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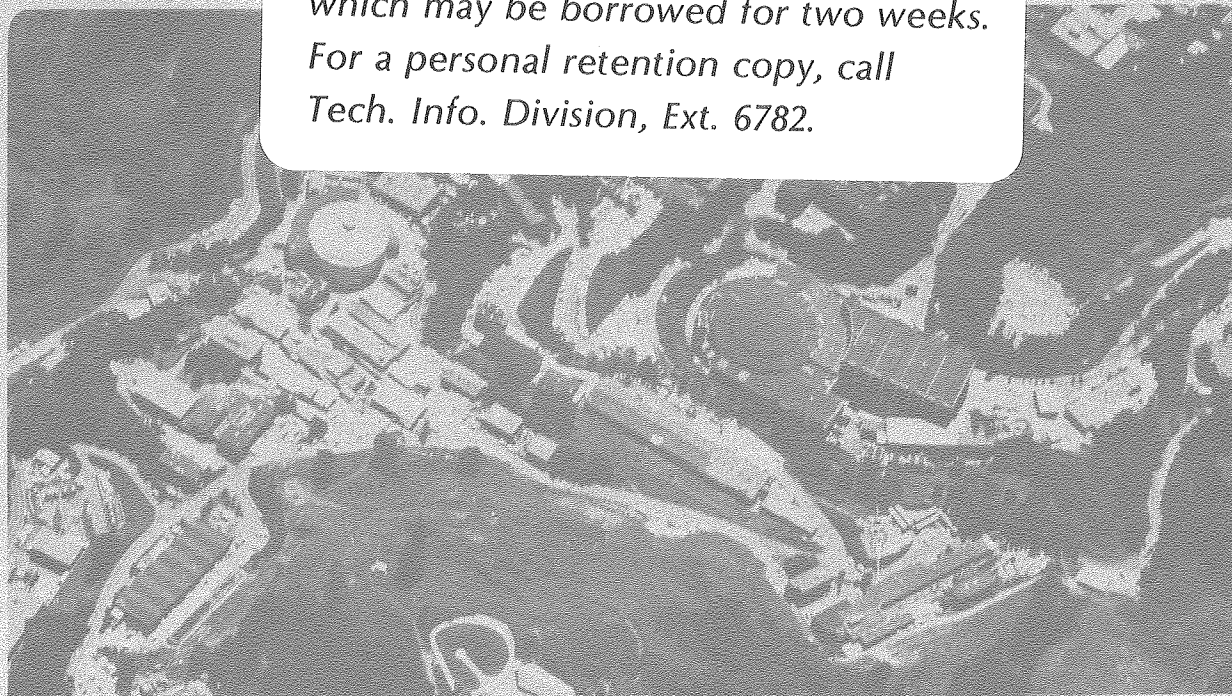
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Nonlinear Models in $2 + \epsilon$ Dimensions

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ABSTRACT

A generalization of the nonlinear σ model is considered. The field takes values in a compact manifold M and the coupling is determined by a Riemannian metric on M . The model is renormalizable in $2 + \epsilon$ dimensions, the renormalization group acting on the infinite dimensional space of Riemannian metrics. Topological properties of the β -function and solutions of the fixed point equation $R_{ij} - \alpha g_{ij} = \nabla_i v_j + \nabla_j v_i$, $\alpha = \pm 1$ or 0 , are discussed.

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Polyakov¹ several years ago studied the renormalization of the $O(N)$ -invariant nonlinear σ model in $2 + \epsilon$ dimensions in the low temperature regime dominated by small fluctuations around ordered states. He found an infrared unstable fixed point at a temperature of order ϵ . The unstable renormalization group trajectory gives a model critical system in its scaling limit or, equivalently, a Euclidean quantum field theory.² In two dimensions the model is asymptotically free.

I describe here³ a more general model to which Polyakov's approach is appropriate: a field $\phi(x)$ taking values in a compact manifold M , governed by the action

$$S(\phi) = \Lambda^\epsilon \int dx \frac{1}{2} T^{-1} g_{ij}(\phi(x)) \delta_\mu \phi^i(x) \delta_\mu \phi^j(x), \quad (1)$$

where Λ^{-1} is the short distance cutoff. The dimensionless coupling $T^{-1} g_{ij}$ is a Riemannian metric on M .⁴ The standard non-linear σ models have M a homogeneous space and g_{ij} an invariant metric.

