Results for the intrabeam scattering growth
Rates for a bi-gaussian distribution

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Abstract

This note lists results for the intrabeam scattering growth rates for a bi-gaussian distribution. The derivation of these results will be given in a future note.

Introduction

This note finds results for the intrabeam scattering growth rates for a bi-gaussian distribution.

The bi-gaussian distribution is interesting for studying the possibility of using electron cooling in RHIC. Studies done using the SIMCOOL program [1] indicate that in the presence of electron cooling, the beam distribution changes so that it develops a strong core and a long tail which is not described well by a gaussian, but may be better described by a bi-gaussian. Being able to compute the effects of intrabeam scattering for a bi-gaussian distribution would be useful in computing the effects of electron cooling, which depend critically on the details of the intrabeam scattering.

Gaussian distribution

Before defining the bi-gaussian distribution, the gaussian distribution will be reviewed.
\( N f(x, p) \) gives the number of particles in \( d^3x d^3p \), where \( N \) is the number of particles in a bunch. For a gaussian distribution, \( f(x, p) \) is given by

\[
f(x, p) = \frac{1}{\Gamma} \exp[-S(x, p)]
\]

\( S = S_x + S_y + S_s \)

\[
S_x = \frac{1}{\epsilon_x} \epsilon_x(x_p, p_x / p_0)
\]

\[
x_p = x - D(p - p_0) / p_0
\]

\[
p_x / p_0 = p_x / p_0 - D(p - p_0) / p_0
\]

\[
\epsilon_x(x, x') = \gamma_{xy}x^2 + 2\alpha_{xy}x'x + \beta_{xy}x'^2
\]

\[
S_y = \frac{1}{\epsilon_y} \epsilon_y(y, p_y / p_0)
\]

\[
\epsilon_y(y, y') = \gamma_{yy}y^2 + 2\alpha_{yy}yy' + \beta_{yy}y'^2
\]

\[
S_s = \frac{1}{2\sigma_x^2}(s - s_c)^2 + \frac{1}{2\sigma_p^2}(p - p_0) / p_0
\]

\[
S_s = \frac{1}{\epsilon_s} \epsilon_s(s, p - p_0) / p_0
\]

\[
\beta_s = \sigma_s / \sigma_p
\]

\[
\epsilon_s = 2\sigma_s \sigma_p
\]

\[
S_s = \frac{1}{\epsilon_s} \epsilon_s(s - s_c, (p - p_0) / p_0)
\]

\[
\Gamma = \int d^3x d^3p \exp[-S(x, p)]
\]

\[
\Gamma = \pi^3 x^2 \epsilon_y \epsilon_s p_0^3
\]

\[
\epsilon_i = \langle \epsilon_i(x, p) \rangle \quad i = x, y, s
\]

\( D \) is the horizontal dispersion. \( D' = dD/ds \). \(< > \) indicates an average over all the particles in a bunch.

**Growth rates for a Gaussian distribution**

In the following, the growth rates are given in the Rest Coordinate System, which is the coordinate system moving along with the bunch. Growth rates
are given for \(< p_i p_j >\). From these one can compute the growth rates for
\(< \epsilon_i >\) using the relations given at the end of this note.

\[
\frac{1}{p_0^2} \frac{d}{dt} < p_i p_j > = \frac{N}{\Gamma} \int d^3 \Delta \exp[-R] C_{ij}
\]

\[
C_{ij} = \frac{2\pi}{p_0^2} \left( \frac{r_0}{2\bar{\beta}c} \right)^2 (|\Delta|^2 \delta_{ij} - 3\Delta_i \Delta_j) 2\bar{\beta}c \ln[1 + (2\bar{\beta}^2 b_{\max}/r_0)^2]
\]

\[
\bar{\beta} = \beta_0 \gamma_0 \Delta/p_0,
\]

\[
r_0 = Z^2 c^2/M c^2
\]

\[
R = R_x + R_y + R_s
\]

\[
R_x = \frac{2}{\beta_x \epsilon_x} \left[ \gamma^2 D^2 \Delta_x^2 + (\beta_x \Delta_x - \gamma \bar{D} \Delta_y)^2 \right]/p_0^2
\]

\[
\bar{D} = \beta_x D + \alpha_x D
\]

\[
R_y = \frac{2}{\beta_y \epsilon_y} \left[ \gamma^2 \Delta_y^2/p_0^2 \right]
\]

\[
R_s = \frac{2}{\beta_s \epsilon_s} \left[ \beta_s^2 \gamma^2 \Delta_s^2/p_0^2 \right]
\]

The integral over \(d^3 \Delta\) is an integral over all possible values of the relative
momentum for any two particles in a bunch. \(\bar{\beta}_0, \gamma_0\) are the beta and gamma
 corressponding to \(p_0\), the central momentum of the bunch in the Laboratory
 Coordinate System. \(\gamma = \gamma_0\)

