SUPERCONDUCTIVITY

APPROVED:

Ira C. Roberts
Major Professor

James L. Carrico
Minor Professor

L. L. Miller
Director of the Department of Physics

Jack Johnson
Dean of the Graduate Division

SUPERCONDUCTIVITY

THESIS

Presented to the Graduate Council of the North
Texas State Teachers College in Partial
Fulfillment of the Requirements

For the Degree of

MASTER OF SCIENCE

By

M. B. Ward Jr., B. S.

Alvin, Texas

June, 1948

156262

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

CHAPTER I

INTRODUCTION

The phenomenon of superconductivity is perhaps one of the most interesting and certainly one of the most puzzling of the properties of matter at extremely low temperatures, such as are attained only with the aid of liquid helium. Since its original discovery by Onnes in 1911 superconductivity has been the subject of a great number of experiments, first and mainly at Leyden, then at Toronto where a cryogenic laboratory has been operated since 1924, and recently in other laboratories. Many theoretical physicists have attempted to find an explanation for the phenomenon, until recently with very little success, and from the theoretical point of view the problem is still far from an adequate solution.[1]

Measurements of resistances at low temperatures are more difficult to carry out than at high temperatures. Such measurements have been made by H. K. Onnes, Leyden, Holland; Meissner, Charlottenburg, Germany; and J. C. McLennan and C. D. Niven, Toronto, Canada.[2] Numerous elements and

---

[1] H. G. Smith and J. O. Wilhelm, Reviews of Modern Physics, VII (1935) 238.

[2] Philosophical Magazine, IV (August, 1927) 21.

alloys were tested and the results showed that the usual statement that resistivity is proportional to the absolute temperature is far from true in most cases. In some instances the resistance abruptly vanishes at a few degrees above the absolute temperature of zero. This extremely low resistance, or extremely high conductivity, at these very low temperatures has been called "superconductivity" by Onnes. The resistance of lead is so very close to zero when in this state that when a current of electricity is generated in a lead ring by induction it continues to flow for more than a day with no additional supply of energy. No satisfactory theory to account for this phenomenon has yet been advanced. It could be that "at these low temperatures the crystal lattices of certain metals so arrange themselves as to leave free channels, with no retarding fields, for electronic flow, and so very little energy is lost."[3] Yet, it is not conclusively established that there is no change in crystal structure, although any such change must be of a secondary nature since there is no change in volume. There is no evidence of any abrupt change in the coefficient of thermal expansion. There is no change in the photoelectric effect, from which it may be concluded that there is no change in the surface work function.[4]

---

[3]Vernon A. Suydam, Electricity and Electromagnetism, p. 212.

[4]Smith and Wilhelm, *op. cit.*, p. 240.

Other properties which have been found to be unchanged are the absorption of β -particles and slow electrons, and the torsion constant.[5]

From all this it may be concluded that the transition to the superconducting state is one which affects only the conducting electrons in the interior of the metal and that changes in the metallic lattice are entirely secondary. The whole mechanism of conduction and resistance is obscure and so affords room for research.

---

[5] *Ibid.*, p. 240.

CHAPTER II

SOME OF MAXWELL'S AND LONDON'S EQUATIONS

Assume a very small parallelepiped of volume $dx\,dy\,dz$ in a space defined by the coordinate axes x, y, and z. Let it contain electricity with a volume density of ρ charges per cubic centimeter. Let the dielectric constant be k. Then into the face $dy\,dz$ along the x-axis, if the field be $\mathcal{E}_x$ along x, there will enter $k\,\mathcal{E}_x\,dy\,dz$ lines of force. For if $\mathcal{E}_x$ be the field in the absence of a dielectric, the infinitesimal volume will have $k\,\mathcal{E}_x$ lines per square centimeter entering (and/or emerging from) it. Since there are charges inside, the lines of force may increase in number from $k\,\mathcal{E}_x$ at the left-hand face to $k\left(\mathcal{E}_x + \frac{\partial \mathcal{E}_x}{\partial x}dx\right)$ at the other end of $dx\,dy\,dz$, dx centimeters away, for the rate of increase of field is $\frac{\partial \mathcal{E}_x}{\partial x}$. Hence the increase in number of lines of force running along the x direction between the near and far sides of the volume will be

$$k\left(\mathcal{E}_x + \frac{\partial \mathcal{E}_x}{\partial x}dx\right)dy\,dz - k\,\mathcal{E}_x\,dy\,dz = k\frac{\partial \mathcal{E}_x}{\partial x}\,dx\,dy\,dz \qquad (1)$$

Likewise along the y-axis we can write this change as $k\frac{\partial \mathcal{E}_y}{dy}\,dx\,dy\,dz$, and similarly along the z-axis it is represented by $k\frac{\mathcal{E}_z}{\partial z}\,dx\,dy\,dz$. Hence the change in the number of lines entering and leaving the volume along all three axes is

$$k\left(\frac{\partial \mathcal{E}_x}{\partial x} + \frac{\partial \mathcal{E}_y}{\partial y} + \frac{\partial \mathcal{E}_x}{\partial z}\right) dx\, dy\, dz \qquad (2)$$

Now this must equal the number of new lines created in $dx\,dy\,dz$ as the result of the charge density ρ , namely, $4\pi\rho\, dx\, dy\, dz$. Hence we can write

$$\frac{\partial \mathcal{E}_x}{\partial x} + \frac{\partial \mathcal{E}_y}{\partial y} + \frac{\partial \mathcal{E}_z}{\partial z} = \frac{4\pi\rho}{k} \qquad (3)$$

which is called the divergence of E, written $div\,\mathcal{E} = \frac{4\pi\rho}{k}$ or, more simply, $\nabla\cdot\mathcal{E}$. The equation merely states that the lines of force emerging from a region in space can come only from charges, and if there is any change in field intensity along x, y and z in $dx\,dy\,dz$, it is caused by charges in the amount given.

Now in a magnetic field in a volume element $dx\,dy\,dz$ carrying ρ poles per cubic centimeter, since the magnetic induction $\mathcal{B} = \mu\mathcal{H}$ where μ is the permeability, it can be shown in complete analogy with the above dielectric case, that if H_x, H_y and H_z are the components of the magnetic field along x, y and z,

$$\left(\frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z}\right) = \frac{4\pi\rho}{\mu} = \nabla\cdot\mathcal{H} \qquad (4)$$

where ρ is now the density of magnetic poles.[1] This equation is merely the magnetic analogue of the electrical field equation and states that a magnetic field is changed only by the presence of poles. In a space free from matter

[1] L. B. Loeb, Electricity and Magnetism, p. 395.

there are no poles and $\rho = 0$. The equation then states that as many lines leave as enter the region. Where no poles exist, div $\overline{H} = 0$, and similarly if no free charges exist in space div $\overline{E} = 0$.

