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AN ELEMENTARY TREATMENT OF GAMMA-RAY HEATING AND
GAMMA-RAY DOSAGE IN INHOMOGENEOUS REACTORS

by

William Primak

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AN ELEMENTARY TREATMENT OF GAMMA-RAY HEATING AND GAMMA-RAY DOSAGE IN INHOMOGENEOUS REACTORS

by

William Primak

ABSTRACT

An analysis is made of the heat produced by the absorption of gamma rays in a sample placed into a reactor. It is clearly shown that enormous local (space) variations in gamma flux exist in current reactors. An application to the Hanford reactors is treated in some detail. Although the estimates obtained may be good to but a factor of two, it is clearly shown that in most cases the major portion of the heating is due to (n,γ) reactions within the sample itself, and in some cases to the gamma rays generated in liners and cans, and not from the gamma rays generated in fission, nor from the moderator. Some implications of these results are discussed, among the most important being their application to radiation chemistry in reactors.

Several mathematical results for absorption and generation of gamma rays in various bodies which should be useful in estimating dosage rates for samples irradiated in reactors are given (Appendix II). A method for making the calculations for an arbitrary absorption law are given (Appendix III). This method may be used with the true absorption law for gamma rays or even for the calculation of the absorption of the energy of particles possessing a range law of absorption, e.g., β -rays or protons.

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I am indebted to O. C. Simpson for several critical discussions of the material presented in this report; to E. J. Hart for assistance in interpreting the radiation chemistry; to G. Blanch (Inst. Num. Anal., UCLA), J. Robinson, and M. K. Brachman for assistance with some mathematical matters; and to J. R. Gilbreath for assistance in editing. That the function arising in the cylinder (I.1) is $J_1(B)$ was pointed out to me by O. C. Simpson. He had encountered the function in another problem and with the assistance of D. Gaarder had tabulated it in the range of small values of the argument (unpublished). Paragraphs 6.1 and 6.2 are due almost entirely to J. Robinson and M. K. Brachman. J. Robinson referred me to Hammersley's paper.

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PREFACE

During the course of preparation of several in-pile experiments designed to go into a Hanford reactor¹ it was realized that the heating of materials placed into this reactor might prove troublesome. The first problem encountered was the construction of a device for measuring fission heating.² It was originally intended to construct the device as an aluminum cylinder about 2 cm in diameter with a small amount of U^{235} at its center and to measure the temperature rise produced by heat transfer across the air gap between the cylinder and the wall of the containing vessel. It was soon realized that the heat generated in the aluminum would be far greater than that in the uranium, and so the uranium holder was altered to contain a minimum amount of other metal.

Shortly after this a NEPA creep apparatus which was inserted into the Hanford reactor reached a temperature much higher than expected.³ Since the writer was concerned with the design of another creep apparatus intended for insertion into the Hanford reactor, he decided to make a detailed investigation of the source of the heating effects. The results proved quite surprising to him and contained implications of more general interest than the particular experiments which stimulated the investigation. Since some of the results do not seem to be widely recognized, it was thought wise to collect them in the form of a report.

After the investigations reported here were substantially completed, the writer's attention was drawn to an experimental report by Richardson⁴ who realized the existence of some of the effects considered here⁵ but did not treat them very fully. Unfortunately, his experiments were performed in such a geometry that it is impractical to interpret them without involved graphical integration. From the results of the present investigation it would seem desirable to perform experiments like Richardson, Allen, and Boyle's⁴ in a channel parallel to the rods, using a much longer sample and calorimeter in a very simple container.

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SECTION 0

INTRODUCTION

0.0 General. Neutrons are absorbed to some extent by all materials with, among other effects, the generation of gamma rays. The calculation of the consequent gamma-ray flux gives rise to a simple class of problems of radiative transfer since the cross sections for gamma-ray generation are not related to the gamma-ray absorption coefficients. The absorption gives rise to ionization which can be dissipated as heat, chemical change, light emission, etc. This report is concerned mainly with the heat produced by the absorption of gamma rays in a material placed into a neutron reactor.

0.1 A material placed into an inhomogeneous reactor may be heated by the absorption of gamma rays generated in several major sources: (a) in the fuel rods and control rods by neutron capture, fission, and beta decay; (b) in the moderator, liners, jackets, shields, etc. by neutron capture and beta decay; and (c) within the material itself by neutron capture and beta decay.* An accurate calculation of these effects would require a knowledge of the gamma-ray spectrum, an analysis of the absorption which considered back scattering and the degradation of the gamma-ray energy, and an analysis of the heating which considered the dependence of the ionization on the medium and the gamma-ray energy. These difficulties are avoided here by guessing mean absorption coefficients and mean values for the ionization. The estimates are, therefore, only very approximate, though probably much better than just orders of magnitude.

0.2 The method of making the calculation. The gamma rays are treated as if they behaved like light passing through an absorbing medium. The sources are treated as possessing a source strength (Mev/sec), an intensity (Mev/sec steradian), and a flux (Mev/sec cm²). It is assumed that Lambert's law does not apply, and that the flux from unit area is independent of angle (i.e., that the brightness of a surface does depend on the angle at which it is viewed). The gamma-ray absorption is assumed to follow an exponential law. The absorption coefficients are estimates since the absorption coefficients given in the handbooks refer to the removal of the primary ray from the beam, rather than the absorption of the

*In a fast neutron reactor a significant contribution might arise from inelastic scattering of neutrons (perhaps 10 to 20 % of the total gamma-ray flux in the fast neutron regions of the reactor). In the reactors considered here in detail, the contribution from this source is insignificant and would merely increase the estimate of the gamma-ray energy per fission very slightly. The subject is, accordingly, neglected.

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energy; and since, further, the absorption coefficient changes as one proceeds from the source because, in general, the secondary radiation will possess a different absorption coefficient than the primary gamma rays. It is assumed that all the energy of beta rays and electrons is dissipated in a range short compared to the dimensions of the objects treated here.

0.3 The heating produced in a sample depends on the flux and the absorption coefficient. In the Compton range the absorption coefficient is proportional to the mass of material and depends on the gamma-ray energy.⁶ Since it will be necessary to calculate the heat developed by gamma rays from different sources (and hence different energies), it is convenient to convert the fluxes which are calculated into the heating which they will produce in a gram of material which is quite transparent. A unit of radiation dosage is thus required which is defined in terms of energy absorption. The roentgen-equivalent-physical (rep) is here adapted for this purpose by slightly altering its definition⁷ to the following one: the dosage which will produce 83 ergs/g of the particular material. The dosage rate is then:

$$\left[\text{Dosage} \left(\frac{\text{rep}}{\text{sec}} \right) \right] = \left[\text{Flux} \left(\frac{\text{Mev}}{\text{sec cm}^2} \right) \right] \left[1.6(10^{-6}) \left(\frac{\text{erg}}{\text{Mev}} \right) \right] \left[\mu \left(\frac{1}{\text{cm}} \right) \right] \left[\frac{1}{\rho} \left(\frac{\text{cm}^3}{\text{g}} \right) \right] \left[\frac{1}{83} \left(\frac{\text{rep g}}{\text{erg}} \right) \right] \alpha,$$

$$= \text{Flux } 1.93(10^{-8}) \frac{\mu}{\rho} \alpha,$$

where μ/ρ is the usual mass absorption coefficient and the units are stated in parentheses. The fraction α , the part of the energy removed from the primary beam which is absorbed, varies from about 1/2 to 2/3 in the range of Compton energies of interest. The value 2/3 is used in most of the calculations given in this report. Because μ/ρ for hydrogen is higher than for the light elements, the dosage rate in water exposed to a given flux is greater. It is assumed in this report that the dosage rate in water is 93/83 that in the light elements.⁷

0.4 Terminology. Several terms are employed which are not customary. The gamma power generated in unit volume is called the generation. That portion of the gamma-ray power generated in a body which emerges is termed the emergence; the portion which is absorbed, the absorption. The emergent power per unit area is called the emergent flux. The emergent flux is therefore the brightness of a unit area integrated over the hemisphere. Flux, as used in this report, always refers to the gamma-ray flux unless it is otherwise qualified. The flux per steradian is termed the intensity. In writing the units of intensity, steradian is often omitted and should be understood. The flux per unit length divided by 4π is termed the intensity of a line source. Some of the intensities and fluxes are given in dosage units and termed dosage intensities or dosage fluxes. They are then to be understood as that intensity or that flux which would yield the dosage given. To conform with mathematical practice it was often found necessary to use the

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same mathematical symbols to stand for differing quantities in different sections of this report. The reader should observe care in interpreting the following symbols: B, E, α , Z, r, R, θ , ϕ , x, y, A, T, N, u, η , D, and others. The inverse trigonometric functions are indicated by a capital A prefix.

0.5 Reactors considered. The results calculated here for CP-3 refer to the original loading with natural uranium rods and to the central thimble arrangement then used. The Hanford results are computed for a thermal neutron flux of $1.8(10^{13})$ in the graphite at the cell edge (¶1.4). No account is taken of the difference in cross sections for thermal and pile neutrons.

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SECTION 1

THE GAMMA-RAY DOSAGE DUE TO FUEL RODS

1.0 General. The emergence (10.4) from the Hanford and the CP-3 fuel rod is calculated. Then from an experimental determination of the gamma-ray flux in the central thimble of CP-3, the intensity of the CP-3 rods is calculated, and from this, ~~the intensity of the Hanford rods.~~ No account is taken of the slightly different gamma-ray spectrum which would be observed for the respective sources.

1.1 Estimate of the fraction of the gamma-ray energy reaching the surface of a Hanford fuel rod. The rate of generation of gamma rays within the fuel rod is taken to be $S(r)$ per cm^3 proportional to the flux within the rod, $\phi(r)$. The slow neutron flux in a Hanford fuel rod, graphed as a function of radius, is given by Fox, Montgomery, and Nather.⁸ A parabola, $\phi(r) = [0.7r^2 + 6.6](10^{12})$, was fitted to this curve; then $S(r) = s_0(0.7r^2 + 6.6)$. Then, using the coordinates shown in Figure 1, the gamma-ray energy which is generated in unit length of rod (assumed to be very long) and which reaches the surface is:

$$\frac{s_0}{4\pi} \int_0^1 dz \int_0^R r dr (0.7r^2 + 6.6) \int_0^{2\pi} d\alpha \int_0^\pi 2d\phi \int_0^{\pi/2} 2e^{-\beta x} \sec \theta \cos \theta d\theta.$$

It is convenient to change the radius variable, r , to λR where R is the radius of the rod, 1.8 cm at Hanford. Then integration over z and α gives

$$2s_0 R^2 \int_0^1 (2.27 \lambda^2 + 6.6) \lambda d\lambda \int_0^\pi d\phi \int_0^{\pi/2} e^{-B} \sec \theta \cos \theta d\theta,$$

where

$$B = \beta x = \beta R (\sqrt{1 - \lambda^2 \sin^2 \theta} - \lambda \cos \theta).$$

For the range of interest, the integral,

$$\int_0^{\pi/2} e^{-B} \sec \theta \cos \theta d\theta,$$

is given within a few per cent by $0.700 e^{-1.08B} + 0.283 e^{-2.17B}$ (see ¶ 6.5). From this point the integration was carried forward by numerical quadrature,

dividing the intervals according to the steepness of the function in that range and using the 2nd or 3rd degree Simpson Rule, whichever applied to the number of intervals at hand. The value of β was taken to be 1.3. The integration was also performed graphically using a planimeter and found to check to better than one per cent. The steps in the integration may be stated formally as follows:

$$2s_0 R^2 \int_0^1 (2.27\lambda^2 + 6.6) \lambda \, d\lambda \left\{ 0.7 \int_0^\pi e^{-1.08B} \, d\phi + 0.283 \int_0^\pi e^{-2.17B} \, d\phi \right\};$$

let

$$\int_0^\pi e^{-1.08B} \, d\phi = \pi\phi_1 \text{ and } \int_0^\pi e^{-2.17B} \, d\phi = \pi\phi_2 ,$$

then there is obtained

$$2s_0 R^2 \int_0^1 (2.27\lambda^2 + 6.6) \lambda \, d\lambda \left[0.7\pi\phi_1(\lambda) + 0.283\pi\phi_2(\lambda) \right] ,$$

or

$$2s_0 \pi R^2 \left\{ \int_0^1 (1.59\phi_1 + 0.643\phi_2) \lambda^3 \, d\lambda + \int_0^1 (4.62\phi_1 + 1.87\phi_2) \lambda \, d\lambda \right\} .$$

Thus the gamma-ray energy which reaches the surface of the rod is $2s_0 \pi R^2(0.84)$. The total generation per unit length of rod is

$$s_0 \int_0^1 dz \int_0^R (0.7r^2 + 6.6) r \, dr \int_0^{2\pi} d\phi ,$$

or $2s_0 \pi R^2(3.87)$. The fraction of the generated gamma rays which reach the surface is therefore 22 per cent.

1.2 Estimate of the fraction of the gamma-ray energy reaching the surface of a CP-3 rod. The calculation presented here is for the natural uranium rods constituting the original loading of CP-3. For the enriched uranium-aluminum alloy rods in the present loading β is small, and the

calculation can be performed in the manner of §3.3. For the natural uranium rods β is assumed to be 1.3 as before. The value of R was 1.4. The generation is assumed to be $\underline{s}$ per unit volume. Then the emergent energy due to unit length of rod is:

$$\frac{s}{4\pi} \int_0^1 dz \int_0^R F(r) r dr \int_0^{2\pi} d\alpha \int_0^\pi 2d\phi \int_0^{\pi/2} 2e^{-\beta x \sec \theta} \cos \theta' d\theta .$$

Proceeding as before, this becomes, successively

$$2s R^2 \int_0^1 F(\lambda) \lambda d\lambda \int_0^\pi d\phi \int_0^{\pi/2} e^{-B \sec \theta} \cos \theta d\theta ,$$

$$2s R^2 \int_0^1 F(\lambda) \lambda d\lambda \int_0^\pi d\phi \left[0.7e^{-1.08B} + 0.283e^{-2.17B} \right] , \text{ and}$$

$$2s\pi R^2 \left\{ \int_0^1 F(\lambda) \left[0.7\phi_1(\lambda) + 0.283\phi_2(\lambda) \right] \lambda d\lambda \right\} .$$

The ratio of the fractional depression of the neutron flux of the CP-3 rods as compared to the Hanford rods is the ratio of the zero order modified Bessel functions of the first kind at the radius. Hence, the fractional depression in the CP-3 rod is $1/1.2$ compared to $1/1.3$ at Hanford. Fitting a parabola to this flux distribution, there is obtained $y = 1 + 0.102r^2$. This is normalized to give unit generation in unit volume of the rod. The normalization factor is

$$\int_0^{1.4} 2\pi y r dr = 1.1 .$$

Hence, $F(r) = 0.91 + 0.0927r^2$. Then we get for the final result $2s\pi R^2(0.131)$. The generation in unit length is $2s\pi R^2(0.5)$. Hence, 26 per cent of the generated gamma rays reach the surface. This result was checked by graphical integration using a planimeter.

1.3.1 The effective intensity of the Hanford rods. The equivalent gamma-ray dosage rate in one of the small tubes of the central thimble of CP-3 was measured by means of a chemical reaction by E. J. Hart⁹ who

reported it to be 513 roentgens/sec at a pile power of 250 kw in a dilute aqueous solution of density about 1.03 g/cm³, having converted the chemical yield with a factor equivalent to the assumption of 6.75(10⁷) Mev/roentgen cm³. The yield was then equivalent to the absorption of

$$\left[513 \left(\frac{\text{r}}{\text{sec}} \right) \right] \left[6.75(10^7) \left(\frac{\text{Mev}}{\text{r cm}^3} \right) \right] \left[\frac{1}{1.03} \left(\frac{\text{cm}^3}{\text{g}} \right) \right] = \left[33.8 (10^9) \left(\frac{\text{Mev}}{\text{g sec}} \right) \right]$$

of gamma rays. However in actual fact this involved a fast neutron contribution. An estimate follows.

1.3.2 The fast neutron energy dissipated in water placed into Hole 12 of the X-10 reactor was determined by Richardson⁴ to be 8.01(10⁻⁴) cal/sec g at a thermal flux of 0.764(10¹²) neutrons/cm² sec,¹⁰ and he considered it would be 1.17 times as great per thermal neutron at a non-water-cooled point in the X-10 reactor.¹¹ The epi-cadmium flux was determined by Jones, Clarke, and Overman¹² to be 29 per cent of the thermal flux in the X-10 reactor. The heating per epi-cadmium neutron (n_{ϵ}) becomes then:

$$1.17 \left[8.01(10^{-4}) \left(\frac{\text{cal}}{\text{sec g}} \right) \right] \left[\frac{1}{0.29 \times 0.764(10^{12})} \left(\frac{\text{cm}^2 \text{ sec}}{\text{n}} \right) \right] \left[2.62(10^{13}) \left(\frac{\text{Mev}}{\text{cal}} \right) \right] =$$

$$\left[0.111 \left(\frac{\text{Mev cm}^2}{\text{n}_{\epsilon}} \right) \right] .$$

The epi-cadmium flux in the central thimble of CP-3 is said to have been 2(10¹¹) at 250 kw.¹³ If the effectiveness of the epi-cadmium flux in CP-3 and in X-10 be the same, and if the effectiveness of these neutrons in producing chemical reaction be the same as that of alpha particles,¹⁴ 1/3 that of gamma rays,¹⁵ the amount of reaction caused by the fast neutrons in Hart's experiment becomes the equivalent of the absorption of

$$\left[0.111 \left(\frac{\text{Mev cm}^2}{\text{n}_{\epsilon}} \right) \right] \left[2(10^{11}) \left(\frac{\text{n}_{\epsilon}}{\text{cm}^2 \text{ sec}} \right) \right] \frac{1}{3} = \left[7.4(10^9) \left(\frac{\text{Mev}}{\text{g sec}} \right) \right]$$

of gamma rays.