The above 3-dimensional integral can be reduced to a 2-dimensional
 integral by integrating over \(|\Delta|\) and using \(d^3 \Delta = |\Delta|^2 d|\Delta| \sin \theta d\theta d\phi\).

\[
\frac{1}{p_0^2} \frac{d}{dt} < p_i p_j > = \frac{N}{\Gamma} 2\pi p_0^3 \left( \frac{r_0}{2\gamma_0^2 \beta_0^2} \right)^2 2\beta_0 \gamma_0 c \int \sin \theta d\theta d\phi \ (\delta_{ij} - 3g_i g_j)
\]

\[
\frac{1}{F} \ln \left[ \frac{\dot{C}}{C} \right]
\]

\[
g_3 = \cos \theta = g_s
\]

\[
g_1 = \sin \theta \cos \phi = g_x
\]

\[
g_2 = \sin \theta \sin \phi = g_y
\]

\[
\dot{C} = 2\gamma_0^2 \beta_0^2 b_{\max}/r_0
\]

\[
F = R/(|\Delta|/p_0)^2
\]
\[ F = F_x + F_y + F_s \]
\[ F_x = \frac{2}{\beta_x \epsilon_x} [\gamma^2 D^2 g_x^2 + (\beta_x g_x - \gamma D g_s)^2] \]
\[ F_y = \frac{2}{\beta_y \epsilon_y} \beta_y^2 g_y^2 \]
\[ F_s = \frac{2}{\beta_s \epsilon_s} \beta_s^2 \gamma^2 g_s^2 \]

(4)

**Bi-Gaussian distribution**

The bi-gaussian distribution will be assumed to have the form given by the following.

\[ N f(x, p) \] gives the number of particles in \( d^3 x d^3 p \), where \( N \) is the number of particles in a bunch. For a bi-gaussian distribution, \( f(x, p) \) is given by

\[ f(x, p) = \frac{N_a}{N} \frac{1}{\Gamma_a} e^{\exp[-S_a(x, p)]} + \frac{N_b}{N} \frac{1}{\Gamma_b} e^{\exp[-S_b(x, p)]} \]

(5)

In the first gaussian, to find \( \Gamma_a, S_a \) then in the expressions for \( \Gamma, S \), given above for the gaussian distribution, replace \( \epsilon_x, \epsilon_y, \epsilon_s \) by \( \epsilon_{xa}, \epsilon_{ya}, \epsilon_{sa} \). In the second gaussian, in the expressions for \( \Gamma, S \), replace \( \epsilon_x, \epsilon_y, \epsilon_s \) by \( \epsilon_{xb}, \epsilon_{yb}, \epsilon_{sb} \). In addition, \( N_a + N_b = N \). This bi-gaussian has 7 parameters instead of the three parameters of a gaussian.

**Growth rates for a Bi-Gaussian distribution**

In the following, the growth rates are given in the Rest Coordinate System, which is the coordinate system moving along with the bunch. Growth rates are given for \( \langle p_i p_j \rangle \). From these one can compute the growth rates for \( \langle \epsilon_i \rangle \) using the relations given at the end of this note.

\[ \frac{1}{p_0^2} \frac{d}{dt} \langle p_i p_j \rangle = N \int d^3 \Delta C_{ij} \left[ \left( \frac{N_a}{N} \right)^2 \frac{\exp(-R_a)}{\Gamma_a} + \left( \frac{N_b}{N} \right)^2 \frac{\exp(-R_b)}{\Gamma_b} \right] \]

\[ + 2 \frac{N_a N_b}{N^2} \frac{\Gamma_c}{\Gamma_a \Gamma_b} \exp(-T) \]

\[ C_{ij} = \frac{2\pi}{p_0^2} (r_0/2\beta^2)^2 (|\Delta|^2 \delta_{ij} - 3 \Delta_i \Delta_j) 2\beta c \ln[1 + (2\beta^2 b_{max}/r_0)^2] \]

