CONVERGENCE TESTS FOR INFINITE SERIES

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CHAPTER I

INTRODUCTION

The field of infinite series is so large that any investigation into that field must necessarily be limited to a particular phase. It would be impossible to cover the entire subject of infinite series; nor would it be possible to list all convergence tests that have ever been used. Rather, an attempt has been made to develop a number of tests having a wide range of applications. Particular emphasis has been placed on tests for series of positive terms.

It is assumed that the reader is familiar with the number system and its development. This will allow an immediate treatment of infinite series.

Definition 1.1: If to each positive integer 1, 2, 3, ..., corresponds a definite real number \( x_n \), then the numbers \( x_1, x_2, x_3, \ldots, x_n, \ldots \) are said to form a sequence.

Definition 1.2: A sequence \( (x_n) \) is said to be bounded if a constant \( K \) exists, such that the inequality \( |x_n| \leq K \) is satisfied for every \( n \).

Definition 1.3: A sequence \( (x_n) \) is said to be monotone increasing if \( x_n \leq x_{n+1} \) for every \( n \); monotone decreasing, if \( x_n \geq x_{n+1} \) for every \( n \).

Definition 1.4: A sequence \( (x_n) \) is called a null sequence if, for every positive number \( \varepsilon \), a number \( n_0 \geq n_0(\varepsilon) \) may always be assigned, such that the inequality \( |x_n| < \varepsilon \) is fulfilled for every \( n > n_0 \).
Theorem 1.1: If $(X_n)$ is a null sequence, and $(A_n)$ any bounded sequence, then the numbers $X_n = A_n X_n$ also form a null sequence.

Theorem 1.2: If $(X_n)$ is a null sequence and the terms of the sequence $(X'_n)$, for every $n$ beyond a certain value $W$, satisfy the condition $|X'_n| \leq |X_n|$, or, more generally, the condition $|X'_n| \leq K |X_n|$ in which $K$ is an arbitrary (fixed) positive number, then $(X'_n)$ is also a null sequence.

Theorem 1.3: If $(X_n)$ is a null sequence, then every sub-sequence $(X'_n)$ of $(X_n)$ is a null sequence.

Theorem 1.4: Let an arbitrary sequence $(X_n)$ be separated into two subsequences $(X'_n)$ and $(X''_n)$ so that, therefore, every term of $(X_n)$ belongs to one and only one of these subsequences. If $(X'_n)$ and $(X''_n)$ are both null sequences, then so is $(X_n)$ itself.

Theorem 1.5: If $(X_n)$ is a null sequence and $(X'_n)$ an arbitrary rearrangement of it, then $(X'_n)$ is also a null sequence.

Theorem 1.6: If $(X_n)$ is a null sequence and $(X'_n)$ is obtained from it by any finite number of alterations, then $(X'_n)$ is also a null sequence.

Theorem 1.7: If $(X'_n)$ and $(X''_n)$ are two null sequences and if the sequence $(X_n)$ is so related to them that from a certain $M$ onwards $X'_n \leq X_n \leq X''_n$ ($n > m$) then $(X_n)$ is also a null sequence.

Theorem 1.8: If $(X_n)$ and $(X'_n)$ are two null sequences, then $(Y_n) = (X_n + X'_n)$ is also a null sequence.

Theorem 1.9: If $(X_n)$ and $(X'_n)$ are null sequences, then so is $(Y'_n) = (X_n - X'_n)$.
Theorem 1.10: If $a > 0$, and $A_1 > 0$ then $\frac{k}{\sqrt[3]{A}} \geq \frac{k}{\sqrt[3]{A_1}}$ according as $A \geq A_1$.

Theorem 1.11: If $a > 0$, then the numbers $X_n = \frac{n}{\sqrt[3]{A}} - 1$ form a null sequence.

**Definition 1.5:** If $(X_n)$ is a given sequence, and if it is related to a definite number $\xi$ in such a way that $(X_n - \xi)$ forms a null sequence, then we say that the sequence $(X_n)$ converges to $\xi$; in notation, $X_n \rightarrow \xi$ or $\lim_{n \rightarrow \infty} X_n = \xi$.

**Definition 1.6:** Every sequence which is not convergent in the above sense, is called divergent.

**Definition 1.7:** If the sequence $(X_n)$ has the property that, given an arbitrary large positive number $G$, another number $n_0$ can always be assigned such that, for every $n > n_0$, $X_n > G$, then we shall say that $(X_n)$ diverges to $+\infty$.

**Definition 1.8:** If the sequence $(X_n)$ has the property that, given an arbitrary negative $G$ (large in absolute value), another number $n_0$ can always be assigned such that, for every $n > n_0$, $X_n < -G$, then we shall say that $(X_n)$ diverges to $-\infty$.

**Definition 1.9:** If two sequences $(X_n)$ and $(Y_n)$, not necessarily convergent, are so related to one another that the quotient $\frac{X_n}{Y_n}$ tends, as $n \rightarrow +\infty$, to a definite finite limit different from zero, then we shall say that the two sequences are asymptotically proportional and write briefly $X_n \sim Y_n$. If in particular this limit is 1, then we say that the two sequences are asympt-
totically equal and write, more expressly $X_n \cong Y_n$.

Theorem 1.12: A convergent sequence determines its limit quite uniquely.

Theorem 1.13: A convergent sequence $(X_n)$ is invariably bounded. Also, if $|X_n| \leq K$, then for the limit $\xi$ we have $|\xi| \leq K$.

Theorem 1.13a: $X_n \to \xi$ implies $|X_n| \to |\xi|$.

Theorem 1.14: If a convergent sequence $(X_n)$ has all its terms different from zero, and if its limit $\xi$ is also $\neq 0$, then the sequence $(\frac{1}{X_n})$ is bounded.

Theorem 1.15: If $(X'_n)$ is a subsequence of $(X_n)$, then $X_n \to \xi$ implies $X'_n \to \xi$.

Theorem 1.16: If the sequence $(X_n)$ can be divided into two subsequences of which each converges to $\xi$, then $(X_n)$ itself converges to $\xi$.

Theorem 1.17: If $(X'_n)$ is an arbitrary rearrangement of $X_n$, then $X_n \to \xi$ implies $X'_n \to \xi$.

Theorem 1.18: If $X_n \to \xi$ and $(X''_n)$ results from $(X_n)$ by a finite number of alterations, then $X'_n \to \xi$.

Theorem 1.19: If $X'_n \to \xi$ and $X''_n \to \xi$, and if the sequence $(X_n)$ is so related to the sequence $(X'_n)$ and $(X''_n)$ that from some place onwards, (i.e. for every $n \geq m$, say), $X'_n \cong X_n \cong X''_n$, then $X_n \to \xi$.

Theorem 1.20: $X_n \to \xi$ and $Y_n \to \theta$ always implies $(X_n + Y_n) \to \xi + \theta$ and the corresponding statement holds for term by term addition of any fixed number - say $P$ - of convergent sequences.

Theorem 1.20a: $X_n \to \xi$ and $Y_n \to \theta$ always implies $(X_n - Y_n) \to \xi - \theta$. 
Theorem 1.21: \( X_n \rightarrow \xi \) and \( Y_n \rightarrow \theta \) always implies \( X_n Y_n \rightarrow \xi \theta \),

and the corresponding statement holds for term by term multiplication of any fixed number - say \( P \) - of convergent sequences.

