THE BUCKLING OF A UNIFORMLY COMPRESSED PLATE WITH INTERMEDIATE SUPPORTS

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THESIS

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1. INTRODUCTION

This problem has been selected from the mathematical theory of elasticity. We consider a rectangular plate of thickness h, length a, and width b. A right-hand system of rectilinear coordinates is chosen with the coordinate axes parallel to the edges of the plate, the x-axis being parallel to the longest edge and the y-axis being parallel to an end-face. The origin is taken as the geometric center of an end-face of the plate (Fig. 1). This causes the xy-plane to coincide with the neutral plane, that is, the plane midway between, and parallel to, the two largest faces of the plate. The plate is subjected to compressive forces parallel to the x-axis, uniformly distributed along the y-axis in the interval $(-\frac{1}{2}b, \frac{1}{2}b)$ and along the line $x = a$, $z = 0$ from $y = -\frac{1}{2}b$ to $y = \frac{1}{2}b$. These forces act in the neutral plane and give the plate a tendency to buckle.

In this paper, an expression for the displacement, $z = w(x,y)$, of the neutral plane from its original position is obtained as a solution of the usual linearized partial differential equation applicable to plate theory. However, this problem differs from other plate problems in that it is assumed that there are two intermediate supports located on the edges of the plate parallel to the compressive forces (hence parallel to the x-axis). Specifically, they are