1.3.3 It would then seem that in Hart's experiment gamma rays were absorbed at the rate $(33.8 - 7.4)(10^9) = 26.4 \text{ Mev/g sec}$. The gamma-ray dosage rate was then

$$\left[26.4 \left(\frac{\text{Mev}}{\text{g sec}} \right) \right] \left[1.6(10^{-6}) \left(\frac{\text{ergs}}{\text{Mev}} \right) \right] \left[\frac{1}{83} \left(\frac{\text{rep g}}{\text{ergs}} \right) \right] = \left[508 \left(\frac{\text{rep}}{\text{sec}} \right) \right]$$

in water. For the light elements it would have been 83/93 of this value or 450 rep/sec.

1.3.4 It will be shown below (2.2.3) that the dosage due to the liners, moderator, etc. might have been 100 rep/sec, and it is therefore assumed that 350 rep/sec arose from the rods. The geometry assumed for the irradiation is shown in Figure 2.¹⁶ The intensity of line sources about 1.5 meters long (this value was chosen arbitrarily) located at the rod centers which would be required to give a dosage of 350 rep/sec at the irradiation position can be computed by means of the exponential integral $\mathcal{J}(\phi, B)$ given in the Project Handbook.¹⁷ The flux, i_a , at the point distant a centimeters from the source of intensity i_o /cm subtending an angle ϕ of a rod is

$$i_a = 2 \frac{i_o}{a} \mathcal{J}(\phi, B) ,$$

where

$$\mathcal{J}(\phi, B) = \int_0^\pi e^{-B \sec \phi} d\phi, \quad B \text{ is } \sum_j \beta_j t_j, \text{ and where } \beta_j \text{ is the absorption}$$

coefficient for the j^{th} intervening layer and t_j is its thickness. The geometry and the results are given in Table I.

Table I

ILLUMINATION OF E. J. HART'S IRRADIATION IN CP-3

Source (Rod #)	Absorber Thickness			$B = \sum \beta_j T_j$	Distance from line source (cm)	Illumination angle (deg)	$\mathcal{J}_o(\phi, B)$	$\frac{i_a}{i_o}$ (percent)
	Water $\beta = 0.055$ (cm)	Al $\beta = 0.11$ (cm)	SiO ₂ $\beta = 0.088$ (cm)					
1	2.0	0.55	0.2	0.19	5.8	84	1.01	35
2	2.4	0.72	0.2	0.23	9.5	80	0.91	19
3	2.0	0.87	0.2	0.22	12.5	76	0.86	14
4	2.4	0.72	0.2	0.23	9.5	80	0.91	19

FIGURE 2.

GEOMETRY OF CP-3
CENTRAL THIMBLE.

* Location of Hart's Experiment.

A similar calculation was carried out for the rods to a distance of 75 cm, though here, because of their greater distance and their smaller contribution, it was considered sufficient to use their distance to the center of the reactor for the value $\underline{a}$ in the equation. Their contribution is $0.45 i_0$. The dosage, I_a , at the point $\underline{a}$ then becomes

$$I_a = I_0 (0.35 + 0.19 + 0.14 + 0.19 + 0.45) = 350 \text{ rep/sec},$$

since the dosage is proportional to the flux. No allowance was made for the neutron flux distribution which probably rose in the rod zone just beyond the central thimble and fell toward the outer fringes of the region considered. The rods, then, act as line sources of dosages intensity $I_0 \cong 260 \text{ rep/sec cm}$ in dosage units.

1.3.5 The effective intensity of the Hanford rods can now be estimated. It is the effective intensity of the CP-3 rods times the ratio of emergence, volume, and neutron flux,

$$(260) \left(\frac{22}{26} \right) \left(\frac{1.8}{1.4} \right)^2 (18) = 6500 \text{ rep/sec cm}$$

in dosage units. This may be a generous estimate because the average flux in the Hanford rods is probably less than 18 times the average flux in the CP-3 rods.

1.3.6 The gamma-ray energy per fission. If it is assumed that the average neutron flux in the CP-3 rod was $0.65(10^{12})$, the number of fissions per centimeter of rod in the central region was

$$\left[6.5(10^{11}) \left(\frac{n}{\text{cm}^2} \right) \right] \left[3.9(10^{-24}) \left(\frac{\text{cm}^2}{\text{atom}} \right) \right] \left[\frac{115}{235} 6.02(10^{23}) (\text{atoms}) \right] = \left[7.5(10^{11}) \left(\frac{\text{fissions}}{\text{cm}} \right) \right]$$

Assuming $\alpha \frac{\mu}{\rho} = 0.04$ (for 1-Mev gamma rays), the gamma-ray generation was

$$\left[260 \left(\frac{\text{rep}}{\text{sec cm steradian}} \right) \right] \left[\frac{1}{1.93(10^{-8})} \left(\frac{\text{Mev}}{\text{rep g}} \right) \right] \left[\frac{1}{0.04} \left(\frac{\text{g}}{\text{cm}^2} \right) \right] \frac{4\pi}{0.26} = \left[1.6(10^{13}) \left(\frac{\text{Mev}}{\text{cm}} \right) \right],$$

(where 0.26 is the emergence) and hence about 22 Mev/fission, which is in excellent accord with the expected value¹⁸ when the crudeness of the calculation is considered.

1.4 The dosage rate at the sample in the Hanford reactor which is due to the rods. Two locations of simple geometry are possible in the graphite lattice. If the rod be taken as the center of the cell, these are A, the corner of a cell (a hole between four fuel rods), and B, the center of the cell (an emptied fuel channel). (See Figure 3.) The dosage rate is computed with the aid of the function $\mathcal{J}(\frac{\pi}{2}, B)$, taking $\beta = 0.068$ (for graphite) and assuming 1-inch holes in the graphite. Then, for position A:

$$\begin{aligned} I_A &= 2I_0 \ 4 \ \mathcal{J}_0(B)/a, \\ &= \frac{8}{15} I_0 \ \mathcal{J}_0(1.15) \\ &= \frac{8}{15} 6500 \times 0.51 = 1800 \text{ rep/sec.} \end{aligned}$$

for position B:

$$\begin{aligned} I_B &= 2I_0 \ 4 \ [\mathcal{J}_0(1.15)/a_2 + \mathcal{J}_0(1.70)/a_3] \\ &= 8 \times 6500 \left[\frac{0.28}{21} + \frac{0.14}{30} \right] = 940 \text{ rep/sec.} \end{aligned}$$

If a 4-inch hole were present in position A, the dosage rate would be 2500 rep/sec. If liners were present in the holes, the figures would have to be reduced slightly.

FIGURE 3.

GEOMETRY FOR TWO IRRADIATION
LOCATIONS IN THE HANFORD LATTICE.

SECTION 2

THE GAMMA-RAY DOSAGE DUE TO MODERATOR AND LINERS

2.1 The dosage at Hanford due to generation in the moderator. The flux due to generation in a moderator is

$$8 \int_0^{\infty} \int_0^{\pi/2} \int_0^{\pi/2} \frac{s_m r^2 d\theta \sin\theta d\phi}{4\pi r^2} e^{-\beta r} dr = \frac{s_m}{\beta},$$

where s_m is the generation per cm^3 . The dosage is then (10.3):

$$\frac{s_m}{\beta} 1.93(10^{-8}) \frac{\beta}{\rho} = 1.93(10^{-8}) s_m / \rho.$$

If it is assumed that the Hanford moderator possesses a macroscopic thermal neutron absorption cross section of $4(10^{-4}) \text{ cm}^2/\text{cm}^3$ resulting in the production of 4.9-Mev gamma rays, the dosage is

$$\left[4(10^{-4}) \left(\frac{\text{cm}^2}{\text{cm}^3} \right) \right] \left[1.8(10^{13}) \left(\frac{n}{\text{cm}^2 \text{ sec}} \right) \right] \left[4.9 \left(\frac{\text{Mev}}{n} \right) \right] \left[1.93(10^{-8}) \left(\frac{\text{rep g}}{\text{Mev}} \right) \right] \\ \times \left[\frac{1}{1.7} \left(\frac{\text{cm}^3}{\text{g}} \right) \right] = \left[400 \left(\frac{\text{rep}}{\text{sec}} \right) \right].$$

2.2. The dosage at Hanford due to generation in the liners. It is obvious that the flux at the center of a cylindrical shell (Figure 4) possessing no self-absorption is

$$\int_0^{2\pi} \frac{s}{4\pi} \frac{\pi T}{r} r d\theta = \frac{\pi}{2} s T,$$

where $\underline{s}$ is the gamma-ray generation per unit volume, and $\underline{T}$ is the wall thickness. It will be shown later (Section 7) that the flux along the radius of the usual liner is quite constant until close to the wall, where it suddenly rises quite sharply for a thin-walled liner. The average flux (averaged over the interior cross section) is 15-25% greater than the above value. It is also shown that the absorption of the wall follows the same behavior and

FIGURE 4.

COORDINATES FOR THE
CYLINDER SHELL (LINER).

is hence of the order of magnitude βT . For the usual liner it is under 5% and is also neglected. Then for a 1/8-inch aluminum liner placed in a Hanford hole, for which $T = 0.3$, $\Sigma = 0.013 \text{ cm}^2/\text{cm}^3$, the neutron flux is $1.8(10^{13})$, and the total energy is 10 Mev/capture of average energy 5 Mev,^{19, 20} and with an assumed value $\alpha\beta/\rho = 0.018$, the dosage rate becomes

$$\frac{\pi}{2} \left[1.8(10^{13}) \left(\frac{n}{\text{cm}^2 \text{ sec}} \right) \right] \left[0.013 \left(\frac{\text{cm}^2}{\text{cm}^3} \right) \right] \left[10 \left(\frac{\text{Mev}}{n} \right) \right] \left[1.93(10^{-8}) \left(\frac{\text{rep g}}{\text{Mev}} \right) \right] \\ \times \left[0.018 \left(\frac{1}{\text{g}} \right) \right], \text{ or } 400 \text{ rep/sec.}$$

2.2.1 The dosage in the central thimble of CP-3 due to neutron capture in the aluminum. The central thimble of CP-3 was a complicated assembly which is here considered the equivalent of an aluminum cylinder of 3/16 inch wall closed at the bottom with a 3/8 inch thick aluminum plate and containing 6 line sources 0.66 g aluminum/cm and one line source 0.88 g aluminum/cm spaced from the starred tube as shown in Figure 3. The flux due to a line source at a distance R generating s per centimeter is $s/4R$. The total dosage rate from the line sources was computed to be 14 rep/sec. It will be shown (§7.3.3) that the dosage rate within a semi-infinite tube closed by a bottom plate of twice the wall thickness is not very different from the dosage rate in the infinite tube. Since the thimble wall is 3/2 that of the Hanford liner considered and the flux 1/18, the dosage rate due to the tube is 33 rep/sec. Adding the result for the line sources, the total dosage rate due to aluminum is 47 rep/sec.

2.2.2 The dosage in the central thimble of CP-3 due to the heavy water moderator. It is assumed that there is a 0.65-millibarn cross section of deuterium for the production of 6.2 Mev of gamma rays and a 1-millibarn cross section of oxygen for the production of 4.1 Mev of gamma rays.²¹ It is further assumed that the neutron flux was 10^{12} . The generation per cm^3 becomes then $1.51(10^8) \text{ Mev/cm}^3$, and the dosage is therefore 2.6 rep/sec.

2.2.3 The total gamma dosage rate in the central thimble of CP-3. It would thus seem that of the gamma-ray dosage derived from E. J. Hart's measurement (§1.3.3), 450 rep/sec, about 47 rep/sec arose from the

thimble itself, and about 3 rep/sec from the moderator. The rest must have come from the water solvent, impurities, the fuel rods, the control rods, etc. If it is assumed that 50 rep/sec arose from other sources than the fuel rods, 350 rep/sec must be assigned to the fuel rods. The dosage derived from neutron capture in the silica vessel was small, amounting to less than 1.5 rep/sec. The capture gammas of the hydrogen in the solution made a surprising contribution, 25-35 rep/sec (13.5).

SECTION 3

THE DOSAGE DUE TO SELF-ABSORPTION

3.0 General. An exact result can be given for the infinite slab.* However, it can be cast into a convenient form only when the absorption is low. By making the same sort of approximation a useful result can be obtained for the rod and the sphere. In general, the computations become difficult to perform analytically except when the body has a uniform absorption coefficient within its whole interior, has a uniform surface brightness, and has a low absorption. The only bodies meeting these conditions are the large thin slab, the long thin rod, and the small sphere.

3.1 Self-absorption of gamma rays generated in an infinite slab of thickness $2A$. The same result can be arrived at by several alternative methods. In the one given here, the exact result is obtained, and the approximation made at the end. Using the coordinates shown in Figure 5, the emergence arising from unit area of slab can be expressed as

$$\frac{2s}{4\pi} \int_0^A dz \int_0^1 dy \int_0^1 dx \int_0^{2\pi} d\phi \int_0^{\pi/2} \left(e^{-B_1 \sec \theta} + e^{-B_2 \sec \theta} \right) \sin \theta d\theta$$

where $B_1 = (A - z)\beta$, $B_2 = (A + z)\beta$, and s is the generation per unit volume. Integrating over x , y , and ϕ , there is obtained

$$s \int_0^A dz \int_0^{\pi/2} \left(e^{-(A-z)\beta \sec \theta} + e^{-(A+z)\beta \sec \theta} \right) \sin \theta d\theta.$$

One may proceed by integrating first with respect to θ or to z . The former gives rise to an exponential integral $E_3(B)$ which can be integrated if a table of such integrals is at hand.²² This is the shorter course. Alternatively, one may integrate first with respect to z and arrive at the identical result. Then, successively,

*See footnote ¶ 3.3.

FIGURE 5.

COORDINATES FOR SLAB.

$$\begin{aligned}
& s \left\{ \int_0^A dz \int_0^{\pi/2} e^{-\beta A \sec \theta} e^{\beta z \sec \theta} \sin \theta \, d\theta + \int_0^A dz \int_0^{\pi/2} e^{-\beta A \sec \theta} e^{-\beta z \sec \theta} \sin \theta \, d\theta \right\} \\
&= s \left\{ \int_0^{\pi/2} e^{-\beta A \sec \theta} \sin \theta \, d\theta \int_0^A e^{\beta z \sec \theta} dz + \int_0^{\pi/2} e^{-\beta A \sec \theta} \sin \theta \, d\theta \int_0^A e^{-\beta z \sec \theta} dz \right\}, \\
&= s \left\{ \int_0^{\pi/2} e^{-\beta A \sec \theta} \frac{\sin \theta}{\beta \sec \theta} d\theta \left[e^{\beta z \sec \theta} \right]_0^A - \int_0^{\pi/2} e^{-\beta A \sec \theta} \frac{\sin \theta}{\beta \sec \theta} d\theta \left[e^{-\beta z \sec \theta} \right]_0^A \right\}, \\
&= \frac{s}{\beta} \int_0^{\pi/2} \left\{ e^{-\beta A \sec \theta} \left[e^{\beta A \sec \theta} - 1 \right] - e^{-\beta A \sec \theta} \left[e^{-\beta A \sec \theta} - 1 \right] \right\} \sin \theta \cos \theta \, d\theta, \\
&= \frac{s}{\beta} \int_0^{\pi/2} (1 - e^{-2\beta A \sec \theta}) \sin \theta \cos \theta \, d\theta, \\
&= \frac{s}{\beta} \left\{ \frac{1}{2} - \int_1^\infty \frac{e^{-2\beta A y}}{y^3} dy \right\}.
\end{aligned}$$

The integral in the last expression will be recognized as the exponential integral $E_3(2\beta A)$ ^{23, 24} The result may then be written

$$\frac{s}{\beta} \left\{ \frac{1}{2} - E_3(2\beta A) \right\}.$$

The gamma rays generated in the volume considered are $2sA$. The emergent fraction is therefore,

$$\frac{1}{2\beta A} \left\{ \frac{1}{2} - E_3(2\beta A) \right\}.$$

A first-order expansion of this result is desired. Now,

$$E_3(x) = \sum_{\substack{j=0 \\ j \neq 2}}^{\infty} \frac{(-x)^j}{(2-j)j!} + (-1)^3 \frac{x^2}{2!} \left\{ \gamma + \ln x - A_3 \right\},$$

where $A_3 = \frac{1}{1} + \frac{1}{2} = 1.5$ and γ is a Bernoulli number, ~ 0.5772 .

Then,

$$E_3(x) \approx \frac{1}{2} - x + \frac{x^3}{6} - \frac{x^2}{2} (\gamma - 1.5) + \frac{x^2}{2} \ln x;$$

or,

$$\frac{\frac{1}{2} - E_3(x)}{x} \approx 1 - \frac{x^2}{6} + \frac{x}{2} (\gamma - 1.5) + \frac{x}{2} \ln x.$$

Using an approximation based on

$$\lim_{x \rightarrow 0} x \ln x = \lim_{x \rightarrow 0} \frac{d \ln x / dx}{dx^{-1} / dx} = \lim_{x \rightarrow 0} -x,$$

there is obtained for small values of x :

$$\frac{\frac{1}{2} - E_3(x)}{x} \approx 1 + \frac{x}{2} (\gamma - 1.5 - 1) \approx 1 - 0.96x$$

to first order terms.

This is the emergent fraction. The absorbed fraction is $0.96x$ or $1.92\beta A$. This is taken to be $2\beta A$. Several other methods were used to make the approximation and yielded approximately the same result.