Theorem 1.22: \( X_n \rightarrow \xi \) and \( Y_n \rightarrow \theta \) always implies, if every \( X_n \neq 0 \)

and also \( \xi \neq 0 \), \( \frac{Y_n}{X_n} \rightarrow \frac{\theta}{\xi} \).

Theorem 1.22a: \( X_n \rightarrow \xi \) always implies, if every \( X_n \) and also \( \xi \) are

\( \neq 0 \), \( \frac{1}{X_n} \rightarrow \frac{1}{\xi} \).

Theorem 1.23: A monotone bounded sequence is invariably convergent;

a monotone sequence which is not bounded is always divergent.

Theorem 1.24: An arbitrary sequence \( (X_n) \) is convergent if and only if, given \( \varepsilon > 0 \), a number \( n_0 = n_0(\varepsilon) \) can always be assigned, such that for any two indices \( n \) and \( n' \) both greater than \( n_0 \), we have in every case

\[ |X_n - X_{n'}| < \varepsilon. \]

Theorem 1.25: The necessary and sufficient condition for the convergence of the sequence \( (X_n) \) is that, given any \( \varepsilon > 0 \), a number \( n_0 = n_0(\varepsilon) \) can always be assigned so that, for every \( n > n_0 \), and every integer \( k \geq 1 \), we always have

\[ |X_{n+k} - X_n| < \varepsilon. \]
CHAPTER II
SERIES OF POSITIVE TERMS

Definition 2.1: An infinite series is a symbol for a definite sequence of numbers deducible from it, namely the sequence of its partial sums.

Thus \( \sum_{n=0}^{\infty} A_n = A_0 + A_1 + A_2 + \ldots + A_n + \ldots \) and \( S = A_0 + A_1 + A_2 + \ldots + A_n + \ldots \).

\( S_2 = A_0 + A_1 + A_2 + \ldots, \quad \ldots, \quad S_n = A_0 + A_1 + A_2 + \ldots + A_n + \ldots \).

Definition 2.2: \( \sum A_n \) is said to be convergent or divergent, according as the sequence of its partial sums shows the behavior indicated by those names. \( \sum A_n \rightarrow S \) means \( S_n \rightarrow S \).

Theorem 2.1: If \( \sum A_n \) is a series of positive terms then \( \sum A_n \) is either convergent or it is divergent to \( +\infty \).

We have \( S_1 < S_2 < S_3 < \ldots \). Either \( (S_n) \) is bounded or it is not. If it is bounded then it is convergent. If it is unbounded, for every number \( K > 0 \), there exists an \( n \) such that \( S_n > K \). But \( S_n < S_{n+1} < S_{n+2} \ldots \). Hence there exists a number \( N \) such that, for every \( n > N \), \( S_n > K \), i.e., \( \lim_{n \to \infty} S_n = +\infty \).

Theorem 2.2: If \( p \) is any positive integer, then the two series \( \sum_{n=0}^{\infty} A_n \) and \( \sum_{n=p}^{\infty} A_n \) converge and diverge together.
The partial sums of both series are monotone increasing and differ by the constant \((A_0 + A_1 + \ldots + A_{p-1})\). The sequences of partial sums are therefore either bounded, or unbounded, for both series simultaneously.

**Theorem 2.3:** If \(\sum C_n\) is a convergent series with positive terms, then so is \(\sum Y_n C_n\), if the factors \(Y_n\) are any positive, but bounded, numbers.

There exists a number \(K_1 > 0\) such that, for every \(n\), \(S_n < K_1\). There exists a number \(K_2 > 0\) such that, for every \(n\), \(Y_n < K_2\). Therefore \(\sum_{i=1}^{n} Y_n C_i \leq K_1 K_2\) and hence, since the sequence of partial sums are bounded, \(\sum Y_n C_n\) is convergent.

**Theorem 2.4:** If \(\sum d_n\) is a divergent series with positive terms, then so is \(\sum \delta_n d_n\), if the factors \(\delta_n\) are any numbers with a positive lower bound \(\delta\).

For every \(K > 0\) there exists a number \(N\) such that, for every \(n > N\), \(S_n > K\). For every \(n\) there exists a number \(\delta_n \geq \delta\). Therefore \(\sum_{i=1}^{\infty} \delta_n d_i > \delta K\), for every \(n > N\), and hence \(\sum \delta_n d_n\) is unbounded and divergent.

**Theorem 2.5:** If \(\sum A_n\) converges then \(\lim_{n \to \infty} A_n = 0\).

For every \(\epsilon > 0\) there exists a number \(N\) such that, for every \(n > N\),

\[
|A_{n+1} - \epsilon| < \epsilon \text{ or, using } M = N+1, \text{ for every } \epsilon > 0 \text{ there exists a number } M \text{ such that, for every } n > M, |0 - A_n| < \epsilon.
\]
The reader should note that this is only a necessary and not a sufficient condition for convergence. The harmonic series $\sum \frac{1}{n}$ meets this condition and yet it is known to diverge.

Theorem 2.6: Let $\sum C_n$ and $\sum D_n$ be two series with positive terms, the first convergent, the second divergent. If $\sum A_n$ is also a series of positive terms then $\sum A_n$ converges if there exists a number $N_1$ such that, for every $n > N_1$, $A_n \leq C_n$ but diverges if there exists a number $N_2$ such that, for every $n > N_2$, $A_n \geq D_n$.

By theorem 2.1 we need only consider the series $\sum_{n=N_1+1}^{\infty} A_n$ and $\sum_{n=N_2+1}^{\infty} C_n$. If $A_n \leq C_n$ then $A_n = Y_n C_n$ where $Y_n = 1$. Hence by theorem 2.3 $\sum A_n$ converges. We next consider $\sum_{n=N_2+1}^{\infty} A_n$ and $\sum_{n=N_2+1}^{\infty} D_n$. If $A_n \geq D_n$ then $A_n = \delta D_n$ where $\delta \geq 1$. Hence, by theorem 2.4, $\sum A_n$ diverges.

Theorem 2.7: Let $\sum C_n$ and $\sum D_n$ denote respectively a convergent and a divergent series of positive terms. If $\sum A_n$ is a series of positive terms and there exists a number $N_1$ such that, for every $n > N_1$, $\frac{A_{n+1}}{A_n} \leq \frac{C_{n+1}}{C_n}$ then $\sum A_n$ converges. If there exists a number $N_2$ such that, for every $n > N_2$, $\frac{A_{n+1}}{A_n} \geq \frac{D_{n+1}}{D_n}$ then $\sum A_n$ diverges.

We may disregard the first $N_1$ terms. If $\frac{A_{n+1}}{A_n} \leq \frac{C_{n+1}}{C_n}$ then $\frac{A_{n+1}}{C_{n+1}} \leq \frac{A_n}{C_n}$. The sequence of the ratios $Y_n = \frac{A_n}{C_n}$ is monotone decreasing.
and consequently, since all its terms are positive, it is necessarily bounded. This establishes convergence by theorem 2.3. If $\frac{A_{n+1}}{A_n} \geq \frac{D_{n+1}}{D_n}$, then $\frac{A_{n+1}}{D_{n+1}} \geq \frac{A_n}{D_n}$. The ratios $\delta_n = \frac{A_n}{D_n}$ increase monotonically, and since they are constantly positive, they then have a positive lower bound. This establishes divergence by theorem 2.4.