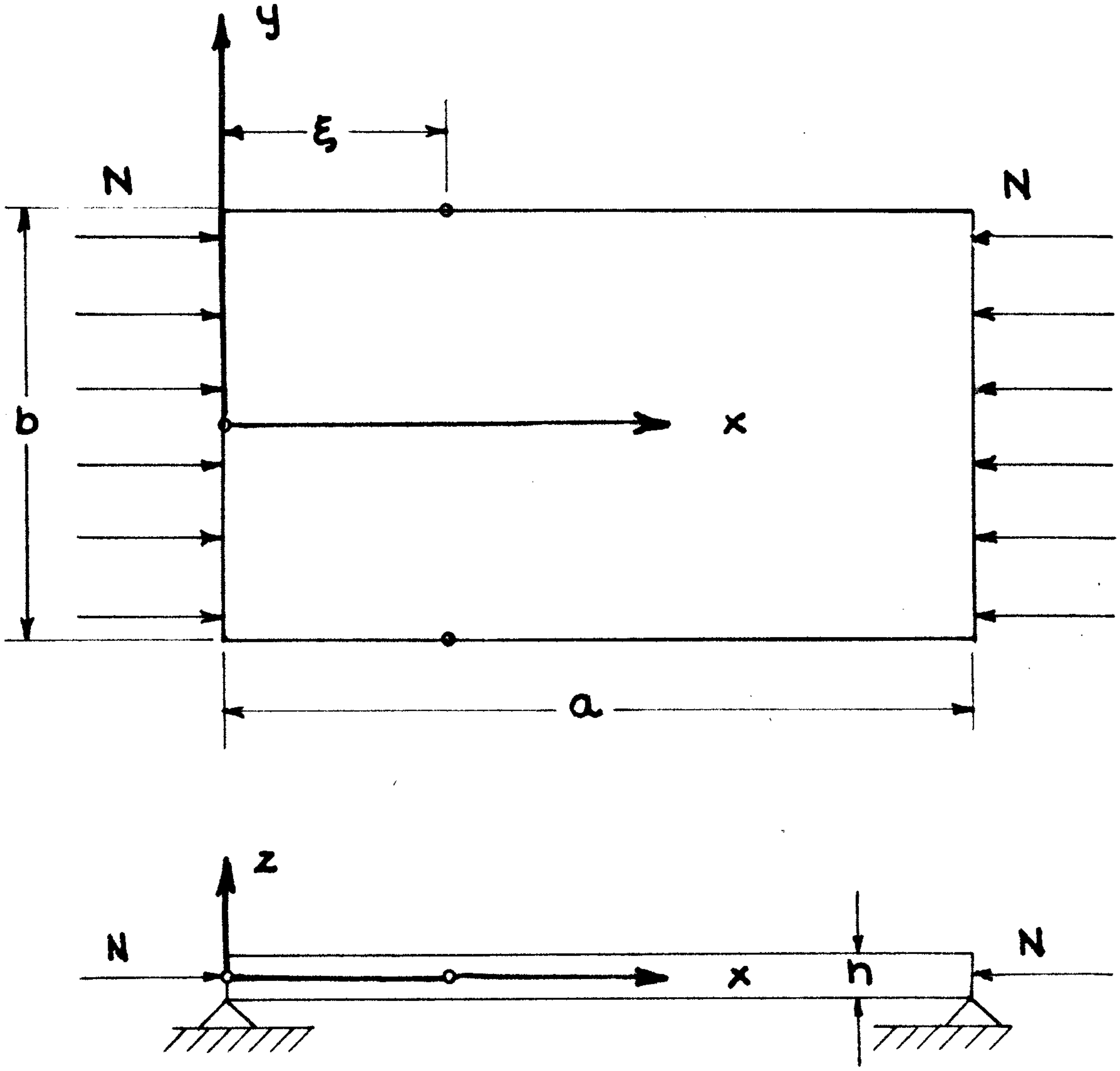

Fig. 1. The undeformed plate

located in the neutral plane at the points ($x = \xi$, $y = \frac{1}{2}b$) and ($x = \xi$, $y = -\frac{1}{2}b$). This gives an additional condition (namely, that $w = 0$ at $(\xi, \frac{1}{2}b)$ and $(\xi, -\frac{1}{2}b)$). From this condition we obtain the critical value of the compressive forces which cause the plate to fail.

The solution of this problem is based on the following assumptions:

A_1. The material is isotropic, i.e., the material has the same elastic properties in all directions.

A_2. The thickness (h) of the plate is small compared with the length (a) and the breadth (b).

A_3. The linearized (small deflection) theory of the theory of elasticity holds.

In order to facilitate the mathematical formulation of the problem, we itemize the following notation:[1]

a: Length of the plate in the x-direction.

b: Breadth of the plate in the y-direction.

h: Thickness of the plate.

w: Displacement of a particle in the z-direction due to an external force, a function of x and y only.

E: Modulus of elasticity in tension and compression, a constant for each material.

ν: Poisson's ratio, the ratio of lateral contraction.

$D = \dfrac{Eh^3}{12(1-\nu^2)}$: Flexural rigidity of the plate.

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[1] S. Timoshenko, Theory of Plates and Shells, p. xi.

σ_x, σ_y: Normal components of stress parallel x-, y-directions.

τ_{xy}, τ_{xz}, τ_{yz}: Shearing stress components.

$M_x = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_x z dz = -D(\frac{\partial^2 \omega}{\partial x^2} + \nu \frac{\partial^2 \omega}{\partial y^2})$: Bending moment per unit length of section of plate perpendicular to x-direction.

$M_y = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_y z dz = -D(\frac{\partial^2 \omega}{\partial y^2} + \nu \frac{\partial^2 \omega}{\partial x^2})$: Bending moment per unit length of section of plate perpendicular to y-direction.

$M_{xy} = -\int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{xy} z dz = D(1 - \nu)\frac{\partial^2 \omega}{\partial x \partial y} = -M_{yx}$: Twisting moment per unit length of section of a plate perpendicular to x-axis.

$Q_x = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{xz} dz$: Shearing force parallel to z-axis per unit length of section of a plate perpendicular to x-axis.

$Q_y = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{yz} dz$: Shearing force parallel to z-axis per unit length of section of a plate perpendicular to y-axis.

N_x, N_y: Normal forces per unit length of sections of a plate perpendicular to x- and y-directions, respectively.

N_{xy}: Shearing force in direction of y-axis per unit length of section of a plate perpendicular to x-axis.

2. FORMULATION OF THE PROBLEM

The well-known, linearized differential equation[2] for the deflection of the neutral plane is

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = -\frac{N}{D}\frac{\partial^2 w}{\partial x^2}. \qquad (1)$$

The equation (1) is applicable to all plate problems in which there are forces in the middle (neutral) plane.