3.2 Self-absorption of gamma rays generated within the small sphere. Using the coordinates shown in Figure 6, the emergence may be expressed as:

$$\int_0^R r^2 dr \int_0^\pi \sin \alpha d\alpha \int_0^{2\pi} d\zeta \frac{s}{4\pi} \int_0^{2\pi} d\phi \int_0^\pi e^{-\beta z} \sin \theta d\theta,$$

where z is a function of r, θ only. Let $r = \lambda R$. Then

$$z = R (\sqrt{1-\lambda^2 \sin^2 \theta} - \lambda \cos \theta).$$

Integrating over α, ζ , and ϕ :

$$2\pi s R^3 \int_0^1 \lambda^2 d\lambda \int_0^\pi e^{-\beta z} \sin \theta d\theta.$$

If βR is small, $e^{-\beta z} = 1 - \beta z$; then:

$$s 2\pi R^3 \int_0^1 \lambda^2 d\lambda \int_0^\pi \left[1 - \beta R (\sqrt{1-\lambda^2 \sin^2 \theta} - \lambda \cos \theta) \right] \sin \theta d\theta$$

$$= s 2\pi R^3 \left\{ \int_0^1 \lambda^2 d\lambda \int_0^\pi (1 + \lambda \beta R \cos \theta) \sin \theta d\theta - \int_0^1 \lambda^2 d\lambda \beta R \right.$$

$$\left. \int_0^\pi \sqrt{1-\lambda^2 \sin^2 \theta} \sin \theta d\theta \right\}$$

$$= s 2\pi R^3 \left\{ \int_0^1 \lambda^2 d\lambda (2 + 0) - \beta R \int_0^1 \lambda^2 d\lambda \int_0^\pi \sqrt{1-\lambda^2 (1-\cos^2 \theta)} \sin \theta d\theta \right\}$$

FIGURE 6.

COORDINATES FOR THE SPHERE.

$$= 4\pi sR^3 \left\{ \frac{1}{3} - \frac{\beta R}{2} \int_0^1 \lambda d\lambda \int_0^\pi \sqrt{1 - \lambda^2 + \lambda^2 \cos^2 \theta} \lambda \sin \theta d\theta \right\},$$

Let $\lambda \cos \theta = y$;

$dy = -\lambda \sin \theta d\theta$. Then:

$$4\pi sR^3 \left\{ \frac{1}{3} - \frac{\beta R}{2} \int_0^1 \lambda d\lambda \int_{-\lambda}^{+\lambda} \sqrt{1 - \lambda^2 + y^2} dy \right\}.$$

Since the function $1 - \lambda^2 + y^2$ is symmetrical in y :

$$\begin{aligned} & 4\pi sR^3 \left\{ \frac{1}{3} - \beta R \int_0^1 \lambda d\lambda \int_0^\lambda \sqrt{1 - \lambda^2 + y^2} dy \right\} \\ &= 4\pi sR^3 \left\{ \frac{1}{3} - \beta R \int_0^1 \frac{1}{2} \left[\lambda + \frac{1 - \lambda^2}{2} \ln \frac{1 + \lambda}{1 - \lambda} \right] \lambda d\lambda \right\} \\ &= 4\pi sR^3 \left\{ \frac{1}{3} - \frac{\beta R}{2} \left[\frac{1}{3} + \frac{1}{2} \left(\frac{\lambda^2}{2} - \frac{\lambda^4}{4} \right) \ln \frac{1 + \lambda}{1 - \lambda} - \frac{1}{2} \int \left(\frac{\lambda^2}{2} - \frac{\lambda^4}{4} \right) \frac{2}{1 - \lambda^2} d\lambda \right]_0^1 \right\} \\ &= 4\pi sR^3 \left\{ \frac{1}{3} - \frac{\beta R}{2} \left[\frac{1}{3} + \frac{1}{2} \left(\frac{1}{2} - \frac{1}{4} \right) \ln \frac{1 + \lambda}{1 - \lambda} - \frac{1}{4} \int \left(\frac{1}{1 - \lambda^2} - 1 + \lambda^2 \right) dy \right]_0^1 \right\} \\ &= 4\pi sR^3 \left\{ \frac{1}{3} - \frac{\beta R}{2} \left[\frac{1}{3} + \left(\frac{1}{8} - \frac{1}{8} \right) \ln \frac{1 + \lambda}{1 - \lambda} + \frac{\lambda}{4} - \frac{\lambda}{12} \right]_0^1 \right\} \\ &= 4\pi sR^3 \left(\frac{1}{3} - \frac{1}{4} \beta R \right) = \frac{4}{3} \pi sR^3 (1 - 0.75 \beta R). \end{aligned}$$

The gamma-ray energy generated in the sphere is $\frac{4}{3}\pi R^3 s$. Hence, the emergent fraction is $(1 - 0.75\beta R)$ and the absorbed fraction $0.75\beta R$.

3.3 Self-absorption of gamma rays generated within the long thin rod. The emergent portion of the flux generated in unit length of rod is given in §1.1 and §1.2. It is

$$2sR^2 \int_0^1 \lambda \, d\lambda \int_0^\pi d\phi \int_0^{\pi/2} e^{-B \sec \theta \cos \theta} d\theta,$$

where now the generation is assumed constant.* It is shown in Appendix I (§6.4) that for small values of B (<0.3) a good approximation for the θ integral is $1 - \frac{\pi}{2} B + \frac{\pi}{2} B^2$. Then one obtains

$$2sR^2 \int_0^1 \lambda \, d\lambda \int_0^\pi \left[1 - 1.57 \beta R (\sqrt{1 - \lambda^2 \sin^2 \phi} - \lambda \cos \phi) + 1.57 \beta^2 R^2 (1 - \lambda^2 \sin^2 \phi + \lambda^2 \cos^2 \phi - 2\lambda \cos \phi \sqrt{1 - \lambda^2 \sin^2 \phi}) \right] d\phi;$$

and using the expansion for the elliptic integral,

$$\int_0^\pi \sqrt{1 - k^2 \sin^2 \phi} \, d\phi = \pi \left(1 - \frac{k^2}{4} - \frac{3}{64} k^4 - \dots \right),$$

*The volume integral of this paragraph was transformed into an integral over the surface of the cylinder by O. C. Simpson. After this report was written, John Robinson succeeded in integrating the expression by means of a Mellin transform to obtain an exact solution which, by the application of suitable recursion formulae, he cast into the form

$$\frac{2}{3} \left[K_1 I_1 + \beta r (K_1 I_0 - K_0 I_1) - 2\beta r + 2\beta^2 r^2 (K_1 I_1 + K_0 I_0) \right],$$

where $\underline{K}$ and $\underline{I}$ designate the modified Bessel functions of order given by the subscripts and whose argument is βr (nomenclature of Adams.³²) The solution will appear in a forthcoming Quarterly Progress Report of the Chemistry Division, ANL-4833.

the integral may be written

$$\begin{aligned}
 & 2sR^2 \pi \int_0^1 \left[1 - 1.57 \beta R \left(1 - \frac{\lambda^2}{4} - \dots \right) + 1.57 \beta^2 R^2 \right] \lambda d\lambda \\
 &= 2\pi sR^2 \left(\frac{1}{2} - \frac{1.57}{2} \beta R \frac{55}{64} + \frac{1.57}{2} \beta^2 R^2 \right) \\
 &= \pi sR^2 (1 - 1.33 \beta R + 1.57 \beta^2 R^2).
 \end{aligned}$$

The energy generated in the length of rod is πsR^2 . Hence, the emergent fraction is $(1 - 1.3\beta R)$ to terms of first order, and the absorbed fraction is approximately $1.3\beta r$.

3.4 Summary. The self-absorbed fractions of the gamma rays generated in the three simple bodies are thus approximately:

Infinite thin slab of half thickness A:	$2\beta A$,
Long thin rod of radius R:	$1.3\beta R$,
The small sphere of radius R:	$0.75\beta R$.

For βR up to 0.2, these results are probably good to better than 20%. For extremely small bodies of dimensions less than many times the range of the secondary electrons, the absorption will fall below these values, and the problems will have to be solved by combining the range laws with the radiative transfer laws.

3.5 An application.* In E. Hart's experiment gamma rays were generated in the water by the reaction $H(n, \gamma) D$. The cross section is 0.32 barn, and there are 2.18 Mev of capture gammas. Thus there are produced

$$\begin{aligned}
 & \left[0.32(10^{-24}) \left(\frac{\text{cm}^2}{\text{atom}} \right) \right] \left[10^{12} \left(\frac{n}{\text{cm}^2 \text{ sec}} \right) \right] \left[\frac{2}{18} 6.02 (10^{23}) \left(\frac{\text{atoms}}{\text{cm}^3} \right) \right] \left[2.18 \left(\frac{\text{Mev}}{n} \right) \right] \\
 &= \left[4.7 (10^{10}) \left(\frac{\text{Mev}}{\text{cm}^3 \text{ sec}} \right) \right].
 \end{aligned}$$

The geometry was that of a short cylinder of radius about 1 cm. The absorption is therefore something under 1.3β . If β is assumed to be 0.04, the absorption is then about 4%, or there are absorbed on the average about

*Suggested by E. J. Hart.

$1.9(10^9)$ Mev/cm³. This is the amount that would be absorbed from an external flux providing the dosage

$$\left[1.9(10^9) \left(\frac{\text{Mev}}{\text{cm}^3 \text{ sec}} \right) \right] \left[1.6(10^{-6}) \left(\frac{\text{erg}}{\text{Mev}} \right) \right] \left[1 \left(\frac{\text{cm}^3}{\text{g}} \right) \right] \left[\frac{1}{83} \left(\frac{\text{rep g}}{\text{erg}} \right) \right] = \left[37 \left(\frac{\text{rep}}{\text{sec}} \right) \right].$$

The value used for β may be a little high. It would seem, however, that the self-absorbed portion of the gamma rays must have been the equivalent of at least 25 rep/sec.

SECTION 4

THE HEATING IN A SAMPLE

4.0 The example considered. As an example, there is calculated the heating to be expected in a long 1/4 inch diameter iron rod placed in an aluminum container inserted into an empty fuel channel in the flattened zone in the Hanford reactor. It is assumed there is 1/8-inch aluminum around the sample.

4.1 Heating due to externally generated gamma rays. The flux at the sample is: from the rods (case B ¶1.4), 940 rep/sec; from the moderator (¶2.1), 400 rep/sec; and from the liners and container (¶2.2), 400 rep/sec; total 1700 rep/sec. The heating is 83 erg/(sec g rep) or the equivalent of 0.014 watt/g.

4.2 Heating from internal capture. The dominant species in iron is Fe⁵⁶ which would give Fe⁵⁷, a stable isotope, by neutron capture. It is not necessary, therefore, to consider beta-decay in this case. The mass balance gives 8.73 millimass units of gamma rays or 8.1 Mev.²¹ The macroscopic absorption cross section given in the handbooks, 0.21 cm²/cm³, is assumed to be for this process. Then the power released by capture is

$$\left[1.8(10^{13}) \left(\frac{n}{\text{cm}^2 \text{ sec}} \right) \right] \left[0.21 \left(\frac{\text{cm}^2}{\text{cm}^3} \right) \right] \left[8.1 \left(\frac{\text{Mev}}{n} \right) \right] \left[1.6(10^{-6})(10^{-7}) \left(\frac{\text{Joules}}{\text{Mev}} \right) \right] = 4.8 \text{ (w/cm}^3\text{)}.$$

The gamma rays are of average energy 4.5 Mev²⁰ for which an absorption coefficient 0.18 is assumed. For a rod the absorption is 1.3βR (¶3.3) or

$$1.3 \times 0.18 \times 0.32 = 0.075.$$

The power absorbed is then

$$(4.8 \times 0.075)/7.8 = 0.046 \text{ watt/g.}$$

4.3 The total heating. Adding the above results, there is obtained 0.06 watt/g for the heating in this rod.

4.4 The temperature of the rod suspended in air. This is a small amount of heat to transfer to a liquid or solid. If, however, the rod is suspended in air out of contact with the walls of the container and without forced circulation, the temperature rise will be considerable. This may be calculated approximately by a method given by Strong²⁵ and originally due to Langmuir. In the range of interest the function designated φ by Strong is T²(10⁻⁷); hence, the absolute temperature of the rod surface T may be written

$$T = \sqrt{0.45(10^7) W + T_0^2}$$

where W is the heat to be dissipated, 0.018 cal/cm², and T₀ is the absolute temperature of the wall, taken as 300°K. Then T is 410°K, a temperature rise of over 100°C.

SECTION 5

DISCUSSION OF THE RESULTS

5.0 Assumptions and approximations involved. Because of the many approximations which had to be made, the numerical value of these results may not be good to even a factor of two. The major weakness of the calculations lies, of course, in the values chosen for β , the absorption coefficient. The manner of estimating the fast neutron ionization in Hart's sample is questionable but could not affect the results greatly. The existence of control rods and shim rods in the reactor has been completely neglected, and they should prove important sources of gamma rays. The assumption that all the beta ray, Compton, and pair electron energy is absorbed in the body is not always a good one but should not in general affect the results greatly. Despite these weaknesses, a number of important conclusions may be drawn from the results.

5.1 Nature of the CP-3 gamma flux. In the central thimble of CP-3 (original) most of the gamma rays originated in the rods. It seems likely that much of this flux consisted of gamma rays of 1 Mev and under from the reasonable result achieved by the use of the particular absorption coefficient employed.

5.2 Nature of the Hanford gamma flux. In the Hanford reactor on the other hand, using only a minimum of aluminum liners, a large fraction of the gamma-ray flux originates in the liners and moderator. Thus, the spectrum must consist of a large flux of high energy - over 1 Mev. The heat produced in a sample by absorption of these externally generated gamma rays ordinarily is small, amounting to at most only a few hundredths of a watt per gram of material. This is only a fraction of the amount of heat generated in a material with the neutron cross section of iron by internally generated gamma rays from neutron capture.

5.3 Application to controlled temperature experiments. The heat generated by absorption of gamma rays from the sources thus far considered will not account for the high temperatures observed in the NEPA creep experiment to which previous reference was made.³ The NEPA rod was mounted within several concentric shields consisting of stainless steel, ceramic, and Pt-Rh wire. Some of the temperature rise may be accounted for by the insulation introduced; but most of it must have been due to the enormous gamma-ray generation in these shields and the heavy end parts of the apparatus, yielding a gamma-ray flux of perhaps 20 - 50 times the value calculated in §2.2 for an aluminum liner. If gamma-ray flux and heating is a concern, it is essential to make a careful choice of containers and liners.

5.4 Application to radiation chemistry experiments. It is obvious that chemical studies (e.g., decomposition and corrosion) which depend on the gamma flux in a reactor must be interpreted in terms of the liners and vessels used and the location chosen. One must be careful in applying results obtained in locations like the recirculation loops used by the Naval Reactor Division²⁶ to a reactor as a whole, for the average gamma-ray flux in a reactor is quite unrelated to what one may find in one of these loops both with respect to flux and with respect to spectrum.

5.5 The gamma-ray converter. From the results reported here, it would seem possible to design gamma-ray irradiation facilities for the interior of a reactor which would permit the irradiation of a sample mainly in the capture gamma rays of the walls of the facility. This would be a gamma-ray "converter." Thus within an iron cylinder of 1/4-inch wall in a fuel channel at Hanford, one might expect to find a dosage rate of about 400 rep/sec from the rods, about 200 rep/sec from the graphite moderator, and about 10,000 rep/sec from the iron.

APPENDIX I

THE FUNCTIONS $\mathcal{J}_n(B)$

6.0 General. Problems involving the calculation of the emergence or absorption in bodies or media which absorb radiation in an exponential manner give rise to solutions which are the integrals of various expressions involving the exponential function. Thus, in this report the E and $\mathcal{J}$ functions were involved. The E functions are well known.^{22, 23, 24} The $\mathcal{J}$ functions are less familiar and have not usually been treated in the manner used here.

6.1 Definitions. The complete $\mathcal{J}$ functions may be defined as

$$\mathcal{J}_n(B) = \int_0^{\pi/2} e^{-B \sec \phi} \cos^n \phi \, d\phi, \quad (6.1.1)$$

and may be transformed by the substitution $x = \sec \phi$ to

$$\mathcal{J}_n(B) = \int_1^{\infty} \frac{e^{-Bx}}{x^{n+1} \sqrt{x^2-1}} \, dx \quad (6.1.2)$$

$$= \int_1^{\infty} x^{-n} e^{-Bx} \, dA \sec x. \quad (6.1.3)$$

Writing $z = \sin \phi$,

$$\mathcal{J}_n(B) = \int_0^1 e^{-[B/(z^2-1)^{1/2}]} (1-z^2)^{(n-1)/2} \, dz. \quad (6.1.4)$$

$\mathcal{J}_1(B)$ defined in this way is the function termed $J(B)$ by Seeliger and tabulated by him by graphical integration.²⁷ Writing $x = \cosh \theta$,

$$\mathcal{J}_n(B) = \int_0^{\infty} e^{-B \cosh \theta} \operatorname{sech}^{(n+1)} \theta \, d\theta, \quad (6.1.5)$$

which can be seen to be the definition of the functions $Ki_{(n+1)}(B)$ which have been tabulated.²⁸ The incomplete integral,

$$\mathcal{J}_0(B, \phi) = \int_0^\phi e^{-B \sec \theta} d\theta$$

is also tabulated.²⁹

6.2 Some properties of the functions $\mathcal{J}_n(x)$. From equation 6.1.1 it is evident that:

$$\mathcal{J}_n(\infty) = 0, \quad (6.2.1)$$

$$\mathcal{J}_n(0) = \frac{\sqrt{\pi}}{2} \frac{\Gamma[1/2(n+1)]}{\Gamma[1/2 n+1]}, \quad (6.2.2)$$

$$d\mathcal{J}_n(B)/dB = -\mathcal{J}_{(n-1)}(B). \quad (6.2.3)$$

The relation of these functions to the modified Bessel function of the second kind of order zero is seen from equation 6.1.5, for $\mathcal{J}_1(B)$ is $K_0(B)$. Then from equation 6.2.3, it is seen that the $\mathcal{J}_n(B)$ for n integer and > 0 are the $(n+2)$ -fold integrals of $K_0(B)$. Since

$$\begin{aligned} \frac{1}{x^{n+1}} &= \frac{1}{n!} \int_0^\infty t^n e^{-tx} dt, \\ \mathcal{J}_n(B) &= \frac{1}{n!} \int_1^\infty \frac{e^{-Bx}}{\sqrt{x^2-1}} dx \int_0^\infty t^n e^{-tx} dt \\ &= \frac{1}{n!} \int_0^\infty t^n dt \int_1^\infty \frac{e^{-(B+t)x}}{\sqrt{x^2-1}} dx \\ &= \frac{1}{n!} \int_0^\infty t^n \mathcal{J}_1(B+t) dt \\ &= \frac{1}{n!} \int_B^\infty (y-B)^n \mathcal{J}_1(y) dy. \end{aligned} \quad (6.2.4)$$

Thus, in particular for the two $\mathcal{J}$ functions of interest in this report:

$$\begin{aligned}
 \mathcal{J}_0(B) &= \int_B^{\infty} K_0(y) dy \\
 &= \int_0^{\infty} K_0(y) dy - \int_0^B K_0(y) dy \\
 &= \frac{\pi}{2} - \int_0^B K_0(y) dy
 \end{aligned} \tag{6.2.5}$$

which is convenient since the integral of equation 6.2.5 is tabulated;^{30, 31} and

$$\begin{aligned}
 \mathcal{J}_1(B) &= \int_B^{\infty} y K_0(y) dy - \int_B^{\infty} B K_0(y) dy \\
 &= \left[-y K_1(y) \right]_B^{\infty} - B \mathcal{J}_0(B) \\
 &= B \left[K_1(B) - \frac{\pi}{2} + \int_0^B K_0(y) dy \right]
 \end{aligned} \tag{6.2.6}$$

where $K_1(B)$ is a modified Bessel function of the second kind of order one. These expressions are clumsy for use in multiple integrals. Some approximations of value here are developed in the remaining paragraphs of this section. A recursion formula which has been given for these functions²⁸ is of little interest here since it involves four successive $\mathcal{J}$ functions.