**Theorem 2.8:** (Cauchy) If $\sum A_n$ is a series of positive terms and if there exists a number $N_1$ such that, for every $n > N_1$, $\frac{n}{\sqrt[n]{A_n}} \leq A < 1$ then $\sum A_n$ converges. If there exists a number $N_2$ such that, for every $n > N_2$, $\frac{n}{\sqrt[n]{A_n}} \geq 1$, then $\sum A_n$ diverges.

Compare the two series:

(1) $\frac{A_n + A_{n+1} + A_{n+2} + \ldots + A_{n+p}}{n}$

(2) $A_n + A_{n+1} + A_{n+2} + \ldots + A_{n+p}$

Series (2) is known to converge since it is a geometric series with ratio $A$ less than 1. If $\frac{n}{\sqrt[n]{A_n}} \leq A$ then $A_n \leq A^n$. Since series (1) is less than series (2), term by term, it must then converge also. Now compare the two series:

(1) $\frac{A_n + A_{n+1} + \ldots + A_{n+p}}{n}$

(2) $1 + 1 + \ldots + 1$.

Series (2) is known to diverge since it is a geometric series with ratio equal to 1. If $\frac{n}{\sqrt[n]{A_n}} \geq 1$, then $A_n \geq 1$. Since series (1) is greater than series (2), term by term, it must then diverge also.
Theorem 2.9: Let $\sum A_n$ be a series of positive terms. (a) If there exists a number $N_1$ such that, for every $n > N_1$, $\frac{A_{n+1}}{A_n} \leq \lambda < 1$, then $\sum A_n$ converges. (b) If there exists a number $N_2$ such that, for every $n > N_2$, $\frac{A_{n+1}}{A_n} \geq 1$, then $\sum A_n$ diverges.

To prove (a), compare the two series:

1. $A_n + A_{n+1} + A_{n+2} + \ldots$

2. $A_n + \lambda A_n + \lambda^2 A_n + \ldots$

Series (2) converges since it is a geometric series with ratio less than 1.

Then since the given series (1) is less than series (2), term by term, it must converge also. To prove (b) compare the two series:

1. $A_n + A_{n+1} + A_{n+2} + A_{n+3} + \ldots$

2. $A_n + A_n + A_n + A_n + \ldots$

Series (2) diverges since it is a geometric series with ratio equal to 1.

Since series (1) is greater than series (2), term by term, then it must diverge also.

Theorem 2.10: (Cauchy) If $\sum A_n$ is a series whose terms form a positive monotone decreasing sequence $(A_n)$, then it converges or diverges with

$$\sum_{k=0}^{\infty} 2^k A_{2^k} = A_1 + 2A_2 + 4A_4 + 8A_8 + \ldots$$

We denote the partial sums by $S_n$ and $t_k$. If $n < 2^k$,

$$S_n < A_1 + (A_2 + A_3) + \ldots + (A_{2^k} + \ldots + A_{2^k + 1} - 1)$$

$$\leq A_1 + 2A_2 + 4A_4 + \ldots + 2^k A_{2^k}$$

$$\leq t_k,$$
and

(a) \[ S_n < t_k. \]

If \( n > 2^k \),
\[
S_n > A_1 + A_2 + (A_3 + A_4) + \ldots + (A_{2^{k-1} + 1} + \ldots + A_{2^k})
\]
\[
> \frac{1}{2} A_1 + A_2 + 2A_3 + \ldots + 2^{k-1} A_{2^k}
\]
\[
> \frac{1}{2} t_k,
\]

and

(b) \[ 2S_n > t_k. \]

Inequality (a) shows that the sequence \( (S_n) \) is bounded if the sequence \( (t_k) \) is bounded; conversely inequality (b) shows that if \( (S_n) \) is unbounded, so is \( (t_k) \). The two sequences are therefore either both bounded or both unbounded, and therefore the two series under consideration either both converge or both diverge.

SERIES OF ARBITRARY TERMS

Theorem 2.11: A necessary and sufficient condition that \( \sum A_n \) converge is that, for every \( \varepsilon > 0 \), there exists a number \( N \) such that, for every

\( n > N \) and every \( p > 0 \), \[ |A_{n+1} + A_{n+2} + \ldots + A_{n+p}| < \varepsilon. \]

A necessary and sufficient condition for convergence is that for every \( \varepsilon > 0 \) there exists a number \( N \) such that, for every \( n > N \) and every \( m > N \),

\( |S_m - S_n| < \varepsilon \). If we replace \( m \) by \( n + p \) then \( |S_{n+p} - S_n| < \varepsilon \). But \( |A_{n+1} + A_{n+2} + \ldots + A_{n+p}| = |S_{n+p} - S_n| < \varepsilon \).
Theorem 2.12: If we deduce, from a given series \( \sum A_n \), a new series \( \sum A'_n \) by omitting a finite number of terms, prefixing a finite number of terms, or altering a finite number of terms (or doing all three things at once) and now designating afresh the terms of the series so produced by \( A'_0, A'_1, A'_2, \ldots \), then either both series converge or both diverge.

The hypotheses imply that a definite integer \( p \geq 0 \) exists such that, from some place onwards, say for every \( n > N \), we have \( A'_n = A_n + p \). Every portion of the one series therefore is also a portion of the other, provided only its initial index be \( > N + |p| \). Then by the preceding theorem our statement is proved.

Theorem 2.13: Convergent series may be added term by term, i.e., if \( \sum A_n = S \) and \( \sum B_n = T \) then \( \sum (A_n + B_n) = S + T \).

If we denote the partial sums by \( S_n \) and \( T_n \), then we have \( S_n \rightarrow S \) and \( T_n \rightarrow T \). Hence by addition of convergent sequences \( (S_n + T_n) \rightarrow S + T \).

Theorem 2.14: Convergent series may be substracted term by term, i.e., if \( \sum A_n = S \) and \( \sum B_n = T \) then \( \sum (A_n - B_n) = S - T \).

The proof is similar to that of the preceding theorem.

Theorem 2.15: A convergent series may be multiplied by a constant, that is to say, from \( \sum A_n = S \) it follows, if \( C \) is an arbitrary number, that \( \sum (CA_n) = CS \).
The partial sums of the new series are $CS_n$, if those of the old are $S_n$. Hence, since we have multiplied a convergent sequence by a constant, 
\[ \sum (CA_n) = CS. \]

**Theorem 2.16:** If for every $n$, $A_n > 0$, $\lim_{n \to \infty} A_n = 0$, and there exists a number $N$ such that, for every $n > N$, $A_{n+1} \leq A_n$, then the alternating series $A_1 - A_2 + A_3 - A_4 + \ldots$ converges.

