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The Hamiltonian structure and Euler-Poincaré formulation of the Vlasov-Maxwell and gyrokinetic systems

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We present a new variational principle for the gyrokinetic system, similar to the Maxwell-Vlasov action presented in Ref. 1. The variational principle is in the Eulerian frame and based on *constrained* variations of the phase space fluid velocity and particle distribution function. Using a Legendre transform, we explicitly derive the field theoretic Hamiltonian structure of the system. This is carried out with the Dirac theory of constraints, which is used to construct meaningful brackets from those obtained directly from Euler-Poincaré theory. Possible applications of these formulations include continuum geometric integration techniques, large-eddy simulation models and Casimir type stability methods.

I. INTRODUCTION

An inherent difficulty in studying the dynamics of magnetized plasmas is the enormous separation of important time-scales present in many physical systems of interest. Nonlinear gyrokinetic theory has become an indispensable tool in these inquiries, as it removes the fastest time-scales from the system, while keeping much of important physics relevant to turbulent transport[2–4]. A particularly nice way to construct a gyrokinetic theory, pioneered in Refs. 5–7, is to use Lie-transforms to asymptotically change into co-ordinates in which gyro-orbit dynamics are decoupled from the rest of the system. A great advantage of this technique, aside from the entirely systematic and formal procedure, is that the single particle equations are guaranteed to be Hamiltonian, with associated conservation properties. Going further, it is advantageous from both a philosophical and practical standpoint to derive the entire system, including both electromagnetic fields and particles, from a single *field-theoretic* variational principle. These ideas were explored by Sugama[8] and Brizard[9], who derived gyrokinetic action principles starting from Maxwell-Vlasov theories, as well as in previous work in Refs. 10 and 11. Some advantages of this type of formulation are a much simplified derivation of the gyrokinetic Maxwell's equations and exact energy-momentum conservation laws through Noether's theorem. Field theories often admit many different variational principles (e.g., for Maxwell-Vlasov see Refs. 1, 12–15), each with its own advantages and disadvantages. A good example is the difference between Lagrangian and Eulerian actions; the former being constructed in variables that follow particle motion and the latter in variables at fixed points in phase space. It is interesting to explore new types of variational principles, both for the general understanding of the structure of the theory in question and for practical applications that may require an action of a particular form.

In this work, we present a new gyrokinetic action principle in Eulerian co-ordinates, using Euler-Poincaré reduction theory[16,17] on the Lagrangian action in Ref. 8 and 18. In addition, using the reduced Legendre transform and the Dirac theory of constraints, we derive field theoretic Poisson brackets, similar to the Vlasov-Maxwell[19] and Vlasov-Poisson[20] brackets. To our knowledge, this is the first explicit demonstration of the Hamiltonian structure of the gyrokinetic system. Our derivation proceeds from the action principle in Ref. 8 and its geometric formulation[18]. We do not purport to derive a gyrokinetic co-ordinate system, but rather formulate the theory based on a given single particle Lagrangian. In this way, it is trivial to extend concepts to deal with more complex gyrokinetic theories, for instance theories with self consistent, time-evolving background fields[18].

We then use the ideas in Ref. 1 to reduce the Lagrangian action to one in Eulerian co-ordinates, based on symmetry under the particle-relabeling map from Lagrangian to Eulerian variables. The variations for this new action principle in the Eulerian frame are *constrained,* and lead to the *Euler-Poincaré equations,* which are shown to give the standard gyrokinetic Vlasov equation. In some ways the action principle is similar to that of Brizard[9], in that constrained variations must be used, with both theories having a similar form for the variation of the distribution function, F. Nevertheless, there are significant differences, particularly that our principle is formulated in terms of the Eulerian phase space fluid velocity and is in standard 6-D phase space, rather than 8-D extended phase space. The Eulerian gyrokinetic action of Ref. 21 is quite different to that presented here, with unconstrained variations in 12-dimensional extended phase space and the use of Hamilton-Jacobi functions in the action functional. Equipped with the Eulerian action, a reduced Legendre transform is performed[1], leading straightforwardly to a Poisson bracket. However, this bracket must be *reduced* to a constraint submanifold before a meaningful form can be obtained, a process that is performed with the Dirac theory of constraints[22–24]. Finally, we show how to include the electromagnetic fields in the bracket via second application of Dirac theory.

One of the our primary motivations in this work is the possibility of utilizing recent ideas from fluid mechanics to develop advanced numerical tools for gyrokinetics. Of particular importance is the idea of geometric integrators, which are designed to numerically conserve various important geometrical properties of the physical system. For instance, having a numerical algorithm that has Hamiltonian structure can be very important, with profound consequences for the long-time conservation properties[25]. The theory of finite dimensional geometric integrators is relatively well developed[25], including an application to single particle guiding center dynamics[26–28]. However, many aspects of the construction of field-theoretic geometrical integrators are not as well understood, both for practical implementation and the deeper mathematical theory. One approach, which has yielded fruitful results, is to discretize a variational principle and perform variations on the discrete action to derive an integration scheme. Some examples of field theoretic integrators constructed in this way are those for elastomechanics[29,30] electromagnetism[31], fluids and magnetohydrodynamics[32,33] and a particle-in-cell (PIC) scheme for the Vlasov-Maxwell system[34]. The results presented in this work would be used to construct a continuum Eulerian gyrokinetic integrator, since our variational principle is in Eulerian form. Analogously, a variational principle in Lagrangian form is used to construct a Lagrangian (particle-in-cell) integrator[34]. We note that in discretizing a variational principle it is obviously not desirable to be in an extended phase space, unless these extra dimensions can somehow be removed after a discretization. As well as integrators, other potential applications of the formulation presented here are the use in stability calculations with Casimir invariants[35] and the construction of regularized models for large-eddy simulation[36–39].

The article is organized as follows. In Section II we clarify the differences between Eulerian and Lagrangian action principles for kinetic theories and explain the Euler-Poincaré formulation of the Maxwell-Vlasov system[1]. This is done with as little reference to the formal mathematics as possible, with the hope that readers unfamiliar with the concepts of Lie groups and algebras should understand the general structure of the theory. Section III explains the construction of the gyrokinetic variational principle, starting from a given single particle gyrokinetic Lagrangian. We give a brief derivation of the Euler-Poincaré equations and show how these lead to a standard form of the gyrokinetic equations. The Hamiltonian structure is dealt with in Section IV. After formally constructing a Poisson bracket from the Lagrangian, we describe how the Dirac theory of constraints is used to reduce the bracket to a meaningful form. Finally, numerical applications are briefly discussed in Section V and conclusions given in Section VI.

Throughout this article we use cgs units. In integrals and derivatives, $\boldsymbol{z}$ denotes all phase space variables, while $\boldsymbol{x}$ denotes just position space variables. Species labels are left out for clarity and implied on the variables F (or f), m, e, $\boldsymbol{U}$ and $\boldsymbol{M}$, respectively the distribution function, particle mass, particle charge, Eulerian fluid velocity and momenta conjugate to $\boldsymbol{U}$. Summation notation is utilized where applicable, with capital indices spanning $1 \to 6$ and lower case indices $1 \to 3$.

II. EULERIAN AND LAGRANGIAN KINETIC VARIATIONAL PRINCIPLES

When formulating a variational principle for a continuum fluid-type theory, it is very important to specify whether Lagrangian or Eulerian variables are being used. These notions can be confusing in kinetic plasma theories, since one must consider the motion of the *phase-space* fluid. In addition, unlike the Euler fluid equations, the equations of motion for kinetic plasma theories have the same form in Eulerian and Lagrangian co-ordinates. Considering the Vlasov-Maxwell system for simplicity, a Lagrangian description gives the equation of motion at the position of a particle carried along by the flow (simply a physical particle). One formulates a variational principle in terms of the fields $\boldsymbol{x}(\boldsymbol{x}_0, \boldsymbol{v}_0, t)$, $\boldsymbol{v}(\boldsymbol{x}_0, \boldsymbol{v}_0, t)$, which are the current position and velocity of an element of phase space that was initially at $(\boldsymbol{x}_0, \boldsymbol{v}_0)$. The distribution function is of course just carried

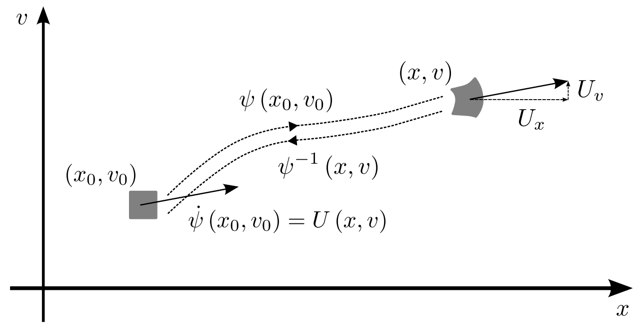

FIG. 1. Illustration of the particle relabelling map, $\psi(\boldsymbol{x}_0, \boldsymbol{v}_0)$ and its inverse for the one-dimensional Vlasov-Poisson system.

along by the Lagrangian co-ordinates, i.e., $f(\boldsymbol{x}(\boldsymbol{x}_0, \boldsymbol{v}_0, t), \boldsymbol{v}(\boldsymbol{x}_0, \boldsymbol{v}_0, t)) = f_0(\boldsymbol{x}_0, \boldsymbol{v}_0)$. This type of formulation is the most natural for a kinetic theory, since it is the logical continuum generalization of the action principle for a collection of particles interacting with an electromagnetic field.

An Eulerian variational principle is formulated in terms of the velocity of the phase space fluid at a fixed point, $\boldsymbol{U}$, without the notion of where phase space density has been in the past. Thus, at a point $(\boldsymbol{x}, \boldsymbol{v})$, the $\boldsymbol{x}$ component of the fluid velocity is simply $\boldsymbol{v}$ (the co-ordinate), while the $\boldsymbol{v}$ component is $\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}/c$. The distribution function f, is *advected* by $\boldsymbol{U}$, meaning it is the solution to the differential equation $\partial_t f = -\mathcal{L}_U f = -\boldsymbol{U} \cdot \nabla f$, where $\boldsymbol{U}$and ∇ are in six-dimensional phase space. An illustration of these concepts is given in Figure 1 for the 1-D Poisson-Vlasov system (in 2-D phase space). Finally, we note that in discussing the distinction between Eulerian and Lagrangian actions, we refer only to the plasma component of the variational principle; the electromagnetic fields are always in Eulerian co-ordinates.

