). Yellow and green horizontal lines specify the values of δ -functional $\hat{i}_{WTKM}^{surf}$ and $\hat{i}_{FBKM}^{surf}$ at $nq_a = 2$. Yellow and green curves represent $\hat{I}_{WTKM}^{edge}(r)$ and $\hat{I}_{FBKM}^{edge}(r)$ calculated using analytical solutions. Eddy current effect is neglected, $\lambda = 0$. (a) Full radial profiles $0 \leq r/a \leq 1$ for three cases, $nq_a = 1.9, 1.99, 1.999$; (b) Edge region $0.95 \leq r/a \leq 1$ for $nq_a = 1.99$. (c) $0.995 \leq r/a \leq 1$ for $nq_a = 1.999$. (d) $0.9995 \leq r/a \leq 1$ for $nq_a = 1.9999$.

The boundary layer becomes evident at $nq_a \rightarrow 2$ with a universal limit $\hat{I}^{edge} \rightarrow 1$ in the plasma core.

Figs. 3b,c,d show the zoomed $\hat{I}^{edge}(r)$ profiles at the plasma edge region together with their asymptotic levels and analytical approximations.

The analytical asymptotic expression, explaining behavior of the surface current $\hat{i}^{surf}$ in Figs. 2,3, can be obtained for both WTKM and free boundary kink modes. For WTKM at $nq \rightarrow m$, and $f \rightarrow 0$, with logarithmic accuracy the behavior of ψ can be represented by

$$\psi \simeq 1 + \frac{\bar{j}_a}{\mu'_a} \frac{x}{a} \ln \frac{|x|}{a}, \quad a\psi' \simeq \frac{\bar{j}_a}{\mu'_a} \ln \frac{|x|}{a}, \quad (25)$$

where the variable x is defined as

$$x \equiv r - a - x_s, \quad f_a + m\mu'_a x_s = 0. \quad (26)$$

Substitution into Eq. (13) gives the asymptotic value for $\hat{i}_{WTKM}^{surf}$ at $nq \rightarrow m$

$$\hat{i}_{WTKM}^{surf}|_{nq_a=m} = -q_a \bar{j}_a = -2 \frac{\bar{j}_a}{\langle \bar{j} \rangle}, \quad (27)$$

where $\langle \bar{j} \rangle$ is the cross-section averaged value of the unperturbed plasma current density.

The same asymptotic solution for ψ allows describing the properties of the edge current excited by *WTKM*

$$\hat{I}_{WTKM}^{edge}(r) = \left(\frac{r\psi'}{m\psi_a} + 1 \right) q_a f_a + (\mu - \bar{j}) q_a \frac{f_a \psi}{f \psi_a} - 1 + \frac{2\lambda^m}{1 - \lambda^m} \quad (28)$$

For $nq \rightarrow m$, the first term in Eq. (11) is negligible because of $f_a \rightarrow 0$. The second term describes the structure of the current in the edge. It decays as $\propto x_s/|x|$ (twice in value at the distance x_s from the boundary). Deeper in the core, $|x| \gg x_s$, the current in the edge I^{edge} reaches its asymptotic value

$$\hat{I}^{edge}(r) = -1 + \frac{2\lambda^m}{1 - \lambda^m}. \quad (29)$$

The results of using the approximate solution (25) for $r\psi', \psi$ are presented in Figs. 3b,c,d by the yellow curves, which become identical to the numerical solution for $\hat{I}_{WTKM}^{edge}(r)$ at $m - nq < 0.01$.

The calculations and analytical consideration indicates that a significant bulk current perturbation, due to the kink mode, is localized at the plasma edge within distance of several

$$\Delta_{WTKM}^{edge} = x_s = \frac{\mu_a - n/m}{|\mu'_a|} \simeq \frac{m - nq_a}{nq'_a} \quad (30)$$

from the boundary. Independent of the equilibrium plasma current distribution, when nq approaches m , this bulk current partially compensates the presence of the δ -functional surface current and makes the value of the total edge current (including both surface and the bulk currents) equal to a universal value (29), which is valid for both *WTKM* and *FBKM*.