By theorem 2.12 it will be no restriction to say that $A_{n+1} \leq A_n$, for every $n$. Denote the partial sums by $S_1, S_2, S_3, \ldots$. Consider only the even ones $S_2, S_4, S_6, \ldots$. $S_4 = (A_1 - A_2 + A_3 - A_4) - (A_1 - A_2) = A_3 - A_4$. $S_6 - S_4 = A_5 - A_6 \geq 0$. Hence $S_2 \leq S_4 \leq S_6 \leq \ldots$. In similar fashion we may show that $S_1 \leq S_3 \leq S_5 \leq \ldots$. Also, for every $n$, $S_{2n} \leq S_1$, since $S_{2n} = S_1 - (A_2 - A_3) - (A_4 - A_5) \ldots - (A_{2n-2} - A_{2n-1})$. Hence $S_{2n}$ is monotone increasing and bounded. Denote by $S$ its limit. Let $\varepsilon > 0$ be fixed. Then there exists a number $N_1$ such that, for every $n > N_1$, $|S - S_{2n}| < \varepsilon/2$. Also since $\lim_{n \to \infty} A_n = 0$ there exists a number $N_2$ such that, for every $i > N_2$, $A_i < \varepsilon/2$. Let $M = \max \{2N_1 + 1, N_2\}$ and choose $m > M$. Either $m$ is even or $m$ is odd. Suppose $m$ is even and $m = 2n$. Then $2n > 2N_1 + 1$ and $n > N$. Therefore $|S - S_m| = |S - S_{2n}| < \varepsilon/2 < \varepsilon$. Suppose $m$ is odd. Let $m = 2n + 1$. Then $|S - S_m| = |S - S_{2n+1}| = |S - S_{2n} + S_{2n} - S_{2n+1}| \leq |S - S_{2n}| + |S_{2n} - S_{2n+1}| = |S - S_{2n}| + A_{2n+1} < \varepsilon/2 + \varepsilon/2 = \varepsilon$. 
ABSOLUTE CONVERGENCE

Theorem 2.17: A series $\sum A_n$ is certainly convergent if the series (of positive terms) $\sum |A_n|$ converges. And if $\sum A_n = s$, $\sum |A_n| \leq S$ then $|s| \leq S$.

If $\sum |A_n|$ converges then for every $\varepsilon > 0$ there exists a number $N$ such that, for every $n > N$ and every $p > 0$, $|A_n| + |A_{n+1}| + |A_{n+2}| + \ldots + |A_{n+p}| < \varepsilon$. But $|A_n| + |A_{n+1}| + |A_{n+2}| + \ldots + |A_{n+p}| \leq |A_n| + |A_{n+1}| + \ldots + |A_{n+p}| < \varepsilon$. Hence $\sum A_n$ converges. Also $|s| = |A_0 + A_1 + A_2 + \ldots + A_n| \leq |A_0| + |A_1| + |A_2| + \ldots + |A_n| < S$. Thus $|s| \leq S$.

Definition 2.3: If a convergent series $\sum A_n$ is such that $\sum |A_n|$ also converges, then the first series will be called absolutely convergent, and otherwise non-absolutely convergent.

Theorem 2.18: If $\sum C_n$ is a convergent series of positive terms and if the terms of a given series $\sum A_n$, for every $n > N$, satisfy the condition $|A_n| \leq |C_n|$ or the condition $|\frac{A_{n+1}}{A_n}| \leq |\frac{C_{n+1}}{C_n}|$, then $\sum A_n$ is absolutely convergent.

By the comparison test $\sum |A_n|$ is convergent and hence by the preceding definition $\sum A_n$ is absolutely convergent.
Theorem 2.19: If $\sum A_n$ is an absolutely convergent series and if the factors $\alpha_n$ form a bounded sequence, then the series $\sum \alpha_n A_n$ is also absolutely convergent.

If $\alpha_n$ is bounded then $|\alpha_n|$ is also bounded. $\sum |\alpha_n| A_n = \sum |\alpha_n A_n|$ is then convergent by theorem 2.3 and hence $\sum \alpha_n A_n$ is absolutely convergent.

Theorem 2.20: If $\sum A_n$ is absolutely convergent, and $\sum B_n$ is any rearrangement of $\sum A_n$, then $\sum B_n$ is absolutely convergent and has the same sum as $\sum A_n$.

We shall first prove that $\sum B_n$ is absolutely convergent. Let $\epsilon > 0$ be chosen. Then there exists a number $N$ such that, for every $n > N$ and every $p > 0$, $|A_{n+1}| + |A_{n+2}| + \ldots + |A_{n+p}| < \epsilon$. Each of the numbers $A_0, A_1, \ldots, A_n, A_{n+1}$ is represented in the series $B_0, B_1, \ldots, B_n, \ldots$.

Suppose $B_m$ is the one of the above $A_\alpha$ having the highest rank in the $B$ series. Choose $k > m$, $q > 0$. Then $|B_{k+1}| + |B_{k+2}| + \ldots + |B_{k+q}| \leq |A_{j+1}| + |A_{j+2}| + \ldots + |A_{j+r}|$ where $A_{j+1}$ is the A of smallest subscript represented among the numbers $B_{k+1}, B_{k+2}, \ldots, B_{k+q}$ and $A_{j+r}$ is the one of largest subscript. But $j + 1 > N + 1$ so that $j > N$ and $r > 0$. Therefore $|A_{j+1}| + |A_{j+2}| + \ldots + |A_{j+r}| < \epsilon$. It follows that $\sum B_n$ is absolutely convergent.

If $\sum B_n$ is absolutely convergent then it is also convergent. We shall now show that if $\sum A_n = A$ then $\sum B_n = A$ also. We denote by $S_n$ and $T_n$ the nth
partial sums of $\sum A_n$ and $\sum B_n$ respectively. Let $\varepsilon > 0$ be chosen. Then there exists a number $N_1$ such that, for every $n > N_1$ and every $p > 0$,

$$|A_{n+1}| + |A_{n+2}| + \ldots + |A_{n+p}| < \varepsilon/3.$$

There exists a number $N_2$ such that, for every $n > N_2$ and every $p > 0$,

$$|B_{n+1}| + |B_{n+2}| + \ldots + |B_{n+p}| < \varepsilon/3.$$

There exists a number $N_3$ such that, for every $n > N_3$,

$$|A_n - S_n| < \varepsilon/3.$$  

Let $N = \max\{N_1, N_2, N_3\}$ and choose $n > N$. Then

$$|A_n - T_n| = |A_n - S_n + S_n - T_n| = |A_n - S_n| + |S_n - T_n| < \varepsilon/3 + |S_n - T_n|.$$  

Now

$$|S_n - T_n| = |A_1 + A_2 + \ldots + A_5 + B_1 + B_2 + \ldots + B_7|$$

where $A_1, A_2, A_3$ denote the $A$'s, if any, among $A_0, A_1, A_2, \ldots, A_n$ which are equal to $B$'s with subscripts greater than $n$, and $B_1, B_2, \ldots, B_7$ denote the $B$'s, if any, which are equal to $A$'s with subscripts greater than $n$. Hence

$$|S_n - T_n| < |A_1| + |A_2| + \ldots + |A_3| + |B_1| + |B_2| + \ldots + |B_7|.$$  

We denote by $i + 1$ and $i + q$ the smallest and the largest respectively of the $B$-subscripts corresponding to the integers $u_1, u_2$ through $u_5$. Similarly we denote by $j + 1$ and $j + r$ the smallest and the largest respectively of the $A$-subscripts corresponding to the integers $v_1, v_2$ through $v_7$. Now

$$|S_n - T_n| < |A_{j+1}| + |A_{j+2}| + \ldots + |A_{j+r}| + |B_{i+1}| + |B_{i+2}| + \ldots + |B_{i+q}| < 2 \varepsilon/3.$$  

Since $j + 1 \geq n + 1$, $i + 1 \geq n + 1$, it follows that $|A - T_n| < \varepsilon$ and $\sum B_n = A$.