A. Euler-Poincaré reduction

This section gives a very informal introduction to Euler-Poincaré theory through a brief review of the Vlasov-Maxwell formulation presented in Ref. 1. We purposefully do not use precise mathematics and notation (e.g., the $\diamond$ and $\mathrm{ad}^\star$ operations) so as to introduce the general ideas to readers not familiar with Lie groups and algebras. A more formal exposition of the mathematical foundations can be found in, for example Refs. 1, 16, and 17.

The purpose of the Euler-Poincaré framework is to provide a straightforward method to pass from a Lagrangian to an Eulerian action principle. Mathematically, the important idea is that the field theory dynamics of the plasma take place on the tangent bundle of the infinite dimensional group of symplectic diffeomorphisms, $G = \text{Diff}\left(T\mathbb{R}^3\right)$ and the Lagrangian is symmetric under right action of this same group. For practical purposes, $\psi \in G$ is simply the particle relabeling map, $\psi(\boldsymbol{x}_0, \boldsymbol{v}_0) = (\boldsymbol{x}(\boldsymbol{x}_0, \boldsymbol{v}_0), \boldsymbol{v}(\boldsymbol{x}_0, \boldsymbol{v}_0))$, which maps plasma particles with initial position $(\boldsymbol{x}_0, \boldsymbol{v}_0)$ to their current position $(\boldsymbol{x}, \boldsymbol{v})$. The distribution function, f, lives on a separate vector space $V^\star$, and ψ acts on f on the right such that $f = f_0\,\psi^{-1}$, where f_0 is the initial distribution function. This equation is simply $f(\boldsymbol{x}(\boldsymbol{x}_0, \boldsymbol{v}_0, t), \boldsymbol{v}(\boldsymbol{x}_0, \boldsymbol{v}_0, t)) = f_0(\boldsymbol{x}_0, \boldsymbol{v}_0)$, as discussed above. ψ acts trivially on the electromagnetic potentials, ϕ and $\boldsymbol{A}$, which are located on a separate manifold Q. The phase space fluid velocity in the Lagrangian frame is simply $\dot{\psi}(\boldsymbol{x}_0, \boldsymbol{v}_0)$, since this is the rate of change of $(\boldsymbol{x}, \boldsymbol{v})$ at the position $(\boldsymbol{x}, \boldsymbol{v})$. In contrast, the Eulerian phase space fluid velocity is $\dot{\psi}\psi^{-1}$, since this operation first takes $(\boldsymbol{x}, \boldsymbol{v})$ back to $(\boldsymbol{x}_0, \boldsymbol{v}_0)$ with ψ^{-1}, then gives the velocity at $(\boldsymbol{x}, \boldsymbol{v})$ with $\dot{\psi}(\boldsymbol{x}_0, \boldsymbol{v}_0)$, see Figure 1. Mathematically, the Eulerian fluid velocity is an element of the Lie algebra, $\mathfrak{g}$, associated with G and the process of Euler-Poincaré reduction takes a Lagrangian on $TG \times V^\star \times TQ$ to a reduced Lagrangian on $\mathfrak{g} \times V^\star \times TQ$

The starting point for the reduction is a Lagrangian Lagrangian for the Vlasov-Maxwell system. For instance,

$$\begin{aligned} L = \sum_s \int d\boldsymbol{x}_0 d\boldsymbol{v}_0\, f_0 \left[\left(\frac{e}{c}\boldsymbol{A}(\boldsymbol{x}) + m\boldsymbol{v}\right)\cdot\dot{\boldsymbol{x}} - \frac{1}{2}m\boldsymbol{v}^2 - e\phi(\boldsymbol{x})\right] \\ + \frac{1}{8\pi}\int d\boldsymbol{x}\left[\left|-\nabla\phi - \frac{\partial\boldsymbol{A}}{\partial t}\right|^2 - |\nabla\times\boldsymbol{A}|^2\right], \end{aligned} \tag{1}$$

which is very similar to the action principle of Low[14]. The Vlasov equation follows from the standard Euler-Lagrange equations for $\psi = (\boldsymbol{x}, \boldsymbol{v})$,

$$\frac{d}{dt}\frac{\delta L}{\delta\dot{\psi}} - \frac{\delta L}{\delta\psi} = 0, \tag{2}$$

along with $f(\boldsymbol{x}, \boldsymbol{v}, t) = f_0(\boldsymbol{x}_0, \boldsymbol{v}_0)$. Maxwell's equations come from the Euler-Lagrange equations for $\boldsymbol{A}$ and ϕ. This Lagrangian is right invariant with respect to the action of G, i.e.,

$$\begin{aligned} L_{f_0}\left(\psi, \dot{\psi}, \phi, \dot{\phi}, \boldsymbol{A}, \dot{\boldsymbol{A}}\right) &= L_{f_0\psi^{-1}}\left(\psi\psi^{-1}, \dot{\psi}\psi^{-1}, \phi, \dot{\phi}, \boldsymbol{A}, \dot{\boldsymbol{A}}\right) \\ &\equiv l\left(\boldsymbol{U}, \phi, \dot{\phi}, \boldsymbol{A}, \dot{\boldsymbol{A}}, F\right), \end{aligned} \tag{3}$$

where $\boldsymbol{U} = \dot{\psi}\psi^{-1} \in \mathfrak{g}$ is the Eulerian fluid velocity, a vector field. In recognizing that the distribution function is actually a phase space *density*, we denote $F = f\,d\boldsymbol{x}\wedge d\boldsymbol{v}$. Treating F as 6-form rather than a scalar changes the form of certain geometrical operations in the Euler-Poincaré equations [Eqs. (6) and (7)] and is very important for the gyrokinetic Euler-Poincaré treatment (see Section III). Practically speaking, to construct the reduced Lagrangian, l, one simply replaces $(\dot{\boldsymbol{x}}, \dot{\boldsymbol{v}})$ with $\boldsymbol{U}$, and considers $\boldsymbol{x}$ and $\boldsymbol{v}$ to be co-ordinates rather than fields. Thus,

$$l = \sum_s \int F\left[\left(\frac{e}{c}\boldsymbol{A} + m\boldsymbol{v}\right)\cdot \boldsymbol{U}_x - \frac{1}{2}m\boldsymbol{v}^2 - e\phi(\boldsymbol{x})\right] + \frac{1}{8\pi}\int d\boldsymbol{x}\left[\left|-\nabla\phi - \frac{\partial \boldsymbol{A}}{\partial t}\right|^2 - |\nabla\times\boldsymbol{A}|^2\right], \tag{4}$$

where $\boldsymbol{U}_x$ denotes the $\boldsymbol{x}$ components of $\boldsymbol{U}$. The equations of motion are derived from the reduced Lagrangian l, by considering how the unconstrained variations of ψ (used to derive the standard Euler-Lagrange equations) translate into *constrained* variations of $\boldsymbol{U}$ and F. This leads to variations of the form

$$\delta\boldsymbol{U} = \frac{\partial\eta}{\partial t} - [\boldsymbol{U},\eta],\ \delta F = -\mathcal{L}_\eta F, \tag{5}$$

where $\eta \in \mathfrak{g}$ (i.e., in the same space as $\boldsymbol{U}$) and vanishes at the endpoints; and $[\,,]$ is the standard Lie bracket, $\boldsymbol{U}.\nabla\eta - \eta.\nabla\boldsymbol{U}$. Evolution of F is given by the advection equation

$$\frac{\partial F}{\partial t} + \mathcal{L}_U F = 0, \tag{6}$$

which arises from the group action on F. Variation of $\int dt\, l$ with $\delta\boldsymbol{U}$ and δF leads to the *Euler-Poincaré equations*,

$$\frac{\partial}{\partial t}\frac{\delta l}{\delta\boldsymbol{U}} = -\mathcal{L}_U\frac{\delta l}{\delta\boldsymbol{U}} + F\nabla\frac{\delta l}{\delta F}, \tag{7}$$

where $\frac{\delta l}{\delta U} \in \mathfrak{g}^\star$ is a 1-form density. We give derivations of Eqs. (5) and (7) in Section III A below. Since F is a 6-form, $\mathcal{L}_U F = \nabla\cdot(F\boldsymbol{U})$ and Eq. (6) is the conservative form of the Vlasov equation (see Section III A for more information). The equations for $\boldsymbol{A}$ and ϕ are just the standard Euler-Lagrange equations. Calculation of Eq. (7) with the Vlasov-Maxwell reduced Lagrangian [Eq. (4)] leads to

$$\boldsymbol{U}_x = \boldsymbol{v},\ \boldsymbol{U}_v = \boldsymbol{E} + \frac{1}{c}\boldsymbol{v}\times\boldsymbol{B}, \tag{8}$$

as expected. The fact that there is no need to solve differential equations for components of $\boldsymbol{U}$ is related to the strong degeneracy in the system (see Sections III and IV below).