As first discovered by Oersted, a magnetic field always accompanies an electric current. A current flowing in a straight wire is said to form circles of magnetic flux about the wire with their planes perpendicular to it. In any one of these circles the direction of $\overline{H}$ has the right hand screw relation to the direction of the current. The field here is not irrotational, the line integral

$$\oint H_s \, ds \tag{5}$$

not being zero for a path surrounding the wire. Surveys have shown to the familar reader that the value of the line integral is directly proportional to the current threading the path of integration. If the factor of proportionality is written "in the form $\frac{4\pi}{c}$, then all physical results relating to the field discovered by Oersted, are summarized in the equation

$$\oint H_s \, ds = \frac{4\pi}{c} i \tag{6}$$

where the path of integration encircles the current in the right handed screw sense."[2] The current flowing across the element of area dS is $i = i_n ds$, then

---

[2]Max Abraham and R. Becker, Classical Theory of Electricity and Magnetism, translated by John Dougall, p. 126.

$$\oint H_s\,ds = i_n\,dS \tag{7}$$

where n denotes the direction normal to the plane of the area dS. Stokes's Theorem states that the line integral of the tangential component of a vector $\bar{H}$ around a closed path is equal to the surface integral of the normal component of curl $\bar{H}$ over the surface enclosed by the path. In symbols, the statement of the theorem as applied to the above equation becomes

$$\oint H_s\,ds = \int_\Sigma \text{curl}_n\,\bar{H}\cdot d\bar{S} \tag{8}$$

or, written in full

$$\begin{aligned}&\oint\left[H_x\cos(x,s) + H_y\cos(y,s) + H_z\cos(z,s)\right]ds = \\ &\int_\Sigma\Big[\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)\cos(x,n) + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right)\cos(y,n) + \\ &\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\cos(z,n)\Big]\,d\sigma\end{aligned} \tag{9}$$

Since $\cos(x,s) = \frac{dx}{ds}$, $\cos(y,s) = \frac{dy}{ds}$ and $\cos(z,s) = \frac{dz}{ds}$, it is obvious that the line integral in the left-hand member can be written as

$$\oint\left(H_x\,dx + H_y\,dy + H_z\,dz\right). \tag{10}$$

Letting $\cos(x,n) = \bar{i}$, $\cos(y,n) = \bar{j}$ and $\cos(z,n) = \bar{k}$, then the right-hand member becomes

$$\int_\Sigma \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix}.\,d\sigma = \int_\Sigma \nabla\times\bar{H}\cdot d\bar{S} \tag{11}$$

where the determinant is the definition of curl $\bar{H}$ or $\nabla\times\bar{H}$. Summarizing, we readily write

$$\begin{aligned}&\frac{4\pi}{c}\,i = \oint\bar{H}\cdot d\bar{s} = \int_\Sigma\nabla\times\bar{H}\cdot d\bar{S} = \int_\Sigma \text{curl}\,\bar{H}\cdot d\bar{S} \\ &\int_\Sigma \text{curl}\,\bar{H}\cdot d\bar{S} = \frac{4\pi}{c}\int_\Sigma \bar{i}\cdot d\bar{S} \\ &\text{curl}\,\bar{H} = \frac{4\pi}{c}\,\bar{i}\end{aligned} \tag{12}$$

or $$\operatorname{curl} \bar{H} = \frac{4\pi}{c} \bar{J} \tag{13}$$

where $\bar{J}$ is the current density, sometimes called the conduction current. We have difficulty when we try to apply the last equation to nonstationary cases. Suppose we have a current flowing in an open circuit, as in the discharge of a condenser. The current starts at the positively charged plate, whose charge diminishes as the current flows to the negatively charged plate and annuls the charge there. Thus we can look upon the condenser plates as sources or sinks of current. Now, if we take the divergence of the last equation, we have

$$\operatorname{div} \operatorname{curl} \bar{H} = \frac{4\pi}{c} \operatorname{div} \bar{J} = 0 \tag{14}$$

since the divergence of any curl is zero, which means that the current is always closed and there are no sources or sinks. Thus we are led to a contradiction.

Maxwell concluded from this that the last equation of curl $\bar{H}$ must be incomplete, and that to the term $\bar{J}$ must be added another term, such that the sum of the two has no divergence. We can determine the value of this term from the equation of continuity for the flow of current. The divergence of $\bar{J}$ measures the flux outward over the surface of unit volume. If there is current flowing outward, the charge within the unit volume must be decreasing, the flux equalling the rate of decrease of charge in unit volume.

Thus we have

$$\operatorname{div} \bar{J} = -\frac{\partial \rho}{\partial t} \tag{15}$$

Taking the charge density equivalent to the divergence of the displacement vector $\bar{D}$:[3]

$$\operatorname{div} \bar{D} = 4\pi\rho \tag{16}$$

we may rewrite the continuity equation, div $\bar{J}$, as

$$\operatorname{div}\left(4\pi \bar{J} + \frac{\partial \bar{D}}{\partial t}\right) = 0 \tag{17}$$

In other words, although div $\bar{J}$ is not zero, the divergence of the quantity $4\pi J + \frac{\partial \bar{D}}{\partial t}$ is always zero, so that this quantity can mathematically be placed equal to a curl. Maxwell made the assumption that the last equation of curl $\bar{H}$ should properly be replaced by

$$\operatorname{curl} \bar{H} = \frac{4\pi}{c}\bar{J} + \frac{1}{c}\frac{\partial \bar{D}}{\partial t} \tag{18}$$

The last factor $\frac{\partial \bar{D}}{\partial t}$ is called the "displacement current" to distinguish it from $\bar{J}$.[4]

Substituting $\bar{D} = k\bar{E}$, where k is the dielectric constant, we have

$$\operatorname{curl} \bar{H} = \frac{4\pi}{c}\bar{J} + \frac{k}{c}\frac{\partial \bar{E}}{\partial t}$$

$$\nabla \times \bar{H} = \frac{1}{c}\left(4\pi \bar{J} + k\dot{\bar{E}}\right) \tag{19}$$

After Oersted's discovery it was natural to look for the production of an electric field by some magnetic means. The phenomenon of electromagnetic induction was discovered by Faraday in London and Henry in Albany, New York. They

---

[3] *Ibid.*, p. 143.

[4] J. C. Slater and N. H. Frank, *Electromagnetism*, p. 84.

showed that if the flux of magnetic induction through a loop of wire is changed, a transient current will flow in the wire. This discovery was included in a mathematical formulation of the theory of electromagnetic induction by F. E. Neumann about ten years later. According to his formulation,[5]

$$IR = -\frac{1}{c}\frac{d}{dt}\int \bar{B}\cdot d\bar{s} \tag{20}$$

In this equation, I is the current which flows in the wire, measured in electrostatic units, R is the resistance in the same units, while $\bar{B}$ is the magnetic induction measured in electromagnetic units and c is a constant which connects the two sets of units.

Now the product of the current flowing in a wire by the resistance between two points of the wire is the difference of potential between those two points, but it is evident that in the case of a closed loop of wire there can be no difference of potential around the whole wire. The effect of the change of the magnetic induction must then be described as an electromotive force. This electromotive force is the integral of an electric field which, however, cannot be represented as the gradient of a potential. If $\bar{E}$ represents the strength of this induced electric field, the above equation then shows that

$$\int \bar{\mathcal{E}}\cdot d\bar{l} = -\frac{1}{c}\frac{d}{dt}\int \bar{B}\cdot d\bar{s} \tag{21}$$

---

[5] W. V. Houston, Principles of Mathematical Physics, p. 227.