For free boundary kink modes, the asymptotic value of $\hat{i}_{FBKM}^{surf}$ and $\hat{I}_{FBKM}^{edge}(r)$ -profile can be obtained based on asymptotic behavior of solution $\eta(r)$ to Eq. (6)

$$\eta \simeq \frac{m}{\bar{\gamma}} \arctan \frac{\bar{\gamma}}{f(r)} + \frac{j'_a}{2\mu'_a f'_a} \ln \frac{\bar{\gamma}^2 + f^2(r)}{a^2 f_a'^2}, \quad f'_a = m\mu'_a. \quad (31)$$

At $f \gg \bar{\gamma}$ this solution reproduces the asymptotic behavior of ψ in Eq. (25)

$$\eta = \frac{m\psi}{f} \rightarrow \frac{1}{\mu'x} + \frac{j'}{a\mu'^2} \ln \frac{|x|}{a} + \mathcal{O}(1). \quad (32)$$

The substitution of this solution (neglecting the logarithmic term) into the boundary condition Eq. (8) leads to the equation determining the asymptotic value of growth rate $\bar{\gamma}$ at $f \rightarrow 0$

$$\dot{\gamma} \equiv \frac{\bar{\gamma}}{f_a}, \quad \frac{\arctan \dot{\gamma}}{\dot{\gamma}} = \frac{2\mu_a - j_a}{2\mu_a} = \frac{aq_a'^c}{2q_a}, \quad (33)$$

with its numerical solution presented in Fig. 4.

FIG. 4. Eigen-value for growth rate $\dot{\gamma} \equiv \bar{\gamma}/f_a$ of FBKM at $nq \rightarrow m$ as a function of the edge shear parameter aq'_a/q_a .

Unlike the case of a uniform current distribution, where $\bar{\gamma}^2 \propto f_a$, in all other cases $\bar{\gamma} \propto f_a$. Because of this, in Eq. (15) for the surface current of FBKM its asymptotic value at $f_a \rightarrow 0$ should be obtained using the solution of Eq. (33), rather than simply neglecting f^2 compared to $\bar{\gamma}^2$. Instead of -2 as estimated in Ref. [7], the asymptotic value of $\hat{i}_{FBKM}^{surf}$ is given by

$$\hat{i}_{FBKM}^{surf} \rightarrow \frac{\dot{\gamma}^2}{\dot{\gamma}^2 + 1} \left(-2 + m - nq_a + \frac{2\lambda(m - nq_a)}{1 - \lambda^m} \right) \simeq \frac{-2\dot{\gamma}^2}{\dot{\gamma}^2 + 1}. \quad (34)$$

The analytical approximation of the FBKM edge current, presented by the be green curves in Figs. 3b,c,d, is based on expression (12)

$$\hat{f}_{FBKM}^{edge}(r) \simeq q_a f \frac{r\eta'}{m\eta_a} - q_a \mu \frac{\eta(r)}{\eta_a} - 1 + q_a f_a + \frac{2\lambda^m q_a f_a}{1 - \lambda^m} \quad (35)$$

with the use of analytical representation (31) for $\eta(r)$. As in the case of WTKM, the analytical representation is practically identical to the numerical solution for $m - nq_a < 0.01$.

Unstable free boundary kink modes creates their own boundary layer with the characteristic width

$$\Delta_{FBKM}^{edge} = \dot{\gamma} x_s \simeq \dot{\gamma} \frac{m - nq_a}{nq_a^c}. \quad (36)$$

Returning to the edge current of the WTKM, Fig. 5 shows the tendencies in the edge current profile depending on the edge current density in an equilibrium configuration. The special case is represented by the value $q_a j_a = 1$ ($j_1 = 1/3$ for the current profile (22)). In this case, the δ -functional surface current is equal to the asymptotic value of the edge current and there is no compensating bulk edge current, $\hat{f}_{WTKM}^{edge}(r) = \hat{i}_{WTKM}^{surf}$. For $q_a \bar{j}_a < 1$ the surface current $\hat{i}_{WTKM}^{surf} < 1$ and the bulk edge current is positive. For $q_a \bar{j}_a > 1$ the surface current $\hat{i}_{WTKM}^{surf} > 1$ and the bulk edge current is negative, compensating the excess $\hat{i}_{WTKM}^{surf} - 1$.

FIG. 5. Normalized edge current $\hat{I}^{edge}(r)$ in $0.999 \leq r/a \leq 1$ for $q_a = 1.9999$, $m = 2$ and different $q_a \bar{j}_a$ for WTKM and FBKM. Color lines have the same meaning as in Fig. 3. (a) $q_a \bar{j}_a = 0$; (b) $q_a \bar{j}_a = 2 - \pi/2$; (c) $q_a \bar{j}_a = 1.00$; (d) $q_a \bar{j}_a = 1.5$.

For FBKM the surface current $\hat{i}_{WTKM}^{surf} = 1$ at $nq_a = m$ for $q_a \bar{j}_a = 2 - \pi/2$, as is shown in Fig. 5b. Together with Fig. 5c it illustrates distinction between the WTKM and FBKM edge current layers.