Correlation functions are generated by the partition function $Z(h) = \int \prod_X d\phi(x) \exp[-S(\phi) + H(\phi)]$ where the a priori measure $d\phi(x)$ is the metric volume element on M and $H(\phi) = \Lambda^{2+\epsilon} \int dx h(x)(\phi(x))$, h being an external field, each $h(x)$ a function on M . The k -fold correlation function takes values in the unit measures on M^k :

$$\langle \phi(x_1) \cdots \phi(x_k) \rangle = Z(0)^{-1} \frac{\delta}{\delta h(x_1)} \cdots \frac{\delta}{\delta h(x_k)} Z(h) /_{h=0} \quad (2)$$

The double expansion in T and ϵ is constructed as a renormalizable perturbation series.⁵ Only fields close to the constants play a

role; $Z(h) = \int dm Z(m, h)$ where $Z(m, h)$ is the sum over small fluctuations around the constant $\phi(x) = m$. A choice of coordinates around each point m in M gives a linear representation for the fluctuations: the linear field $\sigma^i(x)$ is $\phi(x)$ in coordinates around m . The sum over fluctuations becomes

$$Z(m, h) = \int d\sigma \exp[-\tilde{S}(m, \sigma) + \tilde{H}(m, \sigma)]. \quad (3a)$$

$$d\sigma = \prod_x d\sigma(x) \exp[\Lambda^{2+\epsilon} \int dx \log \det J(m, \sigma(x))] \quad (3b)$$

$$\tilde{S}(m, \sigma) = \int dx \frac{1}{2} T^{-1} \tilde{g}_{ij}(m, \sigma(x)) \delta_\mu \sigma^i(x) \delta_\mu \sigma^j(x) \quad (3c)$$

$$\tilde{H}(m, \sigma) = \int dx \tilde{h}(x, m, \sigma(x)). \quad (3d)$$

where $\tilde{g}_{ij}(m, \sigma(x))$ and $\tilde{h}(x, m, \sigma(x))$ are the metric and external field in coordinates around m and $\det J_j^i(m, \sigma(x))$ is the Jacobian of the coordinate map from $\sigma(x)$ to $\phi(x)$. Propagators and vertices come from expansion in powers of σ . Normal coordinates yield:

$$J_j^i(m, \sigma(x)) = \delta_j^i + \frac{1}{6} \sigma^k(x) \sigma^l(x) R_{klj}^i(m) + \dots \quad (4a)$$

$$\tilde{g}(m, \sigma(x)) = g_{ij}(m) + \frac{1}{3} \sigma^k(x) \sigma^l(x) R_{iklj}(m) + \dots \quad (4b)$$

$$\tilde{h}(x, m, \sigma(x)) = \sum_{n=0}^{\infty} \frac{1}{n!} \sigma^{k_1}(x) \dots \sigma^{k_n}(x) \nabla_{k_1} \dots \nabla_{k_n} h(x)(m). \quad (4c)$$

To each constant m corresponds a perturbation series whose vertices are in the most general form required by power counting, so is prima facie renormalizable. But the existence of an underlying nonlinear theory means that the vertices for one constant m determine those for all nearby m' by translation of coordinates and shift of origin. To renormalize the nonlinear theory the renormalized vertices must be made to satisfy an equivalent renormalized invariance. That this can be done is shown in Ref. 3.

Renormalized as dictated by power counting, at a scale set by μ ,

$$g_{ij} = (\Lambda/\mu)^{-\epsilon} g_{ij}^b(\epsilon, \Lambda/\mu, g^R) \quad (5a)$$

$$h(x) = (\Lambda/\mu)^{-2-\epsilon} Z_1(\epsilon, \Lambda/\mu, g^R) h^R(x) + h_1(\epsilon, g) \quad (5b)$$

where g_{ij}^R and $h^R(x)$ are the renormalized coupling and external field, Z_1 is a linear operator on functions on M and h_1 serves to remove quadratic divergences. In the following only renormalized quantities are discussed; the R- superscripts are suppressed.

The partition function satisfies the renormalization group equation

$$\left(\mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} + (\tilde{Y}(g) h(x)) \frac{\partial}{\partial h(x)} \right) Z(h) = 0. \quad (6)$$

The β -function $\beta(g)$ is a vector field on the space of metrics and $\tilde{Y}(g) h(x)$ is a linear vector field on functions.