6.3 An expansion for $\mathcal{J}_0(B)$ and $\mathcal{J}_1(B)$. An expansion follows from the binomial theorem:

$$\begin{aligned}
 \mathcal{J}_n(B) &= \int_1^{\infty} \frac{e^{-Bx}}{x^n} d \operatorname{Asec} x \\
 &= \int_1^{\infty} \frac{e^{-Bx}}{x^n} \left[\frac{1}{x^2} + \frac{1 \cdot 3}{2 \cdot 3x^4} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 5x^6} + \cdots \right] dx \\
 &= \int_1^{\infty} e^{-Bx} \left[\frac{1}{x^{(2+n)}} + \frac{1}{2x^{(4+n)}} + \frac{1 \cdot 3}{2 \cdot 4x^{(6+n)}} + \cdots \right] dx \\
 &= E_{(n+2)} + \frac{1}{2} E_{(n+4)} + \frac{1 \cdot 3}{2 \cdot 4} E_{(n+6)} + \cdots \\
 &= a_0 E_{(n+2)} + a_1 E_{(n+4)} + a_2 E_{(n+6)} + \cdots \\
 &= a_0 E_{(n+2)} + \sum_{m=1}^{\infty} \frac{(2m)!}{(2^m m!)^2} E_{(n+2m+2)} ,
 \end{aligned}$$

an expansion in E functions, which regrettably does not converge usefully in any range of the argument. A useful expansion follows by reducing the expansion to the first exponential integral

$$E_1(x) = \int_1^{\infty} \frac{e^{-Bx}}{x} dx$$

by means of the recursion formula $(n-1) E_n = e^{-B} - B E_{(n-1)}$

Then in particular for $\mathcal{J}_0(B)$,

$$\mathcal{J}_0(B) = a_0 E_2 + a_1 E_4 + a_2 E_6 + \cdots ,$$

$$E_2 = e^{-B} (1) - \frac{B}{1} E_1 ,$$

$$E_4 = e^{-B} \left(\frac{1}{3} - \frac{B}{3 \cdot 2} + \frac{B^2}{3 \cdot 2 \cdot 1} \right) - \frac{B^3}{3!} E_1 ,$$

$$E_6 = e^{-B} \left(\frac{1}{5} - \frac{B}{5 \cdot 4} + \frac{B^2}{5 \cdot 4 \cdot 3} - \frac{B^3}{5 \cdot 4 \cdot 3 \cdot 2} + \frac{B^4}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} \right) - \frac{B^5}{5!} E_1 ,$$

etc. Hence, it is easy to see that

$$f_0(B) = -E_1(B) \left[a_0 B + \frac{a_1}{3!} B^3 + \frac{a_2}{5!} B^5 + \dots \right] \\ + e^{-B} \left[a_0 + a_1 \left(\frac{1}{3} - \frac{B}{3 \cdot 2} + \frac{B^2}{3 \cdot 2 \cdot 1} \right) + a_2 \left(\frac{1}{5} - \frac{B}{5 \cdot 4} + \frac{B^2}{5 \cdot 4 \cdot 3} - \frac{B^3}{5 \cdot 4 \cdot 3 \cdot 2} + \frac{B^4}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} \right) + \dots \right]$$

Since $E_1 = -\gamma - \ln B + \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m \cdot m!} B^m$ (where $\gamma = 0.57715\dots$), the first term can be seen to converge with startling rapidity for $B < 1$. The exponential term can also be arranged in ascending powers of B . The zero-order term is:

$$a_0 + \frac{1}{3} a_1 + \frac{1}{5} a_2 + \dots = 1 + \sum_1^{\infty} \frac{a_m}{2m+1}$$

which is recognized as $\frac{\pi}{2}$.³² The first-order terms are:

$$-B \left[\frac{a_1}{3 \cdot 2} + \frac{a_2}{5 \cdot 4} + \frac{a_3}{7 \cdot 6} + \dots \right] = -B \sum_1^{\infty} \frac{a_m}{(2m+1)(2m)}$$

Now

$$\sum_1^{\infty} \frac{a_m}{(2m+1) 2m} = \sum_1^{\infty} a_m \left(\frac{1}{2m} - \frac{1}{2m+1} \right)$$

The first series is evaluated from

$$\text{Acosh } \frac{1}{x} = \ln \frac{2}{x} - \sum_1^{\infty} \frac{(2m)!}{(2^m m!)^2} \frac{x^{2m}}{2m}$$

$$\text{Acosh } 1 = 0 = \ln 2 - \sum_1^{\infty} \frac{a_m}{2m}$$

Hence the first series is $\ln 2$; the second has already been evaluated. The first-order terms then become

$$-B \left(\ln 2 - \frac{\pi}{2} + 1 \right)$$

The second-order terms are:

$$+B^2 \left(\frac{a_1}{3 \cdot 2 \cdot 1} + \frac{a_2}{5 \cdot 4 \cdot 3} + \frac{a_3}{7 \cdot 6 \cdot 5} + \dots \right)$$

$$= \sum_1^{\infty} \frac{a_m}{(2m+1)(2m)(2m-1)} = \sum_1^{\infty} a_m \left(\frac{1/2}{2m+1} + \frac{1/2}{2m-1} - \frac{1}{2m} \right).$$

Now

$$\sum_1^{\infty} \frac{a_m}{4m^2-1} = 1 - \frac{\pi}{4} = \sum_1^{\infty} a_m \left(\frac{1/2}{2m-1} - \frac{1/2}{2m+1} \right).$$

Then

$$1 - \frac{\pi}{4} = \frac{1}{2} \sum \frac{a_m}{2m-1} - \frac{1}{2} \sum \frac{a_m}{2m+1} = \frac{1}{2} \sum \frac{a_m}{2m-1} - \frac{1}{2} \left(\frac{\pi}{2} - 1 \right),$$

and

$$\frac{1}{2} \sum \frac{a_m}{2m-1} = \frac{1}{2}.$$

Hence, the second-order term is $+B^2 \left(\frac{\pi}{4} - \frac{1}{2} + \frac{1}{2} - \ln 2 \right) = +B^2 \left(\frac{\pi}{4} - \ln 2 \right)$.

The third-order terms are

$$-B^3 \left(\frac{a_2}{5 \cdot 4 \cdot 3 \cdot 2} + \frac{a_3}{7 \cdot 6 \cdot 5 \cdot 4} + \frac{a_4}{9 \cdot 8 \cdot 7 \cdot 6} + \dots \right) = -B^3 \sum_2^{\infty} \frac{(2m-3)! a_m}{(2m+1)!}$$

which converges quite rapidly. The sum of the first six terms of the series is about $-0.00364 B^3$. The fourth-order terms are

$$+B^4 \sum_2^{\infty} \frac{(2m-4)! a_m}{(2m+1)!} \cong 0.00327 B^4.$$

Then one may write

$$J_0(B) = e^{-B} \left[\frac{\pi}{2} - \left(1 + \ln 2 - \frac{\pi}{2} \right) B + \left(\frac{\pi}{4} - \ln 2 \right) B^2 - \dots \right]$$

$$-BE_1(B) \left[1 + \frac{B^2}{2 \cdot 3!} + \frac{1 \cdot 3 B^4}{2 \cdot 4 \cdot 5!} + \dots \right]$$

$$\cong e^{-B} \left[1.571 - 0.1224B + 0.0923B^2 - 0.0036B^3 + 0.0033B^4 - \dots \right]$$

$$-BE_1(B) \left[1 - 0.0833B^2 + 0.00313B^4 + \dots \right],$$

The functions e^{-B} and $E_1(B)$ can be expanded and their respective series multiplied out to give a useful approximation for $\mathcal{J}_0(B)$ for $B < 1$:

$$\mathcal{J}_0(B) \cong (B + 0.0833B^3) \ln B + 1.571 - 1.116B - 0.127B^3 + 0.054B^4 \text{ or}$$

$$\mathcal{J}_0(B) \cong 1.571 - (B + 0.0833B^3) \ln \frac{1}{B} - 1.116B - 0.127B^3 + 0.054B^4 .$$

It is found that the second-order terms are nil. The first expression may be integrated to get $\mathcal{J}_1(B)$:

$$\mathcal{J}_1(B) \cong 1 - 1.571B + (0.5B^2 + 0.021B^4) \ln \frac{1}{B} + 0.808B^2 + 0.095B^4 .$$

There are no third-order terms here.

6.4 Second-order approximations for $\mathcal{J}_0(B)$ and $\mathcal{J}_1(B)$ for small values of B . From §6.3

$$\mathcal{J}_0(B) \cong \frac{\pi}{2} - B \ln \frac{1}{B} - 1.116B \quad (1a)$$

$$\mathcal{J}_1(B) \cong 1 - \frac{\pi}{2} B + \frac{1}{2} B^2 \ln \frac{1}{B} + 0.808B^2 . \quad (1b)$$

These expressions are rather cumbersome for some purposes. A more convenient set of expressions may be obtained by observing that in the region where the second-order term begins to make an appreciable contribution to the function $\mathcal{J}_1(B)$, an average value of $\frac{1}{2} \ln \frac{1}{B}$ is about 0.763. Then one may write

$$\mathcal{J}_1(B) \cong 1 - \frac{\pi}{2} B + \frac{\pi}{2} B^2 , \quad (2b)$$

and by differentiation

$$\mathcal{J}_0(B) \cong \frac{\pi}{2} (1 - 2B) . \quad (2a)$$

The usefulness of these approximations is indicated in Table II.

Table II

COMPARISON OF APPROXIMATIONS TO THE FUNCTIONS $\mathcal{J}_n(B)$

B	$\mathcal{J}_0(B)$	1a	2a	$\mathcal{J}_1(B)$	1b	2b
0.01		1.514	1.54			
0.05	1.3652	1.365	1.41	0.9272	0.925	0.925
0.1	1.2286	1.229	1.26	0.8625	0.863	0.859
0.15	1.1179	1.119	1.10	0.8039	0.806	0.799
0.2	1.0237	1.027	0.943	0.7505	0.751	0.749
0.3	0.8688	0.876	0.647	0.6561	0.661	0.670
0.5	0.6437	0.667		0.5064	0.511	0.606
0.75				0.371	0.381	0.71

It is seen that (2b) is a good approximation (better than 2%) up to $B = 0.3$. The expression (2a) is less useful. It is good to better than 2% up to about $B = 0.01$ and from $B = 0.1$ to 0.2 , but in the range $B = 0.01$ to 0.1 it is not good to better than 3.5%.

6.5 Approximations for $\mathcal{J}_0(B)$ and $\mathcal{J}_1(B)$ at large values of B . At large values of B the exponential character of the functions dominates, hence their curves drawn on semilogarithmic paper approach straight lines which lie below the graph of the function e^{-B} because of the denominators. If one draws at the end of the range of interest a tangent to the functions $\mathcal{J}_0(B)$ or $\mathcal{J}_1(B)$ plotted on semilogarithmic paper and plots the differences between the tangent and the function on the same paper, one obtains a curve which is nearly a straight line to rather small values of the argument. Thus, it is possible to approximate the functions by the sum of two exponentials. This was done, drawing the tangent at $B = 5$, with the following results:

$$\mathcal{J}_0(B) \cong 0.85 e^{-1.09B} + 0.72 e^{-3.27B}$$

$$\mathcal{J}_1(B) \cong 0.700 e^{-1.08B} + 0.283 e^{-2.17B}$$

The values used for the first function, $\mathcal{J}_0(B)$, were taken from a graph calculated by P. Morrison and which was certainly somewhat wrong at small values of the argument, while the values of the second, $\mathcal{J}_1(B)$, were taken from §6.2. These approximations are best near the point of tangency and

poorest at $B = 0$. The ordinates given by these approximations are good to several per cent or better; hence, when they are used for integration (for which their form is very convenient), the results should be at least this good. However, they do not give quite the correct slopes, for $-\frac{\partial \mathcal{J}_1(B)}{\partial B}$ should be $\mathcal{J}_0(B)$, and it is not quite so in these approximations.

APPENDIX II

SEVERAL APPLICATIONS OF THE EXPONENTIAL ABSORPTION LAW

7.0 Lambert's law. If a body generates uniformly and isotropically, then one can see into it to the "effective depth,"

$$\int_0^{\infty} e^{-B\rho} d\rho = \frac{1}{\beta}$$

The distance of the logarithmic decrement. At a given angle (see Figure 7) one sees the surface area proportional to $R d\theta \sec \theta$. The generating volume which one sees is proportional to $\frac{1}{\beta} \cos \theta$. Hence, the brightness observed is proportional to

$$R d\theta \sec \theta \frac{1}{\beta} \cos \theta = \frac{R}{\beta} d\theta$$

and the surface obeys Lambert's law. Since for gamma rays β is < 0.5 for most things of interest and generally is < 0.2 , the law would apply only to bodies many centimeters thick (several logarithmic decrements).^{*} From the previous discussion of the cylindrical shell (§ 2.2) it is obvious that if Lambert's law applied, the flux would be uniform throughout the interior, and indeed also for any shape consisting of parallel elements; it is less obvious that this would be true for other shapes. It is now clear that for gamma rays this is true only for a small hole in a large generating body.

7.1.1 Flux due to self-generation in a spherical body neglecting self-absorption. The flux at a point "P" within a spherical body (Figure 8) is:

$$i = \frac{s}{4\pi} \int_0^{2\pi} \int_0^{\pi} \int_0^z \frac{\rho^2 d\rho}{\rho^2} \sin \theta d\theta d\phi$$

^{*}This view of the cosine law may be extended to a small hole in a cavity emitting particles where it would hold if the cavity were large compared to the mean free path of the particle. A hole in a cavity emitting light would behave in this manner if the interior walls did. The behavior of a diffuse reflector is not so obvious since the particles of the body are not illuminated isotropically and uniformly and hence do not emit isotropically and uniformly.

FIGURE 7.

THE HIGHLY ABSORBING WALL.

FIGURE 8.

THE SPHERICAL BODY.

It can be seen from Figure 9 that from 0 to $\frac{\pi}{2}$:

$$z = \sqrt{R^2 - r^2 \sin^2 \theta} + r \cos \theta = \psi + \xi,$$

and from $\frac{\pi}{2}$ to π :

$$z' = \sqrt{R^2 - r^2 \sin^2 \theta'} - r \cos \theta' = \psi' - \xi.$$

Since the integral is a function of the limits only,

$$i = \frac{s}{2} 2 \int_0^{\pi/2} \psi \sin \theta d\theta,$$

a result of the direction of origin of the flux being immaterial.

$$\begin{aligned} i &= s \int_0^{\pi/2} \sqrt{R^2 - r^2 + r^2 \cos^2 \theta} \sin \theta d\theta \\ &= sr \int_0^{\pi/2} \sqrt{\frac{R^2 - r^2}{r^2} + \cos^2 \theta} \sin \theta d\theta \\ &= \frac{s}{2} \left[R + \frac{R^2 - r^2}{2r} \ln \frac{R + r}{R - r} \right]. \end{aligned}$$

Let $y = \frac{r}{R},$

$$i = \frac{sR}{2} \left[1 + \frac{1-y^2}{2y} \ln \frac{1+y}{1-y} \right].$$

At the center and at the periphery this expression is indeterminate; hence applying L'Hospital's rule

$$\begin{aligned} \lim_{y \rightarrow 0,1} i &= \lim_{y \rightarrow 0,1} \frac{sR}{2} \left[\left(1 + \frac{d}{dy} \ln \frac{1+y}{1-y} \right) / \frac{d}{dy} \frac{2y}{1-y^2} \right] \\ &= \lim_{y \rightarrow 0,1} \frac{sR}{2} \left[1 + \frac{1-y^2}{1+y^2} \right]. \end{aligned}$$

FIGURE 9.

THE "z" FUNCTION.