**Definition 2.4:** A convergent infinite series which remains convergent, with unaltered sum, under any rearrangement, shall be called unconditionally convergent. A convergent series, on the other hand, whose behavior as to convergence can be altered by rearrangement for which therefore the
order of the terms must be taken into account, shall be called conditionally convergent. Thus by the above theorem every absolutely convergent series is unconditionally convergent.

**Theorem 2.21:** Every non-absolutely convergent series is only conditionally convergent.

We need only prove that by a suitable rearrangement of a non-absolutely convergent series \( \sum A_n \), we can deduce a divergent series \( \sum A'_n \). We do so as follows; the terms of the series \( \sum A_n \) which are \( \geq 0 \) we denote, in the order in which they occur in \( \sum A_n \), by \( p_1, p_2, p_3, p_4 \ldots \). Those which are \( < 0 \) we denote similarly by \(-q_1, -q_2, -q_3 \ldots \). Then \( \sum p_n \) and \( \sum q_n \) are series of positive terms. Of these, one at least must diverge. For if both were convergent, say with sums \( P \) and \( Q \), then we should have, for every \( n \), \( |A_0| + |A_1| + \ldots + |A_n| \leq P + Q \) and thus \( \sum A_n \) would be absolutely convergent which contradicts our hypothesis. Suppose \( \sum p_n \) diverges. Let us consider a series of the form \( p_1^+ p_2^+ \ldots + p_{m_1}^+ q_1^- p_{m_1 + 1}^+ p_{m_1 + 2}^+ \ldots + p_{m_x}^- q_2^+ p_{m_x + 1}^+ \ldots \) in which we have alternately a group of positive terms followed by a single negative term. This series is clearly a rearrangement of the given series \( \sum A_n \) and will be denoted by \( \sum A'_n \). Now since \( \sum p_n \) was assumed to diverge, and its partial sums are therefore unbounded, we can first choose \( m_1 \) so large that \( p_1^+ p_2^+ \ldots + p_{m_1}^+ > 1 + q_1^- \), then \( m_2 > m_1 \) so large that \( p_1^+ p_2^+ \ldots + p_{m_1}^+ + \ldots + p_{m_2}^+ > 2 + q_1^- + q_2^+ \) and, generally \( m_t > m_{t-1} \) so large that \( p_1^+ p_2^+ \ldots + p_{m_t}^+ > \ldots + q_t^- + q_{t+1}^+ \).
\[ p_{m_t} > t \cdot q_1 \cdot q_2 \cdot \ldots \cdot q_t \text{ with } t = 3, 4 \ldots. \] But \( \Sigma A_n' \) is then clearly divergent; for each of these partial sums of this series whose last term is a negative term \( -q_t \) of \( \Sigma A_n' \), is by the above \( > t \) \((t = 1, 2, 3 \ldots)\). Since \( t \) may stand for every positive integer, the partial sums of \( \Sigma A_n' \) are unbounded and \( \Sigma A_n' \) itself is divergent.

**Theorem 2.22:** If \( \Sigma A_n \) is absolutely convergent and \( \Sigma B_n \) is convergent, and if \( \Sigma A_n = A \) and \( \Sigma B_n = B \), then \( \Sigma P_n \) where \( P_n = A_0 B_n + A_1 B_{n-1} + \ldots + A_n B_0 \) is convergent and \( \Sigma P_n = AB \).

We choose \( K_1 > |B| \). There exists a number \( K_2 > 0 \) such that, for every \( n \), \( |A_n| < K_2 \). Denote by \( T_n \) the \( n \)th partial sum of \( \Sigma B_n \). Then \( T_n \) is a convergent sequence and there exists a number \( K_3 > 0 \) such that, for every \( n \), \( |T_n| < K_3 \). Let \( \varepsilon \) be any positive number. Denote by \( S_n \) the \( n \)th partial sum of \( \Sigma A_n \). Since \( \Sigma A_n = A \) there exists a number \( N_1 \) such that, for every \( n > N_1 \), \( |A - S_n| < \varepsilon/3K_1 \). Since \( \Sigma |A_n| \) converges there exists a number \( N_2 \) such that, for every \( n > N_2 \), and every \( p > 0 \), \( |A_{n+1}| + |A_{n+2}| + \ldots + |A_{n+p}| < \varepsilon/3K_3 \). Choose \( k > \max \{N_1, N_2\} \). Since \( \Sigma B_n = B \) there exists a number \( N_3 > 0 \) such that, for every \( n > N_3 \), \( |B - T_n| < \varepsilon/3kK_2 \). Let \( V_n \) be the \( n \)th partial sum of \( \Sigma P_n \) and consider \( |AB - V_n| \). Then \( |AB - V_n| \leq |P_0 + P_1 + \ldots + P_n| < |A - S_n| + |B - T_n| < \varepsilon \). Choose \( n > k + N_3 \). Then \( n - k > N_3 > 0 \). Then \( |AB - V_n| \leq |(A_0 + A_1 + \ldots + A_k)B + A_0[(T_n - B) + A_1(T_{n-1} - B) + \ldots + A_k(T_n - B)] + A_{k+1}T_n + \ldots + A_nT_0| \leq |S_k - A| \cdot |B| + |A_k(T_{n-k} - B) + A_{k+1}T_{n-k\ldots} + \ldots + A_nT_0| \leq |S_k - A| \cdot |B| \)
\[ + |A_0| \cdot |T_n - B| + \cdots + |A_k| \cdot |T_{n-k+1}| + |A_{k+1}| \cdot |T_{n-k}| + \cdots \]
\[ + |A_n| \cdot |B_0| < \frac{\varepsilon}{3K_1} \cdot K_1 + K_2 \cdot \frac{\varepsilon}{3kK_2} + K_2 \cdot \frac{\varepsilon}{3kK_2} + \cdots + K_2 \cdot \frac{\varepsilon}{3kK_2} \]
\[ + |A_{b+1}| \cdot K_3 + |A_{k+2}| \cdot K_3 + \cdots + |A_n| \cdot K_3 = 2 \frac{\varepsilon}{3} + K_3 \cdot \frac{\varepsilon}{3K_3} = \varepsilon. \]

Hence if we choose a positive integer \( M > \max \left[ \frac{N_1}{n}, \frac{N_2}{m} \right] \) then for every \( n > M \), \( |AB - V_n| < \varepsilon \). Hence \( \sum P_n = AB \).

**Theorem 2.23:** Given a series \( \sum_{n=0}^{\infty} A_n \), whose terms \( A_n \) are expressible in the form \( A_n = x_n - x_{n+1} \), where \( x_n \) is the term of a convergent sequence of known limit \( \xi \), then \( \sum_{n=0}^{\infty} A_n = x_0 - \xi \).

Let \( S_n \) be the nth partial sum. Then \( S_n = (x_0 - x_1) + (x_1 - x_2) + \cdots \)
\[ + (x_n - x_{n+1}) = x_0 - x_{n+1}. \] Since \( x_n \to \xi \), the theorem follows.
CHAPTER III

SERIES OF POSITIVE TERMS

Theorem 3.1: If $X_1, X_2, \ldots, X_n, \ldots$ are arbitrary positive terms, we always have
\[ \lim X_{n+1}/X_n \leq \lim \frac{n}{n+1} \leq \lim \frac{n}{X_n} \leq \lim X_n. \]

The inner inequality is obvious since the lower limit of any sequence is always ≤ upper limit. The two outer inequalities are so nearly similar that we shall prove only the right one. Let \[ \lim \frac{n}{X_n} = \lambda, \]
\[ \lim X_{n+1}/X_n = \lambda'. \]
Then the statement reduces to $\lambda \leq \lambda'$. If $\lambda' = +\infty$, there is nothing to prove. However, if $\lambda' < +\infty$ we choose $\varepsilon > 0$ and an integer $p$ such that, for every $m \geq p$, $X_{m+1}/X_m < \lambda' + \frac{\varepsilon}{\lambda}$ by definition of an upper limit. We now write these inequalities for every $m = p, p+1, p+2, \ldots, n-1$, and multiply all these inequalities together.

