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Chiping Chen
Principal Investigator
Plasma Science and Fusion Center
Massachusetts Institute of Technology
Cambridge, MA 02139

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Final Report
Theoretical and Computational Investigation of
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Under the auspices of this grant, we carry out vigorous theoretical and computational investigations of periodically focused intense charged-particle beams in parameter regimes relevant to the development of advanced high-brightness, high-power accelerators for high-energy physics research.

The breakthroughs and highlights in our research in the period from April 1, 2010 to March 30, 2013 were:

- a) Theory and simulation of adiabatic thermal Child-Langmuir flow (Chen, Rizzato and Pakter, 2011; Chen, Akylas, Barton, et al. 2012; Mok, 2013);*
- b) Particle-in-cell simulations of adiabatic thermal beams in periodic solenoidal focusing field (Barton, Field, Lang and Chen, 2012; Chen, Barton, Field and Lang 2012; Chen, Akylas, Barton, et al. 2012);*
- c) Dynamics of charged particles in an adiabatic thermal beam equilibrium in a periodic solenoidal focusing field (Wei and Chen, 2011);*
- d) Training of undergraduate researchers and graduate student in accelerator and beam physics.*

While detailed descriptions of our research results are provided in Appendix - Compendium of Publications at the end of this report, a brief introduction and summary is as follows.

Most accelerator applications require high-brightness beams. This is true for high-energy accelerators such as linear colliders. It is also true for energy recovery linacs (ERLs) and free electron lasers (FELs) such as x-ray free electron lasers (XFELs).

Earlier in this research program, we developed paraxial kinetic and warm-fluid equilibrium theories for a thermal charged-particle beam in periodic focusing fields (Samokhvalova, Zhou and Chen, 2007 and 2009; Samokhvalova, 2008; Zhou, Samokhvalova and Chen, 2008). While our kinetic and warm-fluid beam equilibrium theories are equivalent to our warm-fluid equilibrium theory, our kinetic beam equilibrium theory provides detailed information about the electron distribution in phase space. We found good agreement between our theories and the recent experimental measurements of the electron density profiles at University of Maryland Electron Ring (UMER) (Bernal, Quinn, Reiser and O'Shea, 2002; Zhou, Samokhvalova and Chen, 2008; Samokhvalova, 2008).

Recently, we have found evidence for the control of chaotic particle motion using adiabatic thermal beams (Wei and Chen, 2011). We have confirmed the theoretical predictions of the adiabatic thermal beam equilibrium by means of particle-in-cell (PIC) simulations (Barton, Field, Lang and Chen, 2012; Chen, Barton, Field and Lang, 2012; Chen, Akylas, Barton, et al, 2012). Our discovery of the adiabatic thermal beams has filled a major gap in the understanding of periodically focused beams, making an important contribution to beam physics.

Most recently, we have recognized the need to further advance our understanding of particle sources which would enable the particle accelerator community to improve beam brightness. We have begun to re-examine the Child-Langmuir flow (Chen, Rizzato and Pakter, 2011; Chen, Akylas, Barton, et al, 2012; Mok, 2013).

1. Theory and simulation of adiabatic Child-Langmuir flow (Chen, Rizzato and Pakter, 2011; Chen, Akylas, Barton, et al., 2012; Mok, 2013)

Recently, there are renewed interests in the research and development of high-brightness dc thermionic electron or ion guns. The renewed interests arise from a) applications of high-brightness dc electron guns in x-ray free electron laser applications, and b) utilizations of high-brightness dc electron guns in the realization of adiabatic thermal beam equilibria in periodic focusing fields.

The 1D cold-fluid Child-Langmuir (C-L) flow is an important aspect of dc thermionic gun theory. It corresponds to the cold-fluid equilibrium of a charged-particle flow between two plates with an electrostatic potential bias.

The 1D adiabatic thermal C-L flow discussed we have studied corresponds to a warm-fluid equilibrium of a charged-particle flow between two plates with an electrostatic potential bias under the adiabatic equation of state (Chen, Rizzato and Pakter, 2011). A warm-fluid description of the 1D adiabatic thermal C-L flow has been developed. A self-consistent Poisson equation has been derived. In the limit of zero temperature, the self-consistent Poisson equation recovers the corresponding Poisson equation for the cold-fluid C-L flow. An effort has been initiated to validate the theory via self-consistent simulation. Good agreement has been found between the approximation of the adiabatic equation of state and self-consistent simulation.

We have considered the non-relativistic 1D C-L flow under the influence of a finite temperature profile between two conducting plates located at $z=0$ and $z=d$. The adiabatic warm-fluid equations in cgs units are:

$$mnV \frac{\partial V}{\partial z} = -qn \frac{\partial \phi}{\partial z} - \frac{\partial p}{\partial z}, \quad (1.1)$$

$$\frac{\partial nV}{\partial z} = 0, \quad (1.2)$$

$$\partial^2 \phi / \partial z^2 = -4\pi qn, \quad (1.3)$$

$$\frac{\partial}{\partial z} \left(\frac{p}{n^3} \right) = 0, \quad (1.4)$$

$$p = k_B nT, \quad (1.5)$$

where n , V , ϕ , p and T are the equilibrium density, flow velocity, electrostatic potential, longitudinal pressure and temperature profiles, respectively, k_B is the Boltzmann constant, and m and q are the rest mass and charge of the charged particle, respectively.

Equation (1.4) is the one-dimensional adiabatic equation of state. It is a statement of entropy conservation. For any small segment of the flow, the square of the normalized rms thermal emittance, $\varepsilon_n^2 = \gamma^2 \beta^2 \varepsilon^2$, is proportional to p/n^3 or T/n^2 , where ε is the unnormalized rms thermal emittance, $\gamma = (1 - V^2/c^2)^{-1/2} \cong 1$ is the relativistic mass factor, and $\beta = V/c$ with c being the speed of light in vacuum. Therefore, the adiabatic equation of state assures that the normalized rms thermal emittance of any small segment of the flow is conserved.

It is readily shown from Eqs. (1.1), (1.4) and (1.5) that

$$\frac{\partial}{\partial z} \left(\frac{1}{2} mV^2 + q\phi + \frac{3p}{2n} \right) = 0, \quad (1.6)$$

Equation (1.6) is an important conservation law in addition to charge conservation, i.e., $J = qnV = \text{const.}$ and entropy conservation, i.e., $p/n^3 = \text{const.}$ The solution to Eq. (1.6) is

$$\frac{1}{2}mV^2 + q\phi + \frac{3k_B}{2q^2} \frac{T}{n^2} \frac{J^2}{V^2} = C, \quad (1.7)$$

where the constant C is defined by

$$C = \frac{1}{2}mV^2(0) + \frac{3}{2}k_B T_c, \quad (1.9)$$

T_c is the emitter temperature, and use has been made of the boundary condition $\phi(0) = 0$.

At $C = C_{\text{crit}} = 3k_B T_c$, we have

$$V(0) = \left[\frac{3k_B T_c}{m} \right]^{1/2}, \quad (1.10)$$

$$n(0) = \frac{J}{q \left[\frac{3k_B T_c}{m} \right]^{1/2}}, \quad (1.11)$$

$$V = \left\{ \frac{-q\phi + 3k_B T_c}{m} + \left[\left(\frac{-q\phi}{m} \right) \left(\frac{-q\phi + 6k_B T_c}{m} \right) \right]^{1/2} \right\}^{1/2}, \quad (1.12)$$

$$\frac{\partial^2 \phi}{\partial z^2} = \frac{-4\pi J}{\left\{ \frac{-q\phi + 3k_B T_c}{m} + \left[\frac{q\phi}{m} \left(\frac{q\phi - 6k_B T_c}{m} \right) \right]^{1/2} \right\}^{1/2}}. \quad (1.13)$$

The boundary conditions for Poisson's equation (1.13) are

$$\phi(0) = 0 = \phi'(0) \quad (1.14)$$

and

$$\phi(d) = \Phi_d, \quad (1.15)$$

where prime denotes the derivative with respect to z . Note that $\phi'(0) = 0$ is the condition for space-charge-limited emission.

Figure 1.1 shows ϕ/Φ_d versus z/d for a) cold-fluid C-L flow and b) adiabatic thermal C-L flow with $\hat{J} \equiv J/J_{CL} = 2.6$ and $\hat{T}_c \equiv k_B T_c / (-q\Phi_d) = 0.1$, where $J_{CL} \equiv (\sqrt{2}mc^3 / 9\pi qd^2) (-q\Phi_d / mc^2)^{3/2}$ is the current density of the cold-fluid C-L flow.

One-dimensional self-consistent simulations have been performed to verify the theoretical predictions. In the simulations, planar charged sheets have been used. They have a full Maxwellian distribution with temperature T_c shifted by $V(0)$ in the velocity space at $z=0$. When a charged sheet strikes the plate at $z=d$, it is replaced by a new charged sheet at the emitter ($z=0$). The newly injected charged sheets are taken from the shifted full Maxwellian distribution.

Fig. 1.1 Plots of the normalized potential ϕ/Φ_d versus the normalized distance z/d for a) cold-fluid C-L flow with $\hat{J} = 1$ and $\hat{T}_c = 0$ and b) adiabatic thermal C-L flow with $\hat{J} = 2.6$ and $\hat{T}_c = 0.1$ (from Chen, Rizzato and Pakter, 2011).

Fig. 1.2 Comparison of $Tn^2(0)/T_c n^2$ versus the normalized distance z/d from simulation with theory for the choice of system parameters corresponding to $k_B T_c / m \omega_p^2 d^2 = 10$, $V(0)/\omega_p d = 5.48$ and $-q\Phi_d / m \omega_p^2 d^2 = 10$ (from Chen, Rizzato and Pakter, 2011).

Preliminary simulation results have been obtained. The approximation of the adiabatic equation of state, i.e., Eq. (1.4), has been found in good agreement with self-consistent simulation in the high-temperature regime with $k_B T_c / m \omega_p^2 d^2 \gg 1$. Here, $\omega_p^2 = 4\pi q^2 \bar{n} / m$ is the plasma frequency associated with the average particle density $\bar{n} = d^{-1} \int_0^d n(z) dz$. The plots of

$Tn^2(0)/T_c n^2$ versus the normalized distance z/d in Fig. 3 show the comparison between theory and simulation for the choice of system parameters corresponding to: $k_B T_c / m \omega_p^2 d^2 = 10$, $-q\Phi_d / m \omega_p^2 d^2 = 10$, $V(0)/\omega_p d = 5.48$. In the simulation, 50000 charged sheets have been used. Within the statistical fluctuations, the quantity $Tn^2(0)/T_c n^2$ is conserved. The simulation results in Fig. 1.2 validate the approximation of the adiabatic equation of state, i.e., Eq. (1.4).

They also confirm that the normalized (longitudinal) rms thermal emittance of any small segment of the flow is conserved.

Since the presentation initial results at the 2nd International Particle Accelerator Conference (Chen, Rizzato and Pakter, 2011), we developed our own MatLab code and increased the number of charged sheets in our simulation studies, which, we hope, will ultimately permit us to decrease the value of the parameter $\hat{T}_c$ sufficiently to access the regime of practical interest in which $\hat{T}_c$ is of order 10^{-6} . Several undergraduate researchers (Kevin Burdge, Adrian Jimenez-Galindo, Trung V. Phan, and Tenzin Rabga) initiated the efforts on the development of the Matlab code, and a graduate student (Rachel V. Mok) refined code and carried simulation studies (Mok, 2013).

As an example, Figs. 1.3 and 1.4 show the results of a simulation at the end of one transit time using 2×10^6 charged sheets for the choice of parameters corresponding to: $k_B T_c / (-q\Phi_d) = 0.01$ and $J / J_{CL} = 1.4886$.

Fig. 1.3 Phase space distribution of charged sheets at the end of one transit time as obtained from a simulation. The solid curve is the fluid velocity predicted by theory, whereas the dots are the positions and velocities of charged sheets (from Mok, 2013).

Fig. 1.4 Comparison of $T n^2(0) / T_c n^2$ versus the normalized distance z/d at the end of one transit time as obtained from a simulation. The solid line is predicted by theory, whereas the dots from the simulation (from Mok, 2013).

We have also been extending the adiabatic thermal CL flow from 1D to 2D (Chen, Akylas, Barton, et al., 2012). Because the adiabatic equation of state is not known, we have developed a 2D code to simulate the thermal CL flow in a 2D slab geometry and obtained some initial results which are to be analyzed.

2. Particle-in-cell simulations of adiabatic thermal beams in periodic solenoidal focusing fields (Barton, Field, Lang and Chen, 2012; Chen, Barton, Field and Lang, 2012; Chen, Aklyas, Barton, 2012)

The claim of the existence of the adiabatic thermal beams in periodic and axially-varying solenoidal magnetic focusing fields is an extraordinary one. As such, it requires extraordinary proof and verification.

Under the supervision of the principal investigator, a team of undergraduate researchers (Timothy J. Barton, David Field and Kevin, M. Lang) has performed self-consistent simulations of adiabatic thermal beams in periodic solenoidal magnetic focusing fields. In particular, results have been obtained for adiabatic thermal beams that do not rotate in the Larmor frame.