Therefore, at the center the flux is $i_0 = sR$, and at the periphery $i_R = \frac{1}{2} sR$. The average value of the flux is

$$\begin{aligned}
 i_{\text{avg}} &= \frac{3}{4\pi R^3} \int_0^{2\pi} \int_0^\pi \int_0^R \frac{sR}{2} \left[1 + \frac{1-y^2}{2y} \ln \frac{1+y}{1-y} \right] r^2 dr \sin \theta d\theta d\phi \\
 &= \frac{3sR}{2} \int_0^1 \left[1 + \frac{1-y^2}{2y} \ln \frac{1+y}{1-y} \right] y^2 dy \\
 &= \frac{sR}{2} \left\{ 1 + \frac{3}{2} \left[\left(\frac{y^2}{2} - \frac{y^4}{4} \right) \ln \frac{1+y}{1-y} - \int \left(\frac{y^2}{2} - \frac{y^4}{4} \right) \frac{2 dy}{1-y^2} \right]_0^1 \right\} \\
 &= \frac{sR}{2} \left\{ 1 + \frac{3}{2} \left[\left(\frac{y^2}{2} - \frac{y^4}{4} \right) \ln \frac{1+y}{1-y} - \frac{1}{2} \int \left(\frac{1}{1-y^2} - 1 + y^2 \right) dy \right]_0^1 \right\} \\
 &= \frac{sR}{2} \left\{ 1 + \frac{3}{2} \left[\left(\frac{y^2}{2} - \frac{y^4}{4} \right) \ln \frac{1+y}{1-y} - \frac{1}{4} \ln \frac{1+y}{1-y} + \frac{y}{2} - \frac{y^3}{6} \right]_0^1 \right\} \\
 &= \frac{sR}{2} \left[1 + \frac{3}{4} - \frac{1}{4} \right] = 0.75 sR .
 \end{aligned}$$

This is the value of the flux at about $0.656 R$. This value may be used to obtain the value of the absorption of the body which is, thus (to first-order terms), $0.75\beta R$. It is not surprising that the value is exactly that of § 3.2 since the mathematical operations performed to obtain the present result are the same as those used in the earlier section although the philosophy of the methods is quite different.

7.1.2 Flux due to uniform isotropic self-generation in the spherical body possessing self-absorption. The flux at the point P is now

$$\begin{aligned}
 i &= \frac{s}{2} \int_0^\pi \int_0^z e^{-\beta \rho} d\rho \sin \theta d\theta \\
 &= \frac{s}{2\beta} \int_0^\pi (1 - e^{-\beta z}) \sin \theta d\theta .
 \end{aligned}$$

If $\beta R \ll 1$ (the absorption is very small),

$$\begin{aligned} i &= \frac{s}{2\beta} \int_0^\pi (1 - 1 + \beta z - \frac{1}{2} \beta^2 z^2) \sin \theta \, d\theta \\ &= \frac{s}{2} \int_0^\pi (z - \frac{1}{2} \beta z^2) \sin \theta \, d\theta. \end{aligned}$$

Now as in §7.1.1,

$$\begin{aligned} z \Big|_0^\pi &= (\psi - \xi) \Big|_0^{\pi/2} + (\psi + \xi) \Big|_0^{\pi/2} = 2\psi \Big|_0^{\pi/2} \\ z^2 \Big|_0^\pi &= (\psi - \xi)^2 \Big|_0^{\pi/2} + (\psi + \xi)^2 \Big|_0^{\pi/2} \\ &= (\psi^2 + \xi^2 - 2\psi\xi) \Big|_0^{\pi/2} + (\psi^2 + \xi^2 + 2\psi\xi) \Big|_0^{\pi/2} \\ &= 2(\psi^2 + \xi^2) \Big|_0^{\pi/2}. \end{aligned}$$

Hence

$$i = s \int_0^{\pi/2} \left[\psi - \frac{1}{2} \beta (\psi^2 + \xi^2) \right] \sin \theta \, d\theta.$$

The first term of the integral is the solution obtained in §7.1.1. The second term is:

$$-\frac{\beta s}{2} \int_0^{\pi/2} (R^2 - r^2 \sin^2 \theta + r^2 \cos^2 \theta) \sin \theta \, d\theta$$

$$\begin{aligned}
&= -\frac{\beta s}{2} \left[R^2 + r^2 \int_0^{\pi/2} (2 \cos^2 \theta - 1) \sin \theta \, d\theta \right] \\
&= -\frac{\beta s}{2} \left(R^2 - r^2 + \frac{2}{3} r^2 \right) = -\frac{\beta s}{2} \left(R^2 - \frac{1}{3} r^2 \right) .
\end{aligned}$$

The total solution is

$$i = \frac{sR}{2} \left[1 - \beta R \left(1 - \frac{y^2}{3} \right) + \frac{1 - y^2}{2y} \ln \frac{1+y}{1-y} \right] .$$

At the center this is seen to be $i_0 = sR (1 - \frac{1}{2} \beta R)$, while at the periphery $i_R = \frac{sR}{2} (1 - \frac{2}{3} \beta R)$. Thus, the absorption is relatively greater at the periphery, as it should be, for the rays reaching such a point must traverse, on the average, a longer path thru the material. The average value of the first terms has been computed. The average value of the last is

$$\begin{aligned}
&= -\frac{3sR}{2} \beta R \int_0^1 \left(1 - \frac{y^2}{3} \right) y^2 \, dy \\
&= -\frac{3sR}{2} \beta R \left[\frac{y^3}{3} - \frac{y^5}{15} \right]_0^1 = sR \, 0.4 \, \beta R .
\end{aligned}$$

The average value of the flux is then

$$i_{\text{avg}} = sR (0.75 - 0.4 \beta R)$$

which is the flux a very little closer to the center than for the sphere without self-absorption.

7.1.3 Flux in the hollow interior bounded by the generating spherical shell possessing no self-absorption. One may refer to Figure 10 and to the previous paragraphs for the nomenclature.

$$\begin{aligned}
i &= \frac{s}{4\pi} \int_0^{2\pi} \int_0^{\pi} \int_{z_1}^{z_2} \frac{\rho^2 \, d\rho \, \sin \theta}{\rho^2} \, d\theta \, d\phi \\
&= \frac{s}{2} \int_0^{\pi} (z_2 - z_1) \sin \theta \, d\theta
\end{aligned}$$

FIGURE 10.

THE $(z_2 - z_1)$ FUNCTION.

Since

$$\begin{aligned} (z_2 - z_1) \int_0^{\pi} &= (\psi_2 + \xi - \psi_1 - \xi) \int_0^{\pi/2} + (\psi_2 - \xi - \psi_1 + \xi) \int_0^{\pi/2} \\ &= 2(\psi_2 - \psi_1) \int_0^{\pi/2}, \end{aligned}$$

then

$$\begin{aligned} i &= s \int_0^{\pi/2} \left(\sqrt{R_2^2 - r^2 \sin^2 \theta} - \sqrt{R_1^2 - r^2 \sin^2 \theta} \right) \sin \theta \, d\theta \\ &= \frac{s}{2} \left\{ R_2 \left[1 + \frac{1-y_2^2}{2y_2} \ln \frac{1+y_2}{1-y_2} \right] - R_1 \left[1 + \frac{1-y_1^2}{2y_1} \ln \frac{1+y_1}{1-y_1} \right] \right\}, \end{aligned}$$

where $y_2 R_2 = r = y_1 R_1$. Let $T = R_2 - R_1$, then

$$i = \frac{s}{2} \left[T + \frac{1-y_2^2}{2y_2} R_2 \ln \frac{1+y_2}{1-y_2} - \frac{1-y_1^2}{2y_1} R_1 \ln \frac{1+y_1}{1-y_1} \right].$$

At $y = 0$ the expression becomes indeterminate. Applying L'Hospital's rule,

$$\lim_{y \rightarrow 0} i_0 = \lim_{y \rightarrow 0} \frac{s}{2} \left[T + R_2 \frac{1-y_2^2}{1+y_2^2} - R_1 \frac{1-y_1^2}{1+y_1^2} \right] = sT.$$

At the inner wall the third term becomes indeterminate, but on applying L'Hospital's rule it can be seen to vanish. The flux at the inner wall is then:

$$\begin{aligned} i_{R_1} &= \frac{s}{2} \left[(R_2 - R_1) + R_2 \frac{1-y_2^2}{2y_2} \ln \frac{1-y_2}{1+y_2} \right] \\ &= \frac{1}{2} s R_2 (1-y) \left[1 + \frac{1+y}{2y} \ln \frac{1-y}{1+y} \right], \text{ (where } y \equiv y_2); \\ &= \frac{1}{2} s T \left(1 + \frac{1+y}{2y} \ln \frac{1-y}{1+y} \right) = \frac{1}{2} s T \left[1 + \frac{1+y}{y} \operatorname{Atanh} y \right]. \end{aligned}$$

Its ratio to the central flux, $i_0 = s R_2 (1 - y)$, is

$$i_{R_1}/i_0 = \frac{1}{2} \left(1 + \frac{1+y}{2y} \ln \frac{1-y}{1+y} \right) = \frac{1}{2} \left(1 + \frac{1+y}{y} \operatorname{Atanh} y \right).$$

For useful shells $y \sim 1$ and hence,

$$i_R/i_0 \approx \frac{1}{2} (1 + 2 \operatorname{Atanh} y); \quad y = R_1/R_2; \quad R_2 = R_1 + T$$

which is 1.5 to 3.5 times the central flux. The average value of the flux is

$$i_{\text{avg}} = 3 \frac{4\pi s}{4\pi R_1^3} \frac{s}{2} \int_0^{R_1} \left\{ R_2 \left[1 + \frac{1-y_2^2}{2y_2} \ln \frac{1+y_2}{1-y_2} \right] - R_1 \left[1 + \frac{1-y_1^2}{2y_1} \ln \frac{1+y_1}{1-y_1} \right] \right\} r^2 dr$$

Since $y_1 R_1 = r = y_2 R_2$, this becomes

$$\begin{aligned} i_{\text{avg}} &= \frac{3s}{2R_1^3} \left\{ R_2^4 \int_0^{R_1/R_2} \left(1 + \frac{1-y_2^2}{2y_2} \ln \frac{1+y_2}{1-y_2} \right) y_2^2 dy \right. \\ &\quad \left. - R_1^4 \int_0^1 \left(1 + \frac{1-y_1^2}{2y_1} \ln \frac{1+y_1}{1-y_1} \right) y_1^2 dy \right\} \\ &= \frac{3s}{2R_1^3} \left\{ R_2^4 \left\{ \frac{y_2^3}{3} + \frac{1}{2} \left[\left(\frac{y_2^2}{2} - \frac{y_2^4}{4} - \frac{1}{4} \right) \ln \frac{1+y_2}{1-y_2} + \frac{y_2}{2} - \frac{y_2^3}{6} \right] \right\} \right. \\ &\quad \left. - \frac{R_1^4}{2} \right\} \end{aligned}$$

Writing y for R_1/R_2 this becomes

$$i_{\text{avg}} = \frac{3s}{2R_1^3} \left\{ R_2^4 \left[\frac{y^3}{4} + \frac{y}{4} + \frac{1}{2} \left(\frac{y^2}{2} - \frac{y^4}{4} - \frac{1}{4} \right) \ln \frac{1+y}{1-y} \right] - \frac{R_1^4}{2} \right\}.$$

Either of the following equivalent assumptions may be used to obtain a first-order approximation:

$$y^n = (1 - \epsilon)^n \approx 1 - n\epsilon, \text{ or } (R+T)^n \approx R^n + nR^{n-1} T.$$

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Both yield good results even for wall thickness $0.2R$. The former is the more convenient. Then the multiplier of the logarithm becomes

$$\begin{aligned} \frac{1}{4}(1 - 2\epsilon) - \frac{1}{8}(1 - 4\epsilon) - \frac{1}{8} &= 0, \text{ and} \\ i_{\text{avg}} &= \frac{3s}{2} R_2 \left[\frac{1}{y^3} \left(\frac{y^3}{4} + \frac{y}{4} \right) - \frac{y}{2} \right] = \frac{3s}{2} \frac{R_2}{4y^2} (1 + y^2 - 2y^3) \\ &= \frac{3s}{2} \frac{R_2}{4y^2} (1 + 1 - 2\epsilon - 2 + 6\epsilon) = \frac{3s}{2} \frac{R_2 \epsilon}{y^2} . \end{aligned}$$

Since $\epsilon R_2 = R_2 (1 - y) = T$,

$$i_{\text{avg}} = \frac{3}{2} s \frac{T}{y^2} ,$$

which is somewhat greater than $\frac{3}{2}$ central flux.

7.1.4 Flux in a generating spherical shell neglecting self-absorption.

From Figure 8 it can be seen that the flux at $\underline{r}$ in the shell is

$$\begin{aligned} i &= \frac{s}{4\pi} \int_0^{2\pi} \int_0^\pi \int_0^u \frac{1}{\rho^2} \rho^2 d\phi \sin \theta d\theta d\phi \\ &= \frac{s}{2} \int_0^\pi u \sin \theta d\theta . \end{aligned}$$

From Figure 11,

$$\begin{aligned} u \Big|_0^\pi &= z_1 \Big|_0^{\pi/2} + z_2 \Big|_0^{\text{Asin } \frac{R_1}{r}} + (z_4 - z_3) \Big|_0^{\text{Asin } \frac{R_1}{r}} + z_4 \Big|_0^{\pi/2} \text{Asin } \frac{R_1}{r} \\ &= z_1 \Big|_0^{\pi/2} + z_4 \Big|_0^{\pi/2} - (z_3 - z_2) \Big|_0^{\text{Asin } \frac{R_1}{r}} . \end{aligned}$$

FIGURE II.

THE "z" FUNCTIONS
IN THE SHELL.

Since $z_1 = \psi_1 - \xi$, $z_2 = \xi - \psi_2$, $z_3 = \psi_2 + \xi$, $z_4 = \psi_4 + \xi$, where ψ_n refers to the point z_n , and since $\psi_1 = \psi_4$, then

$$u \Big|_0^\pi = 2\psi_1 \Big|_0^{\pi/2} - 2\psi_2 \Big|_0^{\pi/2} \quad \text{Asin } \frac{R_1}{r}$$

$$\text{Hence, } i = s \int_0^{\pi/2} \sqrt{R_2^2 - r^2 \sin^2 \theta} \sin \theta d\theta - \int_0^{\text{Asin } \frac{R_1}{r}} \sqrt{R_1^2 - r^2 \sin^2 \theta} \sin \theta d\theta.$$

The first integral is the result given in §7.1.1. The second integral may be transformed in the same manner used for the first by letting $\cos \theta = w$. Then it becomes

$$\begin{aligned} & r \int_{\frac{\sqrt{r^2 - R_1^2}}{r}}^1 \sqrt{\frac{R_1^2 - r^2}{r^2} + w^2} dw \\ &= \frac{r}{2} \left[w \sqrt{\frac{R_1^2 - r^2}{r^2} + w^2} + \frac{R_1^2 - r^2}{r^2} \ln \left(w + \sqrt{\frac{R_1^2 - r^2}{r^2} + w^2} \right) \right]_1^{\frac{\sqrt{r^2 - R_1^2}}{r}} \\ &= \frac{r}{2} \left[\frac{R_1}{r} + \frac{R_1^2 - r^2}{r^2} \ln \frac{r + R_1}{\sqrt{r^2 - R_1^2}} \right] = \frac{r}{2} \left[\frac{R_1}{r} + \frac{1}{2} \left(\frac{R_1^2}{r^2} - \frac{r^2}{r^2} \right) \ln \frac{1 + \frac{R_1}{r}}{1 - \frac{R_1}{r}} \right]. \end{aligned}$$

Let $x = \frac{R_1}{r}$. Then the expression becomes

$$\frac{r}{2} \left[x - (1 - x^2) \text{Atanh } x \right].$$

Then, writing $y = \frac{r}{R_2}$, the flux at $\underline{r}$ is

$$i = \frac{sR_2}{2} \left[1 + \frac{1 - y^2}{y} \text{Atanh } y \right] - \frac{sr}{2} \left[x - (1 - x^2) \text{Atanh } x \right].$$

At the inner wall $x = 1$, $r = R_1$, $y = \frac{R_1}{R_2} \equiv y_0$, hence,

$$i_{R_1} = \frac{sT}{2} \left[1 + \frac{1+y_0}{y_0} \operatorname{Atanh} y_0 \right].$$

At the outer wall $r = R_2$, $x = y_0$, hence

$$i_{R_2} = \frac{sT}{2} \left[1 + (1+y_0) \operatorname{Atanh} y_0 \right].$$

Thus the flux has its maximum value at the inner wall. In a thin shell ($y_0 = 0.8$ to 0.98) the flux is nearly constant, and since $y_0 = 1 - \epsilon$ is near unity, one may write for the average flux

$$i_{\text{avg}} \cong \frac{sT}{2} \left(1 + \frac{1+y}{\sqrt{y}} \right) \operatorname{Atanh} y = \frac{sT}{2} \left[1 + \frac{2 - \epsilon}{1+0.5\epsilon} \operatorname{Atanh} (1 - \epsilon) \right].$$

In the range $y_0 = 0.8$ to 0.98 the average flux varies from about $1.6 sT$ to $2.6 sT$. The absorption of the shell (in terms of the half thickness) varies from $3.2 \beta \frac{T}{2}$ to $5.6 \beta \frac{T}{2}$.

7.2.1 Flux due to self-generation in the infinite cylinder of small self-absorption. The nomenclature is given in Figure 12 and in the paragraphs on the sphere.

$$i = \frac{s}{\pi} \int_0^\pi \int_0^{\pi/2} \int_0^{z(\phi) \sec \theta} \frac{1}{\rho^2} e^{-\beta \rho} \rho^2 d\rho \cos \theta d\theta d\phi$$

$$= \frac{s}{\pi \beta} \int_0^\pi \int_0^{\pi/2} \left(1 - e^{-\beta z(\phi) \sec \theta} \right) \cos \theta d\theta d\phi$$

$$= \frac{s}{\pi \beta} \int_0^\pi \left[1 - \mathcal{J}_1(\beta z) \right] d\phi.$$

$$\mathcal{J}_1(\beta z) \cong 1 - \frac{\pi}{2} \beta z + \frac{\pi}{2} \beta^2 z^2 \text{ for } \beta z \text{ small.}$$

$$i = \frac{s}{2} \int_0^\pi (z + \beta z^2) d\phi.$$

FIGURE 12.