To obtain the Hamiltonian or *Lie-Poisson* form of the equations, one performs a reduced Legendre transform as,

$$h = \langle \boldsymbol{M}, \boldsymbol{U} \rangle + \int d\boldsymbol{x}\, \boldsymbol{A} \cdot \frac{\delta l}{\delta \dot{\boldsymbol{A}}} - l, \tag{9}$$

where $\boldsymbol{M} = \delta l / \delta \boldsymbol{U}$ and the inner product $\langle\, , \rangle$ is integration over phase space [see Eq. (31)]. (A thorough treatment of the degeneracies of the system is given in Ref. 1.) It is then straightforward to show that

$$\begin{aligned}
\{\Gamma, \Theta\}_{LP} = &- \sum_s \int dz \boldsymbol{M} \cdot \left[\frac{\delta \Gamma}{\delta \boldsymbol{M}}, \frac{\delta \Theta}{\delta \boldsymbol{M}} \right] \\
&+ \sum_s \int dz F \left(\frac{\delta \Theta}{\delta \boldsymbol{M}} \cdot \nabla \frac{\delta \Gamma}{\delta F} - \frac{\delta \Gamma}{\delta \boldsymbol{M}} \cdot \nabla \frac{\delta \Theta}{\delta F} \right) \\
&- 4\pi c \int d\boldsymbol{x} \left(\frac{\delta \Gamma}{\delta \boldsymbol{A}} \cdot \frac{\delta \Theta}{\delta \boldsymbol{E}} - \frac{\delta \Theta}{\delta \boldsymbol{A}} \cdot \frac{\delta \Gamma}{\delta \boldsymbol{E}} \right),
\end{aligned} \tag{10}$$

is an infinite dimensional Poisson bracket for the system; that is, $\dot{\boldsymbol{M}} = \{\boldsymbol{M}, h\}$, $\dot{F} = \{F, h\}$, $\dot{\boldsymbol{E}} = \{\boldsymbol{E}, h\}$ and $\dot{\boldsymbol{A}} = \{\boldsymbol{A}, h\}$ are formally the same as the Euler-Poincaré equations (using the generalized Legendre transform of Ref. 1), and the Jacobi identity is satisfied. Nevertheless, this manifestation of the bracket has major problems. In particular, the meaning of functional derivatives with respect to the $\boldsymbol{M}$ variables can be unclear, since these are constrained due to the linearity of the Lagrangian in $\boldsymbol{U}$. To overcome these problems and formulate a meaningful bracket on the space of plasma densities and electromagnetic fields, we use the Dirac theory of constraints[22,40]. A very brief overview of this is given in the appendix for the convenience of the reader.

The relevant constraints are

$$\begin{aligned}
\Phi_i =& M_i - F \left(\frac{e}{c} A_i + m v_i \right) = 0, \quad i = 1 \to 3, \\
\Phi_j =& M_j = 0, \quad j = 4 \to 6.
\end{aligned} \tag{11}$$

We form the constraint matrix $C_{IJ}(z, z') = \left\{ \Phi_i(z), \Phi_j(z) \right\}$ and construct the inverse according to

Eq. (A2). This comes out to be,

$$C_{IJ}^{-1}(z,z') = \frac{1}{mF(z)}\delta(z-z')\delta_{ss'}$$
$$\times\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & \frac{e}{mc}B_z & -\frac{e}{mc}B_y \\ 0 & -1 & 0 & -\frac{e}{mc}B_z & 0 & \frac{e}{mc}B_x \\ 0 & 0 & -1 & \frac{e}{mc}B_y & -\frac{e}{mc}B_x & 0 \end{pmatrix}, \tag{12}$$

which is simply the single particle Poisson matrix (multiplied by $\delta(z-z')/F$). We will see a similar connection to the single particle Poisson bracket in the reduction of the gyrokinetic bracket. We then use Eq. (A3) and restrict the functionals Γ and Θ to not depend $\boldsymbol{M}$, i.e., $\delta\Gamma/\delta\boldsymbol{M} = 0$. Heuristically, this can be understood as the requirement that all of the F dependence in the functionals is explicit and thus contained in $\delta/\delta F$. Including any $\delta/\delta\boldsymbol{M}$ term (e.g., by using the chain rule for $\boldsymbol{M}[F]$) would count this dependence twice. This argument breaks down for $\boldsymbol{M}_v$, as these variables are constrained to zero and the functional derivatives are undefined. However, all terms involving $\delta/\delta\boldsymbol{M}_v$ cancel when the full calculation of Eq. (A3) is carried out, so there is no issue.

The final result, including a change of variables from $\boldsymbol{A}$ to $\boldsymbol{B}$, is the Poisson bracket for the Mawell-Vlasov system,

$$\begin{aligned}\{\Gamma,\Theta\} = &\sum_s \frac{1}{m}\int dzF\left(\frac{\partial\Gamma_F}{\partial\boldsymbol{x}}\cdot\frac{\partial\Theta_F}{\partial\boldsymbol{v}} - \frac{\partial\Theta_F}{\partial\boldsymbol{x}}\cdot\frac{\partial\Gamma_F}{\partial\boldsymbol{v}}\right) \\ &+ \sum_s \frac{e}{cm^2}\int dzF\boldsymbol{B}\cdot\frac{\partial\Gamma_F}{\partial\boldsymbol{v}}\times\frac{\partial\Theta_F}{\partial\boldsymbol{v}} \\ &+ 4\pi\sum_s\frac{e}{m}\int dz\left[\Theta_F\frac{\partial}{\partial\boldsymbol{v}}\cdot\left(F\frac{\delta\Gamma}{\delta\boldsymbol{E}}\right) - \Gamma_F\frac{\partial}{\partial\boldsymbol{v}}\cdot\left(F\frac{\delta\Theta}{\delta\boldsymbol{E}}\right)\right] \\ &+ 4\pi c\int d\boldsymbol{x}\left(\frac{\partial\Gamma}{\partial\boldsymbol{E}}\cdot\nabla\times\frac{\delta\Theta}{\delta\boldsymbol{B}} - \frac{\partial\Theta}{\partial\boldsymbol{E}}\cdot\nabla\times\frac{\delta\Gamma}{\delta\boldsymbol{B}}\right).\end{aligned} \tag{13}$$

In the case where $\delta\Gamma/\delta\boldsymbol{E}$ has no $\boldsymbol{v}$ dependence this bracket is identical to that calculated in Ref. 19 via alternative methods. The derivation above explicitly shows the link between this and the work of Ref. 1. Somewhat more detail is given for the derivation of the gyrokinetic bracket (see Section IV), which proceeds in a very similar manner.

Euler-Poincaré reduction is perhaps more natural when applied to fluid systems. In this case, there are fewer degeneracies and the Lagrangian/Eulerian distinction is more obviously relevant (e.g. one does not *measure* phase space fluid velocities, in contrast to the Eulerian fluid velocity of a fluid system). Applying the procedure to ideal magnetohydrodynamics leads to the 1962 action principle of Newcomb[41]. Recently, similar ideas have been applied to general reduced fluid and hybrid models in plasma physics[42,43].

III. GYROKINETIC VARIATIONAL PRINCIPLE

Our starting point is the geometric approach to gyrokinetic theory advocated in Ref. 18. The general idea is to construct a field theory, including electromagnetic potentials, from the particle Poincaré-Cartan 1-form, γ. This approach is conceptually very simple; once the interaction of quasi-particles with the electromagnetic field is specified, particle and field equations follow in straightforward and transparent way via the Euler-Lagrange equations. With any desired approximation (e.g., expansion in gyroradius), energetically self-consistent equations are easily obtained without necessitating the use of the pullback operator. The use of these ideas in gyrokinetic simulation has been advocated in, for instance Refs. 44 and 47. In this article we consider the particle 1-form γ as given, its derivation can be found in Refs. 2, 8, and 18 among other works.

The Poincaré-Cartan form γ in 7-D phase space, P, (including time) defines particle motion through Hamilton's equation,

$$i_\tau d\gamma = 0, \tag{14}$$

which is derived from stationarity of the action $\mathcal{A}_{sp} = \int \gamma$. Here τ is a vector field whose integrals define particle trajectories (including the time component) and i denotes the inner product. Note that γ is essentially just $L_{sp}dt$, where L_{sp} is the standard Lagrangian; that is, for $\gamma = \gamma_\alpha dz^\alpha - Hdt$, the Lagrangian is simply $L_{sp} = \gamma_\alpha \dot{z}^\alpha - H$. To construct a field theory, γ is used to define the Louiville 6-form,

$$\Omega_T = -\frac{1}{3!} d\gamma \wedge d\gamma \wedge d\gamma. \tag{15}$$

The Liouville theorem of phase space volume conservation is then simply, $\mathcal{L}_\tau \Omega_T = 0$. Introducing the distribution function of particles in phase space f, the field theory action for the interaction of a field of particles with the electromagnetic field is

$$\mathcal{A} = 4\pi \int f\Omega_T \wedge \gamma + \int dx \mathcal{L}_{EM}, \tag{16}$$

where $\mathcal{L}_{\mathcal{EM}}$ is the electromagnetic Lagrangian density. In this action γ is in the Lagrangian frame. For example, in cartesian position and velocity space, $f\Omega$ is in $(\boldsymbol{x}_0, \boldsymbol{v}_0)$ co-ordinates, while γ is in $(\boldsymbol{x}(\boldsymbol{x}_0, \boldsymbol{v}_0), \boldsymbol{v}(\boldsymbol{x}_0, \boldsymbol{v}_0))$ co-ordinates.

Unless general relativity is important, one can choose τ to be of the form $\tau = \partial/\partial t + \tau_Z$, where τ_Z has no time component, and consider 6-D phase space. Defining Ω to be the $d\boldsymbol{X} \wedge d\boldsymbol{P}$ component of Ω_T (i.e., no $\wedge dt$), the $d\boldsymbol{X} \wedge d\boldsymbol{P}$ component of $\mathcal{L}_\tau \Omega_T$ is the standard Liouville theorem of phase space volume conservation,

$$\frac{\partial}{\partial t}\Omega + \mathcal{L}_{\tau_Z}\Omega = 0. \tag{17}$$

We can simplify the variational principle by considering γ to be $L_{sp}dt$ and carrying out the wedge product. This type of procedure provides a generalization of the original variational principle of Low[14] [e.g., Eq. (1)] to arbitrary particle-field interaction.