The boundary conditions associated with the above differential equation for this particular problem may be formulated as follows:

1. <u>Along the edges $x = 0$ and $x = a$</u>: Since all applied forces are normal to these edges, no displacement is produced here. Thus, $w = 0$. Moreover, the curvature in the y-direction, $\frac{\partial^2 w}{\partial y^2}$, is also zero. We also note that along these edges

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right) = 0, \text{ so that}$$

$$w = \frac{\partial^2 w}{\partial x^2} = 0. \qquad (2)$$

2. <u>Along the edges $y = \pm\frac{1}{2}b$</u>: Here we have $M_y = 0$; consequently,

$$\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2} = 0. \qquad (3)$$

Suppose now that the edges $y = \pm\frac{1}{2}b$ are free along the entire length. If this were the case,[3] then

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[2] <u>Ibid.</u>, p. 314

[3] <u>Ibid.</u>, p. 92.

$$V = \frac{\partial^3 w}{\partial y^3} + (2-\nu)\frac{\partial^3 w}{\partial x^2 \partial y} = 0,$$

where V denotes a vertical force along the free edge. However, at the points $(\xi, \pm\frac{1}{2}b)$ of the edges $y = \pm\frac{1}{2}b$, it is evident that there exist vertical forces in the plate, say $V(\xi, \pm\frac{1}{2}b) = \pm k$. We put $V = 0$ except in a very short interval $(\xi-\epsilon, \xi+\epsilon)$ in which V increases in such a way that[4]

$$\lim_{\epsilon \to 0} \int_{\xi-\epsilon}^{\xi+\epsilon} V(x)dx = k,$$

a finite quantity.

In accordance with accepted notation, we write

$$V = \delta(x-\xi)k, \qquad \int_0^a \delta(x-\xi)dx = 1. \tag{4}$$

Hence we have

$$\frac{\partial^3 w}{\partial y^3} + (2-\nu)\frac{\partial^3 w}{\partial x^2 \partial y} = \pm\delta(x-\xi)k. \tag{5}$$

3. <u>At the points $x = \xi$, $y = \pm\frac{1}{2}b$:</u> At these points the plate is rigidly attached. Thus

$$w(\xi, \pm\tfrac{1}{2}b) = 0. \tag{6}$$

Recapitulating, our boundary conditions are:

$x = 0,\ x = a$:	$w = \frac{\partial^2 w}{\partial x^2} = 0,$	(2)
$y = \pm\frac{1}{2}b$:	$\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} = 0,$	(3)
	$\frac{\partial^3 w}{\partial y^3} + (2-\nu)\frac{\partial^3 w}{\partial x^2 \partial y} = \pm\delta(x-\xi)k,$	(5)
$x = \xi,\ y = \pm\frac{1}{2}b$:	$w = 0.$	(6)

[4] A. G. Webster, <u>Partial Differential Equations of Mathematical Physics</u>, p. 109.

3. SOLUTION OF THE PROBLEM

Following the usual procedure, we assume the solution of (1) in series form:[5]

$$w = \sum \sin a_n x \; f_n(y), \quad a_n = \frac{n\pi}{a}, \tag{7a}$$

where $f_n(y)$ represents a set of functions to be determined. It is apparent that such a function satisfies boundary conditions (2). To determine $f_n(y)$ substitute (7) in (1). This gives

$$\sum a_n^4 \sin a_n x \; f_n(y) - 2\sum a_n^2 \sin a_n x \; f_n''(y) + \sum \sin a_n x \; f_n^{IV}(y) - \frac{N}{D}\sum a_n^2 \sin a_n x \; f_n(y) = 0, \tag{8a}$$

or

$$\sum \sin a_n x \left\{ f_n^{IV}(y) - 2a_n^2 f_n''(y) + (a_n^4 - \frac{N}{D}a_n^2) f_n(y) \right\} = 0. \tag{8b}$$

Since (8b) must be identically satisfied for all x, we may write

$$f_n^{IV}(y) - 2a_n^2 f_n''(y) + (a_n - \frac{N}{D}a_n^2) f_n(y) = 0, \tag{9}$$

a fourth order ordinary differential equation for $f_n(y)$.

The auxiliary equation, readily obtained from (9), is

$$R^4 - 2a_n^2 R^2 + (a_n^4 - \frac{N}{D}a_n^2) = 0. \tag{10}$$

The solutions of (10) are

$$R_1 = \sqrt{a_n^2 + \sqrt{\frac{N}{D}a_n^2}} = \alpha_n, \qquad R_2 = -\sqrt{a_n^2 + \sqrt{\frac{N}{D}a_n^2}} = -\alpha_n, \tag{11}$$

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[5] Throughout this paper, $\sum$ shall be taken to mean $\sum_1^\infty$.