4
\[\tilde{\beta} = \beta_0 \gamma_0 \Delta/p_0\]
\[r_0 = Z^2 e^2/M c^2\]
\[\frac{1}{\epsilon_{ic}} = \frac{1}{2} \left( \frac{1}{\epsilon_{ia}} + \frac{1}{\epsilon_{ib}} \right) \quad i = x, y, s\]
\[\frac{1}{\epsilon_{id}} = \frac{1}{2} \left( \frac{1}{\epsilon_{ia}} - \frac{1}{\epsilon_{ib}} \right)\]
\[r_0 = Z^2 e^2/M c^2\]
\[\Gamma_a = \pi^3 \epsilon_{sa} \epsilon_{xa} \epsilon_{ya} p_0^3\]
\[R_a = R_{xa} + R_{ya} + R_{sa}\]
\[R_{xa} = \frac{2}{\beta_x \epsilon_x} \left[ \gamma^2 D^2 \Delta_x^2 + (\beta_x \Delta_x - \gamma \tilde{D} \Delta_x)^2 \right]/p_0^2\]
\[\tilde{D} = \beta_x D' + \alpha_x D\]
\[R_{ya} = \frac{2}{\beta_y \epsilon_y} \beta_y^2 \Delta_y^2 / p_0^2\]
\[R_{sa} = \frac{2}{\beta_s \epsilon_s} \beta_s^2 \gamma^2 \Delta_s^2 / p_0^2\]
\[T = T_x + T_y + T_s\]
\[T_x = R_{xc} - R_{xd}\]
\[T_y = R_{yc} - R_{yd}\]
\[T_s = R_{sc} - R_{sd}\]
\[R_{xd} = 2 \left\{ \frac{[-\gamma D \Delta_s]^2}{(\beta_x \epsilon_{xc}^2 / \epsilon_{xc})} \right\} \]
\[R_{yd} = \frac{2 \beta_y}{\epsilon_{yd} / \epsilon_{yc}} \Delta_y^2\]
\[R_{sd} = \frac{2 \beta_s}{\epsilon_{sd} / \epsilon_{sc}} \Delta_s^2\]
\[\Delta_i = \Delta_i / p_0 \quad i = x, y, s\] (6)

\[R_a, R_b, R_c\] are each the same as \(R_a\) given above except that \(\epsilon_{ia}\) are replaced
by $\epsilon_{ia}, \epsilon_{ib}, \epsilon_{ic}$ respectively.

The above 3-dimensional integral can be reduced to a 2-dimensional integral by integrating over $|\Delta|$.

\[
\frac{1}{p_0^2} \frac{d}{dt} < p_i p_j > = 2\pi p_0^3 \left( \frac{r_0}{2\gamma_0^2 \beta_0^2} \right)^2 2\beta_0 \gamma_0 c \int \sin \theta d\theta d\phi (\delta_{ij} - 3g_i g_j)
\]

\[
N\left[ \frac{N_a}{N} \right]^2 \frac{1}{\Gamma_a F_a} \ln \frac{\hat{C}}{F_a} + \left( \frac{N_b}{N} \right)^2 \frac{1}{\Gamma_b F_b} \ln \frac{\hat{C}}{F_b} + 2 \frac{N_a N_b}{N^2} \frac{\Gamma_c}{\Gamma_a \Gamma_b} \frac{1}{G} \ln \frac{\hat{C}}{G}
\]

\[
g_3 = \cos \theta = g_s \\
g_1 = \sin \theta \cos \phi = g_x \\
g_2 = \sin \theta \sin \phi = g_y \\
\hat{C} = 2\gamma_0^2 \beta_0^2 b_{\text{max}}/r_0
\]

\[
F_i = R_i/(|\Delta|/p_0)^2 \quad i = a, b, c \\
G = T/(|\Delta|/p_0)^2
\]

(7)

$F_a, F_b, F_c$ are each the same $F$ that was defined for the Gaussian distribution except that the $\epsilon_i$ are replaced by $\epsilon_{ia}, \epsilon_{ib}, \epsilon_{ic}$ respectively.

The above results for the growth rates for a bi-gaussian distribution are expressed as an integral which contains 3 terms, each of which is similar to the one term in the results for the gaussian distribution. These three terms may be given a simple interpretation. The first term represents the contribution to the growth rates due to the scattering of the $N_a$ particles of the first gaussian from themselves, the second term the contribution due to the scattering of the $N_b$ particles of the second gaussian from themselves, and the third term the contribution due to the scattering of the $N_a$ particles of the first gaussian from the $N_b$ particles of the second gaussian.
Emittance growth rates

One can compute growth rates for the average emittances, \( <\epsilon_i> \) in the Laboratory Coordinate System, from the growth rates for \( <p_ip_j> \) in the Rest Coordinate System. In the following, \( dt \) is the time interval in the Laboratory System and \( d\tilde{t} \) is the time interval in the Rest System. \( dt = \gamma d\tilde{t} \)

\[
\frac{d}{dt}\epsilon_x = \frac{\beta_x}{\gamma}\frac{d}{d\tilde{t}} <\frac{p_x^2}{p_0^2}> + \frac{D^2 + \ddot{D}^2}{\beta_x}\frac{d}{d\tilde{t}} <\frac{p_x^2}{p_0^2}> - 2\dot{D}\frac{d}{d\tilde{t}} <\frac{p_x p_s}{p_0^2}>
\]

\[
\frac{d}{dt}\epsilon_y = \frac{\beta_y}{\gamma}\frac{d}{d\tilde{t}} <\frac{p_y^2}{p_0^2}>
\]

\[
\frac{d}{dt}\epsilon_s = \beta_s\gamma\frac{d}{d\tilde{t}} <\frac{p_s^2}{p_0^2}>
\]  

(8)

I thank I. Ben-Zvi for his comments and encouragement. The results given above were found using the results given in references [2,3,4]. The derivation of the results is given in Ref.[5]

References