Thus for $n > p$ we have $X_n/X_p < (\lambda' + \varepsilon/2)^{n-p}$ or $X_n < X_p (\lambda' + \varepsilon/2)^{n-p}$ or $X_n < X_p (\lambda' + \varepsilon/2)^{p}(\lambda' + \varepsilon/2)^{-p}$. We denote the constant term $X (\lambda' + \varepsilon/2)^{-p}$ by $A$. Then $\frac{n}{X_n} < \frac{n}{A} (\lambda' + \varepsilon/2)$. Since $A > 0$, $\frac{n}{A} \to 1$ and thus the right hand member $\to (\lambda' + \varepsilon/2)$. We can now choose $n_0 > p$ so that, for every $n > n_0$, $X_n < (\lambda' + \varepsilon/2)^{n} < X_{n_0} \frac{n}{A} (\lambda' + \varepsilon/2)$. Therefore $\frac{n}{X_n} < \lambda' + \varepsilon$ or $\lambda < \lambda' + \varepsilon$. Hence $\lambda \leq \lambda'$. 

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Definition 3.1: Given two convergent series of positive terms, 
\[ \sum C_n = S \text{ and } \sum C'_n = S', \]
whose partial sums are denoted by \( S_n \) and \( S'_n \) and the corresponding remainders by \( S - S_n = R_n \) and \( S' - S'_n = R'_n \). We say that the second converges more or less rapidly than the first, according as \( \lim \frac{R'_n}{R_n} \leq 0 \) or \( \lim \frac{R'_n}{R_n} = +\infty \). If the limit of this ratio exists and has a finite positive value, or if it be known merely that its lower limit \( > 0 \) and its upper limit \( < +\infty \), then the convergence of the two series will be said to be of the same kind. In any other case a comparison of the rapidity of convergence of the two series is impracticable.

Definition 3.2: If \( \sum d_n \) and \( \sum d'_n \) are two divergent series of positive terms, whose partial sums are denoted by \( S_n \) and \( S'_n \), respectively, the second is said to diverge more or less rapidly than the first according as \( \lim \frac{S'_n}{S_n} = +\infty \) or \( \lim \frac{S'_n}{S_n} = 0 \). If the upper and lower limits of this ratio are finite and positive, then the divergence of both series will be said to be of the same kind. In any other case we shall not compare the two series in respect to rapidity of divergence.

Theorem 3.2: If \( \frac{C'_n}{C_n} \to 0 \ ( +\infty ) \), then \( \sum C'_n \) converges more (less) rapidly than \( \sum C_n \).

In the first case, for any \( \varepsilon > 0 \), we choose \( n_0 \) so that for every \( n > n_0 \) we have \( C'_n < \varepsilon C_n \). We then also have \( \frac{r'_n}{r_n} = \frac{C'_{n+1} + C'_{n+2} + \cdots}{C_{n+1} + C_{n+2} + \cdots} \).
\[ \frac{C_{n+1} + \cdots}{C_{n+1} + \cdots} = \varepsilon. \quad \text{Hence } r'_n / r_n \to 0. \quad \text{The second case reduces to the first by interchanging the series.} \]

**Theorem 3.3:** If \( d'_n / d_n \to 0 \) \( ( + \infty ) \) then \( \sum d'_n \) diverges less (more) rapidly than \( \sum d_n \).

If \( d'_n / d_n \to 0 \) then \( \frac{d'_1 + d'_2 + \cdots + d'_n}{d_1 + d_2 + \cdots + d_n} = \frac{S'_n}{S_n} \to 0 \). This proves the theorem.

**Theorem 3.4:** Given the sequence of series \( \sum \frac{1}{n^\alpha} \), \( \sum \frac{1}{n (\log n)^\alpha} \), \( \sum \frac{1}{n \log n (\log \log n)^\alpha} \), \ldots When \( p \) is fixed each of these series will converge or diverge less and less rapidly as the exponent \( \alpha \) approaches unity ( \( > 1 \) in the first case and \( \leq 1 \) in the second case.) Also each of these series will converge or diverge less and less rapidly, as \( p \) increases, whatever positive value may be given to the exponent \( \alpha \). ( \( > 1 \) in the first case, \( \leq 1 \) in the second).

When \( p \) is fixed, theorems 3.2 and 3.3 are easily applied. The second case will need a little more explanation. We divide the generic term of the \( (p+1) \)st series with the exponent \( \alpha' \) by the corresponding term of the \( p \)th series with the exponent \( \alpha \). By this method we obtain

\[
\frac{(\log pn)^\alpha}{\log pn (\log pn+1)^\alpha'}. \quad \text{In the case of divergent series, } \alpha \text{ and } \alpha' \text{ are}
\]
positive and $\leq 1$. We may then write the expression in the form

$$\frac{1}{(\log_p n)^{1-\alpha} (\log_{p+1} n)^{\alpha}}$$

which tends to 0 as $n$ increases. In the case of convergent series, $\alpha$ and $\alpha'$ are $> 1$. We may then write the expression in the form

$$\frac{(\log_p n)^{\alpha-1}}{(\log_{p+1} n)^{\alpha}}$$

By repeated differentiation of numerator and denominator this expression can be shown to tend to $\infty$ as $n$ increases. This proves all that is needed.

The gradation in the rapidity of the convergence and divergence of these series enables us to deduce complete scales of convergence and divergence tests by introducing these series as comparison series in theorems 2.6 and 2.7.

I If $A_n \left\{ \begin{array}{c} \leq \n \log n \ldots \log_{p-1} n (\log_p n)^{\alpha} \\ \geq \n + 1 \end{array} \right\}$ \( \frac{1}{n \log n \ldots \log_{p-1} n (\log_p n)^{\alpha}} \) with \( \begin{array}{c} \alpha > 1 \\ \alpha \leq 1 \end{array} \), then \( \begin{array}{c} C \\ D \end{array} \).

II If $\frac{A_{n+1}}{A_n} \left\{ \begin{array}{c} \leq \n \log n \ldots \log_{p-1} n (\log_p n)^{\alpha} \\ \geq \n + 1 \end{array} \right\}$ \( \frac{\log n}{\log_{n+1}} \ldots \frac{\log_{p-1} n}{\log_{p-1} (n+1)} \)

\( \cdot \left( \frac{\log_p n}{\log_{p+1} (n+1)} \right)^{\alpha} \) with \( \begin{array}{c} \alpha > 1 \\ \alpha \leq 1 \end{array} \), then \( \begin{array}{c} C \\ D \end{array} \).

These criteria will be referred to as the logarithmic tests of the first and second kinds.