V. SURFACE CURRENT SUPPLY FROM THE PLASMA CORE

The surface currents, calculated from condition $B_{norm} = 0$, serve as a virtual “super-conducting” casing for the plasma core in WTKM, providing its equilibrium. At the same time the force $\vec{i}^{surf} \times \mathbf{B}$ is acting on the surface currents themselves. In the case of WTKM, this force is applied to the wall. In the case of FBKM, the force would be applied to the plasma surface, unless the surface current are flowing along the field lines, as is noticed in Ref. [7].

The surface currents described previously, and in particular from solution of Eq. (6), represent the jump in poloidal component of magnetic field across the plasma boundary. They have no sources

$$\vec{i}^{surf} = (\nabla I \times \mathbf{n}), \quad \nabla_{\omega, \phi} \cdot \vec{i}^{surf} = 0, \quad (37)$$

where $I = I(\omega, \phi)$ is the stream function of $\vec{i}^{surf}$ and $\mathbf{n}$ is the unit normal vector to the plasma surface. For a single FBKM

$$\vec{i}^{surf} = i^{surf} \left(\mathbf{e}_\phi + \frac{nq}{mR} \mathbf{e}_\omega \right). \quad (38)$$

Some apparent paradox is that because of periodicity condition for $I(\omega, \phi)$ these surface currents always intersect the magnetic field lines (unless $nq = m$).

Let us remind that Eq. (6) is obtained by applying the operator $\nabla \times$ to the linearized equations of motion

$$\rho \frac{\partial \mathbf{V}}{\partial t} + \nabla p = (\mathbf{j} \times \mathbf{B}), \quad (39)$$

and, thus, contains not a complete information. In addition, there is a compression of the toroidal magnetic field, which generates the poloidal current in the core

$$\mu_0 \mathbf{j}^{pol} = (\nabla \tilde{B} \times \mathbf{e}_\phi), \quad \mu_0 j_r^{pol} = \frac{1}{r} \frac{\partial \tilde{B}}{\partial \omega}, \quad \mu_0 j_\omega^{pol} = -\frac{\partial \tilde{B}}{\partial r}. \quad (40)$$

This current is small in comparison with the toroidal component of the perturbed current, but is essential for the force balance because it is interacting with the strong longitudinal field. The explicit expression for j_r^{pol} can be obtained from the projection of equation of motion on ignorable direction ($\mathbf{e}_\phi$ in our approximation)

$$j_r^{pol} B_\omega = j_\omega \tilde{B}_r, \quad B_\omega \tilde{B} = j_\omega \psi. \quad (41)$$

The radial component of the equation of motion

$$\rho \gamma^2 \xi + p'_r = ((\mathbf{j} \times \mathbf{B}) \cdot \mathbf{e}_r), \quad (42)$$

after integration across the plasma surface gives the force balance in terms of the surface currents

$$-p = ((\vec{i} \times \mathbf{B}) \cdot \mathbf{n}) = ((\vec{i}^{surf} \times \mathbf{B}) \cdot \mathbf{n}) + ((\vec{i}_\omega \times \mathbf{B}_\phi) \cdot \mathbf{n}). \quad (43)$$

In contrast to $\vec{i}^{surf}$, the current $\vec{i}_\omega$ have a source in the normal component of the bulk radial current $j_r \mathbf{e}_r$ flowing to the plasma boundary from the core

$$\mu_0 i_\omega \equiv \mu_0 \int_{a-\geq}^a j_\omega dr = -\mu_0 \int^\omega d\omega \int_{a-\geq}^a (r j_r)'_r dr = \tilde{B}, \quad (44)$$

which is consistent with the contribution of i_ω into the force balance across the plasma surface [10]

$$((\vec{i}_\omega \times \mathbf{B}_\phi) \cdot \mathbf{n}) = \frac{B \tilde{B}}{\mu_0} = \frac{B^2|_c - B^2|_v}{2\mu_0}. \quad (45)$$

VI. SUMMARY

Calculation of the structure of the edge plasma region perturbed by the external kink modes (WTKM and FBKM) revealed the boundary current layers, consisting from the δ -functional surface currents and the edge localized plasma bulk current perturbations. For non-uniform current the total edge current has a universal value, meaning that the surface current is partially compensated by the bulk edge current. The analytical expressions for the marginal cases $nq_a \simeq m$ are obtained for both growth rates, surface currents of WTKM and FBKM and for the edge current profiles. In particular, it is shown that for non-uniform plasma current distribution the growth rate of FBKM is proportional to $m - nq_a$, rather than $\sqrt{m - nq_a}$ as in the case of uniform current. It is also explained that in the case of free boundary kink mode the force acting on the surface currents, excited by the plasma deformation, is compensated by the force acting on the poloidal component of the surface current, which has a source in the plasma core in the form of the perturbation of the toroidal magnetic field.