We now specialize to a general gyrokinetic form for the particle Lagrangian,

$$\gamma = \frac{e}{c}\boldsymbol{A}^\dagger(\boldsymbol{X}) \cdot d\boldsymbol{X} + \frac{mc}{e}\mu d\theta - H dt, \tag{18}$$

with

$$\boldsymbol{A}^\dagger(\boldsymbol{X}) = \boldsymbol{A} + \frac{mc}{e}u\boldsymbol{b} - \frac{mc^2}{e^2}\mu\left(\boldsymbol{R} + \frac{1}{2}\boldsymbol{b}\,\boldsymbol{b}\cdot\nabla\times\boldsymbol{b}\right). \tag{19}$$

Here, $\boldsymbol{X}$ is the gyrocenter position, u the gyrocenter parallel velocity co-ordinate, μ the conserved magnetic moment and θ the gyrophase. The vector field $\boldsymbol{A}(\boldsymbol{X})$ is the vector potential of the background magnetic field and $\boldsymbol{b}(\boldsymbol{X})$ is the background magnetic field unit vector. These fields will not be considered variables in the field theory action. The vector $\boldsymbol{R}(\boldsymbol{X}) = \nabla\boldsymbol{e}_1 \cdot \boldsymbol{e}_2$, where $\boldsymbol{e}_1(\boldsymbol{X}) \perp \boldsymbol{e}_2(\boldsymbol{X}) \perp \boldsymbol{b}(\boldsymbol{X})$, is necessary for gyrogauge invariance of the Lagrangian, i.e., invariance with respect to a change in the definition of the θ co-ordinate. Eq. (18) is accurate to first order in ϵ_B, the ratio of the gyroradius to the scale length of the magentic field[2]. The single particle Hamiltonian, $H = \frac{1}{2}mu^2 + \mu B(\boldsymbol{X}) + H_{gy}$, contains both the guiding center contribution, $\frac{1}{2}mu^2 + \mu B(\boldsymbol{X})$, and the gyrocenter contribution from the fluctuating fields, H_{gy}. For most of this article H_{gy} will be taken to be a general function of $(\boldsymbol{X}, u, \mu)$. Different forms exist in the literature, depending on desired accuracy and fluctuation model used. For instance, in Ref. 8, H_{gy} is given to second order in ϵ_δ (the ratio of the magnitudes of the fluctuating fields, ϕ_1 and $\boldsymbol{A}_1$, to the background field) as

$$H_{gy} = e\left\langle \psi\left(\boldsymbol{X}+\boldsymbol{\rho}\right)\right\rangle + \frac{e^2}{2mc^2}\left\langle \left|\boldsymbol{A}\left(\boldsymbol{X}+\boldsymbol{\rho}\right)\right|^2\right\rangle \\ - \frac{e}{2}\left\langle\left\{\tilde{S}_1, \tilde{\psi}\right\}\right\rangle, \tag{20}$$

where $\psi = \phi_1 - \frac{1}{c}\boldsymbol{v}\cdot\boldsymbol{A}_1$, $\boldsymbol{\rho}$ is difference between the particle and gyrocenter positions, and $\langle\rangle$ denotes an average over θ. The tilde in $\left\langle\left\{\tilde{S}_1, \tilde{\psi}\right\}\right\rangle$ denotes the gyrophase dependent part of a function and S_1 is a gauge function associated with the first order gyrocenter perturbation, $\left\langle \psi\left(\boldsymbol{X}+\boldsymbol{\rho}\right)\right\rangle$. Eq. (18) is the standard gyrokinetic single particle Lagrangian in *Hamiltonian* form[2], meaning all the fluctuating field perturbations are in the Hamiltonian part (dt component) of γ. This is the form most suitable for computer simulation[2,44] and also has the advantage of having the same Poisson structure as the guiding center equations.

To form a Lagrangian field theory we first calculate the phase space component ($d\boldsymbol{X}\wedge du\wedge d\mu\wedge d\theta$ component) of the volume element $\Omega = -\frac{1}{3}d\gamma\wedge d\gamma\wedge d\gamma$. This is simply $B_\parallel^\dagger/m = \boldsymbol{b}\cdot\boldsymbol{B}^\dagger/m$, with $\boldsymbol{B}^\dagger = \nabla\times\boldsymbol{A}^\dagger$, i.e., the standard guiding center Jacobian. In co-ordinates, the variational principle Eq. (16) is then simply,

$$\begin{aligned}\mathcal{A} &= \int dt\, L_{GK} \\ &= \sum_s \int dt \int d\boldsymbol{X}_0\wedge du_0\wedge d\mu_0\wedge d\theta_0 \frac{1}{m}B_\parallel^\dagger f_0 \\ &\times\left[\frac{e}{c}\boldsymbol{A}^\dagger\cdot\dot{\boldsymbol{X}} + \frac{mc}{e}\mu\dot{\theta} - H\right] + \int dt\, L_{EM},\end{aligned} \tag{21}$$

which is essentially the original gyrokinetic variational principle of Sugama[8]. L_{EM} should be chosen for the Ampére-Poisson system (to remove fast time-scale electromagnetic waves) as

$$L_{EM} = \frac{1}{8\pi}\int d\boldsymbol{x}\left(\left|\nabla\phi_1\right|^2 - \left|\nabla\times\left(\boldsymbol{A}+\boldsymbol{A}_1\right)\right|^2\right), \tag{22}$$

where $\boldsymbol{x} = \boldsymbol{X}+\boldsymbol{\rho}$.

A. Eulerian Gyrokinetic variational principle

We proceed in the reduction of the gyrokinetic variational principle, Eq. (21), in a very similar way to the Vlasov-Maxwell case (Section II). The advected parameter is the 6-form $f\Omega = d\boldsymbol{X} \wedge du \wedge d\mu \wedge d\theta\, B_{\|}^{\dagger} f/m \equiv \hat{F} d\boldsymbol{X} \wedge du \wedge d\mu \wedge d\theta$. $\hat{F}$ is often considered a function for simplicity of notation, but it is understood that operations should be carried out as for a 6-form, e.g., $\mathcal{L}_{\boldsymbol{U}} \hat{F} = \nabla \cdot \left(\hat{F}\boldsymbol{U}\right)$ rather than $\mathcal{L}_{\boldsymbol{U}} \hat{F} = \boldsymbol{U} \cdot \nabla \hat{F}$. The connection to the standard distribution function is provided by the Liouville theorem; the advection equation

$$\partial_t (f\Omega) + \mathcal{L}_{\boldsymbol{U}} (f\Omega) = 0, \tag{23}$$

coupled with Liouville's theorem, Eq. (17), implies

$$\partial_t f + \mathcal{L}_{\boldsymbol{U}} f = 0, \tag{24}$$

which is the standard Vlasov equation.

Operating on Eq. (21) on the right with the particle-relabeling map, ψ^{-1}, leads to the reduced Lagrangian

$$\begin{aligned} l_{GK}\left(\boldsymbol{U}, \phi_1, \boldsymbol{A}_1, \hat{F}\right) &= L_{GK}\left(\psi\psi^{-1}, \dot{\psi}\psi^{-1}, \phi_1, \boldsymbol{A}_1, f_0\Omega\psi^{-1}\right) \\ &= \sum_s \int d\boldsymbol{X} du d\mu d\theta\, \hat{F}\left(\frac{e}{c}\boldsymbol{A}^{\dagger} \cdot \boldsymbol{U}_X + \frac{mc}{e}\mu U_\theta - H\right) \\ &\quad + \frac{1}{8\pi}\int d\boldsymbol{x}\left(|\nabla\phi_1|^2 - |\nabla \times (\boldsymbol{A} + \boldsymbol{A}_1)|^2\right), \end{aligned} \tag{25}$$

where $\boldsymbol{U}_X$ and U_θ are the $\boldsymbol{X}$ and θ components of the Eulerian fluid velocity $\boldsymbol{U}$.

The unconstrained variations in the Lagrangian frame, $\delta\psi$ lead to constrained variations in the Eulerian frame by defining[17,41]

$$\boldsymbol{\eta}(z, t) = \delta\psi(z_0, t), \tag{26}$$

or equivalently $\boldsymbol{\eta} = \delta\psi\psi^{-1}$. Recaling $\boldsymbol{U}(z, t) = \dot{\psi}(z_0, t)$, one then calculates $d\boldsymbol{\eta}/dt$ and $\delta\boldsymbol{U}$, giving

$$\delta\dot{\psi}(z_0, t) = \frac{\partial\boldsymbol{\eta}(z, t)}{\partial t} + \dot{z}^j \frac{\partial\boldsymbol{\eta}(z, t)}{\partial z^j}, \tag{27}$$

$$\delta\dot{\psi}(z_0, t) = \delta\boldsymbol{U}(z, t) + \delta z^j \frac{\partial\boldsymbol{U}(z, t)}{\partial z^j}, \tag{28}$$

which is solved for

$$\delta \boldsymbol{U} = \frac{\partial \boldsymbol{\eta}}{\partial t} + \boldsymbol{U} \cdot \nabla_z \boldsymbol{\eta} - \boldsymbol{\eta} \cdot \nabla_z \boldsymbol{U}, \tag{29}$$

giving the variational form stated in Eq. (5). Similarly, using $f\Omega(z) = f_0\Omega_0(z_0)$ and $\delta(f_0\Omega_0) = 0$, one obtains

$$\delta(f\Omega) = -\mathcal{L}_\eta(f\Omega). \tag{30}$$

Using Eqs. (29) and (30), we give a basic derivation of the Euler-Poincaré equations for a general Lagrangian with an advected volume form, $\hat{F}$.