**Theorem 3.5:** (Raabe) For $p > 0$ the logarithmic scale provides another criterion. If $\left[ \frac{A_{n+1}}{A_n} - 1 \right]^n \left\{ \begin{array}{c} \leq -\alpha \leq -1 \\ \geq -1 \end{array} \right\}$, then \( \begin{array}{c} C \\ D \end{array} \).
The C-condition means that, for every sufficiently large \( n \),
\[
A_{n+1}/A_n \leq 1 - \frac{\alpha}{n} \text{ or } n \ A_{n+1} \leq (n - 1) \ A_n - \beta \ A_n \text{ where } \beta = \alpha - 1 > 0.
\]
Hence \((n - 1) \ A_n - n \ A_{n+1} \geq \beta \ A_n > 0\). Therefore \( n \ A_{n+1} \) is the term of a monotone descending sequence, for a sufficiently large \( n \). Since it is constantly positive, it tends to a limit \( \lambda \geq 0 \). The series \( \sum C_n \)
with \( C_n = (n - 1) \ A_n - n \ A_{n+1} \) therefore converges. Since \( C_n \geq \beta \ A_n \)
then \( A_n \leq \frac{1}{\beta} \ C_n \), and the convergence of \( \sum A_n \) immediately follows.
If the D-condition is fulfilled, we have \( A_{n+1}/A_n \geq 1 - \frac{1}{n} \) or \((n - 1) \ A_n - n \ A_{n+1} \leq 0\). Accordingly \( n \ A_{n+1} \) is the term of a monotone increasing sequence and therefore remains greater than a fixed positive number \( \lambda \).
Since \( n \ A_{n+1} > \lambda \), then \( A_{n+1} > \frac{\lambda}{n} \), \( \lambda > 0 \), and the divergence follows immediately.

**Theorem 3.6:** (Schloemilch) If \( n \log \frac{A_{n+1}}{A_n} \in \{ \leq -\alpha < -1 \} \) then \( \{ C \} \).

In the case of divergence \( \log A_{n+1}/A_n \geq -1/n \) or \( A_{n+1}/A_n \geq e^{-\frac{1}{n}} > 1 - 1/n \). Divergence follows by theorem 3.5. In the case of convergence \( \log A_{n+1}/A_n \leq -\alpha/n \) or \( A_{n+1}/A_n \leq e^{-\frac{\alpha}{n}} > 1 - \alpha/n \) if \( \alpha > \alpha' > 1 \). By theorem 3.5 this proves convergence.

**Theorem 3.7:** If, in the logarithmic scale, we choose \( p = 1 \), we obtain another criterion which, omitting the limiting case \( \alpha = 1 \), we may write \( A_{n+1}/A_n = 1 - 1/n - \frac{\alpha}{n} \log n \) with \( \alpha \in \{ \geq \alpha > 1 \} \) then \( \{ C \} \).
We first put the criterion in the following form \[ -1 + (n - 1) \log n \cdot A_n \]
- \[ n \log n \cdot A_{n+1} \begin{cases} \geq \beta A_n \text{ with } \beta > 0 \text{ C} \\ \leq \beta A_n \text{ with } \beta; 0 \text{ D} \end{cases} \]

Since \((n - 1) \log (n - 1) > -1 + (n - 1) \log n\) we have, in the case of convergence, \((n - 1) \log (n - 1) \cdot A_n \log n A_{n+1} \geq \beta A_n\). Accordingly \(n \log n \cdot A_{n+1}\) is the term of a monotone descending sequence and accordingly tends to a limit \(\lambda > 0\). By theorem 2.23, the series whose nth term is \(C_n = (n - 1) \log (n - 1) \cdot A_n\)
- \(n \log n\) \(A_{n+1}\) must converge. As \(A_n \leq \frac{1}{\beta} C_n\), the same is true of \(\sum A_n\).

In the case of divergence we have \((n - 1) \log (n - 1) A_n - n \log n \cdot A_{n+1} \geq \left[ -\beta' + 1 - (n - 1) \log \left(1 + \frac{1}{n-1}\right) \right] A_n \cdot A_n \rightarrow +\infty\), \((n - 1) \log (1 + \frac{1}{n-1}) \rightarrow 1\).

Therefore, the expression in square brackets \(\rightarrow -\beta'\), and is therefore negative for every sufficiently large \(n\). Hence for those \(n\)'s the expression \(n \log n \cdot A_{n+1}\) increases monotonely and consequently remains greater than a certain positive number \(\lambda\). As \(A_{n+1} \geq \frac{\lambda}{n \log n}, \lambda > 0\), it follows that \(\sum A_n\) must diverge.

Theorem 3.3: (Gauss' Test) If the ratio \(A_{n+1}/A_n\) can be expressed in the form \(A_{n+1}/A_n = 1 - \alpha/n - \beta/n^\gamma\) where \(\lambda > 1\), and \(\theta_n\) is bounded, then \(\sum A_n\) converges when \(\alpha > 1\) and diverges when \(\alpha \leq 1\).

When \(\alpha > 1\) or \(\alpha < 1\) theorem 3.5 shows the validity of the assertion.

For \(\alpha = 1\) we write \(A_{n+1}/A_n = 1 - 1/n - 1/n \log n(\frac{\theta n \cdot \log n}{n^{\lambda - 1}})\). Since \(\lambda - 1 > 0\), the factor in parentheses tends to zero for a sufficiently large \(n\).

Thus \(A_{n+1}/A_n = 1 - 1/n\) and the series diverges by theorem 3.7.
Theorem 3.9: (Abel and Dini) If \( \sum_{n=1}^{\infty} d_n \) is an arbitrary divergent series of positive terms, and \( D_n = d_1 + d_2 + \ldots + d_n \) denotes its partial sums, the series \( \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{d_n}{D_n^\alpha} \) converges when \( \alpha > 1 \) and diverges when \( \alpha \leq 1 \).

In the case \( \alpha = 1 \), \( \frac{d_{n+1}}{D_{n+1}} + \ldots + \frac{d_{n+k}}{D_{n+k}} > \frac{d_{n+k}}{D_{n+k}} = 1 - \frac{D_n}{D_{n+k}} \). By hypothesis \( D_n \to \infty \). We therefore choose \( k = k_n \), for each \( n \), so that \( \frac{D_n}{D_{n+k}} < 1/2 \). Therefore \( a_{n+1} + a_{n+2} + \ldots + a_{n+k_n} > 1/2 \).

Hence the series diverges for \( \alpha = 1 \). The proof of convergence for \( \alpha > 1 \) will be proved with the next theorem.

Theorem 3.10: (Pringsheim) The series \( \sum_{n=2}^{\infty} \frac{d_n}{D_n^\lambda} \) where \( d_n \) and \( D_n \) have the same meaning as in theorem 3.9, converges for every \( \lambda > 0 \).

We choose a natural number \( p \) such that \( 1/p < \lambda \). It will then be sufficient to prove the convergence of the above series when the exponent \( \lambda \) is replaced by \( \theta = 1/p \). Since the series \( \sum_{n=2}^{\infty} \frac{1}{D_n^{\theta}} - \frac{1}{D_n^{\theta-1}} \) converges by theorem 2.23, and since \( D_{n-1} \leq D_n \to +\infty \), and since its terms are all positive, it will be sufficient to establish the inequality

\[
\frac{D_n-D_{n-1}}{D_n\cdot D_{n-1}^{\theta-1}} \leq 1/\theta \left( \frac{1}{D_n^{\theta-1}} - \frac{1}{D_n^{\theta}} \right) \quad \text{or} \quad 1 - \frac{D_{n-1}}{D_n} \leq 1/\theta \left( 1 - \frac{D_n^{\theta}}{D_n^{\theta-1}} \right).
\]

If we let \( X = \left( \frac{D_{n-1}}{D_n} \right)^{1/p} \), our inequality becomes \( (1 - X^p) \leq p(1 - X) \).
for every $X$ such that $0 < X \leq 1$. This becomes obvious when we note
that $(1 - X^P) \cdots (1 - X) (1 + X + \ldots + X^{P-1})$. This establishes the theorem.