Since this equation is true for every loop of wire, and since the tangential component of the electric field is continuous when going from one substance to another, the above equation may be regarded as referring to a line integral around a closed curve outside the loop of wire and not only around the wire itself. It is then a natural generalization to treat this last equation as valid when the line integral is taken around any closed curve whatsoever. When this is done according to Stokes's theorem, we have

$$\oint \bar{\mathcal{E}} \cdot d\bar{l} \int_{\Sigma} \operatorname{curl} \bar{\mathcal{E}} \cdot d\bar{\sigma}$$

$$\int_{\Sigma} \operatorname{curl} \bar{\mathcal{E}} \cdot d\bar{\sigma} = -\frac{1}{c}\frac{d}{dt}\int \bar{B} \cdot d\bar{s}$$

$$\operatorname{curl} \bar{\mathcal{E}} = -\frac{1}{c}\frac{d\bar{B}}{dt} = -\frac{1}{c}\dot{\bar{B}} \tag{23}$$

This is sometimes called Faraday's law of induction. Here it appears in the differential form where $\dot{\bar{B}}$ is the symbol for the derivative of $\bar{B}$ with respect to time. Since $B = \mu H$, then

$$\operatorname{curl} \bar{\mathcal{E}} = \frac{\mu}{c}\dot{\bar{H}} \tag{24}$$

Thus Maxwell's equations in Gaussian units are

$$\operatorname{curl} \bar{\mathcal{E}} = -\frac{1}{c}\dot{\bar{B}} \qquad \operatorname{div} \bar{B} = 0$$

$$\operatorname{curl} \bar{H} = \frac{1}{c}(4\pi \bar{J} + \dot{\bar{D}}) \qquad \operatorname{div} \bar{D} = 4\pi\rho$$

Advancement of the first electromagnetic theories of superconductivity was founded on the phenomenon of persistent currents in lead rings, as previously stated, by

H. K. Onnes. On this foundation it was assumed that the conducting electrons are free, classical particles under the influence solely of the electric field, making $\bar{E}$ and $\bar{H}$ proportional. According to this idea, the Maxwell equations for the fields are to be supplemented by the Newtonian force equation for each electron. It is assumed that the dielectric constant and the magnetic susceptibility are unity, since this is very nearly true for the metals in their normally conducting states. The equations are then:

$$\nabla \times \bar{H} = \frac{4\pi}{c}\bar{J} + \frac{1}{c}\dot{\bar{\mathcal{E}}} \qquad \nabla \cdot \bar{\mathcal{E}} = \rho \tag{25}$$

$$\nabla \times \bar{\mathcal{E}} = -\frac{1}{c}\dot{\bar{H}} \qquad \nabla \cdot \bar{H} = 0 \tag{26}$$

$$m(d\bar{v}_i/dt) = -e\bar{\mathcal{E}} \tag{27}$$

The summation of the Newtonian equation, the last equation, over N conducting electrons per unit volume gives the macroscopic equation[6]

$$\bar{\mathcal{E}} = \lambda\dot{\bar{J}} \quad \text{where } \lambda = \frac{m}{Ne^2} = \left(\frac{10^{23}}{N}\right) \cdot 10^{-11} cm^2 \tag{28}$$

At present it is not clear how N is to be interpreted in terms of an "effective" number of free electrons as used in the electronic theory of metals. We may consider equation (28) as the definition of N.[7] This equation is in agreement with the experimental finding that e.m.f.'s are required not only to maintain a current but to change one. Its substitution in the second Maxwell equation results in

$$\frac{\partial}{\partial t}\left(\bar{H} + \lambda \nabla \times \bar{J}\right) = 0$$

---

[6] Emory Cook, Physical Review, LVIII (August, 1940), 357.

[7] F. London, Reviews of Modern Physics, XVII (1945), 311.

or $$\bar{H} + \lambda \nabla \times \bar{J} = H_0(x) \tag{29}$$

where $H_0(x)$ is independent of the time. Now, since λ is very small, $\bar{H}(x,t) - H_0(x)$ is practically zero (unless the derivatives of $\bar{J}$ are extremely large) so that a field $H_0(x)$, present in the medium when it becomes superconducting, is theoretically frozen in for all time. But an internal field different from zero is contrary to the Meissner effect.[8] The normal component of the magnetic field on the surface of an ideal superconductor is always zero; no penetration of the external field is ever observed. To obtain this experimental result, F. London and H. London arbitrarily assumed that the integration constant vanishes,

$$H_0(x) \equiv 0 \qquad \text{(EMPIRICAL)},$$

and based their theory on the stronger magnetic condition[9]

$$\bar{H} + \lambda c \nabla \times \bar{J} = 0 \tag{30}$$

together with the acceleration condition of the previously stated macroscopic equation and the Maxwell equations.

We can next eliminate $\bar{H}$ and $\bar{J}$ between

$$\bar{H} = -\lambda c \nabla \times \bar{J} \qquad \text{and} \qquad \nabla \times \bar{H} = \frac{4\pi}{c} \bar{J}$$

$$\nabla \times \bar{H} = -\lambda c (\nabla \times \nabla \times \bar{J})$$

$$\nabla \times \bar{H} = -\lambda c [-\nabla \cdot \nabla \bar{J} + \nabla (\nabla \cdot \bar{J})]$$

But $$\nabla \cdot \bar{J} = 0$$

so $$\nabla \times \bar{H} = -\lambda c (-\nabla^2 \bar{J}) = \frac{4\pi}{c} \bar{J}$$

---

[8] See Chapter III for the Meissner effect.

[9] E. F. Burton, H. G. Smith, and J. O. Wilhelm, Phenomena at the Temperature of Liquid Helium, p. 300.

$$\nabla^2 \bar{J} = \frac{4\pi}{\lambda c^2} \bar{J} \tag{31}$$

$$\frac{\partial^2 \bar{J}_x}{\partial x^2} + \frac{\partial^2 \bar{J}_y}{\partial y^2} + \frac{\partial^2 \bar{J}_z}{\partial z^2} = \frac{4\pi \bar{J}}{\lambda c^2} \tag{32}$$

If $\bar{J}$ had been eliminated instead of $\bar{H}$ the last equation would be of the same type except $\bar{H}$ would be the vector present. This equation gives then either the distribution current or the penetration of the magnetic field within the superconductor, both decreasing approximately exponentially from the surface.

From the equations

$$H = -\lambda c \nabla \times \bar{J}, \qquad \nabla \times \bar{H} = \frac{4\pi}{c} \bar{J} + \frac{1}{c} \dot{\bar{\mathcal{E}}} \quad \text{and } \bar{\mathcal{E}} = \lambda \dot{\bar{J}}$$

we can eliminate $\bar{H}$ and $\bar{E}$. We have shown

$$\nabla \times \bar{H} = \lambda c \nabla^2 \bar{J}$$

Then

$$\lambda c \nabla^2 \bar{J} = \frac{4\pi}{c} \bar{J} + \frac{1}{c} \dot{\bar{\mathcal{E}}}$$

$$\nabla^2 \bar{J} - \frac{1}{\lambda c^2} \dot{\bar{\mathcal{E}}} = \frac{4\pi}{\lambda c^2} \bar{J}$$

but since

$$\dot{\bar{\mathcal{E}}} = \lambda \frac{\partial^2 \bar{J}}{\partial t^2}$$

we write

$$\nabla^2 \bar{J} - \frac{1}{c^2} \frac{\partial^2 \bar{J}}{\partial t^2} = \frac{4\pi}{\lambda c^2} \bar{J}$$

The solutions of this equation for the stationary state are of the exponential type.

This theory is different from the ordinary electromagnetic theory in one other respect: the chances of any superficial charge or current density is excluded, being replaced by volume densities in a thin layer at the surface. This is a necessary condition for complete continuity of both components of $\bar{E}$ and $\bar{H}$ at the surface instead of simply the normal component of $\bar{H}$ and tangential component of $\overline{E}$. These conditions shall be used in the solutions of the examples in the following chapters.