The order parameter $\tilde{\Phi}(x)$ dual to $h(x)$ takes values in the non-negative unit measures on M . The free energy

$$\Gamma(\Phi) = \max_h [-\log Z(h) + \mu^{2+\epsilon} \int dx (h(x), \Phi(x))] \quad (7)$$

satisfies the renormalization group equation

$$\left(\mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} - (\gamma(g) * \Phi(x)) \frac{\partial}{\partial \Phi(x)} \right) \Gamma(\Phi) = 0 \quad (8)$$

where $\tilde{\gamma}(g) = -(2 + \epsilon) + \gamma(g)$.

To two loops,⁶ using dimensional regularization and renormalizing by minimal subtraction,

$$\beta_{ij}(T^{-1}g) = -\epsilon T^{-1} g_{ij} + R_{ij} + \frac{1}{2} T (R_{ikln} R_{jkl n}) + O(T^2) \quad (9a)$$

$$\gamma(T^{-1}g) = -\frac{1}{2} T \nabla_k \nabla_k + O(T^3). \quad (9b)$$

$R_{ij} = R^k_{ikj}$ is the Ricci tensor and T has been replaced by $2\pi T$.

The renormalization group has meaning only as it acts on the equivalence classes of metric couplings and external fields under reparametrizations (diffeomorphisms) of M . The partition function $Z(h)$ sees no change when both g_{ij} and h are subjected to the same reparametrization, thus no normalization condition can distinguish among members of the same equivalence class. The construction and renormalization of the perturbation series respect this covariance.

The diffeomorphism classes of metrics make up an infinite dimensional manifold (singular at metrics with symmetry),⁷ over which the external fields form a vector bundle. The renormalization group has its fixed points where $h(x)$ vanishes and $\beta(g)$ is an infinitesimal

reparametrization: $\beta_{ij}(g) = \nabla_i v_j + \nabla_j v_i$ for v a vector field on M .

The coefficients β and γ are natural functions of the metric: when g_{ij} is transformed by a reparametrization of M , $\beta_{ij}(g)$ and $\gamma(g)$ undergo the same transformation. In particular, if g is unaffected then so are β and γ . Thus the renormalization group preserves internal symmetry.

Since a homogeneous space has the same geometry at every point, the couplings of any standard model comprise a finite dimensional submanifold of the metrics at one point in M . Group theoretic formulas for renormalization group coefficients are given in Ref. 3.

Global topological information on the β -function for small T is available when M has dimension two and also when M is homogeneous. In both cases the β -function is a gradient through two loops.³

The fixed points correspond to solutions of

$$R_{ij} - \alpha g_{ij} = \nabla_i v_j + \nabla_j v_i, \quad \alpha = \pm 1 \text{ or } 0. \quad (10)$$

Writing the coupling in the form $T^{-1} (g_{ij} + k_{ij})$, T and k_{ij} small, and keeping only terms of topological significance:

$$\beta(T) = \begin{cases} \leftarrow T - \alpha T^2 & \alpha = \pm 1 \\ \leftarrow T - T^3 & \text{when } \alpha = 0, R_{ijkl} \neq 0 \\ \leftarrow T & R_{ijkl} = 0 \end{cases} \quad (11a)$$

$$\beta^{(k)}_{ij} = \frac{1}{2} T \Delta_{\beta}^{(k)}_{ij} \quad \Delta_{\beta} = - \nabla_1 \nabla_1 + 1^{\text{st}} \text{ order} \quad (11b)$$

$$\tilde{\gamma} = - (2 + \epsilon) + \frac{1}{2} T \Delta_{\gamma} \quad \Delta_{\gamma} = - \nabla_i \nabla_i - 2 v^i \nabla_i. \quad (11c)$$

The only meaningful k_{ij} - directions are those transverse to the reparametrizations and to the T- direction. Δ_{β} is an elliptic operator with positive leading part, so the number of unstable or marginal k- directions is always finite. The flat metrics ($R_{ijkl} = 0$) have trivial perturbation theories; in the following they are excluded from the case $\alpha = 0$.