$$R_3 = \sqrt{a_n^2 - \sqrt{\tfrac{N}{D}} a_n^2} = i\sqrt{-a_n^2 + \sqrt{\tfrac{N}{D}} a_n^2} = i\beta_n,$$
$$R_4 = -\sqrt{a_n^2 - \sqrt{\tfrac{N}{D}} a_n^2} = -i\sqrt{-a_n^2 + \sqrt{\tfrac{N}{D}} a_n^2} = -R_3 = -i\beta_n. \quad (11)$$

Consequently, the general solution of (9) is

$$f_n(y) = P_n e^{\alpha_n y} + Q_n e^{-\alpha_n y} + S_n e^{i\beta_n y} + T_n e^{-i\beta_n y}, \quad (12)$$

where P_n, Q_n, S_n, T_n are constants of integration. Thus we have

$$w = \sum \sin a_n x \left\{ P_n e^{\alpha_n y} + Q_n e^{-\alpha_n y} + S_n e^{i\beta_n y} + T_n e^{-\beta_n y} \right\}. \quad (7b)$$

From the boundary conditions (3) we have, for $y = \frac{1}{2}b$ and $0 \leq x \leq a$,

$$\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} = \Big[\alpha_n^2 (P_n e^{\alpha_n \frac{1}{2}b} + Q_n e^{-\alpha_n \frac{1}{2}b}) - \beta_n^2 (S_n e^{i\beta_n \frac{1}{2}b} + T_n e^{-i\beta_n \frac{1}{2}b})$$
$$-\nu a_n^2 (P_n e^{\alpha_n \frac{1}{2}b} + Q_n e^{-\alpha_n \frac{1}{2}b} + S_n e^{i\beta_n \frac{1}{2}b} + T_n e^{-i\beta_n \frac{1}{2}b})\Big] \sin a_n x.$$

Since this relation must hold for all x in the interval $[0, a]$, it is clear that

$$(\alpha_n^2 - \nu a_n^2)(P_n e^{\alpha_n \frac{1}{2}b} + Q_n e^{-\alpha_n \frac{1}{2}b}) - (\beta_n^2 + \nu a_n^2)(S_n e^{i\beta_n \frac{1}{2}b} + T_n e^{-i\beta_n \frac{1}{2}b}) = 0.$$

Similarly, evaluating (3) at $y = -\frac{1}{2}b$, leads to

$$(\alpha_n^2 - \nu a_n^2)(Q_n e^{\alpha_n \frac{1}{2}b} + P_n e^{-\alpha_n \frac{1}{2}b}) - (\beta_n^2 + \nu a_n^2)(T_n e^{i\beta_n \frac{1}{2}b} + S_n e^{-i\beta_n \frac{1}{2}b}) = 0.$$

Adding the last two equations yields

$$(\alpha_n^2 - \nu a_n^2)(P_n + Q_n)(e^{-\alpha_n \frac{1}{2}b} + e^{\alpha_n \frac{1}{2}b})$$
$$-(\beta_n^2 + \nu a_n^2)(T_n + S_n)(e^{i\beta_n \frac{1}{2}b} + e^{-i\beta_n \frac{1}{2}b}) = 0.$$

This relation is certainly satisfied if we choose

$$Q_n = -P_n \quad \text{and} \quad T_n = -S_n,$$

in which case we have

$$f_n(y) = P_n(e^{\alpha_n y} - e^{-\alpha_n y}) + S_n(e^{i\beta_n y} - e^{-i\beta_n y}).$$

Setting $2P_n = A_n$ and $2S_n = B_n$, we have

$$f_n(y) = A_n \sinh\alpha_n y + B_n \sin\beta_n y. \tag{13}$$

We may now write the solution of (1) in the more explicit form

$$w = \sum \sin a_n x(A_n \sinh \alpha_n y + B_n \sin\beta_n y). \tag{7c}$$

It only remains to determine the coefficients A_n and B_n from the boundary conditions (3) and (5). From (3), we have

$$\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2} = \sum \sin a_n x\left\{A_n\alpha_n^2 \sinh\alpha_n \tfrac{1}{2}b - B_n\beta_n^2 \sin\beta_n \tfrac{1}{2}b\right\}$$

$$- \nu a_n^2 \sum \sin a_n x\left\{A_n \sinh\alpha_n \tfrac{1}{2}b + B_n \sin\beta_n \tfrac{1}{2}b\right\} = 0,$$

or

$$\sum \sin a_n x \; (A_n\alpha_n^2 \sinh \alpha_n \tfrac{1}{2}b - B_n\beta_n^2 \sin\beta_n \tfrac{1}{2}b)$$

$$- \nu a_n^2 (A_n \sinh\alpha_n \tfrac{1}{2}b + B_n \sin\beta_n \tfrac{1}{2}b)\} = 0. \tag{14a}$$