CHAPTER III

DIAMAGNETISM OF THE SUPERCONDUCTING STATE AND THE MEISSNER EFFECT

One of the most important experimental developments of the years 1933 to 1936 concerned the effect of a superconductor upon the distribution of an external magnetic field. A very simple corollary of Maxwell's law of induction,

$$\mathrm{curl}\ \mathcal{E} = \frac{1}{c}\frac{d\bar{B}}{dt} = \frac{\mu}{c}\frac{d\bar{H}}{dt}$$

first noticed by Maxwell himself, is that the normal component of magnetic induction $\bar{B}$ must vanish at the boundary of a perfect conductor, for the tangential component of the electric field $\bar{E}$ must vanish, and so must the normal component of curl $\bar{E}$.[1] From this it follows that in general a magnetic field cannot penetrate a superconductor, but due to any change in the external magnetic field, induced currents will be set up in the surface layers so that the normal component of induction remains zero. Or, if a magnetic field previously existed within the body, it cannot be altered by any change in the external field. Obviously this applies to a superconducting metal only as long as the external field

[1] E. F. Burton, H. G. Smith and J. O. Wilhelm, Phenomena at the Temperature of Liquid Helium, p. 289.

is less than the threshold field.[2] An alternative statement, useful in the discussion of experimental results, is that a superconducting body acts as if it had an effective magnetic permeability $\mu = 0$, that is, acts as a perfect diamagnetic substance.

The effect of this on the distribution of magnetic field was applied by von Laue[3] in an attempt to explain the results of de Hass and Voogd on the magnetic interruption of superconductivity in transverse fields. In this experiment, when the magnetic field is parallel to the wire, the normal component is already zero, and the distribution of field is practically unaffected. However, with a transverse magnetic field, the problem is effectively that of a cylinder with magnetic permeability $\mu = 0$, and the distribution of field becomes that of a cylinder "wedged" between the lines of force. "At a point on the surface of the wire the magnetic field is tangential and is given by

$$H = 2H_0 \sin\theta$$

where H_0 is the applied field at a distance."[4] Superconductivity would then begin to be interrupted along the sides

---

[2] Ibid., p. 103: "At any given temperature the magnetic field which brings the resistance back to one-half the value for the normal metal was originally defined as the 'threshold field' for that temperature, denoted by H_t. However, H_t is now usually taken to be the field strength in which the resistance attains its normal value."

[3] H. G. Smith and J. O. Wilhelm, Reviews of Modern Physics, VII (October, 1935), 243.

[4] Ibid., p. 244.

of the wire as soon as $H_o = \frac{3}{4}H_t$ where H_t is the threshold field. According to de Hass and Voogd, the resistance actually began to reappear when the applied field was a little more than $\frac{3}{4}H_t$.

This explanation cannot be considered as entirely satisfactory in the particular case to which it has been applied, for one cannot understand why it should apply to the case of a single crystal. In fact one would expect that even when the magnetic field starts to penetrate the body at the sides there would still remain a cylinder of elliptical cross section where the magnetic field is less than H_t. The observed resistance would remain zero as long as any such cylinder still existed, and actually there should be no difference between the longitudinal and transverse cases under the conditions of the ordinary experiment on the threshold field.

However, this discussion served to draw attention to the diamagnetic properties of superconductors. It may now be considered as definitely established experimentally that when a body is actually in the superconducting state it acts toward any change in a small magnetic field as if its magnetic permeability were zero, or as a diamagnetic substance with susceptibility $-\chi = \frac{1}{4}\pi$.[5] It is unnecessary to discuss in detail the many confirmations of this statement,

[5] *Ibid.*, p. 244.

as most of them have been incidental to the experiments considered.

It would be expected that a superconductor would act as a perfect diamagnetic as long as the magnetic field at the surface nowhere exceeds the threshold field H_t. For larger fields the superconducting state is destroyed, and, since none of the ferromagnetic metals are known to be superconductors, the permeability may be taken equal to unity. This is true at least for the superconducting elements, Pb, Sn and Hg when fairly pure, and is probably true for most of the pure metals. It is not the case for certain alloys, or for tantalum, which have been shown[6] to depart from the ideal perfect diamagnetism under magnetic fields very much smaller than the threshold field. We shall designate as a "normal superconductor" one for which the limiting field in the present sense coincides with the threshold field for the reappearance of resistance.

A clear distinction must be drawn between the diamagnetic properties described above, where the body remains throughout in the superconducting state, and those diamagnetic effects which appear when a metal passes from the normal to the superconducting state in the presence of an external field. The latter case may occur either:[7]

---

[6] E. F. Burton, H. G. Smith and J. O. Wilhelm, _op. cit._, p. 307.

[7] _Ibid._, p. 244.

(A) when the body is cooled through its transition point in a steady magnetic field (as in the original experiments on persistent currents); or

(B) when the temperature is kept constant and the field strength is reduced from a value greater than the threshold field.

Since the deduction of the diamagnetic effect from Maxwell's equations applies only to changes in the magnetic field we should expect, as shown by Lorentz:[8]

In case A: No change in the distribution of magnetic field.

In case B: The effect of superposing on a uniform field H_t an opposing field equal to $-H_t$ at a great distance, but distributed about the body as if the latter were a perfect diamagnetic. The body ought to become permanently magnetized, with a magnetic moment equal to that acquired by a perfect diamagnetic in a field $-H_t$.

The simple form of the Meissner effect is shown in case A, in a spontaneous readjustment of the magnetic field distribution. This diamagnetic effect, therefore, cannot be due to electromagnetic induction of surface currents, but must be the result of some characteristic property of the superconducting metal.

---

[8] Ibid., p. 245.

CHAPTER IV

LAW OF CONSTANT MAGNETIC FLUX

Since the discovery of the Meissner effect and other magnetic effects associated with superconductivity, there has been some question as to the real nature of the so-called "persistent current" which can be set up in a superconducting ring. Until recently these currents have always been studied by measuring the magnetic moment of the ring with a magnetometer, and it was found by Onnes that almost as large a magnetic moment could be obtained with an open ring in which a narrow slot had been cut. In the latter case at least it is more appropriate to speak of permanent magnetization, rather than of persistent current.

However, F. and H. London[1] have pointed out that there is an essential difference between simply and multiply connected bodies (in the mathematical sense). They give the law that for a multiply connected superconductor (ring or closed circuit) the flux through the non-superconducting region must be constant and equal to the flux at the instant when the body became superconducting, regardless of the Meissner effect. If L is the self-inductance of the

---

[1] F. and H. London, Physica, II (1935), 341.

superconducting circuit, this leads at once to the equation

$$Li = \phi - \phi_0$$

where ϕ is the magnetic flux through the circuit due to the external field, and ϕ_0 is the amount of this flux before the transition to the superconducting state. The current i is then actually a macroscopic circulation of charge around the circuit, which we may describe as a "true current" as opposed to the unknown circulation which causes the magnetic moment in the case of a simply connected body.

