Spherical Heat Conduction Verification Problem

David S. Miller

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What follows is the derivation of an analytic solution for a pure heat conduction problem which should be useful for verification purposes. Consider a sphere of radius $R$ at a constant temperature $T_0$. I seek a solution to the homogeneous heat diffusion equation in spherical coordinates (exterior to the hot sphere)

$$\rho C \frac{\partial T}{\partial t} - \frac{1}{r^2} \frac{\partial}{\partial r} \left( \kappa r^2 \frac{\partial T}{\partial r} \right) = 0$$ \hspace{1cm} [1]

subject to the initial and boundary conditions

$$T(t = 0, r > R) = 0$$

$$T(t, r \leq R) = T_0$$ . \hspace{1cm} [2]

$$T(t, r = \infty) = 0$$

In Eq.1, $C$ is the specific heat, $\rho$ is the density, and $\kappa$ is the conduction coefficient. Specify temperature dependent forms for the specific heat and conduction coefficients as

$$\kappa = \kappa_0 T^n$$

$$C = C_0 T^n$$ \hspace{1cm} [3]

where $\kappa_0$ and $C_0$ are constants and $n$ is some exponent not necessarily an integer. If we substitute Eq.3 into Eq.1 and define

$$\Phi = T^{n+1}$$ \hspace{1cm} [4]

we have

$$\rho C_0 \frac{\partial \Phi}{\partial t} - \frac{1}{r^2} \frac{\partial}{\partial r} \left( \kappa_0 r^2 \frac{\partial \Phi}{\partial r} \right) = 0 .$$ \hspace{1cm} [5]

Take the Laplace transform of Eq.5 to get

$$\frac{\partial}{\partial r} \left( r^2 \frac{\partial \Phi(s, r)}{\partial r} \right) - r^2 s \frac{\rho C_0}{\kappa_0} \Phi(s, r) = 0 .$$ \hspace{1cm} [6]

Now the transformed boundary conditions are

$$\Phi(s, r = R) = \frac{\Phi_0}{s} = \frac{T_0^{n+1}}{s} .$$ \hspace{1cm} [7]

$$\Phi(s, r = \infty) = 0$$

This ODE has the solution (for $r > R$)

$$\Phi(s, r) = \frac{\Phi_0 R}{s r} \exp\left\{ - (r - R) \sqrt{s \rho C_0 \kappa_0} \right\} .$$ \hspace{1cm} [8]

Now perform the Laplace inversion to get

$$T(t, r) = \begin{cases} T_0 & \text{for } r \leq R \\ T_0 \left\{ \frac{R}{r} \text{Erfc} \left( 0.5(r - R) \sqrt{\frac{s \rho C_0}{\kappa_0}} \right) \frac{1}{(n+1)} \right\} & \text{for } r > R \end{cases}$$ \hspace{1cm} [9]

For $n = 0$ this is an easy computational problem. But at large $n$ it will strain the diffusion codes ability to accurately resolve the gradients in the material properties.