$$\begin{aligned}
\delta \int dt\, l &= \int dt \left[\left\langle \frac{\delta l}{\delta \boldsymbol{U}}, \delta \boldsymbol{U} \right\rangle_{\mathfrak{g}} + \left\langle \frac{\delta l}{\delta \hat{F}}, \delta \hat{F} \right\rangle_V \right] \\
&= \int dt \left[\left\langle \frac{\delta l}{\delta \boldsymbol{U}}, \left(\frac{\partial \eta}{\partial t} + [\boldsymbol{U}, \eta] \right) \right\rangle_{\mathfrak{g}} + \left\langle \frac{\delta l}{\delta \hat{F}}, \mathcal{L}_\eta \hat{F} \right\rangle_V \right] \\
&= \int dt \sum_s \int d\boldsymbol{X} du d\mu d\theta \left\{ -\frac{\partial}{\partial t} \frac{\delta l}{\delta \boldsymbol{U}} \eta^i - \left[\frac{\delta l}{\delta U^j} \partial_i U^j \right.\right. \\
&\quad \left.\left. + \partial_j \left(\frac{\delta l}{\delta U^i} U^j \right) \right] \eta^i + \eta^i \hat{F} \partial_i \frac{\delta l}{\delta \hat{F}} \right\} \\
&= \int dt \left\langle -\frac{\partial}{\partial t} \frac{\delta l}{\delta \boldsymbol{U}} - \mathcal{L}_U \frac{\delta l}{\delta \boldsymbol{U}} + \hat{F} \nabla \frac{\delta l}{\delta \hat{F}}, \eta \right\rangle_{\mathfrak{g}},
\end{aligned} \tag{31}$$

giving Eq. (7) since η is arbitrary. The two brackets used are defined as $\langle \mu, \xi \rangle_{\mathfrak{g}} = \sum_s \int d\boldsymbol{X} du d\mu d\theta\, \mu_i \xi^i$ between a 1-form density μdz and a vector field ξ; and $\left\langle f, \hat{F} \right\rangle_v = \sum_s \int f \hat{F}$ between a function f and a volume form $\hat{F}$. Integration by parts is used, with boundary terms dropped, in arriving at the third line. Note that $\hat{F}$ sometimes includes the volume element (lines 1 and 2) and sometimes does not (lines 3 and 4). A more precise derivation can be found in Refs. 1 and 16.

It is now simple to write down the equations of motion for $\boldsymbol{U}$, using

$$\begin{aligned}
&\frac{\delta l_{GK}}{\delta \boldsymbol{U}_X} = \frac{e}{c} \hat{F} \boldsymbol{A}^\dagger, \ \frac{\delta l_{GK}}{\delta U_\theta} = \frac{mc}{e} \mu \hat{F} \\
&\frac{\delta l_{GK}}{\delta U_u} = \frac{\delta l_{GK}}{\delta U_\mu} = 0.
\end{aligned} \tag{32}$$

The derivation is carried out without assumptions about the form of $\boldsymbol{U}$ (e.g., lack of θ dependence) and the $\hat{F}$ advection equation [Eq. (23)] is used to cancel time derivatives. We illustrate the general form of the calculation with the $\delta l/\delta U_X^i$ component of Eq. (7),

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{e}{c}\hat{F}A_i^\dagger\right) &= -\frac{e}{c}A_i^\dagger\frac{\partial}{\partial Z^J}\left(\hat{F}U^J\right) - \frac{e}{c}\hat{F}U^J\frac{\partial A_i^\dagger}{\partial Z^J}\\ &-\frac{e}{c}\hat{F}A_j^\dagger\frac{\partial U^j}{\partial X^i} - \frac{mc}{e}\mu\hat{F}\frac{\partial U_\theta}{\partial X^i} + \hat{F}\frac{\partial}{\partial X^i}\left(\frac{e}{c}\boldsymbol{A}^\dagger\cdot\boldsymbol{U}_X\right.\\ &\left.+\frac{mc}{e}\mu U_\theta - \frac{1}{2}mu^2 - \mu B - H_{gy}\right).\end{aligned}\tag{33}$$

The first two terms in Eq. (33) add to zero due to the advection equation, while the terms involving U_θ cancel. Rearranging and expanding the divergence term leads to

$$\frac{e}{c}\boldsymbol{U}_X\times\boldsymbol{B}^\dagger - mU_u\boldsymbol{b} - \mu\nabla B - \nabla H_{gy} = 0,\tag{34}$$

which gives

$$U_u = -\frac{\boldsymbol{B}^\dagger}{mB_\parallel^\dagger}\cdot\left(\mu\nabla B + \nabla H_{gy}\right)\tag{35}$$

and

$$\boldsymbol{U}_X = \frac{\boldsymbol{B}^\dagger}{B_\parallel^\dagger}\boldsymbol{U}_X\cdot\boldsymbol{b} + \frac{c}{eB_\parallel^\dagger}\boldsymbol{b}\times\left(\mu\nabla B + \nabla H_{gy}\right)\tag{36}$$

when $\boldsymbol{B}^\dagger\cdot$ and $\boldsymbol{b}\times$ are applied respectively. Similarly, the other $\delta l_{GK}/\delta U^J$ equations give

$$U_\mu = 0,\tag{37}$$

$$\boldsymbol{b}\cdot\boldsymbol{U}_X = u + \frac{1}{m}\frac{\partial H_{gy}}{\partial u},\tag{38}$$

$$U_\theta = \frac{eB}{mc} + \frac{e^2}{mc^2}\boldsymbol{U}_X\cdot\frac{\partial\boldsymbol{A}^\dagger}{\partial\mu} + \frac{e}{mc}\frac{\partial H_{gy}}{\partial\mu},\tag{39}$$

and Eq. (38) is combined wiht Eq. (36) to give $\boldsymbol{U}_X$ in terms of co-ordinates. The form of Eqs. (35)-(39) is identical to the standard Lagrangian equations for $\left(\dot{\boldsymbol{X}}, \dot{u}, \dot{\mu}, \dot{\theta}\right)$ because of the linearity of the Lagrangian in $\boldsymbol{U}$. Note that various terms have different origins in the Lagrangian and Eulerian derivations; for instance, $\partial\boldsymbol{X}/\partial t = 0$ in the Eulerian derivation (it is just a co-ordinate), while this is not true in the Lagrangian case.

Equipped with the solution for $\boldsymbol{U}$ in terms of phase space co-ordinates, the gyrokinetic Vlasov equation is Eq. (23), or in co-ordinates,

$$\partial_t \hat{F} + \nabla \cdot \left(\boldsymbol{U} \hat{F} \right) = 0. \tag{40}$$

Maxwell's equations follow from the standard Euler-Lagrange equations for $\boldsymbol{A}$ and ϕ,

$$\frac{\delta l_{GK}}{\delta \boldsymbol{A}_1} = \frac{\delta l_{GK}}{\delta \phi_1} = 0, \tag{41}$$

since $\dot{\phi}_1$ and $\dot{\boldsymbol{A}}_1$ do not appear in l_{GK}. These lead to the gyrokinetic Maxwell's equations for ϕ_1 and $\boldsymbol{A}_1$,

$$\frac{1}{4\pi} \nabla^2 \phi_1 (\boldsymbol{x}) = -\frac{\delta \mathcal{H}}{\delta \phi_1 (\boldsymbol{x})} \tag{42}$$

$$\frac{1}{4\pi} \nabla \times \nabla \times \boldsymbol{A}_1 (\boldsymbol{x}) = -\frac{\delta \mathcal{H}}{\delta \boldsymbol{A}_1 (\boldsymbol{x})} - \frac{1}{4\pi} \nabla \times \boldsymbol{B}, \tag{43}$$

where $\mathcal{H} \equiv \sum_s \int d\boldsymbol{X} du d\mu d\theta \hat{F} H$.

IV. THE HAMILTONIAN FORMULATION AND GYROKINETIC POISSON BRACKETS

We now perform the generalized Legendre transform of l_{GK} and use the corresponding Hamiltonian formulation to construct the Poisson brackets for the gyrokinetic system. One complication is the degeneracy in the Lagrangian that arises from the lack of quadratic dependence on $\boldsymbol{U}$, $\dot{\boldsymbol{A}}$ and $\dot{\phi}$. This issue is discussed in detail Refs. 1 and 45 and those same arguments apply to the gyrokinetic case.

For the moment, we formulate a bracket on the space of plasma densities (see Section IV B) and carry out a Legendre transform in $\boldsymbol{U}$ by defining

$$\boldsymbol{M} = \frac{\delta l_{GK}}{\delta \boldsymbol{U}}. \tag{44}$$

This type of formulation treats the gyrokinetic Poisson-Ampére equations [Eqs. (42) and (43)] as *constraints* on the motion of $\hat{F}$, rather than dynamical equations in their own right. The gyrokinetic

Hamiltonian is defined, as for a standard Legendre transform, as

$$
\begin{aligned}
h_{GK} &= \langle \boldsymbol{M}, \boldsymbol{U} \rangle_{\mathfrak{g}} - l_{GK}\left(\boldsymbol{U}, \hat{F}\right) \\
&= \sum_s \int d\boldsymbol{X} du d\mu d\theta \Big[\boldsymbol{M} \cdot \boldsymbol{U} - \hat{F} \Big(\frac{e}{c} \boldsymbol{A}^{\dagger} \cdot \boldsymbol{U}_X \\
&+ \frac{mc}{e} \mu U_{\theta} - H) \Big] - \frac{1}{8\pi} \int d\boldsymbol{x} \left(|\nabla \phi_1|^2 - |\nabla \times (\boldsymbol{A} + \boldsymbol{A}_1)|^2 \right).
\end{aligned}
\tag{45}
$$

It is easy to show that with this Hamiltonian

$$
\begin{aligned}
\{\Gamma, \Theta\}_{LP} = &- \left\langle \boldsymbol{M}, \left[\frac{\delta \Gamma}{\delta \boldsymbol{M}}, \frac{\delta \Theta}{\delta \boldsymbol{M}} \right] \right\rangle_{\mathfrak{g}} \\
&+ \left\langle \hat{F}, \frac{\delta \Theta}{\delta \boldsymbol{M}} \cdot \nabla \frac{\delta \Gamma}{\delta \hat{F}} - \frac{\delta \Gamma}{\delta \boldsymbol{M}} \cdot \nabla \frac{\delta \Theta}{\delta \hat{F}} \right\rangle_V
\end{aligned}
\tag{46}
$$

is a valid Poisson bracket (see Sec. II A and Ref. 1). To evaluate functional derivatives of h_{GK} [Eq. (45)], one should obtain the Green's function solutions for ϕ_1 and $\boldsymbol{A}_1$, for instance

$$
\phi_1(\boldsymbol{x}) = \sum_s \int d\boldsymbol{X}' du' d\mu' d\theta' K(\boldsymbol{x}|z') \hat{F}(z'),
\tag{47}
$$

from the gyrokinetic Poisson-Ampére equations, and insert these into h_{GK}, see Refs. 20 and 45. For practical calculation, this is the same as neglecting the electromagnetic part of h_{GK} in the functional derivative. In the same way as the Maxwell-Vlasov system (Sec. II A), the manifestation of the bracket in Eq. (46) is not well defined due to the constraints on $\boldsymbol{M}$. In the next section, the Dirac theory of constraints (see appendix) is used to reduce Eq. (46) to a bracket of the space of densities $\hat{F}$.