For such beams, the theoretical predictions of the rms beam envelope, the conservation of the rms thermal emittance, the adiabatic equation of state, and the Debye length have been verified in the PIC simulations (Barton, Field, Lang and Chen, 2012; Chen, Barton, Field and Lang, 2012; Chen, Aklyas, Barton, et al., 2012) using the MIT Two-Dimensional Periodic Focused Beam (PFB2D) PIC code. Results of a typical PFB2D simulation are shown in Figs. 1-3 for the parameters: $S^2 \kappa_z(s) = \sigma_0^2 [1 + \cos(2\pi s/S)]^2$, $\sigma_0 = 80^\circ$, $SK/4\epsilon_{th} = 7.0$, $N_p = 1 \times 10^6$, $(S/4\epsilon_{th})^{1/2} R = 5.0$, integration step size $\Delta s/S = 0.01$, and mesh size $(S/4\epsilon_{th})^{1/2} \Delta \tilde{x} = 0.02$.

The normalized rms beam radius $(4\epsilon_{th}S)^{-1/2} r_{brms}$ from the PFB2D simulation is plotted in Fig. 2.1 as a function of the normalized distance s/S . Within the statistical error of $\pm 1\%$, the result agrees with the matched beam envelope solution. The transverse momentum distribution continues to be Gaussian, as shown in Fig. 2.2. The beam density distribution maintains a plateau distribution as shown in Fig. 2.3. The characteristic scale over which the beam density falls is the Debye length λ_D , which, in this example, has a normalized value of $(4\epsilon_{th}S)^{-1/2} \lambda_D = 0.14$. In Fig. 2.4, the evolutions of the relative rms thermal emittances $\epsilon_{\tilde{x}rms}/\epsilon_{\tilde{x}rms}(0)$ and $\epsilon_{\tilde{y}rms}/\epsilon_{\tilde{y}rms}(0)$ in the PFB2D simulation are shown in solid and dashed curves, respectively, as a function of the normalized distance s/S . Here, $\epsilon_{\tilde{x}rms}(0) = \epsilon_{th}$ and $\epsilon_{\tilde{y}rms}(0) = \epsilon_{th}$ are the rms emittances in the $\tilde{x}$ - and $\tilde{y}$ -directions of the Larmor frame at $s = 0$, respectively.

Fig. 2.2 Plot of the normalized rms beam radius $(4\epsilon_{th}S)^{-1/2} r_{brms}$ versus the normalized distance s/S as obtained from the PIC simulation (from Barton, Field, Lang and Chen, 2012).

Fig. 2.2 Plots of the probability density distributions $\rho(\tilde{x}')$ versus $(S/4\epsilon_{th})^{1/2}\tilde{x}'$ and the probability density distributions $\rho(\tilde{y}')$ versus $(S/4\epsilon_{th})^{1/2}\tilde{y}'$ at $s/S = 20$. The dotted curves are from the PFB2D simulation, whereas the dashed curves are from the theoretical predictions (from Barton, Field, Lang and Chen, 2012).

Fig. 2.3 Plot of the normalized beam density $(4\epsilon_{th}S/N_b)n_b$ versus the normalized radius $(4\epsilon_{th}S)^{-1/2}r$ at $s/S = 20$ (from Barton, Field, Lang and Chen, 2012).

Fig. 2.4 Plots of the relative rms thermal emittances $\epsilon_{\tilde{x}rms} / \epsilon_{\tilde{x}rms}(0)$ and $\epsilon_{\tilde{y}rms} / \epsilon_{\tilde{y}rms}(0)$ versus the normalized distance s/S from the PIC simulation (from Barton, Field, Lang and Chen, 2012).

Both rms thermal emittances are conserved within the statistical error of $\pm 0.3\%$. These results are in agreement with the theoretical prediction of the conservation of rms thermal emittances, and validate the adiabatic equation of state. The adiabatic thermal beam equilibrium has been found to be stable.

3. Dynamics of charged particles in an adiabatic thermal beam equilibrium in a periodic solenoidal focusing field (Wei and Chen, 2011)

We published important results of dynamics of charged particles in the adiabatic thermal beam (Wei and Chen 2011), and compared with those in the well-known Kapchinskij-Vladimirskij (KV)-type beam equilibrium (Kapchinskij and Vladimirskij, 1959; Chen, Pakter and Davidson, 1997; Davidson and Qin, 2001). The results are summarized in Fig. 3.1 It has been found that the widths of nonlinear resonances in the adiabatic thermal beam equilibrium are narrower than those in the KV-type beam equilibrium, and that chaotic particle motion is almost eliminated in the adiabatic thermal beam equilibrium.

Fig. 3.1 Close ups of Poincare surface-of-section maps of charged-particle trajectories in a) KV-type beam equilibrium and b) adiabatic thermal beam equilibrium for the choice of system parameters corresponding to $\omega_b = 0$, $\sigma_0 = 80^\circ$, $SK/4\varepsilon_{th} = 7.0$, $S\sqrt{\kappa_z(s)} = (2/3)^{1/2}\sigma_0[1 + \cos(2\pi s/S)]$. Here, the normalized radial momentum is $(S/4\varepsilon_{th})^{1/2}dr/ds$ and the normalized radius is $r/\sqrt{4\varepsilon_{th}S}$ (from Wei and Chen, 2011).

The importance of these results is twofold: Firstly, the elimination of chaotic particle motion provides a further numerical proof that the scaled transverse Hamiltonian (Zhou, Samokhvalova and Chen, 2008) is a very good approximate constant of motion. This approximate constant of motion and the exact contact of motion of the canonical angular momentum assure that the motion of charged particles is approximately integrable in the four-dimensional phase space of the adiabatic thermal beam equilibrium. Secondly, the elimination of chaotic particle motion may provide valuable insight into how to control chaotic particle motion, halo formation and beam loss in beam transport channels. In particular, if adiabatic thermal beams are stable, they may be free from chaotic particle motion, beam halo formation, and beam loss.

4. Training of undergraduate researchers and graduate student in science and technology of accelerators and charged-particles beams

Under the guidance of the principal investigator (PI) of the grant, several undergraduate student and a graduate student carried out research in beam physics.

In 2010 and 2011, one MIT undergraduate student (Haofei Wei), who was a member of the 5-member US Physics Team and won a Silver medal in the International Physics Olympiad before entering MIT, carried out simulation studies of the phase space of an adiabatic thermal beam in a periodic solenoidal focusing field through the MIT Undergraduate Research Opportunity Program. The results of his research were published in Physical Review (Wei and Chen, 2011).

In 2011 and 2012, three MIT undergraduate students, Timothy J. Barton, David Field and Kevin Lang carried out research in the science and technology of accelerators and charged-particle beams through the MIT Undergraduate Research Opportunity Program. They were selected among eleven applicants interviewed by PI. All of the three undergraduate students had strong educational background. Before entering MIT, Mr. David Field was a two-time (2009 and 2010) member of the 5-member US Physics Teams, and won Silver and Bronze medals in the International Physics Olympiad. Mr. Kevin Lang was also a two-time (2008 and 2009) member of the 24-member US Physics Team. While they were a rising sophomore or junior, they were ready to make discoveries in the science and technology of accelerators and charged-particle beams.

Working together as a research team under the supervision of the principal investigator, the undergraduate researchers developed a MATLAB version of the PFB2D PIC code and used it to verify the theoretical predictions of adiabatic thermal beams in periodic solenoidal focusing fields. The results of their research were published in Physical Review (Barton, Field, Lang and Chen, 2012).

In 2012 and 2013, four MIT undergraduate students, Kevin Burdge, Adrian Jimenez-Galindo, Trung V. Phan, and Tenzin Rabga initiated the efforts on the development of the Matlab code for simulation studies of adiabatic thermal Child-Langmuir flow.

Ms. Rachel V. Mok, a graduate student and research assistant, focused on her research effort on adiabatic thermal C-L flow (Mok, 2013).

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Appendix - Compendium of Publications

1. "Simulation of Adiabatic Thermal Beams in a Periodic Solenoidal Focusing Field," T.J. Barton, T.J., D.M. Field, K.M. Lang, and C. Chen, Phys. Rev. ST Accel. Beams **15**, 124201 (2012).
2. "Simulation Studies of Adiabatic Thermal Beams in a Periodic Solenoidal Focusing Field," C.Chen, T.J. Barton, D.M. Field, and K.M. Lang, Proc. 2012 Int. Part. Accel. Conf., 3003 (2012).
3. "Simulation of Periodically Focused, Adiabatic Thermal Beams," C. Chen, T.R. Akylas, T.J. Barton, D.M. Field, K.M. Lang, and R.V. Mok, Advanced Accelerator Concepts, AIP Conf. Proc. **1507**, 676 (2012).
4. "Dynamics of Charged Particles in an Adiabatic Thermal Beam Equilibrium," H. Wei and C. Chen, Phys. Rev. ST Accel. Beams **14**, 024201 (2011)
5. "One-Dimensional Adiabatic Child-Langmuir Flow," C. Chen, R. Pakter, and F.B. Rizzato, Proc. 2011 Int. Part. Accel. Conf., 694 (2011).

Simulation of adiabatic thermal beams in a periodic solenoidal magnetic focusing field

T. J. Barton, D. M. Field, K. M. Lang, and C. Chen

Plasma Science and Fusion Center, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA

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Self-consistent particle-in-cell simulations are performed to verify earlier theoretical predictions of adiabatic thermal beams in a periodic solenoidal magnetic focusing field [K. R. Samokhvalova, J. Zhou, and C. Chen, *Phys. Plasmas* **14**, 103102 (2007); J. Zhou, K. R. Samokhvalova, and C. Chen, *Phys. Plasmas* **15**, 023102 (2008)]. In particular, results are obtained for adiabatic thermal beams that do not rotate in the Larmor frame. For such beams, the theoretical predictions of the rms beam envelope, the conservations of the rms thermal emittances, the adiabatic equation of state, and the Debye length are verified in the simulations. Furthermore, the adiabatic thermal beam is found to be stable in the parameter regime where the simulations are performed.

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I. INTRODUCTION

Thermal beam equilibrium is desirable in high-brightness charged-particle beam systems where space-charge effects play an important role in beam dynamics [1–3]. In particular, thermal beam equilibrium is a useful beam configuration, because there is no emittance growth if the beam is appropriately matched and if three-dimensional effects are not important. It is also a useful beam configuration for minimizing beam loss. However, achieving thermal beam equilibrium in a practical system, in general, requires innovations in particle source design, beam matching, and system optimization, engineering and fabrication, especially for the injector of a particle accelerator where space-charge effects are often most significant. Indeed, few true thermal beam equilibria have been achieved in space-charge-dominated high-brightness beam systems, despite the growing needs of high-brightness beams in accelerators for x-ray free electron lasers, high-energy and nuclear physics research, and high-energy-density physics research.

The simplest example of thermal beam equilibrium is the rigid-rotor thermal beam equilibrium in a uniform magnet field [4–6]. This equilibrium describes a continuous charged-particle beam undergoing rigid rotation about its axis while propagating along the uniform magnet field. The temperature is isotropic in the beam frame in which the net momentum of the beam vanishes. It is a generalization of the Brillouin flow in which thermal effects are negligibly small [7]. When the beam is space-charge dominated, the equilibrium beam density is uniform in the beam core but drops off exponentially at the beam edge in a few Debye lengths. When the beam is emittance dominated, the

thermal equilibrium beam density is Gaussian in the directions transverse to beam propagation. The distribution of the momentum relative to the fluid average is always Gaussian. Approximate rigid-rotor thermal beam equilibria were demonstrated experimentally [8], confirming certain theoretical predictions such as the beam focusing conditions and the angular velocity of beam rotation. They also revealed distortions in phase space.

Thermal beam equilibrium has been found theoretically in the smooth-focusing approximation for beam propagation in a periodic focusing field such as a periodic solenoidal magnetic focusing field or an alternating-gradient quadrupole magnetic focusing field [1–3]. For the periodic solenoidal magnetic field, the thermal beam equilibrium in the smooth-focusing approximation is essentially the rigid-rotor thermal beam equilibrium in an equivalent uniform magnet field whose strength is equal to the rms average of the periodic solenoidal magnetic focusing field. For the alternating-gradient quadrupole magnetic focusing field, the thermal beam equilibrium in the smooth-focusing approximation describes a continuous charged-particle beam focused by an effective axisymmetric focusing potential that has a quadratic dependence on the transverse displacement. The density and momentum distributions of the thermal beam equilibrium in the smooth-focusing approximation are similar to the density and momentum distributions for the rigid-rotor thermal beam equilibrium in the Larmor frame. In general, the smooth-focusing approximation is valid when the vacuum phase advance is sufficiently small and the rms beam envelope is approximately a constant, as shown in perturbation theory [9,10].

