DEPARTMENT OF ENERGY RESEARCH PROGRAM SUMMARY PROGRESS REPORT

Project Title: Analysis of Self-Similar Solutions of Multidimensional Conservation Laws

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PI Institution: CUNY – CSI

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PI Address: Staten Island, New York 10314

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The funding period of this grant is three years, beginning September 1, 2009. This is the third and concluding progress report. Before the proposal was funded, both Tesdall and Keyfitz moved to different universities. For administrative reasons, the proposal was funded as two separate grants, and hence two separate reports are being filed. However, much of the research, accomplished and planned, is joint.

This report is cumulative. Background on the project is contained in earlier reports and in the project proposal.

1 Report on Research Progress

The papers numbered 2 and 3 below appeared in print during the period between the dates of the last report (May 24, 2011) and this report (August 31, 2012). The paper numbered 4 below changed in status from “submitted” to “to appear” during this same period. In addition, another paper, Reference 5 below, was prepared, submitted and accepted during this period, and preparation of Reference 6 was begun.

In total, this award produced five journal articles, and another article which will be completed and submitted to a journal soon.

The first paper, Reference 1 below, combines numerical analysis of a simple problem for a simple model system of conservation laws with analysis of a linearized version of the problem. This problem describes a situation in which a rarefaction wave reflects off a sonic line, and the main result of the numerical solution obtained in the paper is that this reflection produces a shock. Our analysis of the linearized problem is what leads us to expect the presence of this shock, and our numerical computations confirm this feature. This phenomenon - the reflection of a rarefaction wave off a sonic line to form a shock wave - appears to be a generic feature of solutions of transonic flow problems, including Guderley Mach reflection and flow over a transonic airfoil.

The question of where the shock forms - on the sonic line, or in the supersonic region - in transonic flow problems such as the one considered in Reference 1 is open. No proof exists, and numerical calculations have been inconclusive until recently: most of these appear to show the shock beginning on the sonic line, and the numerical solution depicted in Reference 1 also appears to show the shock beginning on, or extremely close to, the sonic line. In order to answer this question, the PI performed very detailed computations in Reference 2.

These computations show that the shock forms strictly inside the supersonic region due to the coalescence of compression waves, as originally proposed by Guderley. The possibility of a shock forming exactly on the sonic line is not excluded by these numerical results, however. In order to provide some analytical backing to these computations, Keyfitz and Tesdall have begun looking at shock formation in the steady transonic small disturbance equations. Surprisingly, we were able to devise a counterexample (which appears to be highly non-typical) which shows that a shock can form exactly at the sonic line. In ongoing work, we hope to be able to show that if a shock does form exactly on the sonic line, that the formation mechanism is again the coalescence of compression waves in the supersonic region (as it is in the case when the shock forms in the supersonic region).

In another direction, Tesdall and John Hunter have recently completed a study of a problem that describes the diffraction of a weak, self-similar shock near a point where its strength approaches zero and the shock turns continuously into an expansion. This situation occurs, for example, when a weak plane shock hits a rigid semi-infinite wall at normal incidence. This work, contained in Reference 3 below, is motivated by the fact that the phenomenon of a shock that propagates into a constant state and diffracts self-similarly into an expansion wave is not specific to the screen problem - it is likely to occur in other two-dimensional Riemann problems for hyperbolic systems of conservation laws. Numerical solutions obtained in Reference 3 appear to show an intriguing result: the shock dies out and reaches zero strength exactly at the sonic line. The disappearance of a diffracting shock at a sonic point, observed here, contrasts with the formation of a shock in transonic flow due to the coalescence of compression waves. As discussed above and shown in Reference 2, such shocks typically form at supersonic points.

The problem considered in Reference 1 was motivated, in part, by the similarity of its solution to that of Guderley Mach reflection: one basic feature both solutions share is a sonic line across which the solution is continuous. We have studied an even simpler problem in Reference 4 which also has continuous flow at the sonic line. In it, we have posed a perturbation of a one-dimensional transonic problem which has a unique solution. Using Fourier transformation, we have found some solutions to this perturbed problem. We have also found that not all perturbations lead to solvable problems. An additional contribution of the paper is to demonstrate a simple, but new, (to our knowledge) solution of the steady transonic small disturbance equation.

Preliminary numerical solutions given in Reference 3 appear to show that the shock in the shock-diffraction problem dies out exactly at a sonic point. In Reference 5 below, Tesdall and Hunter give a detailed description of the numerical method used in Reference 3, and present highly refined solutions which provide additional evidence that the shock disappears exactly on the sonic line, where the self-similar solutions change type, rather than in the supersonic region. In addition, these solutions for the self-similar diffraction of a shock into an expansion wave are compared and contrasted with solutions for shock formation in steady and self-similar transonic flows.

In recent work, Hunter and Tesdall have begun a study of glancing weak Mach reflection; this is contained in Reference 6 below. The glancing limit of shock reflection is the limit in which the wedge angle tends to zero with the shock strength fixed. In this limit, regular reflection cannot occur, and the reflection pattern is unknown. The main question is what happens near the triple point, and using the method of matched asymptotic expansions we

have formulated a problem for the unsteady transonic small disturbance equations which is valid in this region. This work is currently in progress.