This result is identical with that obtained by classical consideration of a resistanceless circuit, that the e.m.f. is always zero, or

$$\frac{d\phi}{dt} - L\frac{di}{dt} = 0$$

CHAPTER V

SPHERE IN UNIFORM MAGNETIC FIELD

Let the current density be $\bar{J}$, and the density of charge ρ , these being given functions of position and of the time. Confining these to the case of propagation of field in empty space, so that $k=\mu=1$. The field is therefore defined by the equations

$$\left.\begin{aligned} &\operatorname{curl}\bar{H}=\frac{4\pi}{c}\bar{J}+\frac{1}{c}\dot{\bar{\mathcal{E}}} &&\text{(a)}\\ &\operatorname{div}\bar{\mathcal{E}}=4\pi\rho &&\text{(b)}\end{aligned}\right\}\qquad \left.\begin{aligned} &\operatorname{curl}\bar{\mathcal{E}}=-\frac{1}{c}\dot{\bar{H}} &&\text{(c)}\\ &\operatorname{div}\bar{H}=0 &&\text{(d)}\end{aligned}\right\}.\text{(1)}$$

From (a) and (b) it follows in the first place that $\bar{J}$ and cannot be quite arbitrarily prescribed, but that the equation of continuity

$$\operatorname{div}\bar{J}=-\dot{\rho} \tag{1e}$$

must be satisfied everywhere and always. We can satisfy (1d) identically by introducing the vector potential $\bar{A}$ and the scalar potential ϕ as in all vector analysis, so that

$$\bar{H}=\operatorname{curl}\bar{a} \tag{2}$$

With this value of $\bar{H}$, (1c) states that $\bar{\mathcal{E}}+\frac{1}{c}\dot{\bar{a}}$ is irrotational. Hence this equation is satisfied if we put

$$\bar{\mathcal{E}}=-\frac{1}{c}\dot{\bar{a}}-\operatorname{grad}\phi. \tag{3}$$

When we insert these values of $\bar{E}$ and $\bar{H}$ in (1a,b) they become $\quad \operatorname{curl}\operatorname{curl}\bar{a}+\frac{1}{c^2}\ddot{\bar{a}}+\frac{1}{c}\operatorname{grad}\dot{\phi}=\frac{4\pi}{c}\bar{J},$

and $$-\frac{1}{c}\,\mathrm{div}\,\dot{\bar{a}} - \nabla^2\phi = 4\pi\rho$$

Now (1a) only specifies the curl of the vector $\bar{A}$. Its sources are still at our disposal. We define them by laying down the condition

$$\mathrm{div}\,\bar{a} = -\frac{1}{c}\,\dot{\phi} \tag{4}$$

The distribution of a magnetic field which is uniform at great distances shall be found about a superconductor as an example of the type of solution one gets in stationary problems. By the London theory the vector potential $\bar{A}$ and the scalar potential ϕ may be selected so that they are proportional to the current densities and charge, respectively, as will be seen by equations (15) and (16).

We have shown equation (30) on page 13 and it follows that its differential form is

$$\lambda\,\mathrm{curl}\,\dot{\bar{J}} = -\frac{1}{c}\dot{\bar{H}} \tag{5}$$

From this we obtain by integration between limits

$$\mathrm{curl}(\bar{J}-\bar{J}_o) = -\frac{1}{\lambda c}(\bar{H}-\bar{H}_o) \tag{6}$$

where $\bar{J}$ and $\bar{H}$ are the current density and field when the body was still above its transition point, and can be put equal to zero in many cases.

By eliminating $\bar{\Gamma}$ between equation (13) page 8 and equation (30) page 13 in their differential forms and using

$\mathrm{div}\,\bar{J} = 0$ and $\bar{H}_o = \frac{4\pi}{c}\bar{J}_o$

we obtain with complete generality

$$\nabla^2(\bar{J}-\bar{J}_o) = \frac{4\pi}{\lambda c^2}(\bar{J}-\bar{J}_o) \tag{7}$$

and similarly

$$\nabla^2(\bar{H}-\bar{H}_0)=\frac{4\pi}{\lambda c^2}(\bar{H}-\bar{H}_0) \tag{8}$$

We shall refer to this electrodynamical treatment as the "acceleration theory" of superconductivity. It probably leads to the correct results as regards any changes in current or magnetic field, which take place after the body has passed into the superconducting state. However, it is definitely in disagreement with experiment, in general, when the transition takes place in the presence of an external magnetic field or an internal current. For according to equations (7) and (8) the original current density $\bar{J}$ and field $\bar{H}$ should remain unaltered through the transition, which is contrary to the experimental evidence of the Meissner effect.[1]

In view of this discrepancy F. and H. London have developed a new system of electromagnetic equations for superconductors, in which they abandon the fundamental equation

$$\mathcal{E}=\lambda\dot{J} \tag{9}$$

of the acceleration theory, but retain the result (8) in the stronger form[1]

$$\nabla^2\bar{H}=\frac{4\pi}{\lambda c^2}\bar{H} \tag{10}$$

This now applies to the whole field, including that present before the transition, and agrees with the experiments on the Meissner effect (at least for "normal"

---

[1]H. G. Smith and J. O. Wilhelm, Reviews of Modern Physics, VII (October, 1935) 264.

superconductors). This, and the equation

$$\nabla^2 \bar{J} = \frac{4\pi}{\lambda c^2} \bar{J} \tag{11}$$

which necessarily accompanies it, can be considered to arise by assuming the general truth of the equation

$$\bar{H} = -\lambda c \text{ curl } \bar{J} \tag{12}$$

in place of equation (f) which applies only to the time derivatives of these vectors. Now, however, equation (9) holds only in the differential form

$$\text{curl}(\lambda \dot{\bar{J}} - \bar{\mathcal{E}}) = 0 \qquad \text{or} \qquad \lambda \dot{\bar{J}} - \bar{\mathcal{E}} = \text{grad } \nu \tag{13}$$

where ν is an undertermined function. If we put $\nu = \lambda c^2 \rho$ equation (13) becomes

$$\bar{\mathcal{E}} = \lambda (\dot{\bar{J}} - c^2 \text{grad} \rho) \tag{14}$$

which is taken as a second fundamental equation.

Eliminating $\bar{H}$ from the equations (2) and (12) we have

$$\bar{H} = \text{curl } \bar{a} = -\lambda c \text{ curl } \bar{J}$$

$$\bar{a} = -\lambda c \bar{J} \tag{15}$$

Substituting equations (1e) and (4) in (15) after taking the divergence of both sides, we have

$$\nabla \cdot \bar{a} = -\lambda c \nabla \cdot \bar{J}$$

$$-\frac{1}{c} \dot{\phi} = -\lambda c (-\dot{\rho})$$

$$\phi = -\lambda c^2 \rho \tag{16}$$

Suppose a sphere of radius a is taken in a homogeneous magnetic field H_0 directed along the polar axis (using spherical coordinates R, θ, ϕ). The vector potential $\bar{A}$ due to a uniform magnetic field is

$\bar{a} = \frac{H_o}{2} R \sin\theta \, \bar{i}_\phi$ ($\bar{i}_r$, $\bar{i}_\theta$, $\bar{i}_\phi$, are the unit vectors)

as will be shown.

This particular vector potential is that of a uniform field H_o in the z-direction (polar axis is spherical coordinates). Thus we seek a vector $\bar{A}$, such that $\bar{H} = H_o\bar{k} = \text{curl } \bar{A}$, $\bar{k}$ being the unit vector in the z-direction. It is much easier to solve this problem in rectangular coordinates rather than spherical coordinates, and then to transform to spherical coordinates. Thus we seek a solution to the set of equations

$$H_x = 0 = \frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}$$

$$H_y = 0 = \frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x}$$

$$H_z = H_o = \frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}$$

with the additional condition that

$$\text{div } \bar{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z} = 0$$

The specification of the curl and divergence of $\bar{A}$ determines $\bar{A}$ uniquely except for an additive constant which is unimportant. (In the London theory $\bar{A}$ is determined uniquely by setting it proportional to the current density $\bar{J}$.)