The thermal bunched-beam equilibrium has also been discussed in a moving, axisymmetric focusing field that provides both longitudinal focusing and axisymmetric transverse focusing for the beam [1]. In general, it describes an axisymmetric charged-particle bunch characterized by its rms beam radius and rms bunch length. The temperature is isotropic in the beam frame. When the

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bunch is space-charge dominated, the equilibrium beam density is a uniform density in the core of the bunch but drops off exponentially at the edge of the bunch in a few Debye lengths. When the bunch is emittance dominated, the equilibrium beam density is an axisymmetric Gaussian distribution. The thermal bunched-beam equilibrium is stable [9] inside a perfectly conducting circular beam tunnel.

Recently, the adiabatic thermal beam equilibrium has been predicted theoretically in a periodic solenoidal magnetic focusing field [11–13]. In general, an adiabatic thermal equilibrium corresponds to a spatially varying equilibrium state in the zero-order approximation of the Boltzmann equation. In the zero-order approximation, it is assumed that the distribution function is locally Maxwell-Boltzmann, and that the density, temperature, average flow velocity, and self-electric and self-magnetic fields may vary in space. In particular, the existence of the adiabatic thermal beam equilibrium was shown in the framework of kinetic theory and equivalent warm-fluid theory. In the warm-fluid theory of the adiabatic thermal beam equilibrium [11,12], warm-fluid equations were solved in the paraxial approximation. The equation of state was adiabatic. The rms beam envelope, the density and flow velocity profiles, and the self-consistent Poisson equation were derived. In the kinetic theory of the adiabatic thermal beam equilibrium [12,13], the thermal beam distribution function was constructed using the approximate and exact invariants of motion, i.e., a scaled transverse Hamiltonian and the angular momentum. By taking statistical averages, all of the equations in the warm-fluid theory were recovered, including the adiabatic equation of state, the rms beam envelope, the density and flow velocity profiles, and the self-consistent Poisson equation.

It was shown in a numerical study that chaotic particle motion is almost completely eliminated in the adiabatic thermal beam equilibrium [14]. The importance of this result is twofold: First, the elimination of chaotic particle motion provides a further numerical proof that the scaled transverse Hamiltonian is a very good approximate constant of motion. This approximate constant of motion and the exact constant of motion of the canonical angular momentum assure that the motion of charged particles is approximately integrable in the four-dimensional phase space of the adiabatic thermal beam equilibrium. Second, the elimination of chaotic particle motion may provide valuable insight into how to control chaotic particle motion, halo formation, and beam loss in beam transport channels. In particular, if adiabatic thermal beams are stable, they may be used to control chaotic particle motion, beam halo formation, and beam loss.

Some evidence was found that the theoretical prediction of the beam density is supported by the experimental data from a high-intensity beam experiment at the University of Maryland Electron Ring (UMER) [15]. However, agreement between the theory and the experiment was found

only for a short propagation distance that was less than one focusing period [12,13]. The lack of agreement between the theory and the experiment may be due to the fact that the beam was not created under the conditions required for the thermal beam equilibrium. Indeed, thermal beam equilibrium generation is still an important topic of research in beam physics.

The discovery of the adiabatic thermal beam equilibrium was an important advance in beam physics, overcoming the shortcoming of the Kapchinskij-Vladimirskij (KV)-type beam equilibrium in a periodic solenoidal magnetic focusing field [1–4,9,16,17]. The KV-type beam equilibrium has a singular (δ -function) distribution in the four-dimensional phase space. Such a δ -function distribution gives a uniform density profile across the beam in the transverse directions, and a transverse temperature profile which peaks on axis and decreases in a quadratic manner to zero on the edge of the beam. Because of the singularity in the distribution functions, KV beam equilibria are not likely to occur in real physical systems and cannot provide realistic models for theoretical and experimental studies and simulations except in the zero-temperature limit. In KV beam equilibria, there is no relation between temperature and density, and therefore, there is no equation of state. For example, the KV-type beam equilibrium model cannot be used to explain the beam tails in the radial distributions observed in the UMER experiment [15]. Furthermore, if one attempts to describe a realistic beam using the concept of an equivalent KV beam [1], the quadratic temperature profile induces an artificially repulsive pressure force, resulting in not only an unrealistically lower density of the beam on the axis but also unrealistic beam dynamics in phase space [15].

In this paper, we report results of two-dimensional (2D) particle-in-cell (PIC) simulations which further validate the theoretical predictions of the adiabatic thermal beam equilibrium [11,13]. In particular, we discuss simulation results for adiabatic thermal beams that do not rotate in the Larmor frame. For such beams, the simulation results verify the theoretical predictions of the rms beam envelope, the conservations of the rms thermal emittances, the adiabatic equation of state, and the Debye length. They also show beam stability in the parameter regime where the simulations are performed.

The organization of the present paper is as follows. In Sec. II, we discuss the 2D PIC model for adiabatic thermal beams in a periodic solenoidal magnetic focusing field, and describe the initialization of an adiabatic thermal beam. In Sec. III, we discuss results of 2D PIC simulations, compare them with theoretical predictions, and find good agreement between simulation and theory. In Sec. IV, we present the conclusion and discuss directions for future investigation.

II. PARTICLE-IN-CELL MODEL

We study self-consistent charged-particle dynamics in the adiabatic thermal equilibrium of an intense charged-particle

beam propagating with constant axial velocity $\beta_b c \hat{e}_z$ in the periodic solenoidal magnetic focusing field,

$$\mathbf{B}^{\text{ext}} = B_z(s) \hat{e}_z - \frac{1}{2} \frac{dB_z(s)}{ds} (x \hat{e}_x + y \hat{e}_y), \quad (1)$$

where $s = z$ is the axial coordinate, $B_z(s + S) = B_z(s)$ is the axial magnetic field, S is the fundamental periodicity length of the focusing field, and c is the speed of light in vacuum.

The paraxial approximation is made under the following assumptions: (1) $r_{brms} \ll S$, where r_{brms} is the rms beam radius, and (2) $\nu/\gamma_b^3 \beta_b^2 \ll 1$, where $\nu = q^2 N_b / mc^2$ is the Budker parameter of the beam, q and m are the particle charge and rest mass, respectively, $N_b = \int_0^\infty n_b(r, s) 2\pi r dr = \text{const}$ is the number of charged particles per unit axial length, and $\gamma_b = (1 - \beta_b^2)^{-1/2}$ is the relativistic mass factor.

The basic equations in the 2D PIC model are expressed in cgs units as

$$\frac{d^2 \tilde{\mathbf{x}}_{\perp i}}{ds^2} + \kappa_z(s) \tilde{\mathbf{x}}_{\perp i} + \frac{q}{\gamma_b^3 m \beta_b^2 c^2} \frac{\partial \phi(\tilde{\mathbf{x}}_{\perp i}, s)}{\partial \tilde{\mathbf{x}}_{\perp i}} = 0, \quad (2)$$

$$\frac{\partial^2 \phi(\tilde{\mathbf{x}}_{\perp i}, s)}{\partial \tilde{\mathbf{x}}_{\perp i}^2} = -4\pi q n_b(\tilde{\mathbf{x}}_{\perp i}, s), \quad (3)$$

where $i = 1, 2, \dots, N_p$, $\sqrt{\kappa_z(s)} = qB_z(s)/2\gamma_b m \beta_b c^2$, ϕ is the electrostatic potential, n_b is the beam density, and $\tilde{\mathbf{x}}_{\perp i}$ is the transverse position of the i th macroparticle in the Larmor frame, i.e., $\tilde{x} = x \cos \varphi - y \sin \varphi$ and $\tilde{y} = x \sin \varphi + y \cos \varphi$ with $\varphi = \int_0^s \sqrt{\kappa_z(s)} ds$.

In the 2D PIC simulations, the focusing field is chosen to be sinusoidal with $S\sqrt{\kappa_z(s)} = (2/3)^{1/2} \sigma_0 [1 + \cos(2\pi s/S)]$, where σ_0 is the vacuum phase advance. Poisson's equation (3) is solved using a successive over relaxation algorithm in a grounded circular perfect conducting pipe of radius R centered on the z axis on a square mesh. The value of R is chosen to be several times that of the maximum value of the rms beam radius.

The 2D PIC simulations discussed in this paper are limited to a class of adiabatic thermal beams that do not rotate in the Larmor frame. For such a class of beams, the macroparticles are loaded at $s = 0$ according to the initial distribution function [12,13]

$$f(\tilde{\mathbf{x}}_{\perp}, \tilde{\mathbf{x}}'_{\perp}, s = 0) = n_b(\tilde{r}, 0) \exp\left[-\frac{\gamma_b m \beta_b^2 \tilde{\mathbf{x}}_{\perp}^2}{2k_B T_{\perp}(0)}\right], \quad (4)$$

where $\tilde{r} = (\tilde{x}^2 + \tilde{y}^2)^{1/2}$, prime denotes derivative with respect to s , k_B is the Boltzmann constant, and $T_{\perp}(0)$ and $n_b(\tilde{r}, 0)$ are the initial beam temperature and density, respectively. The density $n_b(\tilde{r}, 0)$ is given by

$$n_b(\tilde{r}, 0) = \frac{4\pi C \varepsilon_{th}^2}{r_{brms}^2(0)} \exp\left[-\frac{K \tilde{r}^2}{8\varepsilon_{th}^2} - \frac{\tilde{r}^2}{r_{brms}^2(0)} - \frac{q\phi(\tilde{r}, 0)}{\gamma_b^2 k_B T_{\perp}(0)}\right], \quad (5)$$

where C is a constant determined by $N_b = \int_0^\infty n_b(\tilde{r}, s) 2\pi \tilde{r} d\tilde{r}$, $K \equiv 2q^2 N_b / \gamma_b^3 m \beta_b^2 c^2$ is the generalized beam perveance, the initial electrostatic potential $\phi(\tilde{r}, 0)$ is determined from the Poisson equation

$$\frac{1}{\tilde{r}} \frac{\partial}{\partial \tilde{r}} \left(\tilde{r} \frac{\partial}{\partial \tilde{r}} \right) \phi(\tilde{r}, 0) = -4\pi q n_b(\tilde{r}, 0), \quad (6)$$

and $r_{brms}(0)$ and $r'_{brms}(0) = 0$ are the initial conditions of the periodic rms beam envelope $r_{brms}(s) = r_{brms}(s + S)$ that solves the rms beam envelope equation [12,13]:

$$\frac{d^2 r_{brms}}{ds^2} + \kappa_z(s) r_{brms} - \frac{K}{2r_{brms}} = \frac{4\varepsilon_{th}^2}{r_{brms}^3}. \quad (7)$$

The trajectories of the macroparticles are integrated using the standard leapfrog method. The 2D PIC algorithm is implemented in a MATLAB version of the MIT 2D periodically focused beam (PFB2D) code which was used extensively in simulation studies of high-brightness charged-particle beams in periodic focusing fields [18].

III. SIMULATION RESULTS

Two-dimensional PIC simulations were performed to verify the theoretical predictions of the following [11–13]: (a) rms beam envelope; (b) conservation of rms thermal emittances; (c) adiabatic equation of state; (d) density, velocity, and temperature profiles; and (e) Debye length. Results of a typical simulation are shown in Figs. 1–4 for the parameters: $S^2 \kappa_z(s) = (2/3) \sigma_0^2 [1 + \cos(2\pi s/S)]^2$, $\sigma_0 = 80^\circ$, $SK/4\varepsilon_{th} = 7.0$, $N_p = 1 \times 10^6$, $(4\varepsilon_{th} S)^{-1/2} R = 5.0$, integration step size $\Delta s/S = 0.01$, and mesh size $(4\varepsilon_{th} S)^{-1/2} \Delta \tilde{x} = 0.02$.

The normalized rms beam radius $(4\varepsilon_{th} S)^{-1/2} r_{brms}$ from the simulation is plotted in Fig. 1 as a function of the normalized distance s/S . Within the statistical error

FIG. 1. Plot of the normalized rms beam radius $(4\varepsilon_{th} S)^{-1/2} r_{brms}$ versus the normalized distance s/S .

FIG. 2. Plots of the probability density distributions $\rho(\tilde{x}')$ versus $(S/4\epsilon_{th})^{1/2}\tilde{x}'$ and the probability density distributions $\rho(\tilde{y}')$ versus $(S/4\epsilon_{th})^{1/2}\tilde{y}'$ at $s/S = 20$. The dotted curves are from the simulation, whereas the dashed curves are from the theoretical predictions.

of $\pm 1\%$, the result agrees with the matched beam envelope solution of Eq. (7).

The transverse momentum distribution continues to be Gaussian, as illustrated in Fig. 2. The beam density distribution maintains a plateau distribution as shown in Fig. 3 in which the normalized beam density $(4\epsilon_{th}S/N_b)n_b$ is plotted as a function of the normalized radius $(4\epsilon_{th}S)^{-1/2}r$ at $s/S = 20$. The dotted curves are from the simulation, whereas the dashed curves are from the theoretical predictions. The characteristic scale over which the beam density

FIG. 3. Plot of the normalized beam density $(4\epsilon_{th}S/N_b)n_b$ versus the normalized radius $(4\epsilon_{th}S)^{-1/2}r$ at $s/S = 20$.