A complete treatment of the geometry of the Poisson-Vlasov system, with the electric field as a constraint, is given in Ref. 45. Many similar ideas will apply to the gyrokinetic system, with complications arising from the nonlocal nature of the theory[18] and larger constraint space (ϕ_1 and $\boldsymbol{A}_1$ rather than just ϕ). We reiterate that there are two sets of constraints we consider here; the constraints on $\boldsymbol{M}$ variables, similar to the Maxwell-Vlasov system, and the constraints due to ϕ_1 and $\boldsymbol{A}_1$, which are the gyrokinetic Poisson-Ampére equations. We first deal with the $\boldsymbol{M}$ constraints, eliminating these variables entirely, then explain how to include ϕ_1 and $\boldsymbol{A}_1$ in Section IV B.

A. Gyrokinetic Poisson bracket

There are six constraints given by

$$\Phi_I(z) = M_I(z) - \frac{\delta l_{GK}}{\delta U^I(z)} = 0, \tag{48}$$

with the functional derivatives as listed in Eq. (32). One then forms the constraint matrix $C_{IJ}(z,z') = \{\Phi_I(z), \Phi_J(z')\}$ with the Poisson bracket of Eq. (46) using

$$\begin{aligned}
\frac{\delta\Phi_I(z)}{\delta M_J(\bar{z})} &= \delta_I^J \delta(z-\bar{z})\,\delta_{ss'},\ \frac{\delta\Phi_u(z)}{\delta\hat{F}(\bar{z})} = \frac{\delta\Phi_\mu}{\delta\hat{F}} = 0, \\
\frac{\delta\Phi_i(z)}{\delta\hat{F}(\bar{z})} &= -\frac{e}{c}A_i^\dagger(\bar{z})\,\delta(z-\bar{z})\,\delta_{ss'}, \\
\frac{\delta\Phi_\theta(z)}{\delta\hat{F}(\bar{z})} &= -\bar{\mu}\,\delta(z-\bar{z})\,\delta_{ss'}.
\end{aligned} \tag{49}$$

Dropping boundary terms in integrations and inserting the constraint equations (after calculation of the brackets) leads to the very simple form,

$$\begin{aligned}
C_{IJ}(z,z') &= \hat{F}\delta(z-z')\,\delta_{ss'} \\
&\times \begin{pmatrix}
0 & -\frac{e}{c}B_z^\dagger & \frac{e}{c}B_y^\dagger & -mb_x & \frac{mc}{e}W_x & 0 \\
\frac{e}{c}B_z^\dagger & 0 & -\frac{e}{c}B_x^\dagger & -mb_y & \frac{mc}{e}W_y & 0 \\
-\frac{e}{c}B_y^\dagger & \frac{e}{c}B_x^\dagger & 0 & -mb_z & \frac{mc}{e}W_z & 0 \\
mb_x & mb_y & mb_z & 0 & 0 & 0 \\
-\frac{mc}{e}W_x & -\frac{mc}{e}W_y & -\frac{mc}{e}W_z & 0 & 0 & -\frac{mc}{e} \\
0 & 0 & 0 & 0 & \frac{mc}{e} & 0
\end{pmatrix},
\end{aligned} \tag{50}$$

where all functions are of the $\boldsymbol{z}$ variable and $\boldsymbol{W} = \boldsymbol{R} + \frac{1}{2}\boldsymbol{b}\,\boldsymbol{b}\cdot\nabla\times\boldsymbol{b}$. Because of the simple form in $\boldsymbol{z}'$, this matrix is easy to invert according to Eq. (A2) giving,

$$C_{IJ}^{-1}(z,z') = \frac{1}{\hat{F}B_{\parallel}^{\dagger}}\delta(z-z')\,\delta_{ss'} \times \begin{pmatrix} 0 & \frac{c}{e}b_z & -\frac{c}{e}b_y & \frac{1}{m}B_x^{\dagger} & 0 & \frac{c}{e}\hat{W}_x \\ -\frac{c}{e}b_z & 0 & \frac{c}{e}b_x & \frac{1}{m}B_y^{\dagger} & 0 & \frac{c}{e}\hat{W}_y \\ \frac{c}{e}b_y & -\frac{c}{e}b_x & 0 & \frac{1}{m}B_z^{\dagger} & 0 & \frac{c}{e}\hat{W}_z \\ -\frac{1}{m}B_x^{\dagger} & -\frac{1}{m}B_y^{\dagger} & -\frac{1}{m}B_z^{\dagger} & 0 & 0 & \frac{1}{m}W^{\dagger} \\ 0 & 0 & 0 & 0 & 0 & \frac{e}{mc}B_{\parallel}^{\dagger} \\ -\frac{c}{e}\hat{W}_x & -\frac{c}{e}\hat{W}_y & -\frac{c}{e}\hat{W}_z & -\frac{1}{m}W^{\dagger} & -\frac{e}{mc}B_{\parallel}^{\dagger} & 0 \end{pmatrix}, \tag{51}$$

where again functions are of the $\boldsymbol{z}$ variable, $\hat{\boldsymbol{W}} \equiv \boldsymbol{b}\times\boldsymbol{W}$ and $W^{\dagger} \equiv \boldsymbol{B}^{\dagger}\cdot\boldsymbol{W}$. Of course, this matrix is nothing but the single particle gyrokinetic Poisson matrix[2] as was the case for the Maxwell-Vlasov system. Restricting the functionals Γ and Θ to not depend on $\boldsymbol{M}$ (see Sec. II A) and using

$$\left\{\Gamma[\hat{F}], \Phi_J(z)\right\} = \hat{F}(z)\frac{\partial}{\partial z^J}\frac{\delta\Gamma}{\delta\hat{F}}, \tag{52}$$

the field theory gyrokinetic Poisson bracket is simply,

$$\{\Gamma,\Theta\}_{DB} = \left\langle \hat{F}, \left\{\frac{\delta\Gamma}{\delta\hat{F}}, \frac{\delta\Theta}{\delta\hat{F}}\right\}_{sp}\right\rangle_V. \tag{53}$$

Here $\{\,,\}_{sp}$ is the single particle Poisson bracket structure

$$\begin{aligned} \{f,g\}_{sp} = &-\frac{c\boldsymbol{b}}{eB_{\parallel}^{\dagger}}\cdot\nabla f\times\nabla g + \frac{\boldsymbol{B}^{\dagger}}{mB_{\parallel}^{\dagger}}\cdot\left(\nabla f\frac{\partial g}{\partial u} - \nabla g\frac{\partial f}{\partial u}\right) \\ &+\frac{c\hat{\boldsymbol{W}}}{e}\cdot\left(\nabla f\frac{\partial g}{\partial\theta} - \nabla g\frac{\partial f}{\partial\theta}\right) + \frac{W^{\dagger}}{m}\left(\frac{\partial f}{\partial u}\frac{\partial g}{\partial\theta} - \frac{\partial g}{\partial u}\frac{\partial f}{\partial\theta}\right) \\ &+\frac{e}{mc}\left(\frac{\partial f}{\partial\mu}\frac{\partial g}{\partial\theta} - \frac{\partial f}{\partial\theta}\frac{\partial g}{\partial\mu}\right). \end{aligned} \tag{54}$$

We note that, as for the Maxwell Vlasov system, the $\delta/\delta M_u$ and $\delta/\delta M_\mu$ terms cancel in a full calculation, so there is no issue with these being undefined. The field theory bracket, Eq. (53), is of exactly the form one would expect based on the Poisson-Vlasov bracket[20] and Maxwell-Vlasov bracket[19] (Eq. (13) without $\delta/\delta\boldsymbol{E}$ and $\delta/\delta\boldsymbol{B}$ terms). It is aesthetically pleasing to see this type of structure emerge from the entirely systematic procedure applied above. The reduced Hamiltonian to be used with Eq. (53) is simply Eq. (45) with constraints on $\boldsymbol{M}$ inserted explicitly,

$$h = \sum_s \int d\boldsymbol{X} du d\mu d\theta \hat{F} H - \frac{1}{8\pi} \int d\boldsymbol{x} \left(|\nabla\phi_1|^2 - |\nabla \times (\boldsymbol{A} + \boldsymbol{A}_1)|^2 \right). \tag{55}$$

It is easy to show that $\partial_t \hat{F} = \left\{ \hat{F}, h \right\}$ is just the conservative form of the gyrokinetic Vlasov equation, Eq. (40).

B. Inclusion of electromagnetic fields

The bracket, Eq. (54), does not include electromagnetic field equations, meaning the gyrokinetic Maxwell's equations, Eqs. (42) and (43), must be specified as separate constraints on the motion to obtain a closed system. Here, we illustrate how to explicitly include the electromagnetic potentials in the bracket for a simplified gyrokinetic system. This procedure also works to extend the simple Poisson-Vlasov bracket[20,45] to include the motion of ϕ. The general technique is to add a Poisson-Ampére canonical bracket to the gyrokinetic bracket [Eq. (54)] and perform a reduction on this extended bracket with the Dirac theory of constraints. It is important to recognize that this is only valid because the full constraint matrix would be block diagonal if the reduction were performed in one-step from an original bracket that included electromagnetic and plasma components (i.e., Eq. (46) with the addition of canonical brackets in $\boldsymbol{A}_1$ and ϕ_1). This condition is satisfied because $\boldsymbol{A}_1$ and ϕ_1 do not appear in the symplectic structure of the original Lagrangian.