The only way this set of equations has been solved here is by inspection. Thus if we let $a_x = -\frac{H_o}{2} y$, $a_y = \frac{H_o}{2} x$ and $a_z = 0$ then we see that the two conditions are satisfied, and that is all we need to know. Thus

$$\bar{a} = -\frac{H_o}{2} y \, \bar{i}_x + \frac{H_o}{2} x \, \bar{i}_y$$

in rectangular coordinates. In spherical coordinates

$$y = r \sin\theta \sin\phi$$

$$x = r \sin\theta \cos\phi$$

$$\bar{i}_x = \sin\theta\cos\phi\,\bar{i}_r + \cos\theta\cos\phi\,\bar{i}_\theta - \sin\phi\,\bar{i}_\phi$$

$$\bar{i}_y = \sin\theta\sin\phi\,\bar{i}_r + \cos\theta\sin\phi\,\bar{i}_\theta + \cos\phi\,\bar{i}_\phi$$

All the $\bar{I}$'s given above are the unit vectors in the direction denoted by the subscript. Substituting these equations into the expression for $\bar{A}$, we obtain

$$\bar{A} = \frac{H_0}{2} r\sin\theta\,\bar{i}_\phi \tag{17}$$

Of course the solution was first written in rectangular coordinates; however, one would expect the solution to be easier in rectangular coordinates since the symmetry of the field is destroyed somewhat by writing the equations in spherical coordinate form. They are simple enough in rectangular coordinates to warrant solution by trial and error methods.

Now to account for the disturbance due to the prescence of a sphere let us first consider the case of a normal sphere in a uniform magnetic field. The field produced by a uniformly magnetized sphere can be obtained easily by elementary methods. Consider concentric spheres of equal radii a, one of which has a uniform volume density ρ of positive magnetic charge and the other an equal volume density of negative charge. If the first is displaced a small distance ℓ relative to the second, every element of positive charge is displaced a distance ℓ from the negative charge with which it originally coincided, forming therewith an elementary magnetic dipole. By this displacement we have arrived at a uniformly magnetized sphere of intensity of magnetization $I = \rho\ell$

Now the field at outside points due to a spherical distribution of magnetic charge is the same as if the entire charge were concentrated at the center of the sphere. Consequently the magnetic field at some point P outside the two spheres is that produced by a dipole of moment

$$M = \frac{4}{3}\pi a^3 \rho l = \frac{4}{3}\pi a^3 I \tag{18}$$

located at the origin.

"In terms of its magnetic moment the potential (outside the sphere) is

$$\Phi = \frac{M\cos\theta}{r^2} \tag{19}$$

The components of the field in the direction of increasing r and θ are"[2]

$$\begin{aligned} H_r &= -\frac{\partial\Phi}{\partial r} = \frac{2M\cos\theta}{r^3} \\ H_\theta &= -\frac{\partial\Phi}{r\,\partial\theta} = \frac{M\sin\theta}{r^3} \end{aligned} \tag{20}$$

where M designates the magnetic moment, the product ρl (Fig. 1), and r is considerably larger than l.

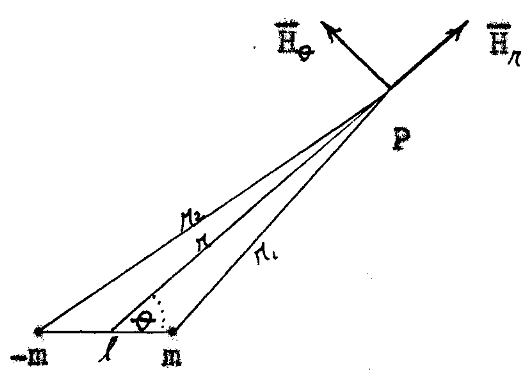

Fig. 1.--Components of magnetic field at point P distance r from center of an elementary magnetic bar.

[2]L. Page and N. I. Adams, Principles of Electricity, p. 129.

To find the field inside the sphere we must remember that the magnetic intensity at a point distant r from the center of a sphere which has a uniform distribution of magnetic charge is that produced by the charge inside a sphere of this radius, so that

$$H = \frac{m}{a^3} r$$

As H is directed along the radius vector, the vector fields $\overline{H}_1$ and $\overline{H}_2$ due to the positive and negative spheres are proportional to the vector distances $\overline{r}_1$ and $\overline{r}_2$ from their respective centers, the first being directed along $\overline{r}_1$ and the second, since the charge is negative, in the direction opposite to $\overline{r}_2$. But the vector sum of $\overline{r}_1$ and $-\overline{r}_2$ is $-\ell$, the separation of the two spheres. Therefore the resultant field is in the negative z direction and has the magnitude

$$H_z = -\frac{m}{a^3}\ell = -\frac{4}{3}\pi\rho\ell = -\frac{4}{3}\pi I \qquad (21)$$

This field is uniform throughout the interior of the sphere. Being in the opposite direction to I it tends to weaken the magnetization of the sphere. It is known as the demagnetizing field.

So far we have been considering a permanent magnet in the form of a uniformly magnetized sphere, such as a spherical steel magnet. Let us now consider a paramagnetic sphere of permeability μ placed in a uniform external field H_o parallel to the z axis. As the field H_o is uniform, the elementary dipoles in the interior of the sphere will be in equilibrium if the sphere becomes uniformly magnetized by induction so as to produce a uniform field H of its own of precisely the same character as that of the permanent spherical magnet just considered. The total field is the resultant of H_o and H. The components of the latter at points outside the sphere are given by (20). Therefore the resultant field is

$$H_r = H_o \cos\theta + \frac{8\pi a^3 I}{3 a^3}\cos\theta \tag{22}$$

in the direction of the radius vector, and

$$H_\theta = -H_o \sin\theta + \frac{4\pi a^3 I}{3 r^3}\sin\theta \tag{23}$$

at right angles thereto in the direction of increasing θ.

Inside the sphere the field is in the z direction and equal to the sum of H_o and the internal field (21). It is

$$H_z = H_o - \frac{4\pi I}{3} \tag{24}$$

When we pass from one magnetic medium to another the components of B normal to the surface of separation and the components of H parallel to the surface are the same in the two media. Setting B $= \mu$ H, and since a = r here, we have from (22) and (24)

$$\left(H_o + \frac{8\pi I}{3}\right)\cos\theta = \mu\left(H_o - \frac{4\pi I}{3}\right)\cos\theta$$

for the normal components of B, or

$$I = \left(1 - \frac{\mu - 1}{\mu + 2}\right)H_o \tag{25}$$

Equation (25), then, specifies the intensity of magnetization produced in the paramagnetic sphere by the impressed field H_o.

Substituting (25) in (24) the resultant field in the interior of the sphere is seen to be

$$H_z = \left(1 - \frac{\mu - 1}{\mu + 2}\right) H_o \tag{26}$$

which is less than H_o if μ is greater than unity, due to the fact that the induced poles on the surface of the sphere give rise to a field in the interior opposed to H_o. The field is the demagnetizing field.

The magnetic induction inside the sphere is

$$B_z = \mu H_z = \frac{3\mu}{\mu+2} H_o = \frac{3}{1+\frac{2}{\mu}} \tag{27}$$

which is greater or less than H_o according as μ is greater or less than unity. Therefore lines of induction are crowded together in a paramagnetic sphere, $\mu > 1$, and in a diamagnetic sphere, $\mu < 1$, the lines of induction are spread apart.

In the ideally pure superconductor the magnetic induction is equal to zero as long as the external magnetic field at the surface is less than the threshold field strength. This is the same thing as saying that a superconductor in a magnetic field acts as if its magnetic permeability were equal to zero. Thus, equation (27) becomes zero and equation (26) becomes

$$H_z = \frac{3}{2} H_o \tag{28}$$

CHAPTER VI

TRANSMISSION THROUGH A THIN SHEET AT NORMAL INCIDENCE

The development of the preceding theory has been for the stationary state; no attention has been given to problems of the non-stationary type. In accordance with the London equations the effect of increasing the frequency of an electromagnetic wave is to increase the penetration depth of the fields, until at frequencies above a "critical frequency", the wave will be propagated through the superconductor undamped. This theory does not take into account two factors:

(1) The bound electrons. The effect due to the bound electrons is negligible except at frequencies in the range of the resonant frequencies of the atoms.