FIG. 4. Plots of the relative rms thermal emittances $\epsilon_{\tilde{x}rms}/\epsilon_{\tilde{x}rms}(0)$ and $\epsilon_{\tilde{y}rms}/\epsilon_{\tilde{y}rms}(0)$ versus the normalized distance s/S .

falls is the Debye length λ_D , which, in this example, has a normalized value of $(4\epsilon_{th}S)^{-1/2}\lambda_D = 0.14$. There is good agreement between the simulation and theory.

In Fig. 4, the evolutions of the relative rms thermal emittances $\epsilon_{\tilde{x}rms}/\epsilon_{\tilde{x}rms}(0)$ and $\epsilon_{\tilde{y}rms}/\epsilon_{\tilde{y}rms}(0)$ in the simulation are shown in solid and dashed curves, respectively, as a function of the normalized distance s/S . Here, $\epsilon_{\tilde{x}rms}(0)$ and $\epsilon_{\tilde{y}rms}(0)$ are the rms emittances in the $\tilde{x}$ and $\tilde{y}$ directions at $s = 0$, respectively. Both rms thermal emittances are conserved within the statistical error of $\pm 0.3\%$. These results are in agreement with the theoretical prediction of the conservation of rms thermal emittances, and validate the adiabatic equation of state [11,12].

In the parameter regimes where simulation studies were performed, we did not find any indication of beam instability. The results shown in Figs. 1–4 are typical; that is, the Gaussian transverse momentum distributions, the plateau beam density distribution, the temperature, and rms beam radius repeat themselves as the beam propagates through every period in the focusing field.

While a complete proof of the stability of adiabatic thermal beams is not available, the present simulations are consistent with the previously reported simulations showing that a beam with a semi-Gaussian distribution (i.e., a distribution that is Gaussian in the momentum space and uniform in the configuration space) suffers from considerably less emittance growth than a beam with a KV distribution [19].

IV. CONCLUSION

Two-dimensional particle-in-cell simulations were performed to verify the earlier theoretical predictions of adiabatic thermal beams in a periodic solenoidal magnetic focusing field. Results were obtained for adiabatic thermal beams that do not rotate in the Larmor frame. For such beams, the theoretical predictions of the rms beam envelope, the conservations of the rms thermal emittances, the adiabatic equation of state, and the Debye length were verified in the 2D PIC simulations. Furthermore, the

adiabatic thermal beam was found to be stable in the parameter regime where the simulations were performed.

The generation of an adiabatic thermal beam in a practical beam device will be an important topic of future research in beam physics. In general, achieving thermal beam equilibrium in a practical system will still require innovations in particle source design, beam matching, and system optimization, engineering and fabrication, especially for the beam injector where space-charge effects are most significant.

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SIMULATION STUDIES OF ADIABATIC THERMAL BEAMS IN A PERIODIC SOLENOIDAL FOCUSING FIELD*

C. Chen[#], T.J. Barton, D.M. Field and K.M. Lang, MIT, Cambridge, MA 02139, U.S.A.

Abstract

Two-dimensional (2D) particle-in-cell (PIC) simulations are performed to verify earlier theoretical predictions of adiabatic thermal beams in a periodic solenoidal magnetic focusing field. In particular, results are obtained for adiabatic thermal beams that do not rotate in the Larmor frame. For such beams, the theoretical predictions of the rms beam envelope, the conservations of rms thermal emittances, the adiabatic equation of state, and the Debye length are verified in the PIC simulations.

INTRODUCTION

Adiabatic thermal beam equilibrium was discovered recently in a periodic solenoidal magnetic focusing field [1-3]. In particular, the existence of the adiabatic thermal beam equilibrium was shown in the frameworks of kinetic theory and equivalent warm-fluid theory.

In the warm-fluid theory of the adiabatic thermal beam equilibrium [1,2], warm-fluid equations were solved in the paraxial approximation. The equation of state was adiabatic. The rms beam envelope, the density and flow velocity profiles, and the self-consistent Poisson equation were derived.

In the kinetic theory of the adiabatic thermal beam equilibrium [3], the thermal beam distribution function was constructed using the approximate and exact invariants of motion, i.e., a scaled transverse Hamiltonian and the angular momentum. By taking statistical averages, all of the equations in the warm-fluid theory were recovered, including the adiabatic equation of state, the rms beam envelope, the density and flow velocity profiles, and the self-consistent Poisson equation.

Effects of the beam perveance, emittance and rotation on the beam envelope and density distribution were examined. Good agreement was found [3] between theory and a recent high-intensity beam experiment performed at the University of Maryland Electron Ring (UMER) [4].

The phase space for charged-particle motion in the adiabatic thermal beam equilibrium was analyzed [5] and compared with that of the KV-type beam equilibrium [6-8]. It was found that the widths of nonlinear resonances in the adiabatic thermal beam equilibrium are narrower than those in the KV-type beam equilibrium. Numerical evidence is presented, indicating the almost complete absence of chaotic particle motion in the adiabatic thermal beam equilibrium.

The discovery of the adiabatic thermal beam equilibrium was an important advance in beam physics,

overcoming the shortcoming of the Kapchinskij-Vladimirskij (KV) type equilibrium in a periodic solenoidal magnetic focusing field [6-8]. The KV type equilibrium has a singular (δ -function) distribution in the four-dimensional phase space. Such a δ -function distribution gives a uniform density profile across the beam in the transverse directions, and a transverse temperature profile which peaks on axis and decreases quadratically to zero on the edge of the beam. Because of the singularity in the distribution functions, these beam equilibria are not likely to occur in real physical systems and cannot provide realistic models for theoretical and experimental studies and simulations except in the zero-temperature limit. For example, the KV equilibrium model cannot be used to explain the beam tails in the radial distributions observed in the recent high-intensity beam experiment [4]. In contrast, the measured density distribution matches that of the adiabatic thermal beam equilibrium [3,4].

In this paper, results of two-dimensional (2D) particle-in-cell (PIC) simulations are presented, which further validate the theoretical predictions of the adiabatic thermal beam equilibrium. In particular, results are obtained for adiabatic thermal beams that do not rotate in the Larmor frame. For such beams, the theoretical predictions of the rms beam envelope, the conservations of rms thermal emittances, the adiabatic equation of state, and the Debye length are verified in the simulations.

PARTICLE-IN-CELL MODEL

We study charged-particle dynamics in the adiabatic thermal equilibrium of an intense charged-particle beam propagating with constant axial velocity $\beta_b c \hat{e}_z$ in the periodic solenoidal magnetic focusing field

$$\mathbf{B}^{ext} = B_z(s) \hat{e}_z - \frac{1}{2} \frac{dB_z(s)}{ds} (x \hat{e}_x + y \hat{e}_y), \quad (1)$$

where $s = z$ is the axial coordinate, $B_z(s + S) = B_z(s)$ is the axial magnetic field, S is the fundamental periodicity length of the focusing field, and c is the speed of light in vacuum.

The paraxial approximation is made under the following assumptions : 1) $r_{brms} \ll S$, where r_{brms} is the rms beam radius, and 2) $v / \gamma_b^3 \beta_b^2 \ll 1$, where $v = q^2 N_b / mc^2$ is the Budker parameter of the beam, q and m are the particle charge and rest mass, respectively, $N_b = \int_0^\infty n_b(r, s) 2\pi r dr = \text{const}$ is the number of charged particles per unit axial length, and $\gamma_b = (1 - \beta_b^2)^{-1/2}$ is the relativistic mass factor.

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[#]chenc@psfc.mit.edu

The basic equations in the 2D PIC model are expressed in cgs units as [3]

$$\frac{d^2 \tilde{\mathbf{x}}_{\perp i}}{ds^2} + \kappa_z(s) \tilde{\mathbf{x}}_{\perp i} + \frac{q}{\gamma_b^3 m \beta_b^2 c^2} \frac{\partial \phi(\tilde{\mathbf{x}}_{\perp i}, s)}{\partial \tilde{\mathbf{x}}_{\perp i}} = 0, \quad (2)$$

$$\frac{\partial \phi(\tilde{\mathbf{x}}_{\perp i}, s)}{\partial \tilde{\mathbf{x}}_{\perp i}} = -4\pi q n_b(\tilde{\mathbf{x}}_{\perp i}, s), \quad (3)$$

where $i = 1, 2, \dots, N_P$, $\sqrt{\kappa_z(s)} = qB_z(s)/2\gamma_b m \beta_b c^2$, ϕ is the electrostatic potential, n_b is the beam density, and $\tilde{\mathbf{x}}_{\perp i}$ is the transverse position of the i th macroparticle in the Larmor frame, i.e., $\tilde{x} = x \cos \varphi - y \sin \varphi$ and

$$\tilde{y} = x \sin \varphi + y \cos \varphi \quad \text{with} \quad \varphi = \int_0^s \sqrt{\kappa_z(s)} ds.$$

In the 2D PIC simulations, the focusing field is chosen to be sinusoidal with $S\sqrt{\kappa_z(s)} = \sigma_0 [1 + \cos(2\pi s/S)]$, where σ_0 is the vacuum phase advance. Poisson's equation (3) is solved using a successive over relaxation (SOR) algorithm in a grounded circular perfect conducting pipe of radius R centered on the z -axis on a square mesh. The value of R is chosen to be several times that the maximum value of the rms beam radius.

The 2D PIC simulations discussed in this paper are limited to a class of adiabatic thermal beams that do not rotate in the Larmor frame. For such a class of beams, the macroparticles are loaded at $s = 0$ according to the initial distribution function

$$f(\tilde{\mathbf{x}}_{\perp}, \tilde{\mathbf{x}}'_{\perp}, s=0) = n_b(\tilde{r}, 0) \exp \left\{ -\frac{\gamma_b m \beta_b^2 \tilde{\mathbf{x}}_{\perp}^2}{2k_B T_{\perp}(0)} \right\}, \quad (4)$$

where $\tilde{r} = (\tilde{x}^2 + \tilde{y}^2)^{1/2}$, prime denotes derivative with respect to s , k_B is the Boltzmann constant, and $T_{\perp}(0)$ and $n_b(\tilde{r}, 0)$ are the initial beam temperature and density, respectively. The density $n_b(\tilde{r}, 0)$ is given by

$$n_b(\tilde{r}, 0) = \frac{4\pi C \varepsilon_{th}^2}{r_{brms}^2(0)} \exp \left\{ -\frac{K \tilde{r}^2}{8\varepsilon_{th}^2} - \frac{\tilde{r}^2}{r_{brms}^2(0)} - \frac{q\phi(\tilde{r}, 0)}{\gamma_b^2 k_B T_{\perp}(0)} \right\}, \quad (5)$$

where C is a constant determined by $N_b = \int_0^\infty n_b(\tilde{r}, s) 2\pi \tilde{r} d\tilde{r}$, $K \equiv 2q^2 N_b / \gamma_b^3 m \beta_b^2 c^2$ is the generalized beam perveance, the initial electrostatic potential $\phi(\tilde{r}, 0)$ is determined from the Poisson equation

$$\frac{1}{\tilde{r}} \frac{\partial}{\partial \tilde{r}} \left(\tilde{r} \frac{\partial}{\partial \tilde{r}} \right) \phi(\tilde{r}, 0) = -4\pi q n_b(\tilde{r}, 0), \quad (6)$$

and $r_{brms}(0)$ and $r'_{brms}(0) = 0$ are the initial conditions of the periodic rms beam envelope $r_{brms}(s) = r_{brms}(s + S)$ that solves the rms beam envelope equation

$$\frac{d^2 r_{brms}}{ds^2} + \kappa_z(s) r_{brms} - \frac{K}{2r_{brms}} = \frac{4\varepsilon_{th}^2}{r_{brms}^3}. \quad (7)$$

The trajectories of the macroparticles are integrated using the standard leapfrog method.

The 2D PIC algorithm is implemented in a Matlab version of the MIT 2D Periodically Focused Beam (PFB2D) code which was used extensively in simulation studies of high-brightness charged-particle beams in periodic focusing fields [9]. The trajectories of the macroparticles are integrated using the standard leapfrog method.

RESULTS

Two-dimensional PIC simulations are performed to verify the theoretical predictions on the following:

- rms beam envelope;
- conservations of rms thermal emittances;
- adiabatic equation of state;
- density, velocity, and temperature profiles; and
- Debye length.

Results of a typical PFB2D simulation are shown in Figs. 1-5 for the parameters: $\sigma_0 = 80^\circ$, $SK/4\varepsilon_{th} = 7.0$, $N_P = 1 \times 10^6$, $(S/4\varepsilon_{th})^{1/2} R = 5.0$, integration step size $\Delta s/S = 0.01$, and mesh size $(S/4\varepsilon_{th})^{1/2} \Delta \tilde{x} = 0.02$.

The focusing parameter $S^2 \kappa_z(s) = \sigma_0^2 [1 + \cos(2\pi s/S)]^2$ is plotted in Fig. 1 as a function of the normalized distance s/S . The normalized rms beam radius $(S/4\varepsilon_{th})^{1/2} r_{brms}$ from the PFB2D simulation is plotted in Fig. 2 as a function of the normalized distance s/S . Within the statistical error of $\pm 1\%$, the result agrees with the matched beam envelope solution of Eq. (7).