For clarity, we use with a simplified electrostatic system in the drift kinetic limit, with $H = e\phi_1 + m |\delta \boldsymbol{u}_E|^2 / 2$ where $\delta \boldsymbol{u}_E = c (\boldsymbol{b} \times \nabla \phi_1) / B$. We also assume quasineutrality, which amounts to neglecting the $\int d\boldsymbol{x} |\nabla\phi|^2 / 8\pi$ term in the Lagrangian, and set $\boldsymbol{W}$ to zero[46]. The Hamiltonian for the system is

$$\begin{aligned} h = &\int d\boldsymbol{X} \phi(\boldsymbol{X}) \Pi(\boldsymbol{X}) \\ &+ \sum_s \int d\boldsymbol{X} du d\mu d\theta \hat{F} \left(\frac{m}{2} u^2 + \mu B + e\phi + \frac{mc^2}{2B^2} |\nabla_\perp \phi|^2 \right), \end{aligned} \tag{56}$$

and the unreduced Poisson bracket

$$\{\Gamma, \Theta\} = \left\langle \hat{F}, \left\{ \frac{\delta \Gamma}{\delta \hat{F}}, \frac{\delta \Theta}{\delta \hat{F}} \right\}_{sp} \right\rangle_V + \int d\boldsymbol{X} \left(\frac{\delta \Gamma}{\delta \phi} \frac{\delta \Theta}{\delta \Pi} - \frac{\delta \Theta}{\delta \phi} \frac{\delta \Gamma}{\delta \Pi} \right), \tag{57}$$

where $\Pi = \delta l / \delta \dot{\phi}$ is the variable canonically conjugate to ϕ. This model is the electrostatic version of the simplified gyrokinetic system in Refs. 44 and 47. Physically, the $m \left|\delta \boldsymbol{u}_E\right|^2 / 2 = mc^2 / 2B^2 \left|\nabla_\perp \phi\right|^2$ term in the Hamiltonian is the polarization drift in the drift kinetic limit[5,47]. $\nabla_\perp$ indicates a gradient with respect to a co-ordinate system locally perpendicular to the background magnetic field (we are neglecting derivatives of $\boldsymbol{b}$). Unlike in the previous section, ϕ is now considered a separate field in the Hamiltonian, and Poisson's equation should not be used to evaluate functional derivatives.

The constraints are

$$\begin{aligned} \Phi_1 &= \frac{\delta h}{\delta \phi} = \sum_s \int du d\mu d\theta \left[e\hat{F} - mc^2 \nabla_\perp \cdot \left(\frac{\hat{F}}{B^2} \nabla_\perp \phi \right) \right], \\ \Phi_2 &= \Pi, \end{aligned} \tag{58}$$

where $\Pi = 0$ since $\delta l / \delta \dot{\phi} = 0$. $\Phi_1 = 0$ is the gyrokinetic Poisson equation; this constraint arises as a secondary Dirac constraint that is necessary to satisfy $\dot{\Phi}_2 = 0$, see Ref. 23 for more information. Using Eq. (57) the constraint matrix is,

$$C_{IJ}(\boldsymbol{X}, \boldsymbol{X}') = \begin{pmatrix} 0 & C \\ -C & 0 \end{pmatrix}, \tag{59}$$

where

$$C = \frac{\delta \Phi_1(\boldsymbol{X})}{\delta \phi(\boldsymbol{X}')} = -c^2 \nabla'_\perp \cdot \left[\frac{\hat{n}(\boldsymbol{X}')}{B^2(\boldsymbol{X}')} \nabla'_\perp \delta(\boldsymbol{X} - \boldsymbol{X}') \right] \tag{60}$$

with $\hat{n} = \sum_s \int du d\mu d\theta m \hat{F}$ and $\nabla'_\perp$ indicating the derivative is with respect to $\boldsymbol{X}'_\perp$. The inverse matrix, chosen to satisfy Eq. (A2), is

$$C^{-1}_{IJ}(\boldsymbol{X}, \boldsymbol{X}') = \begin{pmatrix} 0 & -C^{-1} \\ C^{-1} & 0 \end{pmatrix}, \tag{61}$$

where

$$C^{-1}(\boldsymbol{X},\boldsymbol{X}') = -\frac{1}{c^2}\nabla_\perp^{-1}\cdot\left[\frac{B^2(\boldsymbol{X})}{\hat{n}(\boldsymbol{X})}\nabla_\perp^{-1}\delta(\boldsymbol{X}-\boldsymbol{X}')\right]. \tag{62}$$

The Dirac bracket is constructed using

$$\begin{aligned}\{\Gamma,\Phi_1(\boldsymbol{X})\} &= c^2\nabla_\perp\cdot\left(\frac{\hat{n}}{B^2}\nabla_\perp\frac{\delta\Gamma}{\delta\Pi}\right)\\ &\quad+\tilde{N}_\Gamma+\frac{mc^2}{e}\nabla_\perp\cdot\left(\frac{\nabla_\perp\phi}{B^2}\tilde{N}_\Gamma\right),\\ \{\Gamma,\Phi_2(\boldsymbol{X})\} &= \frac{\delta\Gamma}{\delta\phi(\boldsymbol{X})},\end{aligned} \tag{63}$$

where

$$\tilde{N}_\Gamma = \sum_s\int du d\mu d\theta\,\frac{1}{m}\nabla\cdot\left(cf\boldsymbol{b}\times\nabla\frac{\delta\Gamma}{\delta\hat{F}}+\frac{e}{m}f\boldsymbol{B}^\dagger\frac{\partial}{\partial u}\frac{\delta\Gamma}{\delta\hat{F}}\right), \tag{64}$$

with the corresponding definition for $\tilde{N}_\Theta$. With Eq. (A3), this leads to

$$\begin{aligned}\{\Gamma,\Theta\}_{DB} &= \sum_s\int d\boldsymbol{X}du d\mu d\theta\hat{F}\left\{\frac{\delta\Gamma}{\delta\hat{F}},\frac{\delta\Theta}{\delta\hat{F}}\right\}_{sp}\\ &\quad+\int d\boldsymbol{X}\nabla_\perp^{-1}\cdot\left(\frac{B^2}{c^2\hat{n}}\nabla_\perp^{-1}\frac{\delta\Theta}{\delta\phi}\right)\left[\tilde{N}_\Gamma+\frac{mc^2}{e}\nabla_\perp\cdot\left(\frac{\nabla_\perp\phi}{B^2}\tilde{N}_\Gamma\right)\right]\\ &\quad-\int d\boldsymbol{X}\nabla_\perp^{-1}\cdot\left(\frac{B^2}{c^2\hat{n}}\nabla_\perp^{-1}\frac{\delta\Gamma}{\delta\phi}\right)\left[\tilde{N}_\Theta+\frac{mc^2}{e}\nabla_\perp\cdot\left(\frac{\nabla_\perp\phi}{B^2}\tilde{N}_\Theta\right)\right].\end{aligned} \tag{65}$$

Here, $\{\,,\}_{sp}$ is the single particle bracket as in Eq. (54) (with $\boldsymbol{W}=0$). Note that in forming Eq. (65), terms involving $\delta/\delta\Pi$ in the Dirac part of Eq. (A3), cancelled with the canonical part of the original bracket, as would be expected.

With the reduced Hamiltonian (Eq. (56) without the first term), the bracket can easily be checked to give the Vlasov equation as $\partial_t\hat{F}(\boldsymbol{X}) = \left\{\hat{F}(\boldsymbol{X}),h\right\}_{DB}$. Noticing that the $\partial/\partial u$ term in $\partial_t\hat{F}$ [see Eq. (40)] integrates to zero, we see that $\sum_s\int du d\mu d\theta\, e\partial_t\hat{F} = -\tilde{N}_h$. This is used in $\partial_t\phi(\boldsymbol{X}) = \{\phi(\boldsymbol{X}),h\}_{DB}$ to show

$$\sum_s\int du d\mu d\theta\left\{e\partial_t\hat{F}-mc^2\nabla_\perp\cdot\left[\frac{1}{B^2}\frac{\partial}{\partial t}\left(\hat{F}\nabla_\perp\phi\right)\right]\right\}=0, \tag{66}$$

which is just the time derivative of Poisson's equation for this gyrokinetic model. Using the procedure presented above there should be no particular obstacle to the construction of brackets for more complex gyrokinetic theories. For instance, one could include finite Larmor radius effects or magnetic fluctuations[48]. However, considering the complexity of the bracket for even a very simple gyrokinetic model, such brackets are unlikely to be of much practical use.

V. NUMERICAL APPLICATIONS

One of the main motivations for this work is the possibility of using similar ideas in a *discrete* context to design continuum geometric integrators for gyrokinetic systems. To elaborate on this idea, here we give a simple explanation of some geometric discretization methods based on recent work in numerical fluid dynamics. The methods described here are just examples from a large array of literature on the subject. Some other techniques can be found in, for instance Refs. 49–53. In addition we remark on how Euler-Poincaré models can be used to formulate sub-grid models for turbulence simulation and some of the challenges associated with extending these ideas to gyrokinetic turbulence.

a. Lagrangian side: discrete Euler-Poincaré equations Conceptually, an obvious way to design a geometric integrator for an Euler-Poincaré system is to directly discretize the Euler-Poincaré variational principle. If one can design discrete variations of the correct form, the entire integrator can be constructed directly from the variational principle as for a standard variational integrator. This approach has recently been successfully applied to develop an integrator for the Euler fluid equations[32] and more complex fluids, including magnetohydrodynamics (MHD)[33]. The utility of such an approach is illustrated by the very nice properties of these schemes. For instance, the MHD scheme[33] exactly preserves $\nabla\cdot\boldsymbol{B} = 0$ and the cross helicity $\int d\boldsymbol{x}\, \boldsymbol{v}\cdot\boldsymbol{B}$. As one consequence of this, there is almost no artificial magnetic reconnection. The symplectic nature of the scheme also leads to other very good long time conservation properties.