THE SOLID CYLINDER.

The behavior of $z(\phi)$ is identical with that of $z(\theta)$ shown in §7.1.2; hence,

$$i = s \int_0^{\pi/2} \sqrt{R^2 - r^2 \sin^2 \phi} \, d\phi - \beta s \int_0^{\pi/2} (R^2 - r^2 \sin^2 \phi + r^2 \cos^2 \phi) \, d\phi .$$

To facilitate the expression of the remaining results, Jahnke and Emde's notation³³ is employed:

$$E = \int_0^{\pi/2} \sqrt{1 - y^2 \sin^2 \phi} \, d\phi$$

$$B = \int_0^{\pi/2} \frac{\cos^2 \phi}{\sqrt{1 - y^2 \sin^2 \phi}} \, d\phi$$

This E should not be confused with the notation used for the exponential integrals nor the B with the notation used for their argument in several parts of this report. Letting $r = yR_1$ as before,

$$i = sR \left[\int_0^{\pi/2} \sqrt{1 - y^2 \sin^2 \phi} \, d\phi - \beta R \int_0^{\pi/2} (1 - y^2 \sin^2 \phi + y^2 \cos^2 \phi) \, d\phi \right]$$

$$= sR \left(E - \frac{\pi}{2} \beta R \right) .$$

From the tables given by Jahnke and Emde:

$$i_0 = \frac{\pi}{2} sR (1 - \beta R) ,$$

and

$$i_R = sR \left(1 - \frac{\pi}{2} \beta R \right) .$$

The average flux is

$$i_{\text{avg}} = \frac{2\pi}{\pi R^2} \int_0^R sR \left(E - \frac{\pi}{2} \beta R \right) r \, dr$$

$$\begin{aligned}
&= sR \left[\int_0^1 E \, 2y \, dy - \frac{\pi}{2} \beta R \right] \\
&= sR \left[\frac{2}{3} y^2 (E+B) - \frac{\pi}{2} \beta R \right]_{y^2=0}^1 \\
&= sR \left[\frac{4}{3} - \frac{\pi}{2} \beta R \right] = sR \frac{\pi}{2} (0.849 - \beta R) ,
\end{aligned}$$

which is the flux at about $0.72 \underline{R}$. This gives the first-order approximation for the absorption of the body as $1.33 \beta R$, which is the same as that obtained previously by another method (§3.3).

7.2.2 Flux in the hollow interior bounded by the generating cylindrical shell possessing no self-absorption. Using the nomenclature of the preceding paragraphs,

$$\begin{aligned}
i &= \frac{s}{\pi} \int_0^\pi \int_0^{\pi/2} \int_{z_1(\phi) \sec \theta}^{z_2(\phi) \sec \theta} d\rho \cos \theta \, d\theta \, d\phi \\
&= \frac{s}{\pi} \int_0^\pi \int_0^{\pi/2} (z_2 - z_1) \sec \theta \cos \theta \, d\theta \, d\phi \\
&= \frac{s}{2} \int_0^\pi (z_2 - z_1) \, d\phi .
\end{aligned}$$

The properties of $\underline{z}$ have been described previously, and therefore

$$i = s \int_0^{\pi/2} \left[\psi_2(\phi) - \psi_1(\phi) \right] d\phi = s (R_2 E_2 - R_1 E_1) .$$

At the center $E_2 = E_1 = \frac{\pi}{2}$; hence $i_0 = \frac{\pi}{2} sT$.

$$\begin{aligned}
\text{At the wall } i_{R_1} &= s \left[R_2 E\left(\frac{R_1}{R_2}\right) - R_1 E(1) \right] \\
&= s R_2 \left[E(y) - y \right] \\
&= s T \frac{E(y) - y}{1 - y} \text{ or } s T \left(1 + \frac{E(y) - 1}{1 - y} \right),
\end{aligned}$$

which for practical tubes ($y = 0.8$ to 0.98) ranges from 1.5 to 2.25 times the central flux. The average flux is

$$i_{\text{avg}} = \frac{2 \pi s}{\pi R_1^2} \int_0^{R_1} (R_2 E_2 - R_1 E_1) r \, dr.$$

Since $r = R_1 y_1 = R_2 y_2$,

$$\begin{aligned}
i_{\text{avg}} &= \frac{s}{R_1^2} \left[R_2^3 \int_0^{\frac{R_1}{R_2}} E_2 2y_2 \, dy_2 - R_1^3 \int_0^1 E_1 2y_1 \, dy_1 \right] \\
&= \frac{s}{R_1^2} \left[R_2^3 \frac{2}{3} x_2 (E_2 + B_2) \Big|_0^{\frac{R_1}{R_2}} - R_1^3 \frac{2}{3} x_1 (E_1 + B_1) \Big|_0^1 \right] \\
&= \frac{s}{R_1^2} \left\{ R_2^3 \frac{2}{3} y^2 \left[E(y) + B(y) \right] - \frac{2}{3} R_1^3 \right\} \\
&= \frac{4}{3} s R_2 \left[\frac{E(y) + B(y)}{2} - y \right],
\end{aligned}$$

and since $R_2(1 - y) = T$, $i_{\text{avg}} = \frac{4}{3} s T \frac{E(y) + B(y) - 2y}{2(1 - y)}$.

For tubes in the range of $y = 0.8$ to 0.98 , i_{avg} ranges from 1.16 to 1.26 times the central flux. Thus the flux must be a very flat function of the radius until very close to the wall where it rises quite sharply.

7.2.3 Flux in the hollow interior bounded by the generating cylindrical shell possessing self-absorption. Using the same nomenclature as in the preceding paragraph,

$$\begin{aligned}
 i &= \frac{s}{\pi} \int_0^\pi \int_0^{\pi/2} \int_{z_1 \sec \theta}^{z_2 \sec \theta} \frac{1}{\rho^2} e^{-\beta(\rho - z_1 \sec \theta)} \rho^2 d\rho \cos \theta d\theta d\phi \\
 &= \frac{s}{\pi\beta} \int_0^\pi \int_0^{\pi/2} \left[e^{-\beta(z_1 - z_1) \sec \theta} - e^{-\beta(z_2 - z_1) \sec \theta} \right] \cos \theta d\theta d\phi \\
 &= \frac{s}{\pi\beta} \int_0^\pi \int_0^{\pi/2} \left(1 - e^{-\beta u \sec \theta} \right) \cos \theta d\theta d\phi \\
 &= \frac{s}{\pi\beta} \int_0^\pi \left[1 - \mathcal{F}_1(\beta u) \right] d\phi .
 \end{aligned}$$

Since $\mathcal{F}_1(\beta u) \approx 1 - \frac{\pi}{2} \beta u + \frac{\pi}{2} \beta^2 u^2$ for small βu ,

$$i = \frac{s}{2} \int_0^\pi (u - \beta u^2) d\phi .$$

The properties of $u \equiv (z_2 - z_1)$ were treated in §7.1.3. The properties of $u^2 \equiv (z_2 - z_1)^2$ are the following:

$$\begin{aligned}
 u^2 \Big|_0^\pi &= (\psi_2 - \psi_1)^2 \Big|_0^{\pi/2} + (\psi_2 - \psi_1)^2 \Big|_0^{\pi/2} \\
 &= 2 (\psi_2^2 + \psi_1^2 - 2\psi_1 \psi_2) \Big|_0^{\pi/2} \\
 &= 2 \left[R_2^2 - r^2 \sin^2 \phi + R_1^2 - r^2 \sin^2 \phi - 2\sqrt{(R_2^2 - r^2 \sin^2 \phi)(R_1^2 - r^2 \sin^2 \phi)} \right]_0^{\pi/2}
 \end{aligned}$$

If the following approximation be made:

$$(R_2^2 - r^2 \sin^2 \phi)(R_1^2 - r^2 \sin^2 \phi) \cong (R_1 R_2 - r^2 \sin^2 \phi)(R_1 R_2 - r^2 \sin^2 \phi)$$

$$\begin{aligned} u^2 \bigg|_0^\pi &\cong 2 \left[R_2^2 + R_1^2 - 2r^2 \sin^2 \phi - 2R_1 R_2 + 2r^2 \sin^2 \phi \right] \bigg|_0^{\pi/2} \\ &= 2(R_2 - R_1)^2 \bigg|_0^{\pi/2} = 2T^2 \bigg|_0^{\pi/2} . \end{aligned}$$

The flux then becomes

$$\begin{aligned} i &= s \int_0^{\pi/2} (\psi_2 - \psi_1 - \beta T^2) d\phi \\ &= s (R_2 E_2 - R_1 E_1 - \frac{\pi}{2} \beta T^2) \end{aligned}$$

From a comparison with the results of §7.2.2,

$$i_0 = \frac{\pi}{2} s T (1 - \beta T)$$

and

$$i_{R_1} = s T \left[\frac{E(y) - y}{1-y} - \frac{\pi}{2} \beta T \right] .$$

The relative effect of absorption is less at the wall than in the center because it is the immediate vicinity of the wall which makes the greatest contribution to this point.

$$i_{\text{avg}} = s T \left[\frac{4}{3} \frac{E(y) + B(y) - 2y}{2(1-y)} - \frac{\pi}{2} \beta T \right]$$

7.2.4 The absorption of a thin cylindrical shell. It can be seen by comparison of §7.2.2 and §7.2.3 that including the self-absorption in calculating the average flux introduces only a second-order term in $\underline{T}$ into the result for the absorption. The self-absorption is therefore neglected in this computation of the first-order term of the absorption, and

the method of §7.1.4 is followed. By comparison of §7.2.2 and §7.1.4 it is obvious that

$$i = s \left[\int_0^{\pi/2} \sqrt{R_2 - r^2 \sin^2 \phi} \, d\phi - \int_0^{A \sin \frac{R_1}{r}} \sqrt{R_1^2 - r^2 \sin^2 \phi} \, d\phi \right].$$

The first integral was computed in §7.2.1. The second is transformed by the substitution

$$\frac{r}{R_1} \sin \phi = \sin \Omega, \quad .$$

when it becomes

$$\frac{R_1^2}{r} \int_0^{\pi/2} \frac{\cos^2 \Omega \, d\Omega}{\sqrt{1 - \left(\frac{R_1}{r}\right)^2 \sin^2 \Omega}}, \quad ,$$

and it can be recognized as the complete elliptic integral $\underline{B}$ given in §7.2.1. The flux may then be written as

$$i = s R_2 E\left(\frac{r}{R_2}\right) - s \frac{R_1^2}{r} B\left(\frac{R_1}{r}\right) .$$

At the inner wall it is

$$i_{R_1} = s \left[R_2 E\left(\frac{R_1}{R_2}\right) - R_1 \right] = s R_2 \left[E(y) - y \right] ,$$

where $y = \frac{R_1}{R_2}$. Since $(1-y) R_2 = T$,

$$i_{R_1} = sT \frac{E(y) - y}{1-y} = sT \left[1 + \frac{E(y) - 1}{1-y} \right]$$

as before. At the outer wall the flux is

$$\begin{aligned} i_{R_2} &= sR_2 E(1) - s \frac{R_1^2}{R_2} B\left(\frac{R_1}{R_2}\right) \\ &= sR_2 E(1) - s R_2 y^2 B(y) \\ &= sT \frac{1 - y^2 B(y)}{1-y} . \end{aligned}$$

Hence, when y ranges from 0.8 to 0.98, the flux at the inner wall ranges from 2.38 sT to 3.53 sT, and at the outer wall from 2.22 sT to 3.51 sT.

Thus, for such tubes the absorption ranges from $4.4 \beta \frac{T}{2}$ to $7 \beta \frac{T}{2}$.

7.3.0 The short tube -- General. Many practical bodies consist of short tubes. The noncentral flux in these bodies is usually a very complicated expression. The radial behavior of practical tubes (length $\gg R$) would not be very different from that found in the foregoing paragraphs. The central flux is now calculated to indicate the axial behavior.

7.3.1 Central flux due to self-generation in the finite cylindrical shell, no self-absorption. From Figure 13.

$$\begin{aligned}
 i_0 &= \frac{2\pi R s T}{4\pi} \int_{-q}^{L-q} \frac{1}{r^2} dq \\
 &= \frac{sTR}{2} \int_{-\text{Atan } \frac{q}{R}}^{\text{Atan } (L-q)} \frac{R \sec^2 \theta}{R^2 \sec^2 \theta} d\theta \\
 &= \frac{sT}{2} \left(\text{Atan } \frac{L-q}{R} + \text{Atan } \frac{q}{R} \right) .
 \end{aligned}$$

It can be seen that the flux will vary by almost a factor of 2 from the end to the middle.

7.3.2 Central flux due to self-generation in the semi-infinite cylindrical shell closed with a hemispherical shell. The central flux in the infinite cylindrical portion is $\frac{\pi}{2}$ sT. It falls to

$$\frac{\pi}{4} sT + \frac{sT}{2} \cong 1.29 sT$$

on the junction plane of the cylinder and hemisphere when their wall thicknesses are the same. It would generally rise slightly as one proceeded into the hemispherical portion to an extent depending on the wall thickness. At the bottom, the contribution from the cylindrical portion is about

$$\frac{sT}{2} \int_{\pi/4}^{\pi/2} d\theta = \frac{\pi}{8} sT ,$$

FIGURE 13.

FINITE CYLINDRICAL SHELL.

and from the spherical portion, 1 sT to 2 sT; hence the flux at the bottom is 1.4 to 2.4 sT. The average flux is thus sensibly constant.

7.3.3 Central flux due to self-generation in a semi-infinite cylindrical shell closed with a flat disc and possessing no self-absorption. The central flux due to a small thickness of the bottom plate is (Figure 14.).

$$\begin{aligned}
 & \frac{s \, dt}{4} \int_0^{\text{Atan } R/x} \frac{2\pi (x \tan \theta)(x \sec^2 \theta) \, d\theta}{x^2 \sec^2 \theta} \\
 &= \frac{s \, dt}{2} \int_0^{\text{Atan } R/x} \tan \theta \, d\theta = \frac{s \, dt}{2} \ln \sec \text{Atan } \frac{R}{x} \\
 &= \frac{s \, dt}{2} \ln \frac{\sqrt{x^2 + R^2}}{x} = \frac{s \, dt}{4} \ln \frac{x^2 + R^2}{x^2} .
 \end{aligned}$$

That due to the whole plate, where $x = v+t$, is

$$\begin{aligned}
 & \frac{s}{4} \int_0^T \ln \frac{(v+t)^2 + R^2}{(v+t)^2} \, dt = \frac{s}{4} \int_v^{v+T} \ln \frac{x^2 + R^2}{x^2} \, dx \\
 &= \frac{s}{4} \left[x \ln \frac{x^2 + R^2}{x^2} + 2R^2 \int \frac{dx}{x^2 + R^2} \right]_0^{v+T} \\
 &= \frac{s}{4} \left\{ (v+T) \ln \frac{(v+T)^2 + R^2}{(v+T)^2} - v \ln \frac{v^2 + R^2}{v^2} + 2R \text{Atan } \frac{v+T}{R} - 2R \text{Atan } \frac{v}{R} \right\} .
 \end{aligned}$$

At the plate $v = 0$, and the result is

$$\frac{s}{4} \left(T \ln \frac{T^2 + R^2}{T^2} + 2R \text{Atan } \frac{T}{R} \right) .$$

If $R \gg T$ the result at $v = 0$ becomes

$$\frac{s}{4} \left\{ 2T \ln \frac{R}{T} + 2R \frac{T}{R} \right\} = \frac{sT}{2} \left(1 + \ln \frac{R}{T} \right) .$$

FIGURE 14.

THE SEMI-INFINITE CYLINDRICAL
SHELL CLOSED WITH A FLAT DISC.

An application of interest might be the central thimble of CP-3 assumed here to consist of a semi-infinite cylinder of 6.3 cm radius and 1/2 cm wall with a 1 cm thick bottom plate. The flux in the cylindrical portion is about $1.57 \frac{T}{2}$ (where $T = 1$ cm) or 0.78s. On the axis at the bottom it is

$$\frac{\pi}{4} \frac{T}{2} + \frac{sT}{2} (1 + \ln 6.3) = 1.8$$

while at about 12 cm from the bottom plate (about $2R$) the flux will fall to about 0.72s (as can be shown by plotting the flux) and will rise gradually until at $15R$ to $20R$ it will not differ sensibly from the value for the infinite cylinder.