Figure 1: Plot of the normalized focusing parameter $S^2 \kappa_z(s)$ versus the normalized distance s/S .

Figure 2: Plot of the normalized rms beam radius $(S/4\varepsilon_{th})^{1/2} r_{brms}$ versus the normalized distance s/S .

Figure 3: Plots of (a) probability density distributions $\rho(\tilde{x}')$ versus $(S/4\epsilon_{th})^{1/2}\tilde{x}'$ and (b) probability density distributions $\rho(\tilde{y}')$ versus $(S/4\epsilon_{th})^{1/2}\tilde{y}'$ at $s/S = 20$. The dotted curves are from the PFB2D simulation, whereas the dashed curves are from the theoretical predictions [1-3].

Figure 4: Plot of the normalized beam density $(4\epsilon_{th}S/N_b)n_b$ versus the normalized radius $(S/4\epsilon_{th})^{1/2}r$ at $s/S = 20$.

Figure 5: Plots of the relative rms thermal emittances $\epsilon_{\tilde{x}rms}/\epsilon_{\tilde{x}rms}(0)$ and $\epsilon_{\tilde{y}rms}/\epsilon_{\tilde{y}rms}(0)$ versus the normalized distance s/S .

The transverse momentum distribution remains to be Gaussian, as illustrated in Fig. 3. It is evident that there is good agreement between the simulation and theory.

The beam density distribution maintains a plateau distribution as shown in Fig. 4 in which the normalized beam density $(4\epsilon_{th}S/N_b)n_b$ is plotted as a function of the normalized radius $(S/4\epsilon_{th})^{1/2}r$ at $s/S = 20$. The dotted curves are from the PFB2D simulation, whereas the dashed curves are from the theoretical predictions. The characteristic scale over which the beam density falls is the Debye length λ_D [1-3], which, in this example, has a normalized value of $(S/4\epsilon_{th})^{1/2}\lambda_D = 0.14$. There is good agreement between the simulation and theory.

In Fig. 5, the evolutions of the relative rms thermal emittances $\epsilon_{\tilde{x}rms}/\epsilon_{\tilde{x}rms}(0)$ and $\epsilon_{\tilde{y}rms}/\epsilon_{\tilde{y}rms}(0)$ in the PFB2D simulation are shown in solid and dashed curves, respectively, as a function of s/S . Here, $\epsilon_{\tilde{x}rms}(0)$ and $\epsilon_{\tilde{y}rms}(0)$ are the rms thermal emittances in the $\tilde{x}$ - and $\tilde{y}$ -directions at $s = 0$, respectively. Both rms emittances are conserved within the statistical error of $\pm 0.3\%$. These results are in agreement with the theoretical predictions of the conservations of rms thermal emittances and adiabatic equation of state.

CONCLUSION

Two-dimensional particle-in-cell simulations were performed to verify the theoretical predictions of adiabatic thermal beams in a periodic solenoidal magnetic focusing field. Results were obtained for adiabatic thermal beams that do not rotate in the Larmor frame. For such beams, the theoretical predictions of the rms beam envelope, the conservations of rms thermal emittances, the adiabatic equation of state, and the Debye length were verified in the 2D PIC simulations.

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Simulation of Periodically Focused, Adiabatic Thermal Beams

C. Chen, T.R. Akylas, T.J. Barton, D.M. Field, K.M. Lang and R.V. Mok

Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

Abstract. Self-consistent particle-in-cell simulations are performed to verify earlier theoretical predictions of adiabatic thermal beams in a periodic solenoidal magnetic focusing field [K.R. Samokhvalova, J. Zhou and C. Chen, *Phys. Plasma* 14, 103102 (2007); J. Zhou, K.R. Samokhvalova and C. Chen, *Phys. Plasma* 15, 023102 (2008)]. In particular, results are obtained for adiabatic thermal beams that do not rotate in the Larmor frame. For such beams, the theoretical predictions of the rms beam envelope, the conservations of the rms thermal emittances, the adiabatic equation of state, and the Debye length are verified in the simulations. Furthermore, the adiabatic thermal beam is found to be stable in the parameter regime where the simulations are performed.

Keywords: Charged-particle beam, periodic focusing, space charge, adiabatic thermal equilibrium

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INTRODUCTION

Thermal beam equilibrium is desirable in high-brightness charged-particle beam systems where space-charge effects play an important role in beam dynamics [1-3]. In particular, thermal beam equilibrium is a useful beam configuration, because there is no emittance growth if the beam is appropriately matched and if three-dimensional effects are not important. It is also a useful beam configuration for minimizing beam loss. However, achieving thermal beam equilibrium in a practical system, in general, requires innovations in particle source design, beam matching, and system optimization, engineering and fabrication, especially for the injector of a particle accelerator where space-charge effects are often most significant. Indeed, few true thermal beam equilibria have been achieved in space-charge-dominated high-brightness beam systems, despite the growing needs of high-brightness beams in accelerators for x-ray free electron lasers, high-energy and nuclear physics research, and high-energy-density physics research.

The simplest example of thermal beam equilibrium is the rigid-rotor thermal beam equilibrium in a uniform magnet field [4-6]. This equilibrium describes a continuous charged-particle beam undergoing rigid rotation about its axis while propagating along the uniform magnet field. The temperature is isotropic in the beam frame in which the net momentum of the beam vanishes. It is a generalization of the Brillouin flow in which thermal effects are negligibly small [7]. When the beam is space-charge-dominated, the equilibrium beam density is uniform in the beam core but drops off exponentially at the beam edge in a few Debye lengths. When the beam is emittance-dominated, the thermal equilibrium beam density is Gaussian in the directions transverse to beam propagation. The distribution of the momentum relative to the fluid average is always Gaussian. Approximate rigid-rotor thermal beam equilibria were demonstrated experimentally [8], confirming certain theoretical predictions such as the beam focusing conditions and the angular velocity of beam rotation. They also revealed distortions in phase space.

Thermal beam equilibrium has been found theoretically in the smooth-focusing approximation for beam propagation in a periodic focusing field such as a periodic solenoidal magnetic focusing field or an alternating-gradient quadrupole magnetic focusing field [1-3]. For the periodic solenoidal magnetic field, the thermal beam equilibrium in the smooth-focusing approximation is essentially the rigid-rotor thermal beam equilibrium in an equivalent uniform magnet field whose strength is equal to the rms average of the periodic solenoidal magnetic focusing field. For the alternating-gradient quadrupole magnetic focusing field, the thermal beam equilibrium in the smooth-focusing approximation describes a continuous charged-particle beam focused by an effective axisymmetric focusing potential that has a quadratic dependence on the transverse displacement. The density and momentum

distributions of the thermal beam equilibrium in the smooth-focusing approximation are similar to the density and momentum distributions for the rigid-rotor thermal beam equilibrium in the Larmor frame. In general, the smooth-focusing approximation is valid when the vacuum phase advance is sufficiently small and the rms beam envelope is approximately a constant, as shown in perturbation theory [9,10].

Thermal bunched-beam equilibrium has also been discussed in a moving, axisymmetric focusing field that provides both longitudinal focusing and axisymmetric transverse focusing for the beam [1]. In general, it describes an axisymmetric charged-particle bunch characterized by its rms beam radius and rms bunch length. The temperature is isotropic in the beam frame. When the bunch is space-charge-dominated, the equilibrium beam density is a uniform density in the core of the bunch but drops off exponentially at the edge of the bunch in a few Debye lengths. When the bunch is emittance-dominated, the equilibrium beam density is an axisymmetric Gaussian distribution. The thermal bunched-beam equilibrium is stable [9] inside a perfectly conducting circular beam tunnel.

Recently, adiabatic thermal beam equilibrium has been predicated theoretically in a periodic solenoidal magnetic focusing field [11-13]. In general, an adiabatic thermal equilibrium corresponds to a spatially-varying equilibrium state in the zero-order approximation of the Boltzmann equation. In the zero-order approximation, it is assumed that the distribution function is locally Maxwell-Boltzmann, and that the density, temperature, average flow velocity, and self-electric and self magnetic fields may vary in space. In particular, the existence of the adiabatic thermal beam equilibrium was shown in the framework of kinetic theory and equivalent warm-fluid theory. In the warm-fluid theory of the adiabatic thermal beam equilibrium [11,12], warm-fluid equations were solved in the paraxial approximation. The equation of state was adiabatic. The rms beam envelope, the density and flow velocity profiles, and the self-consistent Poisson equation were derived. In the kinetic theory of the adiabatic thermal beam equilibrium [12,13], the thermal beam distribution function was constructed using the approximate and exact invariants of motion, i.e., a scaled transverse Hamiltonian and the angular momentum. By taking statistical averages, all of the equations in the warm-fluid theory were recovered, including the adiabatic equation of state, the rms beam envelope, the density and flow velocity profiles, and the self-consistent Poisson equation.

It was shown in a numerical study that chaotic particle motion is almost completely eliminated in the adiabatic thermal beam equilibrium [14]. The importance of this result is twofold: Firstly, the elimination of chaotic particle motion provides a further numerical proof that the scaled transverse Hamiltonian is a very good approximate constant of motion. This approximate constant of motion and the exact contact of motion of the canonical angular momentum assure that the motion of charged particles is approximately integrable in the four-dimensional phase space of the adiabatic thermal beam equilibrium. Secondly, the elimination of chaotic particle motion may provide valuable insight into how to control chaotic particle motion, halo formation and beam loss in beam transport channels. In particular, if adiabatic thermal beams are stable, they may be used to control chaotic particle motion, beam halo formation, and beam loss.

Some evidence was found that the theoretical prediction of the beam density is supported by the experimental data from a high-intensity beam experiment at the University of Maryland Electron Ring (UMER) [15]. However, agreement between the theory and the experiment was found only for a short propagation distance that was less than one focusing period [12,13]. The lack of agreement between the theory and the experiment may be due to the fact that the beam was not created under the conditions required for the thermal beam equilibrium. Indeed, thermal beam equilibrium generation is still an important topic of research in beam physics.

The discovery of the adiabatic thermal beam equilibrium was an important advance in beam physics, overcoming the shortcoming of the Kapchinskij-Vladimirskij (KV)-type beam equilibrium in a periodic solenoidal magnetic focusing field [1-4,9,16,17]. The KV-type beam equilibrium has a singular (δ -function) distribution in the four-dimensional phase space. Such a δ -function distribution gives a uniform density profile across the beam in the transverse directions, and a transverse temperature profile which peaks on axis and decreases in a quadratic manner to zero on the edge of the beam. Because of the singularity in the distribution functions, KV beam equilibria are not likely to occur in real physical systems and cannot provide realistic models for theoretical and experimental studies and simulations except in the zero-temperature limit. In KV beam equilibria, there is no relation between temperature and density, and therefore, there is no equation of state. For example, the KV-type beam equilibrium model cannot be used to explain the beam tails in the radial distributions observed in the UMER experiment [15]. Furthermore, if one attempts to describe a realistic beam using the concept of an equivalent KV beam [1], the quadratic temperature profile induces an artificially repulsive pressure force, resulting in not only a unrealistically lower density of the beam on the axis but also unrealistic beam dynamics in phase space [15].

In this paper, we report results of two-dimensional (2D) particle-in-cell (PIC) simulations which further validate the theoretical predictions of the adiabatic thermal beam equilibrium [11-13]. In particular, we discuss simulation results for adiabatic thermal beams that do not rotate in the Larmor frame. For such beams, the simulation results verify the theoretical predictions of the rms beam envelope, the conservations of the rms thermal emittances, the

adiabatic equation of state, and the Debye length. They show beam stability in the parameter regime where the simulations are performed. We also describe briefly our initiative to explore two-dimensional adiabatic thermal Child-Langmuir flow which may be of critical importance for future design and experimental studies of adiabatic thermal beams.