(2) The normal conducting electrons. The experiments of de Hass and Bremmer[1] showing a sudden diminution in the thermal conductivity upon passing from the non-superconducting to the superconducting state when an external magnetic field is reduced to below the transition value indicate that

---

[1] W. S. de Hass and H. Bremmer, Akademie van Wetenschappen te Amsterdam, XXXIV (1931), 336.

only a certain fraction of the conducting electrons pass over into the superconducting phase, the number increasing with a decrease in temperature. Since nothing quantitative is known about this effect, we shall assume, for simplicity, that all the conducting electrons pass over into the superconducting phase.

One of the best ways to detect any frequency effects is to study the transmission of electromagnetic waves through a thin sheet of superconducting material. Let an infinite, thin sheet of superconducting material of thickness d be imbedded in a medium characterized by the propagation factor k_1 and permeability μ, and let the propagation factor of the superconductor according to the London theory be k_2. Let a plane wave be of normal incidence on the sheet from the left in the z direction, as shown in the following figure:[2]

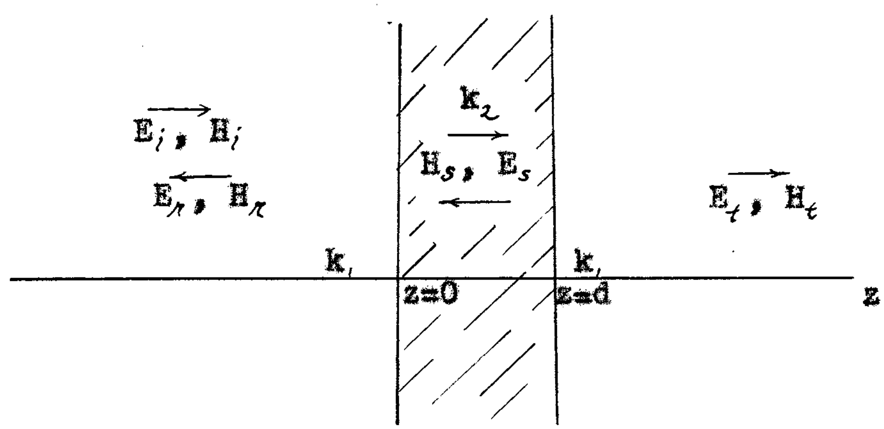

Fig. 2--Reflection and transmission of plane waves by a plane sheet at normal incidence.

This development is given by Stratton as applied to a

[2]J. A. Stratton, Electromagnetic Theory, p. 511.

normal conductor but is applied here to the superconductor.

For normal incidence we need only consider the magnitudes of the field vectors, and so for the incident and reflected waves at the first boundary we can write[3]

$$\mathcal{E}_i = \mathcal{E}_0 e^{jk_1 z - j\omega t} \qquad\qquad \mathcal{H}_i = \frac{k_1 c}{\omega \mu_1}\mathcal{E}_1$$

$$\mathcal{E}_r = \mathcal{E}_i e^{-jk_1 z - j\omega t} \qquad\qquad \mathcal{H}_r = -\frac{k_1 c}{\omega \mu_1}\mathcal{E}_r$$

the equation for H being obtained from the relation

$$\nabla \times \bar{\mathcal{E}} = -\frac{1}{c}\dot{\bar{H}}$$

Since we have a wave transmitted at the first boundary and a wave reflected at the second boundary, we must have for the field within the sheet

$$\mathcal{E}_s = \left(\mathcal{E}_2^+ e^{jk_2 z} + \mathcal{E}_2^- e^{-jk_2 z}\right) e^{-j\omega t}$$

$$\mathcal{H}_s = \left(\mathcal{E}_2^+ e^{jk_2 z} - \mathcal{E}_2^- e^{-jk_2 z}\right)\frac{k_2 c}{c} e^{-j\omega t}$$

where $\mathcal{E}^+$ and $\mathcal{E}^-$ represent the external and internal fields respectively, and for the transmitted wave

$$\mathcal{E}_t = \mathcal{E}_3 e^{jk_1 z - j\omega t} \qquad\qquad \mathcal{H}_t = \frac{k_1 c}{\mu_1 \omega}\mathcal{E}_t$$

Considering the boundary conditions one is led to these four equations:

$$\mathcal{E}_0 + \mathcal{E}_1 = \mathcal{E}_2^+ + \mathcal{E}_2^-$$

$$\mathcal{E}_0 - \mathcal{E}_1 = \mu_1 \frac{k_2}{k_1}\left(\mathcal{E}_2^+ - \mathcal{E}_2^-\right)$$

$$\mathcal{E}_2^+ e^{jk_2 d} + \mathcal{E}_2^- e^{-jk_2 d} = \mathcal{E}_3 e^{jk_1 d}$$

$$\mathcal{E}_2^+ e^{jk_2 d} - \mathcal{E}_2^- e^{-jk_2 d} = \frac{k_1}{\mu_1 k_2}\mathcal{E}_3 e^{jk_1 d}$$

Assuming the medium on either side to be free space the problem will be considerably simplified, and yet will give

---

[3] Ibid., p. 511.

the most significant results. In this case we have

$$k_1^2 = \frac{\omega^2}{c^2} \qquad k_2^2 = \frac{\omega^2}{c^2} - \beta^2$$

Transmission and reflection coefficients T and R are given by the square of the absolute value of the ratios $\frac{\mathcal{E}_1}{\mathcal{E}_0}$ and $\frac{\mathcal{E}_3}{\mathcal{E}_0}$ respectively. Solving for these ratios from the boundary equations, we get

$$\frac{\mathcal{E}_1}{\mathcal{E}_0} = \frac{\left(\frac{k_1-k_2}{k_1+k_2}\right)\left(1-e^{2jk_2d}\right)}{1-\left(\frac{k_1-k_2}{k_1+k_2}\right)^2 e^{2jk_2d}}$$

$$\frac{\mathcal{E}_3}{\mathcal{E}_0} = \frac{4k_1k_2 e^{j(k_2-k_1)d}}{(k_1+k_2)^2-(k_1-k_2)^2e^{2jk_2d}}$$

Two cases become apparent, determined by the conditions $\frac{\omega^2}{c^2} < \beta^2$ and $\frac{\omega^2}{c^2} > \beta^2$. In the first case k_2 is imaginary, while in the second case it is real, while k_1 is always real. Let the notation < and > be used to distinguish between these two cases. If we let $\alpha = \frac{\omega}{\beta c}$ and $\gamma = \beta d$, then after calculations we have

$$R< = \left[1 + \frac{4\alpha^2(1-\alpha^2)}{\sinh^2(\gamma\sqrt{1-\alpha^2})}\right]^{-1} \qquad R> = \left[1 + \frac{4\alpha^2(\alpha^2-1)}{\sin^2(\gamma\sqrt{\alpha^2-1})}\right]^{-1}$$

$$T< = \left[1 + \frac{\sinh^2(\gamma\sqrt{1-\alpha^2})}{4\alpha^2(1-\alpha^2)}\right]^{-1} \qquad T> = \left[1 + \frac{\sin^2(\gamma\sqrt{\alpha^2-1})}{4\alpha^2(\alpha^2-1)}\right]^{-1}$$

Here it is immediately seen that in both cases R + T = 1, which is in keeping with the London theory, showing that Joule heat can only be produced at the surface of a superconductor where a normal component of current density meets a charge density, which is not the case here.