7.4.1 The flux within a thin absorbing spherical shell possessing no self-generation and placed in a region originally of uniform isotropic flux. Using spherical coordinates originating at the point $\underline{P}$, there are in the absence of the absorbing shell $\frac{i}{4\pi}$ rays/steradian entering $\underline{P}$. Because of the shell these are partially absorbed, and in the element $\sin \theta \, d\theta \, d\phi$ there enter only $(i/4\pi) e^{-\beta(z_2-z_1)} \sin \theta \, d\theta \, d\phi$. The flux is no longer uniform, and it is isotropic only at the center. The flux at $\underline{P}$ is

$$j = \frac{i}{4\pi} \int_0^\pi e^{-\beta[z_2(\theta) - z_1(\theta)]} \sin \theta \, d\theta \int_0^{2\pi} d\phi ,$$

and since the shell is thin

$$j = \frac{i}{2} \int_0^\pi [1 - \beta(z_2 - z_1)] \sin \theta \, d\theta .$$

The properties of z_2 and z_1 have been described in §7.1.3, and the solution may be written immediately as

$$j = i \left[1 - \frac{\beta}{2} \left(T + \frac{1-y_2^2}{2y_2} R_2 \ln \frac{1+y_2}{1-y_2} - \frac{1-y_1^2}{2y_1} R_1 \ln \frac{1+y_1}{1-y_1} \right) \right] .$$

Thus the absorption has the same properties as the generated flux. At the center it is βT while at the periphery it is

$$\frac{\beta}{2} \left(T + \frac{1-y^2}{2y} R_2 \ln \frac{1+y}{1-y} \right) , \quad \text{where } y = \frac{R_1}{R_2} ,$$

which in practical cases may be 1.5 to 3 times the absorption at the center. This is reasonable, since except for two rays, the path length through the wall is greater for the rays converging at a point on the wall than for those converging at the center. The average value for the absorption then becomes $\frac{3}{2} \beta \frac{T}{y^2}$ which is a little greater than 3/2 of the absorption at the center. The average flux may therefore be written

$$j_{\text{avg}} \cong i \left(1 - \frac{3}{2y^2} \beta T \right) \cong i (1 - 1.7\beta T).$$

7.4.2 The flux within a thin absorbing cylindrical shell possessing no generation and placed in a region originally of uniform isotropic flux. According to the nomenclature of the preceding paragraphs,

$$\begin{aligned} j &= \frac{i}{4\pi} 2 \int_0^\pi d\phi \int_0^{\pi/2} e^{-\beta [z_2(\theta) - z_1(\theta)] \sec \theta} \cos \theta d\theta d\phi \\ &= \frac{i}{\pi} \int_0^\pi \mathcal{F}_1(\beta u) d\phi \\ &= \frac{i}{\pi} \int_0^\pi \left(1 - \frac{\pi}{2} \beta u + \frac{\pi}{2} \beta^2 u^2 \right) d\phi \\ &= i \left[1 - \beta \int_0^{\pi/2} (u - \beta u^2) d\phi \right]. \end{aligned}$$

This is the same integral as that of §7.2.3 with the absorption coefficient replacing the generation factor. Hence, the flux at the center is

$$j_0 = i \left[1 - \frac{\pi}{2} \beta T (1 - \beta T) \right].$$

At the wall it is

$$j_R = i \left[1 - \beta T \left(\frac{E(y) - y}{1 - y} - \frac{\pi}{2} \beta T \right) \right],$$

which means that the effect of absorption at the wall is 1.5 to 2.5 times as great as in the center. However, most of this additional absorption takes place very close to the wall, for the average flux is

$$j_{\text{avg}} = i \left[1 - \beta T \left(\frac{4}{3} \frac{E(y) + B(y) - 2y}{2(1-y)} - \frac{\pi}{2} \beta T \right) \right],$$

which for practical tubes might mean 8/7 to 5/4 times the absorption at the center; i. e., $j_{\text{avg}} \cong i(1 - 1.2 \frac{\pi}{2} \beta T)$.

7.4.3 The cadmium-shielded vessel. A cadmium-surrounded sample placed in a region of uniform slow neutron flux is an application of ¶7.4.1 and ¶7.4.2. The neutrons are absorbed in a uniform thin layer of the outer surface and generate capture gamma rays. The vessel then absorbs these and behaves as if it were immersed in a region of uniform isotropic flux.

7.5 The flux in a self-generating slab. From Figure 15, the flux at q is

$$\begin{aligned} i &= \frac{s}{4\pi} \int_0^{2\pi} d\phi \int_0^{\pi/2} \sin \theta d\theta \left\{ \int_0^{(T-q) \sec \theta} + \int_0^{q \sec \theta} \right\} e^{-\beta \rho} \frac{\rho^2 d\rho}{\rho^2} \\ &= \frac{s}{2\beta} \int_0^{\pi/2} (2 - e^{-\beta(T-q) \sec \theta} - e^{-\beta q \sec \theta}) \sin \theta d\theta, \end{aligned}$$

which will be recognized as the Gold integral. The notation used by Placzek, le Caine, and Chandrasekhar will be used for this form of exponential integral, and it should not be confused with the notation used in the preceding paragraphs for the elliptic integral of the second kind. Then,

$$i = \frac{s}{\beta} \left\{ 1 - \frac{1}{2} E_2 \left[\beta(T-q) \right] - \frac{1}{2} E_2(\beta q) \right\}.$$

Values of E_2 may be obtained from Placzek's tables.²⁴ The flux in the central plane is

$$i_{1/2} = \frac{s}{\beta} \left[1 - E_2 \left(\frac{\beta T}{2} \right) \right],$$

FIGURE 15.
INFINITE SLAB.

and at the boundary surface,

$$i_T = \frac{s}{\beta} \left[1 - \frac{1}{2} E_2(\beta T) - \frac{1}{2} E_2(0) \right] .$$

Since

$$E_2(0) = 1, \quad i_T = \frac{s}{2\beta} \left[1 - E_2(\beta T) \right] ,$$

the average flux is

$$\begin{aligned} i_{\text{avg}} &= \frac{s}{T\beta} \int_0^T \left\{ 1 - \frac{1}{2} E_2[\beta(T-q)] - \frac{1}{2} E_2(\beta T) \right\} dq \\ &= \frac{s}{\beta T} \left[T + \frac{1}{2\beta} \int_{\beta T}^0 E_2(B_1) dB_1 - \frac{1}{2\beta} \int_0^{\beta T} E_2(B_2) dB_2 \right] , \end{aligned}$$

where

$$\beta(T - q) \equiv B_1 \text{ and } \beta q \equiv B_2 .$$

Since

$$E_{(n+1)}(a) = \frac{1}{n} - \int_0^a E_n(b) db ,$$

$$\begin{aligned} i_{\text{avg}} &= \frac{s}{\beta T} \left\{ T - \frac{1}{\beta} \left[\frac{1}{2} - E_3(\beta T) \right] \right\} \\ &= \frac{s}{\beta} \left\{ 1 - \frac{1}{\beta T} \left[\frac{1}{2} - E_3(\beta T) \right] \right\} . \end{aligned}$$

Then according to the approximation derived in §3.1,

$$i_{\text{avg}} = \frac{s}{\beta} \left[1 - (1 - \beta T) \right] = sT ,$$

which yields the absorption calculated in §3.1.

7.6 The total converter. The thin-walled vessel, black to slow neutrons, which is made of material generating Γ Mev of gamma rays for each slow neutron absorbed, when immersed in a region of slow neutron flux N neutrons/cm² sec, will have a surface source strength of $N\Gamma$ Mev/cm² sec. In general $\Gamma \approx 8$, and the average energy of the individual gamma rays will be a 3-6 Mev for which the absorption $\frac{\mu}{\rho} \alpha = 0.02$ per gram. Neglecting the self-absorption of the vessel, the central dosage is, if the vessel be spherical, $(N\Gamma) 1.93(10^{-8})(0.02)$ rep, and if it be cylindrical, $(1.57 N\Gamma) 1.93(10^{-8})(0.02)$ rep. The noncentral flux, the average flux, and the self-absorption corrections may be computed for the particular geometry of interest by applying the foregoing paragraphs.

APPENDIX III

THE ARBITRARY SELF-ABSORPTION LAW

8.0 The principle of superposition. The calculation of the self-absorption of a body may be performed by the fivefold integration of the absorption law, consisting of the twofold integration over the polar and azimuthal angles originating at the arbitrary point within the body, followed by the threefold integration of the point over the whole body. This is usually an impossible task for the arbitrary absorption law. However, one may proceed by a different route. The self-absorption is a function of the possible path lengths within the body. If one were to possess a distribution function for the possible path lengths within a body, one could superpose the absorption over each of these paths by means of a single integration to obtain the final result. Since only one integration is involved, one could even handle the arbitrary absorption law by resorting to a graphical integration. The calculation of the local flux may, of course, be treated in a similar manner.*†

8.1 The inverse polar derivative as a density of path lengths function. In the plane, the fraction of possible paths that lie in the angle $d\theta$ is $\frac{1}{2\pi} d\theta$. If these paths emanate from the point P (Figure 16) and intersect the curve C at a distance ρ , the path length from the point to the line will have increased by $d\rho$ in the angle $d\theta$. The fraction of path lengths per unit length of path, of the path length ρ , which is contributed by the angle $d\theta$, is then $\frac{1}{2\pi} \frac{d\theta}{d\rho}$.

8.2 The density of possible path lengths to the circumference within the area bounded by a circle. The fraction of path lengths of length u contributed by the point P in the circle (Figure 17) is integrated over the area

*The present approach resulted from an attempt to find a convenient solution to a problem proposed by A. Jaffey of this laboratory.

†These procedures were developed in Feb., 1952 to solve the problem presented in ANL-4763. In May, 1952, there was brought to my attention an article by J. M. Hammersley, Proc. Roy. Soc., London, A210, 93 (1951). Hammersley developed this method previously, and in the present article he gives the solutions for the rod (finite cylinder) and the disc, which are not given in this report. The development presented in this report is somewhat different and very much simpler than the one given by Hammersley. It would seem the geometric distribution functions are well established (e.g., Hammersley's reference to R. Deltheil, "Probabilités Geometric," Gauthier-Villars, Paris, 1926) although some of the particular ones given here may not have been published previously.

FIGURE 16.
THE POLAR DERIVATIVE.

FIGURE 17.
THE CIRCLE

of the circle from which paths of such length can emanate. Expressing the results in units of the circle radius, it is obvious that the shortest path length is zero and the longest one, two; and this defines the complete range of the density function, $N(u)$. If y be the radial variable, path lengths greater than unity can arise only from points $y \geq u - 1$, and path lengths less than unity only from points $y \leq 1 - u$; hence, the range of y is from $|1 - u|$ to 1. The fraction of path lengths of length u arising at y can be seen to be

$$\frac{2 \cdot 2\pi y}{\pi 1^2} = 4y.$$

Then

$$N(u) = \int_{|1-u|}^1 \frac{1}{2\pi} \frac{d\theta}{du} 4y dy.$$

Since

$$u = \sqrt{1 - y^2 \sin^2 \theta} - y \cos \theta,$$

then

$$\theta = \text{Acos} \frac{1 - u^2 - y^2}{2uy},$$

and

$$\frac{d\theta}{du} = \frac{1 + u^2 - y^2}{u \sqrt{-y^4 + 2y^2(1+u^2) - (1-u^2)^2}}.$$

Hence,

$$N(u) = \frac{1}{\pi u} \int_{|1-u|}^1 \frac{(1 + u^2 - y^2) 2y dy}{\sqrt{-y^4 + 2y^2(1+u^2) - (1-u^2)^2}},$$

which is a function of y^2 . Writing $y^2 = z$,

$$N(u) = \frac{1}{\pi u} \int_{(1-u)^2}^1 \frac{(1 + u^2 - z) dz}{\sqrt{-z^2 + 2z(1+u^2) - (1-u^2)^2}},$$

which is seen to be the sum of two integrals, the first being an Asin, and the second the sum of an Asin and the radical of the denominator. The

Asin terms are identical in magnitude and opposite in sign and hence vanish, leaving the radical:

$$N(u) = \frac{1}{\pi u} \sqrt{-z^2 + 2z(1+u^2) - (1-u^2)^2} \Big|_{(1-u)^2}^1.$$

This expression vanishes at the lower limit; hence

$$N(u) = \frac{2}{\pi} \sqrt{1 - \frac{1}{4}u^2} \quad \text{or} \quad \frac{1}{\pi} \sqrt{4-u^2},$$

which will be recognized as the quarter of an ellipse of semimajor axis 2 and semiminor axis $\frac{2}{\pi}$, and

$$\int_0^2 N(u) du = 1 \text{ as required.}$$

8.3 The density of possible path lengths to the surface from points within the region of the sphere. As before the results are expressed in units of the radius. From Figure 18 it is obvious that

$$N(u) = \frac{3}{4\pi l^3} \int_{|1-u|}^1 \frac{1}{2\pi} \frac{d\theta}{du} \frac{2\pi h}{u} \frac{4\pi y^2}{4\pi l^2} dy.$$

The presence of the factor u is understood when it is realized that the density of paths depends not on the circumference $2\pi h$ but on the angle subtended by this circumference. The value of $\frac{d\theta}{du}$ has been given in the preceding paragraph. The value of h follows:

$$1 = h^2 + \left(y + \sqrt{u^2 - h^2}\right)^2,$$

$$\text{and} \quad h = \frac{1}{2y} \sqrt{-y^4 + 2y^2(1+u^2) - (1-u^2)^2} \equiv \frac{R}{2y}.$$

The radical R , is the same encountered in §8.2.

FIGURE 18.

THE SPHERE.

Then

$$\begin{aligned} N(u) &= \frac{3}{4u} \int_{|1-u|}^1 \frac{1+u^2-y^2}{uR} \frac{R}{y} y^2 dy \\ &= \frac{3}{4u^2} \int_{|1-u|}^1 (1+u^2-y^2) y dy, \end{aligned}$$

a function of y^2 . Let $z = y^2$, then

$$\begin{aligned} N(u) &= \frac{3}{8u^2} \int_{(1-u)^2}^1 (1+u^2-z) dz \\ &= \frac{3}{8u^2} \left[(1+u^2)z - \frac{z^2}{2} \right]_{(1-u)^2}^1 \\ &= \frac{3}{4} \left(1 - \frac{1}{4} u^2 \right), \end{aligned}$$

which is a parabola of axis $N(u)$, cutting the $N(u)$ axis at $\frac{3}{4}$ and the u axis at 2, and

$$\int_0^2 N(u) du = 1 \text{ as required.}$$

This yields $3/4$ as the coefficient of the first-order term in the exponential self-absorption, also as required.

8.4.0 The density of path lengths from a point to a line distant p .
From Figure 19 and §8.1.

$$N(u) = \frac{2}{\pi} \frac{d\theta}{du},$$

$$\theta = A \cos \frac{p}{u},$$

and

$$\frac{d\theta}{du} = \frac{p}{u \sqrt{u^2 - p^2}} = \frac{1}{u \sqrt{\frac{u^2}{p^2} - 1}}$$

FIGURE 19.

THE POINT AND
THE LINE.

The range of $N(u)$ is from $\underline{p}$ to ∞ , hence

$$N(u) \Big|_p^\infty = \frac{2}{\pi} \frac{p}{u \sqrt{u^2 - p^2}} .$$

The integral $\int_p^\infty N(u) du = 1$ as required.

8.4.1 The density of path lengths from a point to a line distant p , weighted by the cosine of the angle. Obviously, from §8.4.0

$$N(u) \Big|_p^\infty = \frac{p^2}{u^2 \sqrt{u^2 - p^2}} .$$

The integral $\int_p^\infty N(u) du = 1$, as required.

8.4.2 The density of possible interior path lengths to the surface of an infinite cylinder. It is obvious that the solution sought is the combination of the solutions of §8.2 and §8.4.1. Designating the variable of §8.2 by $\underline{p}$ and the respective distributions by $n(p)$ and $m(u)$

$$N(u) = \int n(p) m(u) dp = \int \frac{1}{\pi} \sqrt{\frac{4-p^2}{u^2-p^2}} \frac{p^2}{u^2} dp .$$

The solution possesses a singularity at $u = 2$, requiring its expression in two parts:

$$N(u) \Big|_{u=0}^2 = \frac{1}{\pi u^2} \int_0^u \sqrt{\frac{4-p^2}{u^2-p^2}} p^2 dp .$$

The substitution $p = u \sin \phi$ shows this to be an elliptic integral.

$$N(u) \Big|_{u < 2} = \frac{2}{\pi} \int_0^{\pi/2} \sin^2 \phi \sqrt{1 - \frac{u^2}{4} \sin^2 \phi} d\phi .$$

The solution is given by W. Gröbner and N. Hofreiter³⁴ and may be expressed in the notation of §7.2.1 as

$$N(u) \Big|_{u < 2} = \frac{2}{\pi} \frac{1}{3} \left[2 E\left(\frac{u}{2}\right) - B\left(\frac{u}{2}\right) \right] .$$

For values of $u > 2$:

$$N(u) \Big|_{u=2}^{\infty} = \frac{1}{\pi u^2} \int_0^2 \sqrt{\frac{4-p^2}{u^2-p^2}} p^2 dp,$$

which is cast into the Legendrian canonical form by the substitution

$$p = 2 \sin \phi ,$$

$$N(u) \Big|_{u > 2} = \frac{16}{\pi u^3} \int_0^{\pi/2} \frac{\cos^2 \phi \sin^2 \phi d\phi}{\sqrt{1 - \frac{4}{u^2} \sin^2 \phi}}$$

The solution is given by Gröbner and Hofreiter³⁵ and, in the notation of §7.2.1, is

$$N(u) \Big|_{u > 2} = \frac{2}{\pi} \frac{1}{3} \frac{2}{u} \left[2 B\left(\frac{2}{u}\right) - E\left(\frac{2}{u}\right) \right] .$$

A tabulation of values computed from Jahnke and Emde's tables of $\underline{B}$ and $\underline{E}$ is given in Table 3 (§8.4.3). The values have not been checked; the last figure given is certainly unreliable.