PARTICLE-IN-CELL MODEL

We study self-consistent charged-particle dynamics in the adiabatic thermal equilibrium of an intense charged-particle beam propagating with constant axial velocity $\beta_b c \hat{\mathbf{e}}_z$ in the periodic solenoidal magnetic focusing field

$$\mathbf{B}^{ext} = B_z(s) \hat{\mathbf{e}}_z - \frac{1}{2} \frac{dB_z(s)}{ds} (x \hat{\mathbf{e}}_x + y \hat{\mathbf{e}}_y), \quad (1)$$

where $s = z$ is the axial coordinate, $B_z(s+S) = B_z(s)$ is the axial magnetic field, S is the fundamental periodicity length of the focusing field, and c is the speed of light in vacuum. The paraxial approximation is made under the following assumptions : 1) $r_{brms} \ll S$, where r_{brms} is the rms beam radius, and 2) $v/\gamma_b^3 \beta_b^2 \ll 1$, where $v = q^2 N_b / mc^2$ is the Budker parameter of the beam, q and m are the particle charge and rest mass, respectively, $N_b = \int_0^\infty n_b(r, s) 2\pi r dr = \text{const}$ is the number of charged particles per unit axial length, and $\gamma_b = (1 - \beta_b^2)^{-1/2}$ is the relativistic mass factor. The basic equations in the 2D PIC model are expressed in cgs units as

$$\frac{d^2 \tilde{\mathbf{x}}_{\perp i}}{ds^2} + \kappa_z(s) \tilde{\mathbf{x}}_{\perp i} + \frac{q}{\gamma_b^3 m \beta_b^2 c^2} \frac{\partial \phi(\tilde{\mathbf{x}}_{\perp i}, s)}{\partial \tilde{\mathbf{x}}_{\perp i}} = 0, \quad (2)$$

$$\frac{\partial^2 \phi(\tilde{\mathbf{x}}_{\perp i}, s)}{\partial \tilde{\mathbf{x}}_{\perp i}^2} = -4\pi q n_b(\tilde{\mathbf{x}}_{\perp i}, s), \quad (3)$$

where $i = 1, 2, \dots, N_p$, $\sqrt{\kappa_z(s)} = q B_z(s) / (2 \gamma_b m \beta_b c^2)$, ϕ is the electrostatic potential, n_b is the beam density, and $\tilde{\mathbf{x}}_{\perp i}$ is the transverse position of the i th macroparticle in the Larmor frame, i.e., $\tilde{x} = x \cos \varphi - y \sin \varphi$ and $\tilde{y} = x \sin \varphi + y \cos \varphi$ with $\varphi = \int_0^s \sqrt{\kappa_z(s)} ds$. In the 2D PIC simulations, the focusing field is chosen to be sinusoidal with $S \sqrt{\kappa_z(s)} = \sigma_0 [1 + \cos(2\pi s / S)]$, where σ_0 is the vacuum phase advance. Poisson's equation (3) is solved using a successive over relaxation (SOR) algorithm in a grounded circular perfect conducting pipe of radius R centered on the z -axis on a square mesh. The value of R is chosen to be several times that the maximum value of the rms beam radius.

The 2D PIC simulations discussed in this paper are limited to a class of adiabatic thermal beams that do not rotate in the Larmor frame. For such a class of beams, the macroparticles are loaded at $s = 0$ according to the initial distribution function [12,13]

$$f(\tilde{\mathbf{x}}_{\perp}, \tilde{\mathbf{x}}'_{\perp}, s = 0) = n_b(\tilde{r}, 0) \exp \left\{ -\frac{\gamma_b m \beta_b^2 \tilde{\mathbf{x}}_{\perp}^2}{2 k_B T_{\perp}(0)} \right\}, \quad (4)$$

where $\tilde{r} = (\tilde{x}^2 + \tilde{y}^2)^{1/2}$, prime denotes derivative with respect to s , k_B is the Boltzmann constant, and $T_{\perp}(0)$ and $n_b(\tilde{r}, 0)$ are the initial beam temperature and density, respectively. The density $n_b(\tilde{r}, 0)$ is given by

$$n_b(\tilde{r}, 0) = \frac{4\pi C \varepsilon_{th}^2}{r_{brms}^2(0)} \exp \left\{ -\frac{K \tilde{r}^2}{8 \varepsilon_{th}^2} - \frac{\tilde{r}^2}{r_{brms}^2(0)} - \frac{q \phi(\tilde{r}, 0)}{\gamma_b^2 k_B T_{\perp}(0)} \right\}, \quad (5)$$

where C is a constant determined by $N_b = \int_0^\infty n_b(\tilde{r}, s) 2\pi \tilde{r} d\tilde{r}$, $K \equiv 2q^2 N_b / \gamma_b^3 m \beta_b^2 c^2$ is the generalized beam perveance, the initial electrostatic potential $\phi(\tilde{r}, 0)$ is determined from the Poisson equation

$$\frac{1}{\tilde{r}} \frac{\partial}{\partial \tilde{r}} \left(\tilde{r} \frac{\partial}{\partial \tilde{r}} \right) \phi(\tilde{r}, 0) = -4\pi q n_b(\tilde{r}, 0), \quad (6)$$

The trajectories and $r_{brms}(0)$ and $r'_{brms}(0) = 0$ are the initial conditions of the periodic rms beam envelope $r_{brms}(s) = r_{brms}(s + S)$ that solves the rms beam envelope equation [12,13]

$$\frac{d^2 r_{brms}}{ds^2} + \kappa_z(s) r_{brms} - \frac{K}{2r_{brms}} = \frac{4\epsilon_{th}^2}{r_{brms}^3}. \quad (7)$$

The trajectories of the macroparticles are integrated using the standard leapfrog method. The 2D PIC algorithm is implemented in a Matlab version of the MIT 2D Periodically Focused Beam (PFB2D) code which was used extensively in simulation studies of high-brightness charged-particle beams in periodic focusing fields [18].

SIMULATION RESULTS

Two-dimensional PIC simulations are performed to verify the theoretical predictions on the following [10-12]: a) rms beam envelope; b) conservation of rms thermal emittances; c) adiabatic equation of state; d) density, velocity, and temperature profiles; and e) Debye length. Results of a typical PFB2D simulation are shown in Figs. 1-4 for the parameters: $S^2 \kappa_z(s) = \sigma_0^2 [1 + \cos(2\pi s/S)]^2$, $\sigma_0 = 80^\circ$, $SK/4\epsilon_{th} = 7.0$, $N_p = 1 \times 10^6$, $(S/4\epsilon_{th})^{1/2} R = 5.0$, integration step size $\Delta s/S = 0.01$, and mesh size $(S/4\epsilon_{th})^{1/2} \Delta \tilde{x} = 0.02$.

The normalized rms beam radius $(S/4\epsilon_{th})^{1/2} r_{brms}$ from the PFB2D simulation is plotted in Fig. 1 as a function of the normalized distance s/S . Within the statistical error of $\pm 1\%$, the result agrees with the matched beam envelope solution of Eq. (7). The transverse momentum distribution continues to be Gaussian, as illustrated in Fig. 2.

The beam density distribution maintains a plateau distribution as shown in Fig. 3 in which the normalized beam density $(4\epsilon_{th} S/N_b) n_b$ is plotted as a function of the normalized radius $(S/4\epsilon_{th})^{1/2} r$ at $s/S = 20$. The dotted curves are from the PFB2D simulation, whereas the dashed curves are from the theoretical predictions. The characteristic scale over which the beam density falls is the Debye length λ_D , which, in this example, has a normalized value of $(S/4\epsilon_{th})^{1/2} \lambda_D = 0.14$. There is good agreement between the simulation and theory.

In Fig. 4, the evolutions of the relative rms thermal emittances $\epsilon_{xrms}/\epsilon_{xrms}(0)$ and $\epsilon_{yrms}/\epsilon_{yrms}(0)$ in the PFB2D simulation are shown in solid and dashed curves, respectively, as a function of the normalized distance s/S . Here, $\epsilon_{xrms}(0)$ and $\epsilon_{yrms}(0)$ are the rms emittances in the $\tilde{x}$ - and $\tilde{y}$ -directions at $s=0$, respectively. Both rms thermal emittances are conserved within the statistical error of $\pm 0.3\%$. These results are in agreement with the theoretical prediction of the conservation of rms thermal emittances, and validate the adiabatic equation of state [11,12].

In the parameter regimes where simulation studies were performed, we did not find any indication of beam instability. The results shown in Figs. 1-4 are typical; that is, the Gaussian transverse momentum distributions, the plateau beam density distribution, the temperature and rms beam radius repeat themselves as the beam propagates through every period in the focusing field.

While a complete proof of the stability of adiabatic thermal beams is not available, the present simulations are consistent with the previously reported simulations showing that a beam with a semi-Gaussian distribution (i.e., a distribution that is Gaussian in the momentum space and uniform in the configuration space) suffers from considerably less emittance growth than a beam with a KV distribution [19].

FIGURE 1. Plot of the normalized rms beam radius $(S/4\epsilon_{th})^{1/2} r_{brms}$ vs. the normalized distance s/S .

FIGURE 2. Plots of the probability density distributions $\rho(\tilde{x}')$ vs. $(S/4\epsilon_{th})^{1/2}\tilde{x}'$ and the probability density distributions $\rho(\tilde{y}')$ versus $(S/4\epsilon_{th})^{1/2}\tilde{y}'$ at $s/S = 20$. The dotted curves are from the PFB2D simulation, whereas the dashed curves are from the theoretical predictions.

FIGURE 3. Plot of the normalized beam density $(4\epsilon_{th}S/N_b)n_b$ vs. the normalized radius $(S/4\epsilon_{th})^{1/2}r$ at $s/S = 20$.

FIGURE 4. Plots of the relative rms thermal emittances $\epsilon_{\tilde{x}rms} / \epsilon_{\tilde{x}rms}(0)$ and $\epsilon_{\tilde{y}rms} / \epsilon_{\tilde{y}rms}(0)$ vs. the normalized distance.

FIGURE 5. Plot of the normalized density $(-9\pi qd^2 / \sqrt{2}\Phi_d)n$ as a function of the normalized coordinates x/d and y/d .

RESEARCH INITIATIVE ON ADIABATIC THERMAL CHILD-LANGMUIR FLOW

We have taken an initiative to explore two-dimensional adiabatic thermal Child-Langmuir flow in an effort to generalize recent theoretical and simulation work on the one-dimensional adiabatic thermal Child-Langmuir flow [20]. In particular, we are developing a two-dimensional PIC code for the geometry of a 2D slab. A crude example of such a 2D PIC simulation is shown in Fig. 5. In Fig. 5, the normalized density $(-9\pi q d^2 / \sqrt{2}\Phi_d)n$ is plotted as a function of the normalized coordinates x/d and y/d after five transit time, where d is the distance between the emitter $y=0$ and the collector $y/d=1$, q is the particle charge, Φ_d is the voltage between the collector and emitter, and the normalized width of the emitter is $x/d=0.5$. The mesh sizes are $\Delta x/d=0.0625$ and $\Delta y/d=0.1$, and the number of macroparticles is about 115200. We are in the process of finding means to control numerical noise in our PIC simulations. Our goals are to study the flow properties, especially the longitudinal and transverse emittances and their coupling, and to improve particle source design and beam matching.

CONCLUSION

Two-dimensional particle-in-cell simulations were performed to verify the earlier theoretical predictions of adiabatic thermal beams in a periodic solenoidal magnetic focusing field. Results were obtained for adiabatic thermal beams that do not rotate in the Larmor frame. For such beams, the theoretical predictions of the rms beam envelope, the conservations of the rms thermal emittances, the adiabatic equation of state, and the Debye length were verified in the 2D PIC simulations. Furthermore, the adiabatic thermal beam was found to be stable in the parameter regime where the simulations were performed. The generation of an adiabatic thermal beam in a practical beam device will be an important topic of future research in beam physics. In general, achieving thermal beam equilibrium in a practical system will still require innovations in particle source design, beam matching, and system optimization, engineering and fabrication, especially for the beam injector where space-charge effects are most significant.

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Dynamics of charged particles in an adiabatic thermal beam equilibrium

Haofei Wei and Chiping Chen

Plasma Science and Fusion Center, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA
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Charged-particle motion is studied in the self-electric and self-magnetic fields of a well-matched, intense charged-particle beam and an applied periodic solenoidal magnetic focusing field. The beam is assumed to be in a state of adiabatic thermal equilibrium. The phase space is analyzed and compared with that of the well-known Kapchinskij-Vladimirskij (KV)-type beam equilibrium. It is found that the widths of nonlinear resonances in the adiabatic thermal beam equilibrium are narrower than those in the KV-type beam equilibrium. Numerical evidence is presented, indicating almost complete elimination of chaotic particle motion in the adiabatic thermal beam equilibrium.

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I. INTRODUCTION

High-brightness beams in particle accelerators and beam devices and facilities are often generated in a regime where space charge plays an important role. Examples of high-brightness electron beams are rf and thermionic photo-injectors, thermionic DC injectors, electron beams in high-power rf sources, etc. Examples of high-brightness ion beams include negative hydrogen ion beams in spallation neutron sources and various ion beams in high energy density physics research.

Exploration of equilibrium states of charged-particle beams and their stability properties is critical to the advancement of basic particle accelerator physics. Of particular concern are emittance growth and beam losses, which are related to the evolution of charged-particle beams in their nonequilibrium states. To minimize emittance growth and control beam losses, it is critical to find and study equilibrium distributions of high-brightness charged-particle beams in accelerators and beam transport systems.

Several kinetic equilibria have been discovered for periodically focused intense charged-particle beams. Well-known equilibria for periodically focused intense beams include the Kapchinskij-Vladimirskij (KV) equilibrium in an alternating-gradient (AG) quadrupole magnetic focusing field [1,2] and the periodically focused rigid-rotor Vlasov equilibrium of the KV type in a periodic solenoidal magnetic focusing field [3,4]. Both of these beam equilibria [1–4] have a singular (δ -function) distribution in the four-dimensional phase space. Such a δ -function distribution gives a uniform density profile across the beam in the transverse directions, and a transverse temperature profile which peaks on axis and decreases quadratically to zero on the edge of the beam. Because of the singularity in the distribution functions, these beam equilibria are not likely to occur in real physical systems and cannot provide realistic models for theoretical and experimental studies and simulations except in the zero-temperature limit. For

example, the KV equilibrium model cannot be used to explain the beam tails in the radial distributions observed in recent high-intensity beam experiments [5]. Recently, adiabatic thermal beam equilibria have been discovered in a periodic solenoidal magnetic focusing field [6–8] and an AG quadrupole magnetic focusing field [8,9]. The measured density distribution [5] matches that of the adiabatic thermal beam equilibrium in a spatially varying solenoidal magnetic focusing field [6,8].