The first requirement in constructing an Euler-Poincaré integrator is a finite dimensional approximation to the diffeomorphism Lie group. In the case of fluids or MHD, the group is that of *volume preserving* diffeomorphisms and a matrix Lie group is constructed to satisfy analogous properties to the infinite dimensional group. For Vlasov-Poisson, Vlasov-Maxwell or a gyrokinetic system, the group is that of *symplectomorphisms*. Thus, for a discretization, a different matrix Lie group than the fluid case should be used, with properties designed to mimic those of the infinite dimensional symplectomorphism group. Using this group one can find the Lie algebra, which will give the form of the space of discrete vector fields (just the Eulerian phase space fluid velocities). Group operations can then be constructed as matrix multiplications as for a standard finite dimensional Lie group, and advected parameters included through the use of discrete exterior calculus. One would then use the *discrete Euler-Poincaré theorem*[54], which gives discrete update equations [in analogy with Eq. (31)] from a discrete reduced Lagrangian. An algorithm of this form can be shown to be symplectic and have similar conservation properties (arising from variants of Noether's theorem) to the continuous system. The final update equations obtained from this method are not as complex as one might expect and would not preclude incorporation into large scale codes. Obviously there are several unanswered questions regarding the application of this method to kinetic plasma systems. First, one must discretize the symplectomorphism group, which may not be trivial. In two phase space dimensions the group is the same as the group of volume preserving diffeomorphisms; however, in higher dimensions the symplectomorphisms form a more restricted class of transformations. The lack of a finite boundary in velocity space may also present issues relating to the discretization of the symplectomorphisms. The degeneracy of the system is another aspect which differs from the fluid system, and the consequences of this in the discrete setting would have to be carefully considered. Finally, for a gyrokinetic system, it would be necessary to remove the θ dimension in some way. This could potentially be done either in the continuous setting or after discrete equations have been obtained.

b. Hamiltonian side: Poisson bracket discretization Another way to form a discrete Hamiltonian system is to directly discretize the Poisson bracket. The general idea is simple, one finds a discrete Hamiltonian functional and discrete bracket that are finite dimensional approximations to the continuous versions. In this way, one discretizes (in phase space) via the method of lines, and reduces the infinite dimensional system to an approximate finite dimensional one. Any symplectic temporal discretization can then be used to ensure the system is discretely Hamiltonian[53]. The difficultly arises in ensuring a correct Hamiltonian discretization of the bracket. This requires antisymmetry and the Jacobi identity to be satisfied, and such a bracket can be very difficult to find in practice. For instance, for the Euler fluid equations, the non-canonical structure complicates matters and a discrete bracket has been found only for simplified cases[55]. An obvious place to start in this endeavor would be the Vlasov-Poisson system, as the structure is much more simple. Generalizations to gyrokinetic systems could then potentially be achieved through Nambu bracket formulations[44].

c. Alpha models and large-eddy simulation Much work has been done in the last decade in the fluids community on so-called *alpha models.* The general idea is to regularize the fluid equations (Navier-Stokes or MHD) at the level of the Euler-Poincaré variational principle, by adding terms into the Lagrangian that include gradients of the fluid velocity. These terms penalize the formation of small scale structures, and can thus be used as a large eddy simulation (LES) model, causing turbulence to dissipate at larger scales[56]. These methods have been shown to have some significant advantages over more traditional LES methods (for instance those based on hyperdiffusion) especially for simulation of MHD turbulence[37,57]. As gyrokinetic turbulence simulation becomes a more mature subject, it is interesting to enquire whether similar alpha models could be formulated for gyrokinetic large eddy simulation.

In fact, alpha models can be *derived* from a standard fluid variational principle, by averaging over small scale fluctuations that are assumed to be advected by the larger scale flow[38,39]. Approaching the gyrokinetic variational principle in a similar way leads to the addition of extra, regularized terms into the gyrokinetic Lagrangian. For instance, following the general ideas in Ref. 38, averaging over perpendicular $\boldsymbol{X}$-space fluctuations of scale length α, and ensuring gauge invariance, we were led to the regularized Lagrangian $l = l_{GK} + l_\alpha$, where

$$l_\alpha = \alpha^2 \int d\boldsymbol{X} du d\mu d\theta \hat{F} \left(\boldsymbol{B}^\dagger \cdot \nabla \times \boldsymbol{U}_X - \nabla_\perp^2 H \right). \tag{67}$$

While this Lagrangian gives well-defined equations of motion, there is a fundamental problem in that it destroys some of the degeneracy in the original system. As a consequence of this, the equations of motion involve solving spatial PDEs for $\boldsymbol{U}$, which would significantly increase computation times, defeating the purpose of an LES. It is not yet clear if it is possible to design a regularization of this type for the gyrokinetic system that retains the redundancy of the $\boldsymbol{U}$ fields and allows one to write down a standard Vlasov equation. We note that gyrokinetic LES has been explored and implemented recently on the GENE code, by adding hyperdiffusive terms in the perpendicular co-ordinates[36,58].

VI. CONCLUDING REMARKS

In this article we have applied the Euler-Poincaré formalism to derive a new gyrokinetic action principle in Eulerian co-ordinates. We start with a single-particle Poincaré-Cartan 1-form, using the theory of Ref. 18 to systematically construct a gyrokinetic field theory action in Lagrangian co-ordinates. The fundamental idea is then to reduce this action using symmetry under the the particle-relabeling map, $(\boldsymbol{x}, \boldsymbol{v}) = \psi(\boldsymbol{x}_0, \boldsymbol{v}_0)$, which takes particles with initial position $(\boldsymbol{x}_0, \boldsymbol{v}_0)$ to their current location $(\boldsymbol{x}, \boldsymbol{v})$[1,16,17]. This process leads to an action functional formulated in terms of the Eulerian phase space fluid velocity, $\boldsymbol{U}$, and the advected plasma density, $\hat{F}$, as well as the standard electromagnetic potentials. In the course of reduction, the arbitrary variations of the Lagrangian fields (used to derive equations of motion) lead to *constrained* variations of the Eulerian fields, $\boldsymbol{U}$ and $\hat{F}$. Because of this, field motion is governed by the *Euler-Poincaré* equations rather than the standard Euler-Lagrange equations. Explicit calculation of the Euler-Poincaré equations for a standard gyrokinetic single particle Lagrangian is shown to give the gyrokinetic Vlasov equation. Since the space of electromagnetic potentials is not altered by $\psi(\boldsymbol{x}_0, \boldsymbol{v}_0)$, the gyrokinetic Poisson-Ampere equations arise from the standard Euler-Lagrange equations for the perturbed potentials.

Using the methodology set out in Ref. 1 we then perform a Legendre transform to derive the Hamiltonian form of the gyrokinetic system. The principal difficulty is the strong degeneracy, which is related to the linearity and lack of time derivatives for certain function variables in the action principle. Physically, this arises from the fact that the plasma distribution function encodes the information about particle phase space trajectories. The degeneracy leads to a Poisson bracket in terms of too many variables; namely, a series of constrained momentum variables canonically conjugate to $\boldsymbol{U}$ as well as the distribution function $\hat{F}$. To reduce the bracket into a well defined form we use the Dirac theory of constraints, which is a systematic way to project a Poisson bracket onto a constraint submanifold when momentum variables are constrained. This leads to an infinite dimensional gyrokinetic Poisson bracket, which takes a natural form based on the single particle bracket. We also demonstrate how this procedure leads to the full, electromagnetic Vlasov-Maxwell bracket as derived in Ref. 19. Since the electromagnetic equations in the gyrokinetic system are really constraints on the motion, we chose to include these in the bracket via a second application of the Dirac theory of constraints. The general method is expounded through construction of the bracket for a simplified electrostatic model with no finite Larmor radius effects. Although the brackets obtained by such an approach are probably to complicated to be of much practical use, it makes for an interesting application of Dirac theory.

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Appendix A: Dirac Constraints

The Dirac theory of constraints or Dirac bracket, is used to generalize the Poisson bracket to Hamiltonian systems with constraints. The original purpose of the theory was to construct quantizable Poisson brackets starting with a degenerate Lagrangian; i.e., a Lagrangian where the momenta are not independent functions of velocities. The theory applies equally well to a bracket that is already in non-canonical form, a realization that can be very useful in the construction of field theoretic brackets[24,40]. For example, in Ref. 23, the non-canonical magnetohydrodynamic bracket is reduced to incorporate the incompressibility constraint. We give a very brief overview of the theory here for the convenience of the reader. More complete treatments can be found in Refs. 22–24, 40, and 59.

We consider an infinite dimensional Hamiltonian system with Poisson bracket $\{\,,\}$, Hamiltonian h and N constraint functions $\Phi_1,\ \Phi_2,\ldots,\Phi_N = 0$. The constraint matrix,

$$C_{ij}\left(z,z'\right) = \left\{\Phi_i,\Phi_j\right\}, \tag{A1}$$

and its inverse, defined using

$$\int dz' C_{ij}\left(z,z'\right) C^{-1}_{jk}\left(z',z''\right) = \delta_{ik}\delta\left(z-z''\right), \tag{A2}$$

are used to form the Dirac bracket,

$$\begin{aligned}\{\Gamma,\Theta\}_{DB} &= \{\Gamma,\Theta\} \\ &\quad - \int dz dz' \left\{\Gamma,\Phi_i\left(z\right)\right\} C^{-1}_{ij}\left(z,z'\right)\left\{\Phi_j\left(z'\right),\Theta\right\}.\end{aligned} \tag{A3}$$

Geometrically, the constraints force motion to lie on a *constraint submanifold,* which inherits the Dirac bracket from the Poisson bracket on the original manifold[59].

In the case where the matrix C is not invertible, Dirac theory gives one or more secondary constraints, which must be included and the constraint matrix recalculated. See Refs. 23 and 59 for more information.

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