Since (14a) must be identically satisfied for all x, we may write

$$A_n(\alpha_n^2 - \nu a_n^2)\sinh\alpha_n \tfrac{1}{2}b - B_n(\beta_n^2 + \nu a_n^2)\sin\beta_n \tfrac{1}{2}b = 0. \tag{14b}$$

From (5), we have

$$\frac{\partial^3 w}{\partial y^3} + (2-\nu)\frac{\partial^3 w}{\partial x^2 \partial y} = \sum \sin a_n x\left\{A_n\alpha_n^3 \cosh\alpha_n \tfrac{1}{2}b - B_n\beta_n^3 \cos\beta_n \tfrac{1}{2}b\right.$$

$$\left. - a_n^2(2-\nu)(\alpha_n A_n \cosh\alpha_n \tfrac{1}{2}b + B_n\beta_n \cos\beta_n \tfrac{1}{2}b)\right\} = \delta(x-\xi)k.$$

Let us denote now the coefficient of $\sin a_n x$ in the above equation by C_n. Then we may write

$$\sum C_n \sin a_n x = \delta(x-\xi)k. \tag{15}$$

Recall now that

$$\int_0^a \sin\frac{n\pi x}{a}\sin\frac{m\pi x}{a}dx = \tfrac{1}{2}a \text{ if } m = n, = 0 \text{ if } m \neq n,$$

and further that

$$\int_0^a \delta(x-\xi)\sin\frac{m\pi x}{a}\,dx = \sin\frac{m\pi\xi}{a}.$$

Then, multiplying (15) by $\sin\frac{m\pi x}{a}$ and integrating, we obtain

$$\int_0^a \sin a_m x \sum C_n \sin a_n x\,dx = \tfrac{1}{2}aC_n = k\sin a_n\xi.$$

Consequently, (15) may be written

$$\sum C_n \sin a_n x = \frac{2k}{a}\sum \sin a_n\xi \sin a_n x.$$

Moreover, the coefficients C_n may be simplified as follows:

$$A_n\alpha_n^3\cosh\alpha_n\tfrac{1}{2}b - B_n\beta_n^3\cos\beta_n\tfrac{1}{2}b - a_n^2(2-\nu)(\alpha_nA_n\cosh\alpha_n\tfrac{1}{2}b + B_n\beta_n\cos\beta_n\tfrac{1}{2}b)$$

$$= A_n\alpha_n(\alpha_n^2 - 2a_n^2 + \nu a_n^2)\cosh\alpha_n\tfrac{1}{2}b - B_n\beta_n(\beta_n^2 + 2a_n^2 - \nu a_n^2)\cos\beta_n\tfrac{1}{2}b$$

$$= A_n\alpha_n\left(a_n^2 + \sqrt{\tfrac{N}{D}a_n^2} - 2a_n^2 + \nu a_n^2\right)\cosh\alpha_n\tfrac{1}{2}b$$

$$- B_n\beta_n\left(-a_n^2 + \sqrt{\tfrac{N}{D}a_n^2} + 2a_n^2 - \nu a_n^2\right)\cos\beta_n\tfrac{1}{2}b$$

$$= A_n\alpha_n(\beta_n^2 + \nu a_n^2)\cosh\alpha_n\tfrac{1}{2}b - B_n\beta_n(\alpha_n^2 - \nu a_n^2)\cos\beta_n\tfrac{1}{2}b.$$

Consequently, boundary condition (5) becomes

$$\sum \sin a_n x\left\{A_n\alpha_n(\beta_n^2 + \nu a_n^2)\cosh\alpha_n\tfrac{1}{2}b - B_n\beta_n(\alpha_n^2 - \nu a_n^2)\cos\beta_n\tfrac{1}{2}b\right\}$$

$$= \frac{2k}{a}\sum \sin a_n x \sin a_n\xi. \qquad (16a)$$

Since (16a) must hold for all x,

$$A_n\alpha_n(\beta_n^2+\nu a_n^2)\cosh\alpha_n\tfrac{1}{2}b$$

$$-\,B_n\beta_n(\alpha_n^2-\nu a_n^2)\cos\beta_n\tfrac{1}{2}b = \frac{2k}{a}\sin a_n\xi. \qquad (16b)$$