There have been many studies of charged-particle dynamics in the KV-type equilibria [10–14]. These studies have shown that the phase space for the KV-type equilibria exhibits rich nonlinear resonances and chaotic seas for charged particles outside the beam envelope [10,11]. If charged particles cross the beam envelope due to perturbations, they may enter chaotic seas to form a beam halo or cause beam losses [12–14].

The purpose of this paper is to show numerically that chaotic particle motion is almost completely eliminated in the adiabatic thermal beam equilibrium. The importance of this result is twofold: First, the elimination of chaotic particle motion provides a further numerical proof that the scaled transverse Hamiltonian defined in Eq. (25) in [6] is a very good approximate constant of motion. This approximate constant of motion and the exact contact of motion of the canonical angular momentum assure that the motion of charged particles is approximately integrable in the four-dimensional phase space of the adiabatic thermal beam equilibrium. Second, the elimination of chaotic particle motion may provide valuable insight into how to control chaotic particle motion, halo formation, and beam loss in beam transport channels. In particular, if adiabatic thermal beams are stable, they may be free from chaotic particle motion, beam halo formation, and beam loss.

The organization of the present paper is as follows. In Sec. II, the basic equations of the adiabatic thermal beam equilibrium are reviewed. The equations of motion for a charged particle are presented. In Sec. III, results of a numerical analysis of charged-particle motion are

discussed. In particular, the Poincaré surface-of-section maps of the motion of charged particles in phase space are generated numerically. Comparisons are made between the KV-type and adiabatic thermal beam equilibria. Numerical evidence is presented for almost complete elimination of chaotic particle motion in the adiabatic thermal beam equilibrium. In Sec. IV, the conclusion of the present paper is presented.

II. THEORETICAL MODEL

We study charged-particle dynamics in the adiabatic thermal equilibrium of an intense charged-particle beam propagating with constant axial velocity $\beta_b c \hat{e}_z$ in the periodic solenoidal magnetic focusing field,

$$\mathbf{B}^{\text{ext}} = B_z(s) \hat{e}_z - \frac{1}{2} \frac{dB_z(s)}{ds} (x \hat{e}_x + y \hat{e}_y), \quad (1)$$

where $s = z$ is the axial coordinate, $B_z(s + S) = B_z(s)$ is the axial magnetic field, S is the fundamental periodicity length of the focusing field, and c is the speed of light in vacuum. The adiabatic thermal beam equilibrium has been derived under the paraxial approximation with the following assumptions: (1) $r_{b \text{ rms}} \ll S$, where $r_{b \text{ rms}}$ is the rms beam radius and (2) $\nu/\gamma_b^3 \beta_b^2 \ll 1$, where $\nu = q^2 N_b / mc^2$ is the Budker parameter of the beam, q and m are the particle charge and rest mass, respectively, $N_b = \int_0^\infty n_b(r, s) 2\pi r dr = \text{const}$ is the number of particles per unit axial length, and $\gamma_b = (1 - \beta_b^2)^{-1/2}$ is the relativistic mass factor.

In the adiabatic thermal beam equilibrium [6–8], the beam density distribution is given by

$$n_b(r, s) = \frac{4\pi C \varepsilon_{\text{th}}^2}{r_{b \text{ rms}}^2(s)} \exp\left\{-\left[\frac{K}{2} + \frac{4\varepsilon_{\text{th}}^2}{r_{b \text{ rms}}^2(s)}\right] \frac{r^2}{4\varepsilon_{\text{th}}^2} - \frac{q}{\gamma_b^2 k_B T_\perp(s)} \phi(r, s)\right\}, \quad (2)$$

and the self-electric potential $\phi(r, s)$ is determined by the Poisson equation,

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \phi}{\partial r} \right) = -4\pi q n_b(r, s), \quad (3)$$

and the free-space boundary conditions. In Eqs. (2) and (3), C is a constant determined by $N_b = \int_0^\infty n_b(r, s) 2\pi r dr$, $K \equiv 2q^2 N_b / \gamma_b^3 m \beta_b^2 c^2$ is the generalized beam perveance, $\varepsilon_{\text{th}} = [k_B T_\perp(s) r_{b \text{ rms}}^2 / 2\gamma_b m \beta_b^2 c^2]^{1/2}$ is the rms thermal emittance in the Larmor frame, $\tilde{x} = x \cos \varphi - y \sin \varphi$ and $\tilde{y} = x \sin \varphi + y \cos \varphi$, where $\varphi = \int_0^s \sqrt{\kappa_z(s)} ds$, $T_\perp(s)$ is the Kelvin temperature of the beam, k_B is the Boltzmann constant, and the rms beam envelope $r_{b \text{ rms}}(s) = r_{b \text{ rms}}(s + S)$ solves the beam envelope equation

$$\frac{d^2 r_{b \text{ rms}}}{ds^2} + \kappa_z(s) r_{b \text{ rms}} - \frac{K}{2r_{b \text{ rms}}} = \frac{4\varepsilon_{\text{th}}^2}{(1 - \omega_b^2) r_{b \text{ rms}}^3}, \quad (4)$$

where $\sqrt{\kappa_z(s)} \equiv q B_z(s) / 2\gamma_b m \beta_b c^2$ and $\omega_b^2 = 1 - (\varepsilon_{\text{th}} / \varepsilon_{\tilde{x} \text{ rms}})^2$ with $\varepsilon_{\tilde{x} \text{ rms}}$ being the rms emittance in the $\tilde{x}$ direction.

III. NUMERICAL ANALYSIS

Figure 1 shows (a) normalized focusing parameter $S^2 \kappa_z(s)$ and (b) normalized rms envelope $r_{b \text{ rms}} / \sqrt{4\varepsilon_{\text{th}} S}$ versus the normalized distance s/S for the choice of system parameters corresponding to $S\sqrt{\kappa_z(s)} = (2/3)^{1/2} \sigma_0 [1 + \cos(2\pi s/S)]$, $SK/4\varepsilon_{\text{th}} = 7.0$, $\omega_b = 0$, and vacuum phase advance $\sigma_0 = 80^\circ$. For $\omega_b = 0$, $\varepsilon_{\text{th}} = \varepsilon_{\tilde{x} \text{ rms}}$ and the KV-type and adiabatic thermal beam equilibria have the same rms beam envelopes.

Figure 2 shows (a) density n_b and (b) radial self-electric field E_r for the KV-type and adiabatic thermal beam equilibria at $s = 0$ with the same choice of system parameters as in Fig. 1. While the self-electric fields of the two beams are similar, there are three important differences: (a) the density in the interior for the adiabatic thermal beam is

FIG. 1. Plots of (a) normalized focusing parameter $S^2 \kappa_z(s)$ and (b) normalized rms envelope $r_{b \text{ rms}} / \sqrt{4\varepsilon_{\text{th}} S}$ versus the normalized distance s/S for the choice of system parameters corresponding to $SK/4\varepsilon_{\text{th}} = 7.0$, $S\sqrt{\kappa_z(s)} = (2/3)^{1/2} \sigma_0 [1 + \cos(2\pi s/S)]$, $\omega_b = 0$, and $\sigma_0 = 80^\circ$.

FIG. 2. Plots of (a) normalized density $n_b/n_{KV}(0,0)$ and (b) normalized radial self-electric field $S^{3/2}KE_r/4\epsilon_{th}^{1/2}qN_b$ versus normalized radius $r/\sqrt{4\epsilon_{th}S}$ in the KV-type beam equilibrium (dashed curve) and the adiabatic thermal beam equilibrium (solid curve) at $s = 0$ for the same choice of system parameters as in Fig. 1. Here, $n_{KV}(0,0)$ is the density of the KV-type beam equilibrium at $s = 0$ and $r = 0$.

higher than that for the KV-like beam; (b) the electric field near the normalized radius $r/\sqrt{4\epsilon_{th}S} \approx 2.0$ has a smooth transition from negative to positive slope for the adiabatic thermal beam, whereas its radial derivative is discontinuous for the KV-type beam equilibrium; (c) the self-electric field near the edge of the adiabatic thermal beam is weaker than that of the KV-type beam. These differences result in significant changes in charged-particle dynamics (see Figs. 3 and 4).

The radial equation of motion of a charged particle in the cylindrical coordinates is

$$\frac{d^2r}{ds^2} + \frac{P_\theta^2}{r^3} + \kappa_z(s)r + \frac{q}{\gamma_b^3 m \beta_b^2 c^2} \frac{\partial \phi(r,s)}{\partial r} = 0, \quad (5)$$

where the canonical angular momentum P_θ is conserved. Figure 3 shows a comparison between the Poincaré surface-of-section maps of charged-particle trajectories in (a) KV-type beam equilibrium and (b) adiabatic thermal

FIG. 3. Poincaré surface-of-section maps of charged-particle trajectories in (a) KV-type beam equilibrium and (b) adiabatic thermal beam equilibrium for $P_\theta = 0$ and the same choice of system parameters as in Fig. 1. Here, the normalized radial momentum is $(S/4\epsilon_{th})^{1/2}dr/ds$ and the normalized radius is $r/\sqrt{4\epsilon_{th}S}$.

beam equilibrium for the choice of system parameters corresponding to $P_\theta = 0$, $S\sqrt{\kappa_z(s)} = (2/3)^{1/2}\sigma_0[1 + \cos(2\pi s/S)]$, $\sigma_0 = 80^\circ$, $\omega_b = 0$, and $SK/4\epsilon_{th} = 7.0$. These parameters correspond to a highly space-charge-dominated beam rotating at the (local) Larmor frequency in a high-strength periodic solenoidal magnetic focusing field. They are generated by plotting (r, P_r) as a trajectory arrives at the lattice points $s/S = 0, 1, 2, \dots, 2000$. For these parameters, the density for the KV-type beam equilibrium drops abruptly at $r/\sqrt{4\epsilon_{th}S} \approx 2.0$, whereas the density for the adiabatic thermal beam equilibrium falls from its flattop value to almost zero between $r/\sqrt{4\epsilon_{th}S} \approx 1.6$ and 2.4 .

For $r/\sqrt{4\epsilon_{th}S} < 2.0$, the phase space is regular in both the KV-type and adiabatic thermal beam equilibria, and the action of a charged particle in the KV-type beam is larger than that in the adiabatic thermal beam, as shown in Fig. 3. The phase space area (action) of a charged particle in the interior of the adiabatic thermal beam is significantly smaller than that of the KV-type beam because the density

FIG. 4. Close-ups of Poincaré surface-of-section maps of charged-particle trajectories for the two cases shown in Fig. 3.

of the adiabatic thermal beam approaches the density of the corresponding cold beam, which is higher than the density of the KV-type beam. In the region $2 \leq r/\sqrt{4\epsilon_{th}\bar{S}} \leq 2.4$, however, there are striking differences between the KV-type and adiabatic thermal beam equilibria, as shown in Fig. 4. Comparing Fig. 4(a) with Fig. 4(b), there are two important differences to note. First, there are chaotic seas in the phase space of the KV-type beam, whereas chaotic motion is almost absent in the phase space of the adiabatic thermal beam equilibrium. Second, the widths of the nonlinear resonances in the adiabatic thermal beam equilibrium are narrower than those in the KV-type beam equilibrium. For example, the width of the nonlinear resonance at $(r/\sqrt{4\epsilon_{th}\bar{S}}, P_\theta) \equiv (2.36, 0)$ is $\Delta r/\sqrt{4\epsilon_{th}\bar{S}} = 0.013$ in the adiabatic thermal beam, whereas the width of the corresponding resonance in the KV-type beam is $\Delta r/\sqrt{4\epsilon_{th}\bar{S}} = 0.021$.

IV. CONCLUSION

We analyzed charged-particle motion in the self-electric and self-magnetic fields of a well-matched, intense charged-particle beam in a period solenoidal magnetic

focusing field. We assumed that the beam is in the state of adiabatic thermal equilibrium. We compared the phase space of the adiabatic thermal beam equilibrium with that of a corresponding KV-type beam equilibrium. We found that the widths of some of the nonlinear resonances in the adiabatic thermal beam equilibrium are narrower than those in the KV-type beam equilibrium. We presented numerical evidence for almost complete elimination of chaotic particle motion in the adiabatic thermal beam equilibrium in a periodic solenoidal magnetic focusing field.