Curves showing T as a function of α for several values of the parameter γ are shown on the following page. The theory may not be valid in the region of the "critical frequency" ($\alpha = 1$) and above, but it is of theoretical interest to see how T varies in this region. At low frequencies the wave is highly damped, so to speak, since the field falls off exponentially within the superconductor. In this region practically all the energy is reflected and transmission through the sheet is negligible except for thicknesses of the order of the penetration depth. However, above the critical point a wave passes through the sheet undamped and the transmission is determined mainly by interference effects due to the boundary surfaces.

It is possible to obtain thin films of thickness 10^{-6} centimeters. One can obtain the value $\gamma = 1$ experimentally. This curve shows considerable transmission in the lower regions of the spectrum, where one might assume the validity of the London theory. It shows a transmission of around 50% for values of α as low as 0.05, which corresponds to a wave length of about 12μ. A calculation of this transmission through a copper sheet of this thickness for the same frequency, using the static value of its conductivity, shows a negligible transmission because of Joule heating. This suggests an experiment to check the validity of the London

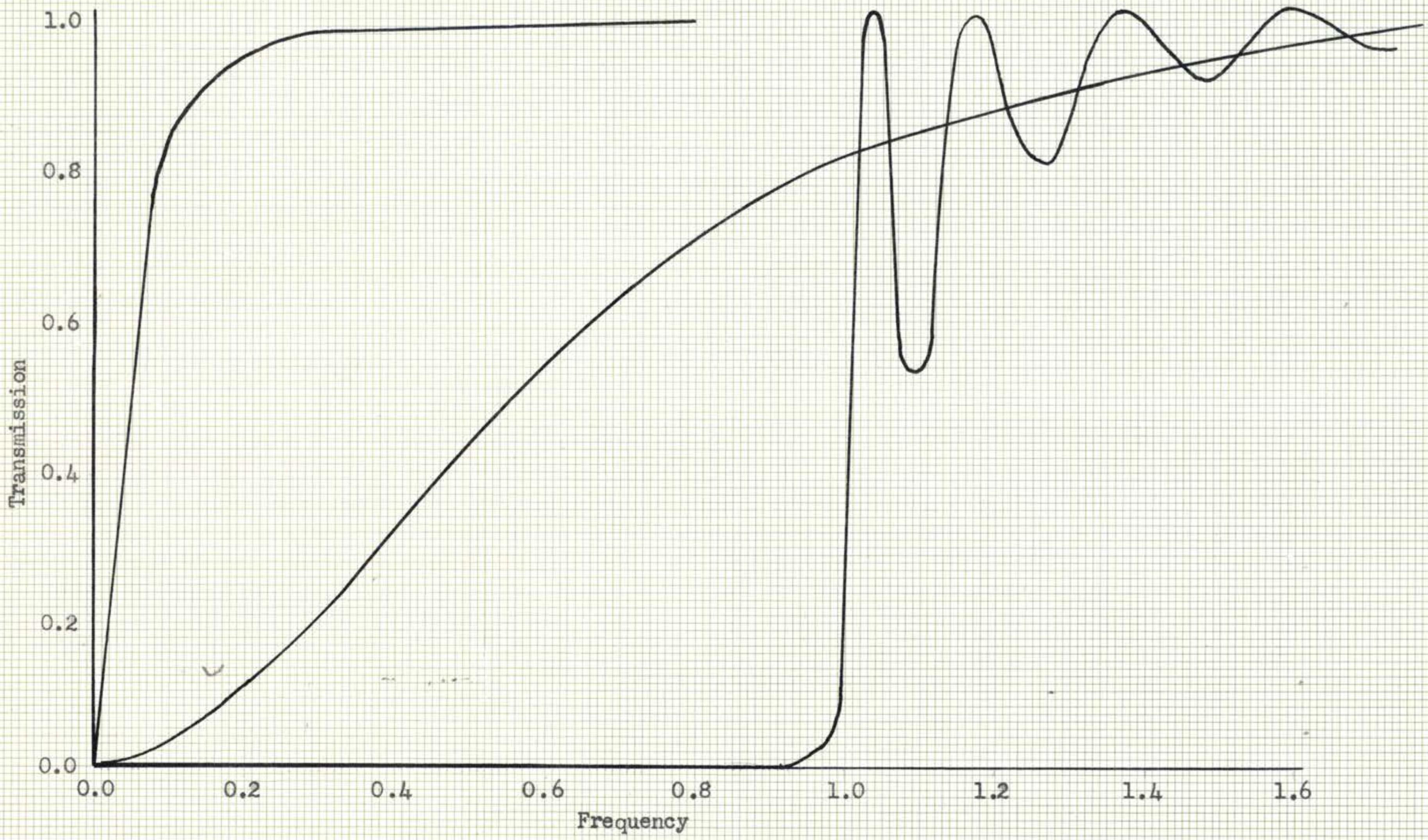

Fig. 3.--Transmission coefficients as a function of frequency

theory by comparing the transmission at ordinary temperatures of a metal known to be a superconductor with the transmission at temperatures below the transition temperature. However, experiments performed at Toronto[3] indicate that there is a minimum thickness for the existence of normal superconductivity, which is given to be around 1μ. If this is the case, then information about the nature of superconductors cannot be obtained by measuring the transmission through thin films, since it is negligible for thicknesses of 1μ and above.

---

[3]E. F. Burton, J. O. Wilhelm and A. D. Misener, Transactions, Royal Society of Canada, XXVII (1934), 65; A. D. Misener and J. O. Wilhelm, Transactions, Royal Society of Canada, XXIX (1935), 29.

BIBLIOGRAPHY

Abraham, Max and Becker, R., Classical Theory of Electricity and Magnetism, translation by John Dougall, London, Blackie and Son, Limited, 1946.

Burton, E. F., Smith, H. G. and Wilhelm, J. O., Phenomena at the Temperature of Liquid Helium, New York, Reinhold Publishing Corporation, 1940.

Burton, E. F., Wilhelm, J. O., and Misener, A. D., Royal Society of Canada, Transactions, Series 3, Sec. 3, XXVII (1943), 65.

Cook, Emory, Physical Review, LVIII (August, 1940), 357-61.

de Hass, W. S. and Bremmer, H., Akademie van Wetenschappen te Amsterdam, No. 3, XXXIV (1931), 336.

Houston, W. V., Principles of Mathematical Physics, New York, McGraw-Hill Book Co., 1934.

Loeb, L. B., Electricity and Magnetism, New York, John Wiley and Sons, Inc., 1947.

London, F., Reviews of Modern Physics, XVII (1945), 311.

London, F. and London, H., Physica, II (1935), 341.

Misener, A. D. and Wilhelm, J. O., Royal Society of Canada, Transactions, Series 3, Sec. 3, XXIX (1935), 5.

Page, Leigh, and Adams, N. I., Principles of Electricity, New York, D. Van Nostrand Company, Inc., 1931.

Philosophical Magazine, IV (August, 1927), 21.

Slater, J. C. and Frank, N. H., Electromagnetism, New York, McGraw-Hill Book Co., 1947.

Smith, H. G. and Wilhelm, J. O., Reviews of Modern Physics, VII (October, 1935), 237-71.

Stratton, J. A., Electromagnetic Theory, New York, McGraw-Hill Book Co., 1941.

Suydam, Vernon A., Electricity and Electromagnetism, New York, D. Van Nostrand Co., 1940.