When $\alpha = 1$ or 0 there is a nontrivial fixed point for $\epsilon > 0$ at $T \approx \epsilon$ or $T \approx \epsilon^{1/2}$, infrared unstable in at least the T- direction. When α is -1 there is a fixed point for $\epsilon < 0$ at $T \approx -\epsilon$, infrared stable in the T- direction. In all three cases there are also trivial fixed points at $T = 0$. No other kind of fixed point at nondegenerate coupling is possible because when the two loop term in the β -function vanishes, i.e. $R_{ikln} R_{jkl n} = \nabla_i w_j + \nabla_j w_i$, then $\int dm R_{ikln} R_{ikln} = 0$, so $R_{ijkl} = 0$.

In two dimensions the trivial and nontrivial fixed points merge at $T = 0$, asymptotically free in the small when $\alpha = 1, 0$ and in the large when $\alpha = -1$. When $\alpha = 0$ $\beta(T)$ vanishes to second order in T , so the approach to freedom is extraordinarily slow.

All known solutions of (10) are actually Einstein metrics ($v^i = 0$). For $\alpha = 1$ there is available only one example which is not locally homogeneous.⁸ Among the homogeneous spaces those admitting just one invariant metric are necessarily Einstein,⁹ but others with less symmetry are known.¹⁰ Some have instability in k- directions, so provide model multicritical points.³ The only known Ricci-flat spaces ($\alpha = 0$)

are the Kahler manifolds of Yau.¹¹ Einstein metrics with $\alpha = -1$ are known in two varieties: the locally symmetric spaces of noncompact type and the Kahler metrics of Yau.¹¹

For $\epsilon > 0$, $\alpha = 0$ or 1 , the long distance physics is qualitatively familiar. Below the critical temperature long distance behavior is governed by the trivial fixed point at $T = 0$, so there is a degenerate set of pure equilibrium states, labelled by the points in M . At $T = 0$ the free energy $\Gamma(\Phi)$ is minimized by the point measures $\Phi_m(x) = \delta_m$. As T increases, the set of minima is still M , but the minimizing order parameters have diffused outward; to lowest order $\Phi_m = \exp(s \Delta_y^*) (\delta_m)$, $s = \frac{1}{2} \log(1 - T/T_c)$. At $T = T_c$ the degeneracy of equilibria disappears, the Φ_m having converged to the unique measure annihilated by γ^* . To lowest order the anomalous dimensions of $\Phi(x)$ are determined by the eigenvalues of Δ_y . Long distance properties for $T > T_c$ are not accessible to perturbation theory, but the system presumably remains disordered.

A solution of (10) with v^i not a gradient would show some novel features: approaching the critical surface, the order parameter would drift as it diffused (because of the term $-2v^i \nabla_i$ in Δ_y) and the anomalous dimensions could be complex.

The $\alpha = -1$ fixed points are analogous to ϕ^4 fixed points near four dimensions, the ϵ expansion probing dimensions below two. The scaling limit in two dimensions is trivial, so it would seem more interesting to attempt an interpretation of the $T = 0$ fixed points as the long distance termini of trajectories originating on a critical surface at nonzero T . Infrared asymptotic freedom implies a correlation

function $\langle \phi(x)\phi(0) \rangle$ decaying as $(\log |x|)^{-\Delta}$ for large $|x|$. But high temperature series for lattice versions of the nonlinear models always show finite correlation lengths, so there must be an intervening phase transition. The locally symmetric $\alpha = -1$ spaces all have non-trivial, nonabelian fundamental groups, allowing topologically stable vortex-like field configurations. Phase transitions due to dissociation of multivortex bound systems might be expected.¹² Other of the $\alpha = -1$ manifolds, being simply connected, call for different mechanisms.

Construction of a nonstandard model requires the bare a priori measure $d\phi(x)$ which avoids nonspontaneous long range ordering. For asymptotically small T it can be calculated from the renormalization group equation for the bare external field. It depends on the method of short distance regularization and differs from the metric volume element whenever h_1 in (5b) is nonzero. The difference is of order T , so in two dimensions the critical a priori measure is exactly the metric volume element. But an infinite number of relevant couplings (the external fields) must be fixed in order to bring the a priori measure to its critical value. In this sense the nonstandard models are unnatural. The standard models have enough internal symmetry to determine the a priori measure uniquely, so for them these issues do not arise.

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