For each n, (14b) and (16b) constitute a set of two linear equations for two unknowns. The solution of each pair is

$$A_n = \frac{2k}{a}\sin a_n\xi\left\{\frac{(\beta_n^2+\nu a_n^2)\sin\beta_n\tfrac{1}{2}b}{\beta_n(\alpha_n^2-\nu a_n^2)^2\sinh\alpha_n\tfrac{1}{2}b\cos\beta_n\tfrac{1}{2}b-\alpha_n(\beta_n^2+\nu a_n^2)^2\cosh\alpha_n\tfrac{1}{2}b\sin\beta_n\tfrac{1}{2}b}\right\}, \qquad (17)$$

and

$$B_n = -\frac{2k}{a}\sin a_n\xi\left\{\frac{(\alpha_n^2-\nu a_n^2)\sinh\alpha_n\tfrac{1}{2}b}{\beta_n(\alpha_n^2-\nu a_n^2)^2\sinh\alpha_n\tfrac{1}{2}b\cos\beta_n\tfrac{1}{2}b-\alpha_n(\beta_n^2+\nu a_n^2)^2\cosh\alpha_n\tfrac{1}{2}b\sin\beta_n\tfrac{1}{2}b}\right\}.$$

Substituting these values for A_n and B_n in (7c) we next seek a condition which satisfies $w(\xi, \tfrac{1}{2}b) = 0$, that is,

$$\frac{2k}{a}\sum\sin^2 a_n\xi\left\{\frac{(\beta_n^2+\nu a_n^2)\sinh\alpha_n\tfrac{1}{2}b\sin\beta_n\tfrac{1}{2}b-(\alpha_n^2-\nu a_n^2)\sinh\alpha_n\tfrac{1}{2}b\sin\beta_n\tfrac{1}{2}b}{\beta_n(\alpha_n^2-\nu a_n^2)^2\sinh\alpha_n\tfrac{1}{2}b\cos\beta_n\tfrac{1}{2}b-\alpha_n(\beta_n^2+\nu a_n^2)^2\cosh\alpha_n\tfrac{1}{2}b\sin\beta_n\tfrac{1}{2}b}\right\} = 0. \qquad (18)$$

Since $\alpha_n > 0$, $\beta_n > 0$, $b > 0$, it follows that $\sinh\alpha_n\tfrac{1}{2}b \neq 0$ and $\cosh\alpha_n\tfrac{1}{2}b \neq 0$. Therefore, (18) is satisfied only if

$$\sin\beta_n\tfrac{1}{2}b = 0,$$

or

$$\beta_n\tfrac{1}{2}b = m\pi, \qquad (19)$$

where m is an integer. Our expression for the displacement curve now becomes

$$w = \frac{2k}{a}\sum\sin a_n x\,\sin a_n\xi\,\sin\frac{2m\pi y}{b}\left\{\frac{1}{\beta_n(\alpha_n^2-\nu a_n^2)}\right\}. \qquad (20)$$

4. CONCLUSIONS

We now change the form of (19) by making use of some algebraic manipulations, as follows:

$$\beta_n = \frac{2m\pi}{b},$$

$$\beta_n^2 = \frac{4m^2\pi^2}{b^2},$$

$$\sqrt{\frac{N}{D}}a_n^2 = \frac{4m^2\pi^2}{b^2} + a_n^2,$$

$$\sqrt{\frac{N}{D}} = \frac{4m^2\pi a}{nb^2} + \frac{n\pi}{a},$$

$$\frac{N}{D} = \pi^2\left(\frac{4m^2 a}{nb^2} + \frac{n}{a}\right)^2,$$

$$\frac{Nb^2}{D} = \pi^2\left(4\frac{m^2 a}{nb} + \frac{nb}{a}\right)^2. \qquad (21)$$

It is now required to find the minimum or critical value of (21). In general, this minimum value is obtained by taking $m = 1$. Then

$$\frac{Nb^2}{D} = \pi^2\left(4\frac{a}{nb} + \frac{nb}{a}\right)^2.$$

Now let

$$\kappa = \left(4\frac{a}{nb} + \frac{nb}{a}\right)^2.$$

A plot of κ against a/b is shown in Figure 2. It should be noted that κ is equal to 16 for a square plate as well as