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APPENDIX A: CORRECTIONS TO CALCULATIONS OF REF. [7]

Fig. 6 represents recalculations (using **Cbcy1**) of data presented in 3 figures of Ref. [7] for the current distribution specified by the safety factor $q(r)$ -profile as

$$q(r) = q_0 + q_1 \frac{r^2}{a^2}. \quad (\text{A1})$$

The following notations from Ref. [7], used in figures, are defined as

$$\tilde{\gamma} \equiv \frac{\bar{\gamma}}{m}, \quad bc(\tilde{\gamma}) \equiv -\frac{\tilde{\gamma}^2 + f_a^2}{m^2} \frac{a\eta'}{\eta} + (2\mu_a - f_a) \frac{f_a}{m}, \quad \frac{a\hat{\sigma}}{\xi_a} \equiv \hat{i}_{FBKM}^{surf}. \quad (\text{A2})$$

FIG. 6. (a) Boundary condition $bc(\tilde{\gamma})$ vs $\tilde{\gamma}$ (recalculated Fig. 1 of Ref. [7]); (b) Growth rate $\tilde{\gamma}^2$ of FBKM $m = 3$ as function of q'' in Eq. (A1) for $q_a = 2.5$ and $q_a = 2.4$ (recalculated Fig. 2 of Ref. [7]); (c) Surface current $-a\hat{\sigma}/\xi_a = -\hat{i}_{FBKM}^{surf}$ for $m = 3$ as function of q'' for $q_a = 2.5$ and $q_a = 2.4$ (recalculated Fig. 3 of Ref. [7]).

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Figure captions:

FIG. 1. (a) Toroidal magnetic field lines punch the plasma surface; (b) Surface currents eliminate B_{norm} : ■ opposite to I_{pl} , ■ in the direction of I_{pl} ; (c) Hiro currents in the wall are always opposite to I_{pl} .

FIG. 2. (a) Growth rate $\hat{\gamma}^2$ of free boundary kink modes; (b) Normalized surface current $-\hat{i}_{WTKM}^{surf}$ (with the opposite sign) for WTKM; (c) Normalized surface currents and $-\hat{i}_{FBKM}^{surf}$ for free boundary kink modes. Stabilizing wall is absent.

FIG. 3. Normalized edge current $\hat{I}^{edge}(r)$ for $q_a j_a = 0.5$, $m = 2$ for WTKM (blue curves) and FBKM (red curves). Yellow and green horizontal lines correspond to δ -functional $\hat{i}_{WTKM}^{surf}$ and $\hat{i}_{FBKM}^{surf}$ at $nq_a = 2$. Yellow and green curves represent $\hat{I}_{WTKM}^{edge}(r)$ and $\hat{I}_{FBKM}^{edge}(r)$ calculated using analytical solutions. Eddy current effect is neglected, $\lambda = 0$. (a) Full radial profiles $0 \leq r/a \leq 1$ for three cases, $nq_a = 1.9, 1.99, 1.999$; (b) Edge region $0.95 \leq r/a \leq 1$ for $nq_a = 1.99$. (c) $0.995 \leq r/a \leq 1$ for $nq_a = 1.999$. (d) $0.9995 \leq r/a \leq 1$ for $nq_a = 1.9999$.

FIG. 4. Eigen-value for growth rate $\dot{\gamma} \equiv \bar{\gamma}/f_a$ of FBKM at $nq \rightarrow m$ as a function of the edge shear parameter aq'_a/q_a .

FIG. 5. Normalized edge current $\hat{I}^{edge}(r)$ in $0.999 \leq r/a \leq 1$ for $q_a = 1.9999$, $m = 2$ and different $q_a \bar{j}_a$ for WTKM and FBKM. Color lines have the same meaning as in Fig. 3. (a) $q_a \bar{j}_a = 0$; (b) $q_a \bar{j}_a = 0.25$; (c) $q_a \bar{j}_a = 1.00$; (d) $q_a \bar{j}_a = 1.5$;

FIG. 6. (a) Boundary condition $bc(\tilde{\gamma})$ vs $\tilde{\gamma}$ (recalculated Fig. 1 of Ref, [7]); (b) Growth rate $\tilde{\gamma}^2$ of FBKM $m = 3$ as function of q'' in Eq. (A1) for $q_a = 2.5$ and $q_a = 2.4$ (recalculated Fig. 2 of Ref, [7]); (c) Surface current $-a\hat{\sigma}/\xi_a = -\hat{i}_{FBKM}^{surf}$ for $m = 3$ as function of q'' for $q_a = 2.5$ and $q_a = 2.4$ (recalculated Fig. 3 of Ref, [7]).

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