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ONE-DIMENSIONAL ADIABATIC CHILD-LANGMUIR FLOW

C. Chen, MIT, Cambridge, MA 02139, U.S.A.

F.B. Rizzato, R. Pakter, IF-UFRGS, Porto Alegre 91501-970, RS, Brazil

Abstract

A theory is presented that describes steady-state one-dimensional Child-Langmuir flow at a self-consistent finite temperature distribution. In particular, warm-fluid equations and adiabatic equation of state are used to derive the self-consistent Poisson equation. The profiles of the charged-particle density, the velocity, the electrostatic potential, the pressure and the temperature are computed. The adiabatic equation of state assures the conservation of normalized (longitudinal) rms thermal emittance of any small segment of the flow. Good agreement is found between the approximation of the adiabatic equation of state and self-consistent simulation.

INTRODUCTION

Recently, there are renewed interests in the research and development of high-brightness dc thermionic electron or ion guns. The renewed interests arise from a) applications of high-brightness dc electron guns in x-ray free electron laser applications [1], and b) utilizations of high-brightness dc electron guns in the realization of adiabatic thermal beam equilibria in periodic focusing fields [2-5].

The 1D cold-fluid Child-Langmuir (C-L) flow [6,7] is an important aspect of dc thermionic gun theory. It corresponds to the cold-fluid equilibrium of a charged-particle flow between two plates with an electrostatic potential bias.

The 1D adiabatic thermal C-L flow discussed in this paper corresponds to a warm-fluid equilibrium of a charged-particle flow between two plates with an electrostatic potential bias under the adiabatic equation of state. A warm-fluid description of the 1D adiabatic thermal C-L flow is presented. A self-consistent Poisson equation is derived. In the limit of zero temperature, the self-consistent Poisson equation recovers the corresponding Poisson equation for the cold-fluid C-L flow. An effort is initiated to validate the theory via self-consistent simulation. To date, good agreement has been found between the approximation of the adiabatic equation of state and self-consistent simulation in the high-temperature regime.

THEORY OF ONE-DIMENSIONAL ADIABATIC THERMAL CHILD-LANGMUIR FLOW

We consider the non-relativistic 1D C-L flow under the influence of a finite temperature profile between two conducting plates located at $z=0$ and $z=d$. The adiabatic warm-fluid equations in cgs units are:

$$mnV \frac{\partial V}{\partial z} = -qn \frac{\partial \phi}{\partial z} - \frac{\partial p}{\partial z}, \quad (1)$$

$$\frac{\partial nV}{\partial z} = 0, \quad (2)$$

$$\partial^2 \phi / \partial z^2 = -4\pi qn, \quad (3)$$

$$\frac{\partial}{\partial z} \left(\frac{p}{n^3} \right) = 0, \quad (4)$$

$$p = k_B nT, \quad (5)$$

where n , V , ϕ , p and T are the equilibrium density, flow velocity, electrostatic potential, longitudinal pressure and temperature profiles, respectively, k_B is the Boltzmann constant, and m and q are the rest mass and charge of the charged particle, respectively.

Equation (4) is the one-dimensional adiabatic equation of state, which can be derived in [8]. It is a statement of entropy conservation. For any small segment of the flow, the normalized rms thermal emittance $\varepsilon_n = \gamma\beta\varepsilon$ is proportional to p/n^3 or T/n^2 , where ε is the unnormalized rms thermal emittance, $\gamma = (1 - V^2/c^2)^{-1/2} \cong 1$ is the relativistic mass factor, and $\beta = V/c$ with c being the speed of light in vacuum. Therefore, the adiabatic equation of state assures that the normalized rms thermal emittance of any small segment of the flow is conserved.

Substituting Eqs. (4) and (5) into Eq. (1) yields

$$\frac{\partial}{\partial z} \left(\frac{1}{2} mV^2 + q\phi + \frac{3p}{2n} \right) = 0, \quad (6)$$

$$\frac{\partial}{\partial z} \left(\frac{1}{2} mV^2 + q\phi + \frac{3}{2} k_B T \right) = 0. \quad (7)$$

Equation (6) or (7) is an important conservation law in addition to charge conservation, i.e., $J = qnV = \text{const.}$

and entropy conservation, i.e., $p/n^3 = \text{const.}$

Because

$$T/n^2 = \text{const.}, \quad (8)$$

from Eqs. (4) and (5), we express Eq. (7) as

$$\frac{1}{2} mV^2 + q\phi + \frac{3k_B}{2q^2} \frac{T}{n^2} \frac{J^2}{V^2} = C, \quad (9)$$

where the constant C is defined by

$$C = \frac{1}{2} mV^2(0) + \frac{3}{2} k_B T_c, \quad (10)$$

T_c is the emitter temperature, and use has been made of the boundary condition $\phi(0) = 0$. The solutions to Eq. (9) are

$$V^2 = (V^2)_\pm = -\frac{(q\phi - C)}{m} \pm \left[\frac{(q\phi - C)^2}{m^2} - \frac{3J^2 k_B T}{q^2 m n^2} \right]^{1/2}. \quad (11)$$

Because $q\phi < 0$ in the gun, the solution of interest is

$$V^2 = (V^2)_+ = -\frac{(q\phi - C)}{m} + \left[\frac{(q\phi - C)^2}{m^2} - \frac{3J^2 k_B T}{q^2 m n^2} \right]^{1/2} \quad (12)$$

which, in the cold limit ($T = 0$), gives

$$V = V_{cold} \equiv \left(\frac{-2q\phi}{m} \right)^{1/2}. \quad (13)$$

In order for V^2 to be real, we must have

$$q\phi - C \leq -\left[\frac{3mJ^2}{q^2} \left(\frac{k_B T}{n^2} \right) \right]^{1/2}. \quad (14)$$

Using the boundary conditions at the emitter, i.e., $\phi(z=0)=0$, $T(z=0)=T_c$ and $n(z=0)=n(0)$, we find that the critical value of C at which the equal sign in Eq. (14) holds is

$$C = C_{crit} \equiv \left\{ \frac{3mJ^2}{q^2} \left[\frac{k_B T_c}{n^2(0)} \right] \right\}^{1/2}. \quad (15)$$

Substituting Eq. (15) and $V(0)=J/qn(0)$ into Eq. (10), we obtain

$$C_{crit} = 3k_B T_c. \quad (16)$$

At $C = C_{crit} = 3k_B T_c$, we have

$$V(0) = \left[\frac{3k_B T_c}{m} \right]^{1/2}, \quad (17)$$

$$n(0) = \frac{J}{q \left[\frac{3k_B T_c}{m} \right]^{1/2}}, \quad (18)$$

$$V = \left\{ \frac{-q\phi + 3k_B T_c}{m} + \left[\left(\frac{-q\phi}{m} \right) \left(\frac{-q\phi + 6k_B T_c}{m} \right) \right]^{1/2} \right\}^{1/2}, \quad (19)$$

$$\frac{\partial^2 \phi}{\partial z^2} = \frac{-4\pi J}{\left\{ \frac{-q\phi + 3k_B T_c}{m} + \left[\frac{q\phi}{m} \left(\frac{q\phi - 6k_B T_c}{m} \right) \right]^{1/2} \right\}^{1/2}}. \quad (20)$$

The boundary conditions for Poisson's equation (20) are

$$\phi(0) = 0 = \phi'(0) \quad (21)$$

and

$$\phi(d) = \Phi_d, \quad (22)$$

where prime denotes the derivative with respect to z . Note that $\phi'(0)=0$ is the condition for space-charge-limited emission. For $-q\phi/3k_B T_c \ll 1$, we may approximate Eq. (20) as

$$\frac{\partial^2 \phi}{\partial z^2} \approx \frac{-4\pi J}{\sqrt{3k_B T_c / m}} \quad (23)$$

which gives the dependence of $\phi \propto z^2$ at very small values of z .

It is useful to scale Eq. (20) in terms of the quantities in the cold-fluid C-L flow, i.e.,

$$\frac{\partial^2 \hat{\phi}}{\partial \hat{z}^2} = \frac{4\sqrt{2}}{9} \frac{\hat{J}}{\sqrt{\hat{\phi} + 3\hat{T}_c + [\hat{\phi}(\hat{\phi} + 6\hat{T}_c)]^{1/2}}}, \quad (24)$$

where

Figure 1: Plots of the normalized potential ϕ/Φ_d versus the normalized distance z/d for a) cold-fluid C-L flow with $\hat{J}=1$ and $\hat{T}_c=0$ and b) adiabatic thermal C-L flow with $\hat{J}=1.15$ and $\hat{T}_c=0.001$. For $k_B T_c=0.1$ eV, the choice of system parameters corresponding to a low voltage diode with $\Phi_d=100$ V.

$$\hat{\phi} \equiv \frac{\phi}{\Phi_d}, \quad \hat{z} \equiv \frac{z}{d}, \quad \hat{T}_c = \frac{k_B T_c}{-q\Phi_d}, \quad \hat{J} = \frac{J}{J_{CL}} \quad (25)$$

$$J_{CL} = \frac{\sqrt{2}mc^3}{9\pi q d^2} \left(\frac{-q\Phi_d}{mc^2} \right)^{3/2} \quad (26)$$

is the current density of the cold-fluid C-L flow in which $\hat{T}_c=0$, $\hat{J}=1$ and $\hat{\phi}=\hat{z}^{4/3}$. The boundary conditions for Eq. (23) are:

$$\hat{\phi}(0) = 0 = \hat{\phi}'(0) \quad \text{and} \quad \hat{\phi}(1) = 1. \quad (27)$$

Figure 1 shows ϕ/Φ_d versus z/d for a) cold-fluid C-L flow and b) adiabatic thermal C-L flow with $\hat{J}=1.15$ and $\hat{T}_c=0.001$. For $k_B T_c=0.1$ eV, the choice of system parameters corresponds to a low-voltage electron diode with $\Phi_d=100$ V. It is interesting to observe that for the choice of system parameters in Fig. 1, the current density of 1D adiabatic thermal C-L flow is 15% higher than the cold-fluid C-L current given in Eq. (33).

Figure 2 shows ϕ/Φ_d versus z/d for a) cold-fluid C-L flow and b) adiabatic thermal C-L flow with $\hat{J}=2.6$ and $\hat{T}_c=0.1$. For $k_B T_c=0.1$ eV, the choice of system parameters corresponds to a low-voltage electron diode with $\Phi_d=1.0$ V.

SELF-CONSISTENT SIMULATION

One-dimensional self-consistent simulations are performed to verify the theoretical predictions. In the simulations, planar charged sheets are used. A detailed description of the simulation model is given in [9].

Figure 2: Plots of the normalized potential ϕ/Φ_d versus the normalized distance z/d for a) cold-fluid C-L flow with $\hat{J}=1$ and $\hat{T}_c=0$ and b) adiabatic thermal C-L flow with $\hat{J}=2.6$ and $\hat{T}_c=0.1$. For $k_B T_c = 0.1$ eV, the choice of system parameters corresponds to a low-voltage electron diode with $\Phi_d = 1.0$ V.

Figure 3: Comparison of $Tn^2(0)/T_c n^2$ versus the normalized distance z/d from simulation with theory for the choice of system parameters corresponding to $k_B T_c / m\omega_p^2 d^2 = 10$, $V(0)/\omega_p d = 5.48$ and $-q\Phi_d / m\omega_p^2 d^2 = 10$.

In contrast to the simulations in [9], which employed a truncated Maxwellian distribution in the velocity space at the emitter ($z=0$), the present simulations employ a full Maxwellian distribution with temperature T_c shifted by $V(0)$ in the velocity space at $z=0$. When a charged sheet strikes the plate at $z=d$, it is replaced by a new charged sheet at the emitter ($z=0$). The newly injected charged sheets are taken from the shifted full Maxwellian distribution.

Preliminary simulation results are obtained. To date, the approximation of the adiabatic equation of state, i.e., Eq. (4) or (8), has been found in good agreement with self-consistent simulation in the high-temperature regime

with $k_B T_c / m\omega_p^2 d^2 \gg 1$. Here, $\omega_p^2 = 4\pi q^2 \bar{n} / m$ is the plasma frequency associated with the average particle density $\bar{n} = d^{-1} \int_0^d n(z) dz$. The plots of $Tn^2(0)/T_c n^2$ versus

the normalized distance z/d in Fig. 3 show the comparison between theory and simulation for the choice of system parameters corresponding to: $k_B T_c / m\omega_p^2 d^2 = 10$, $-q\Phi_d / m\omega_p^2 d^2 = 10$, $V(0)/\omega_p d = 5.48$. In the simulation, 50000 charged sheets are used. Within the statistical fluctuations, the quantity $Tn^2(0)/T_c n^2$ is conserved. The simulation results in Fig. 3 validate the approximation of the adiabatic equation of state, i.e., Eq. (4) or (8). They also confirm that the normalized (longitudinal) rms thermal emittance of any small segment of the flow is conserved.

CONCLUSION

A theory was presented that describes steady-state one-dimensional Child-Langmuir flow at a self-consistent finite temperature distribution. In particular, warm-fluid equations and adiabatic equation of state were used to derive the self-consistent Poisson equation. The profiles of the charged-particle density, the velocity, the electrostatic potential, the pressure and the temperature are computed. An effort was initiated to validate the theory via self-consistent simulation. To date, good agreement was found between the approximation of the adiabatic equation of state and self-consistent simulation in the high-temperature regime.

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