Theorem 3.11: (Cesaro) If $\frac{d_n}{D_n} \to 0$ we have $\frac{d_1}{D_1} + \frac{d_2}{D_2} + \ldots + \frac{d_n}{D_n} \sim \log D_n$.

If $X_n = \frac{d_n}{D_n} \to 0$ we have $\frac{X_n}{\log \frac{D_n}{1 - X_n}} = \frac{\frac{d_n}{D_n}}{\log \frac{D_n}{D_{n-1}}} \to 1$. The undefined number $D_0$ we here assume $= 1$, also replacing the above ratio by 1 for all indices $n$ for which $X_n = 0$. Since $\log D_n \to +\infty$ we then have

$$\frac{\frac{d_1}{D_1} + \frac{d_2}{D_2} + \ldots + \frac{d_n}{D_n}}{\log D_1 + \log \frac{D_2}{D_1} + \ldots + \log \frac{D_n}{D_{n-1}}} = \frac{1}{\log D_n} \left[ \frac{\frac{d_1}{D_1} + \frac{d_2}{D_2} + \ldots + \frac{d_n}{D_n}}{1} \right] \to 1.$$ 

This proves the theorem.

Theorem 3.12: (Dini) If $\sum c_n$ is a convergent series of positive terms, and $r_{n-1} = \sum_{n+1}^\infty$ denotes its remainder after the $(n-1)\text{st}$ term, then $\sum \frac{c_n}{r_n^{\alpha-1}} = \sum \frac{c_n}{(c_n + c_{n+1} + \ldots)^{\alpha}} \begin{cases} \text{converges when } \alpha < 1 \\text{diverges when } \alpha \geq 1 \end{cases}.$

We shall first take the divergent case. For $\alpha = 1$ we have $\frac{c_n}{r_n^{\alpha-1}} \cdots + \frac{c_{n+k}}{r_n^{k-1}} = 1 - \frac{r_{n+k}}{r_{n-1}}$. For every fixed $n$ this value may be made $> 1/2$ by suitable choice of $k$, as $r_n \to 0$ for a sufficiently large $\lambda$. The series must therefore diverge. For $\alpha > 1$ this will also be the case, since $r_n < 1$ for every sufficiently large $n$. If $\alpha < 1$ we may
choose a positive integer $p$ so that $\alpha < 1 - 1/p$. Let $\theta = 1/p$.

Since $r_n < 1$ for $n > N$ it will be sufficient to establish the convergence of the series $\sum \frac{C_n}{(r_{n-1})^{1-\theta}} \equiv \sum \frac{r_{n-1} - r_n}{r_{n-1}} \cdot r_n^{\theta - 1}$. Since $r_n$ tends monotonely to 0, $\sum (r_n^{\theta - 1} - r_n^{\theta})$ is certainly convergent with positive terms. It therefore suffices to show that $\frac{r_{n-1} - r_n}{r_{n-1}} \cdot r_n^{\theta - 1} \leq 1/\theta (r_{n-1}^{\theta} - r_n^{\theta})$. If we let $\left(\frac{r_n}{r_{n-1}}\right)^{1/p} = y$ the relation becomes $(1 - y^p) \leq p(1 - y)$. This relationship is evident, since $0 < y \leq 1$.

**Theorem 3.13:** (Cauchy) Let $\sum_{n=1}^{\infty} A_n$ be a given series of monotonely diminishing terms. If there exists a function $f(x)$ positive and monotone decreasing for $x \geq 1$ for which $f(n) = A_n$ for every $n$ then $\sum A_n$ converges if, and only if, the numbers $J_n = \int_{i}^{n} f(t) \, dt$ are bounded.

Let $K$ be an integer $\geq 2$. Then for $(K-1) \leq t \leq K$ we have $f(t) \geq A_K$, and for $K \leq t \leq K + 1$, $f(t) \leq A_K$. It follows that $\int_{K}^{K+1} f(t) \, dt \leq A_K \leq \int_{K}^{K+1} f(t) \, dt$ $(K = 2, 3, \ldots)$. Assuming these inequalities written down for $K = 2, 3, \ldots, n$, and added, we obtain $\int_{i}^{n} f(t) \, dt \leq A_2 + A_3 + \ldots + A_n = S_n - A_1 < \int_{i}^{n} f(t) \, dt$. From the right hand inequality it follows that when the integrals $J_n$ are bounded, that so are the partial sums of the series. The left hand inequality shows the converse. This proves the theorem.

**Theorem 3.14:** (Ermakoff) If $f(x)$ is related to a given series $\sum A_n$ of positive, monotonely diminishing terms, in the manner described in
the preceding theorem, and also satisfies the conditions there laid down,

\[ \sum A_n \equiv \sum f(n) \begin{cases} \text{converges} & \text{if } \frac{e^{x f(n)}}{f(n)} \leq 1 \\ \text{diverges} & \text{if } \frac{e^{x f(n)}}{f(n)} \geq 1 \end{cases} \]

for every sufficiently large \( x \).

If the first of these inequalities be satisfied for \( x \geq x_0 \), we have for these \( x \)'s

\[ \int_{x_0}^{x} f(t) \, dt = \int_{x_0}^{x} e^t f(e^t) \, dt \leq \theta \int_{x_0}^{x} f(t) \, dt. \]

Consequently

\[ (1 - \theta) \int_{x_0}^{x} f(t) \, dt \leq \theta \left[ \int_{x_0}^{x} f(t) \, dt - \int_{x_0}^{x} f(t) \, dt \right] \]

\[ \leq \theta \left[ \int_{x_0}^{x} f(t) \, dt - \int_{x_0}^{x} f(t) \, dt \right] \]

\[ \leq \theta \left[ \int_{x_0}^{x} f(t) \, dt \right]. \]

Thus the integral on the left, and hence also \( \int_{x_0}^{x} f(t) \, dt \) is for every \( x > x_0 \) less than a certain fixed number. The series \( \sum A_n \) must therefore, converge by the preceding theorem. If, on the other hand, we assume the second inequality satisfied for \( x > x_1 \), we have, for these \( x \)'s

\[ \int_{x_1}^{x} f(t) \, dt \]

\[ = \int_{x_1}^{x} e^t f(e^t) \, dt \geq \int_{x_1}^{x} f(t) \, dt. \]

A comparison of the first and third integrals shows further that \( \int_{x_1}^{x} f(t) \, dt \geq \int_{x_1}^{x} f(t) \, dt \). On the right hand side of this inequality, we have a fixed quantity \( y > 0 \) and the inequality expresses the fact that for every \( n > x_1 \) we can assign \( k_n \) so that, with the same meaning for \( J_n \) as in the preceding theorem, \( J_n \) cannot be bounded and \( J_n \) therefore cannot converge by the preceding theorem.
BIBLIOGRAPHY