8.4.3 Table III

VALUES FOR $N(u)$ FOR THE INFINITE CYLINDER

<u>u</u>		<u>$N(u)$</u>
0		0.500000000
0.2		0.49815
0.4		0.49245
0.6		0.48279
0.8		0.46913
1.0		0.45041
1.2		0.42643
1.4		0.39594
1.6		0.35666
1.8		0.30386
2.0		0.212206591
20/9	2.2222	0.1244
20/8	2.5000	0.07937
20/7	2.8571	0.04984
20/6	3.3333	0.02996
20/5	4.0000	0.01671
20/4	5.0000	0.00817
20/3	6.6667	0.00345
20/2	10.0000	0.00101
20/1	20.0000	0.00012

8.5 The density of possible path lengths to the surface and lying in the infinite slab. It is obvious from §8.4 that the density of path lengths from a point to a plane is

$$N(u) = \frac{1}{2\pi} \frac{p}{u \sqrt{u^2 - p^2}} \frac{2\pi h}{u}.$$

Since

$$h = \sqrt{u^2 - p^2},$$

$$N(u) = \frac{p}{u^2},$$

and the normalization test,

$$\int_p^\infty N(u) du = \left. \frac{p}{u} \right|_p^\infty = 1$$

In proceeding to calculate the density of paths in the slab only one face need be considered since the function is normalized anyway. It must be remembered a singularity exists at $u = T$. Then for the slab,

$$\begin{aligned} N(u) \Big|_{u=0}^T &= \frac{1}{T} \int_0^u \frac{p}{u^2} dp, & N(u) \Big|_{u=T}^\infty &= \frac{1}{T} \int_0^T \frac{p}{u^2} dp, \\ N(u) \Big|_{u=0}^T &= \frac{1}{2T}, & N(u) \Big|_{u=T}^\infty &= \frac{T}{2u^2}. \end{aligned}$$

This is normalized, for

$$\int_0^\infty N(u) du = \int_0^T \frac{1}{2T} du + \int_T^\infty \frac{T}{2u^2} du = 1$$

Since the function may seem strange, its application to the exponential absorption is detailed. The absorption is

$$\begin{aligned} & \int_0^{\infty} N(u) e^{-\beta u} du \\ &= \frac{1}{2T} \int_0^T e^{-\beta u} du + \frac{T}{2} \int_T^{\infty} \frac{e^{-\beta u}}{u^2} du . \end{aligned}$$

If $\underline{u}$ and β be expressed in units of $\underline{T}$, this becomes

$$\begin{aligned} & \frac{1}{2} \int_0^1 e^{-\beta u} du + \frac{1}{2} \int_1^{\infty} \frac{e^{-\beta u}}{u^2} du \\ &= \frac{1}{2\beta} (1 - e^{-\beta}) + \frac{1}{2} E_2(\beta) \\ &= \frac{1}{2\beta} \left[1 - e^{-\beta} + \beta E_2(\beta) \right] , \end{aligned}$$

where E_2 is the Gold integral. Applying the recursion formula for these exponential integrals there is obtained

$$\frac{1}{2\beta} \left[1 - 2 E_3(\beta) \right] .$$

If the units of β be changed to centimeters, one has the results previously developed by other methods.

8.6.0 Nonuniform generation; the generation weighting function.

The form of the absorption equation is $\underline{G} \underline{Q}(u)$ where $\underline{G}$ is an amplitude and $\underline{Q}$ is the absorption function. The fractional absorption is then given by

$$\int \frac{N(u) \underline{G} \underline{Q}(u)}{\underline{G}} du .$$

The functions $N(u)$ found in the preceding sections assumed that the generation was uniform, and hence $N(u)$ was a function of the geometric form of the body only. If the generation is no longer uniform, the density of paths originating at a point must be weighted according to the relative generation at that point. This may be expressed formally in the following manner. If the distribution function for uniform generation is

$$N(u) = \int_{U_1}^{U_2} f(u, y) dy$$

where U_1, U_2 are functions of u only, then the unnormalized distribution function for nonuniform generation is

$$D(u) = \int_{U_1}^{U_2} f(u, y) g(y) dy$$

and $g(y)$ is defined as the generation weighting function. In the class of problems treated in this report, the generation is the result of neutron absorption; the nonuniform generation is the result of a depression of the neutron flux in the interior of the body and is the result of the neutron absorption. For the two bodies of great practical interest, the small sphere and the thin rod, the neutron flux can be expressed quite accurately by a parabola since it expands into a series in which the odd-order terms are absent. Then $g(y) = g_0 + g_1 y^2$, where g_0 and g_1 are constants and y is expressed in units of radii.

8.6.1 Nonuniform generation in the small sphere. From §8.3 and §8.4.0

$$\begin{aligned} D(u) &= \frac{3}{4u^2} \int_{|1-u|}^1 (1+u^2-y^2) (g_0+g_1y^2) y dy \\ &= \frac{3}{8u^2} \int_{(1-u)^2}^1 (1+u^2-z) (g_0+g_1z) dz \\ &= \frac{3}{4} g_0 \left(1 - \frac{1}{4}u^2\right) + \frac{3g_1}{8u^2} \left[\frac{z^2}{2}(1+u^2) - \frac{z^3}{3} \right]_{(1-u)^2}^1 \end{aligned}$$

$$= \frac{3}{4} g_0 \left(1 - \frac{1}{4} u^2\right) + \frac{1}{16} g_1 (12 - 16u + 9u^2 + u^4).$$

$$\int_0^2 D(u) du = g_0 + 0.6g_1 \text{ as required by the geometry; hence}$$

$$N(u) = \frac{3}{4} \frac{g_0}{g_0 + 0.6g_1} \left(1 - \frac{1}{4} u^2\right) + \frac{g_1}{16(g_0 + 0.6g_1)} (12 - 16u + 9u^2 + u^4).$$

8.6.2 The weighted density of path lengths to the circumference in the area bounded by a circle. Applying the weighting function $g(y) = g_0 + g_1 y^2$ to the partially developed result of §8.2 and proceeding as before,

$$\begin{aligned} D(u) &= \frac{1}{\pi u} \int_{|1-u|}^1 \frac{(1+u^2-y^2) 2y (g_0+g_1 y^2) dy}{\sqrt{-y^4+2y^2(1+u^2) - (1-u^2)^2}} \\ &= \frac{1}{\pi u} \int_{(1-u)^2}^1 \frac{(1+u^2-z) (g_0+g_1 z) dz}{\sqrt{-z^2 + 2z(1+u^2) - (1-u^2)^2}} \\ &= g_0 \mathfrak{h}(u) + \frac{g_1}{\pi u} \int_{(1-u)^2}^1 \frac{(1+u^2-z) z dz}{\sqrt{-z^2 + 2z(1+u^2) - (1-u^2)^2}} \end{aligned}$$

where $\mathfrak{h}(u)$ is the result obtained in §8.2 without the weighting function. Writing $\underline{R}$ for the denominator within the integral, the integral becomes

$$\begin{aligned} &(1+u^2) \int_{(1-u^2)}^1 \frac{z dz}{\underline{R}} - \int_{(1-u^2)}^1 \frac{z^2 dz}{\underline{R}} \\ &= (1+u^2) \left[-\underline{R} + (1+u^2) \int \frac{dz}{\underline{R}} \right] + \left[\frac{z}{2} + \frac{6(1+u^2)}{4} \right] \underline{R} \\ &\quad - \frac{3(1+u^2) - (1-u^2)^2}{2} \int \frac{dz}{\underline{R}}, \end{aligned}$$

and since the radical vanishes at the lower limit, the integral becomes

$$\left(1 + \frac{1}{2}u^2\right) u \sqrt{4-u^2} + \left[-\frac{1}{2} (1+u^2)^2 + \frac{1}{2} (1-u^2)^2 \right] \int \frac{dz}{R}.$$

The second term of this last expression is

$$\begin{aligned} & - 2u^2 \int \frac{dz}{(1-u)^2} \\ & = 2u^2 \operatorname{Asin} \left[\frac{-z + (1+u^2)}{[(1+u^2)^2 - (1-u^2)^2]^{1/2}} \right]_{(1-u)^2}^1 = -2u^2 \left(\frac{\pi}{2} - \operatorname{Asin} \frac{u}{2} \right). \end{aligned}$$

Then combining the terms, the result is

$$D(u) = g \mathcal{H}(u) + \frac{g_1}{\pi} \left(1 + \frac{1}{2}u^2\right) \sqrt{4-u^2} = g_1 u \left(1 - \frac{2}{\pi} \operatorname{Asin} \frac{u}{2}\right).$$

The normalization constant, $\int_0^2 D(u) du$, is found to be $g_0 + \frac{1}{2}g_1$ as required.

Hence,

$$N(u) = D(u) / \left(g_0 + \frac{1}{2}g_1\right).$$

8.6.3 Nonuniform generation in the thin rod. The solutions of §8.6.2 and §8.4.1 are combined in the manner of §8.4.2. Designating the distribution derived in §8.6.2 as $\mathcal{L}(p)$, that of §8.4.1 as $\mathcal{M}(u)$, and that of §8.4.2 as $\mathcal{H}(u)$, the distribution desired, $N(u)$, is,

$$\begin{aligned} N(u) &= \int \mathcal{L}(p) \mathcal{M}(u) dp \\ &= \frac{1}{\pi u^2 (g_0 + \frac{1}{2}g_1)} \int \left\{ g_0 p^2 \frac{s}{t} + g_1 p^2 \frac{s}{t} + \frac{g_1 p^4}{2} \frac{s}{t} - \pi g_1 \frac{p^3}{t} + \frac{2g_1 p^3}{t} \operatorname{Asin} \frac{p}{2} \right\} dp, \end{aligned}$$

where $s = \sqrt{4-p^2}$ and $t = \sqrt{u^2-p^2}$.

The solution possesses a singularity at $u = 2$ and must be expressed in two parts. The necessary reduction formulae may be found in Gröbner and Hofreiter,³⁶ and in the notation of Jahnke and Emde³³ the integrals are:

$\underline{H}$	$\int_0^u H dp$	$\int_0^2 H dp$
$p^2 \frac{s}{t}$	$\frac{2}{3} u^2 (2E - B)$	$\frac{2}{3} 2u (2B - E)$
$p^4 \frac{s}{t}$	$\frac{8}{15} u^2 (E + 2u^2 E - 2B - u^2 B)$	$\frac{8}{15} 2u (B + 2u^2 B - u^2 E - 2E)$
$\frac{p^3}{t}$	$\frac{2}{3} u^3$	$\frac{2}{3} u^3 - \frac{2}{3} (u^2 + 2) \sqrt{u^2 - 4}$
$\frac{p^3}{t} A \sin \frac{p}{2}$	$\frac{u^2}{9} (4B + 3u^2 B - 2E)$	$\frac{u^2}{9} \cdot \frac{2}{u} (4E + 3u^2 E - 2B)$ $- \frac{\pi}{2} \cdot \frac{2}{3} (u^2 + 2) \sqrt{u^2 - 4}$
Modulus of E, B	$\frac{u}{2}$	$\frac{2}{u}$

and adding these results

$$N(u) \Big|_{u < 2} = \frac{2}{\pi} \cdot \frac{1}{3} \cdot \frac{1}{g_0 + \frac{1}{2} g_1} \left\{ g_0 (2E - B) + \frac{g_1}{15} (26E + 12u^2 E - 7B + 9u^2 B) - g_1 \pi u \right\}$$

and

$$N(u) \Big|_{u > 2} = \frac{2}{\pi} \cdot \frac{1}{3} \cdot \frac{1}{g_0 + \frac{1}{2} g_1} \left\{ \frac{2}{u} g_0 (2E - B) + \frac{2}{u} \frac{g_1}{15} (26B + 12u^2 B - 7E + 9u^2 E) - g_1 \pi u \right\},$$

or

$$N(u) \Big|^{u < 2} = \frac{1}{g_0 + \frac{1}{2}g_1} \left\{ n(u) + \frac{2}{\pi} \cdot \frac{1}{3} \cdot \frac{g_1}{15} (26E + 12u^2E - 7B + 9u^2B - 15\pi u) \right\}$$

and

$$N(u) \Big|^{u > 2} = \frac{1}{g_0 + \frac{1}{2}g_1} \left\{ n(u) + \frac{2}{\pi} \cdot \frac{1}{3} \cdot \frac{2}{u} \cdot \frac{g_1}{15} (26B + 12u^2B - 7E + 9u^2E - \frac{15}{2}\pi u^2) \right\}$$

$$\equiv \frac{1}{g_0 + \frac{1}{2}g_1} \left\{ g_0 n(u) + g_1 n_1(u) \right\}.$$

Values for $n_1(u)$ are given in Table IV (§8.6.4). These numbers have not been checked, and certainly the last place given is to be considered unreliable.

8.6.4 Table IV

THE COEFFICIENT OF THE PARABOLIC WEIGHTING FACTOR FOR THE
DISTRIBUTION OF PATH LENGTHS IN THE INFINITE CYLINDER

	<u>u</u>	<u>$n_1(u)$</u>
	0	0.500
	0.2	0.380
	0.4	0.285
	0.6	0.216
	0.8	0.170
	1.0	0.145
	1.2	0.138
	1.4	0.144
	1.6	0.155
	1.8	0.161
2 x	$\sqrt{0.85}$ 1.8439	0.160
2 x	$\sqrt{0.90}$ 1.8974	0.155
2 x	$\sqrt{0.95}$ 1.9494	0.145
2 x	$\sqrt{0.98}$ 1.9799	0.135
	2.0	0.123
	20/9 2.2222	0.067
	20/8 2.5000	0.041
	20/7 2.8571	0.026
	20/6 3.3333	0.015
	20/5 4.0000	0.0084
	20/4 5.0000	0.0022
	20/3 6.6667	0.0006

8.6.5 An application. An application of these results is the trivial case of calculating the fraction of the gamma rays emerging from the CP-3 and the HEW rods assuming an exponential absorption law, the problem treated in Section 1 of this report. The data are given in Table V.

Table V

DATA FOR THE CALCULATION OF THE FRACTION OF GAMMA RAYS
EMERGING FROM THE CP-3 AND HEW RODS

	HEW	CP-3
$\phi(r)$ (cm)	$6.6 + 0.7r^2$ $= 1 + 0.106r^2$	$1 + 0.102r^2$
Radius	1.8	1.4
$\phi(\lambda)$ (radii)	$1 + (1.8)^2 0.106r^2$ $= 1 + 0.3434\lambda^2$	$1 + (1.4)^2 0.102r^2$ $= 1 + 0.1999\lambda^2$
g_0	1	1
g_1	0.3434	0.1999
$g_0 + \frac{1}{2} g_1$	1.1717	1.0995
$\frac{1}{g_0 + \frac{1}{2} g_1}$	0.8535	0.9095
β (per cm)	1.3	1.3
β (per radius)	2.34	1.82

The calculations are given in Tables VI and VII.

Table VI

FRACTION OF GAMMA RAYS EMERGING FROM HEW RODS

I u	II 2.34u	III $e^{-2.34u}$	IV $\eta(u)$	V $g_1 n_1(u)$	VI IV + V	VII $\frac{1}{g_0 + \frac{1}{2}g_1}$ VI	VIII III x VII
0	0	1.	0.500	0.172	0.672	0.573	0.573
0.2	0.468	0.626	0.498	0.130	0.628	0.534	0.334
0.4	0.936	0.392	0.492	0.098	0.590	0.504	0.198
0.6	1.404	0.246	0.483	0.074	0.557	0.475	0.117
0.8	1.872	0.154	0.469	0.058	0.527	0.450	0.069
1.0	2.340	0.096	0.450	0.050	0.500	0.427	0.041
1.2	2.808	0.060	0.426	0.047	0.473	0.404	0.024
1.4	3.276	0.038	0.396	0.049	0.445	0.380	0.014
1.6	3.744	0.024	0.357	0.053	0.410	0.350	0.008
1.8	4.212	0.015	0.303	0.055	0.358	0.306	0.005
2.0	4.680	0.009	0.212	0.043	0.255	0.218	0.002

Numerical quadrature of VIII = 21.4 %

Table VII

FRACTION OF GAMMA RAYS EMERGING FROM CP-3 RODS

I u	II 1.82u	III $e^{-1.82u}$	IV $\eta(u)$	V $g_1 \eta_1(u)$	VI IV + V	VII $\frac{1}{g_0 + \frac{1}{2}g_1}$ VI	VIII III x VII
0	0	1.	0.500	0.100	0.600	0.546	0.546
0.2	0.364	0.695	0.498	0.076	0.574	0.522	0.363
0.4	0.728	0.483	0.492	0.057	0.549	0.499	0.241
0.6	1.092	0.336	0.483	0.043	0.526	0.478	0.160
0.8	1.456	0.233	0.469	0.034	0.503	0.457	0.107
1.0	1.820	0.162	0.450	0.029	0.479	0.436	0.071
1.2	2.184	0.113	0.426	0.028	0.454	0.413	0.047
1.4	2.548	0.078	0.396	0.029	0.425	0.387	0.030
1.6	2.912	0.054	0.357	0.031	0.388	0.353	0.019
1.8	3.276	0.038	0.303	0.032	0.335	0.305	0.012
2.0	3.640	0.026	0.212	0.025	0.237	0.216	0.006

Numerical quadrature of VIII = 26.2%

Extending the quadrature might increase the figure in the last place slightly. The results are in agreement with those obtained by the method given in Section I of this report. A less trivial application of the present method may be found in ANL-4763, W. Primak and L. Fuchs, "The Corrosion of Metals on Irradiation."

8.7 The local flux. The methods introduced in this section may be used to obtain the local flux. In the absence of self-absorption the results are not very useful since the law of flux is a simple one, i.e., $\frac{1}{u^2}$, but in the presence of pronounced self-absorption according to an arbitrary law the

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method becomes a valuable one. Thus for the infinite body the distribution of distances about a point is $n(u) = \nu 4\pi u^2$, where ν is an appropriate normalization constant which in a practical problem is the reciprocal of the volume chosen for integration. If the arbitrary absorption law is $Q(u)$, the flux is

$$\int n(u) \frac{1}{u^2} Q(u) du = \nu 4\pi \int Q(u) du = \frac{3}{a^3} \int_0^a Q(u) du,$$

where a is a distance over which the absorption is so great that the contribution from beyond this distance is negligible.

The distribution of distances about a point in the body must be obtained for each body. For the sphere it is particularly simple. For $u < (1-y)$ it is obvious that

$$n(u) = \frac{3}{4\pi} 4\pi u^2 = 3u^2.$$

For $(1-y) < u < (1+y)$ the distances no longer extend over all solid angles, and the distribution is not isotropic. The distribution is (referring to Figure 18 and using ψ for the supplement of θ)

$$n(u) = \frac{3}{4}\pi u^2 \int_0^{2\pi} \int_0^{\xi} \sin \psi d\psi d\phi.$$

It is obvious from the cosine law that

$$\xi = A \cos \frac{u^2 + y^2 - 1}{2yu},$$

the units of distance being the sphere radius. Then,

$$n(u) = \frac{3u^2}{2} \left(1 - \frac{u^2 + y^2 - 1}{2yu}\right) = \frac{3}{4} \frac{u}{y} \left[1 - (u-y)^2\right].$$

It is now clear why the flux will vary only slowly as one moves from the center to the surface of the sphere when the flux law varies steeply with distance. Since self-absorption serves to increase the steepness of the flux law, it serves to flatten the flux distribution even more.

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