2 Key Accomplishments

- Brought large-scale (25×10^6 data points), advanced methods for numerical solution of PDE to bear on the two-dimensional Riemann problem considered in Reference 1, and obtained numerical solution of problem. This numerical solution allowed us to describe key features of the solution.
- Combined analysis with numerical solution cited above to explain these numerical results.
- High-resolution, large-scale numerical computations (Reference 2) show for the first time that a shock forms strictly in the supersonic region.
- Numerical solutions in References 3 and 5 appear to show the disappearance of a diffracting shock at a sonic line.

3 Collaborators at Other Institutions

John K. Hunter	University of California, Davis
Katarina Jegdić	University of Houston, Downtown
Barbara L. Keyfitz	Ohio State University
Kevin R. Payne	University of Milan
Nedyu I. Popivanov	University of Sophia
Richard Sanders	University of Houston

4 Honors and Synergistic Activities

4.1 Honors

I received a prestigious invitation to lecture on the research funded by this grant in May, 2012. This invitation was with full travel support, with the lecture to be given at *International Conference on Conservation Laws and Applications*, Tata Institute of Fundamental Research, Centre for Applicable Mathematics, Bangalore, India, July 1–3, 2013.

I was also appointed to the editorial board of the Journal of Computational Science, May 2012.

4.2 Synergistic Activities

1. Co-organizing the minisymposium *Recent progress in hyperbolic problems: theory and computation*, to be held August 26–30, 2013, at the AMMCS-2013 conference, Waterloo, Ontario, Canada.

2. Co-organizing *3rd New York Conference on Applied Mathematics*, Rensselaer Polytechnic Institute, Troy, NY, October 13, 2012.
3. Organized *Hyperbolic Conservation Laws and Applications*, CUNY Graduate Center, New York, NY, April 26, 2012.
4. Organizing Committee, *CSI Undergraduate Mathematics Lecture Series*, inaugural lecture held at College of Staten Island, February 22, 2012; second in series to be held on November 6, 2012.
5. Co-organized (with Jae-Hun Jung, SUNY-Buffalo) the minisymposium *Computational Methods for Hyperbolic Problems* at the AMMCS-2011 conference, Waterloo, Ontario, Canada, July 25–29, 2011.

5 Publications Produced by this Award

1. A. M. Tesdall and B. L. Keyfitz, *A continuous, two-way free boundary in the unsteady transonic small disturbance equations*, Journal of Hyperbolic Differential Equations, 7 (2010), pp. 317–338.
2. A. M. Tesdall, *High-resolution solutions for the supersonic formation of shocks in transonic flow*, Journal of Hyperbolic Differential Equations, 8 (2011), pp. 485–506.
3. J. K. Hunter and A. M. Tesdall, *On the self-similar diffraction of a weak shock into an expansion wavefront*, SIAM Journal on Applied Mathematics, 72 (2012), pp. 124–143.
4. B. L. Keyfitz, A. M. Tesdall, K. R. Payne, and N. I. Popivanov, *The sonic line as a free boundary*, Quarterly of Applied Mathematics, to appear.
5. A. M. Tesdall and J. K. Hunter, *Self-similar solutions for the diffraction of weak shocks*, Journal of Computational Science, to appear.
6. J. K. Hunter and A. M. Tesdall, *Glancing weak Mach reflection*, in preparation.

6 Presentations Produced by this Award

1. Paris VI (Pierre et Marie Curie), Mathematics, Paris, France, July 2, 2012.
2. HYP 2012, 14th International Conference on Hyperbolic Problems, Padova, Italy, June 25–29, 2012. *Special Session on Wave Patterns Analysis*.
3. BCC-CUNY Mathematics Department Seminar, May 8, 2012.
4. Joint Mathematics Meetings, Boston, Massachusetts, January 4–7, 2012. *Special Session on Nonlinear Hyperbolic Partial Differential Equations*.
5. 2011 DOE (U. S. Department of Energy) Applied Mathematics Program Meeting, Washington, D.C., October 17–19, 2011.

6. AMS Eastern Sectional Meeting, Cornell University, Ithaca, New York, September 10–11, 2011. *Special Session on Mixed Type Equations.*
7. ICIAM 2011 7th International Congress on Industrial and Applied Mathematics, Vancouver, B.C., Canada, July 18–22, 2011. *Special Session on Numerical Aspects of Hyperbolic Equations and Related Problems.*
8. 8th AIMS Conference on Dynamical Systems, Differential Equations and Applications, May 25–28, 2010, Dresden, Germany. “High-Resolution Solutions for Shock Formation in Transonic Flow.”
9. SIAM Conference on Analysis of PDEs, December 7–10, 2009, Miami, Florida. “A two-phase Stefan problem for the unsteady transonic small disturbance equations.”
10. AMS Fall 2009 Eastern Section Meeting, Penn State University, State College, PA Partial differential equations of mixed elliptic-hyperbolic type, III, October 24–25, 2009. “High-resolution solutions for shock formation in transonic flow.”
11. New York Conference on Applied Mathematics, Rochester Institute of Technology (RIT), October 17, 2009, “High-resolution solutions for shock formation in transonic flow.”
12. University at Buffalo, October 16, 2009, “High-resolution solutions for shock formation in transonic flow.”
13. IMA Summer Program on Nonlinear Conservation Laws and Applications, Institute for Mathematics and its Applications (IMA), University of Minnesota, Minneapolis, Minnesota, July 13–July 31, 2009; “High-resolution solutions for weak shock reflection.”
14. International Conference on Applied Analysis and Scientific Computation (ICAASC 2009), Shanghai, China, June 25–28, 2009; “A free boundary problem for the unsteady transonic small disturbance equations.”