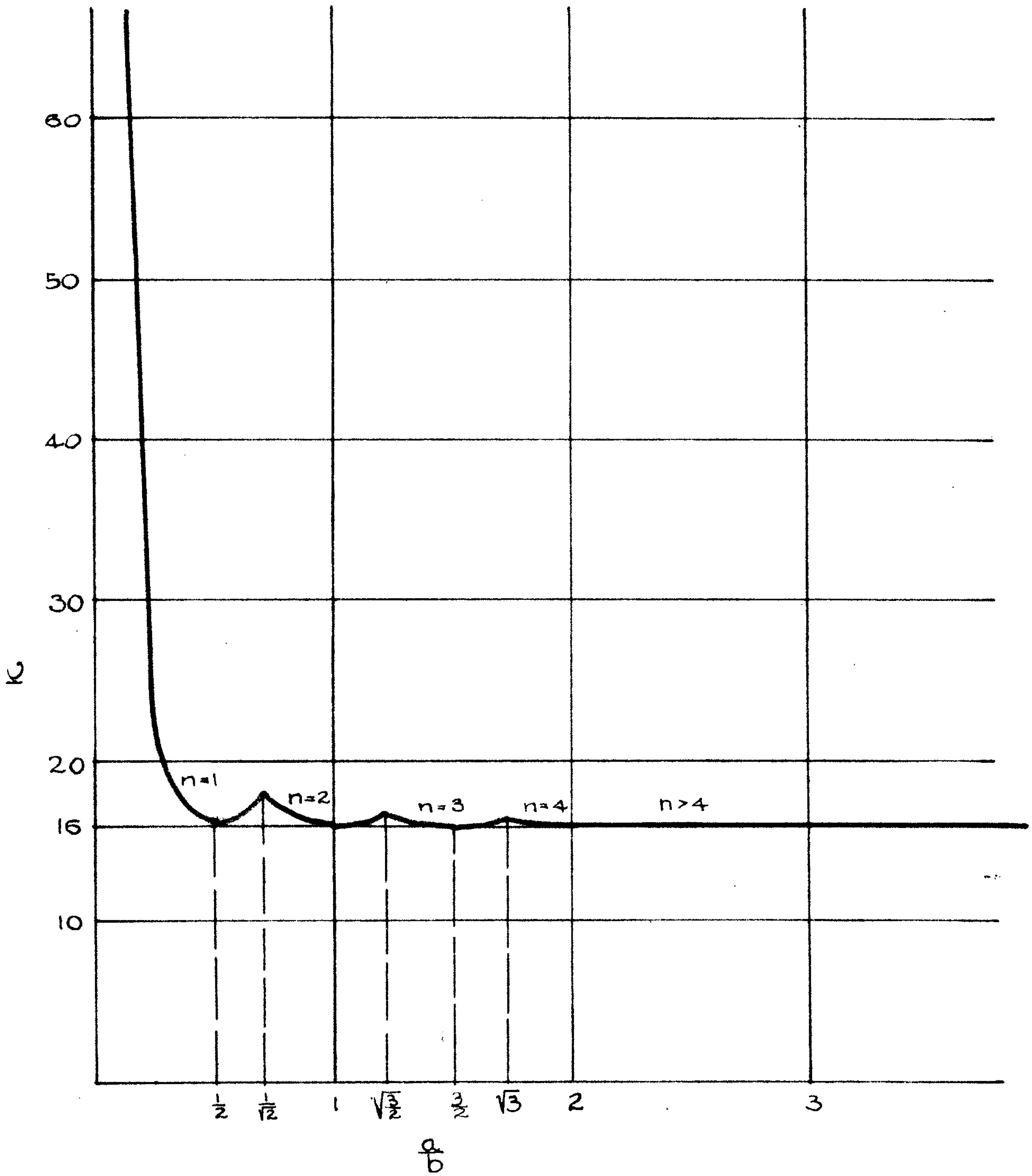

Fig. 2. Values of κ for various $\frac{a}{b}$ and n

for any plate which can be subdivided into an integral number of squares with side $\frac{1}{2}$b. It can also be seen that for long plates k remains practically constant at the value of 16.

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