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**A NOTE ON
SEVEN ANALOGOUS PROPERTIES BETWEEN
STIRLING NUMBERS OF THE FIRST KIND
AND BINOMIAL COEFFICIENTS**

Tommy Wright

MANAGED BY
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Computer Science and Mathematics Division
Mathematical Sciences Section

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**A NOTE ON
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ABSTRACT

This notes gives seven analogous properties between Stirling numbers of the first kind and binomial coefficients.

1. INTRODUCTION

If n and r are both nonnegative integers where $r \leq n$, the **binomial coefficient** $\binom{n}{r}$ is given by

$$\binom{n}{r} \equiv \frac{n!}{r!(n-r)!}. \quad (1)$$

The symbol $\binom{n}{r}$ is called a binomial coefficient because it is the coefficient of the $(r+1)^{th}$ term in the expansion of $(1+x)^n$ by the binomial theorem. Furthermore, these coefficients are the entries in Pascal's triangle. For a recent historical treatment of Pascal's *arithmetical triangle's* roots, which stretch backward before Christ, see Edwards(1987). The binomial coefficient plays a fundamental role in several areas including combinatorics, applied probability, and probability sampling (Knuth, 1981; Graham, Knuth, and Patashnik, 1989; Ross, 1989; Wilf, 1989; and Wright, 1989, 1991).

If n and r are both nonnegative integers where $r \leq n$, the **Stirling Number of the First Kind** $\left[\begin{smallmatrix} n \\ r \end{smallmatrix} \right]$ is defined as

$$\left[\begin{smallmatrix} n \\ r \end{smallmatrix} \right] \equiv \text{the sum of all possible products of } n-r \text{ integers taken from the first } n \text{ positive integers.} \quad (2)$$

For $r = n$, we define $\left[\begin{smallmatrix} n \\ n \end{smallmatrix} \right] \equiv 1$. Note that $\left[\begin{smallmatrix} n \\ 0 \end{smallmatrix} \right] \equiv n!$. Thus if $n = 4$ and $r = 2$, $\left[\begin{smallmatrix} 4 \\ 2 \end{smallmatrix} \right] = 1 \cdot 2 + 1 \cdot 3 + 1 \cdot 4 + 2 \cdot 3 + 2 \cdot 4 + 3 \cdot 4 = 35$. Also $\left[\begin{smallmatrix} 4 \\ 0 \end{smallmatrix} \right] = 4! = 24$ and $\left[\begin{smallmatrix} 4 \\ 4 \end{smallmatrix} \right] = 1$. In general, the number of terms in the sum $\left[\begin{smallmatrix} n \\ r \end{smallmatrix} \right]$ is $\binom{n}{r}$.

In a result analogous to the binomial theorem, it can be shown that

$$\prod_{i=1}^k (i+x) = \sum_{r=0}^k \left[\begin{smallmatrix} k \\ r \end{smallmatrix} \right] x^r. \quad (3)$$

The quantities $\left[\begin{smallmatrix} n \\ r \end{smallmatrix} \right]$ have a triangular arrangement which is similar to Pascal's triangle for the binomial coefficients. (Graham, Knuth, and Patashnik (1989); and Wright(in press))

2. SOME ANALOGOUS PROPERTIES OF THE COEFFICIENTS $\left[\begin{smallmatrix} n \\ r \end{smallmatrix} \right]$ AND $\binom{n}{r}$

In this section, we list several properties of the coefficients $\left[\begin{smallmatrix} n \\ r \end{smallmatrix} \right]$. For each property, an analogous result is noted for Pascal's triangle. The proofs of these properties are straightforward.

Property 1.

$$\left[\begin{smallmatrix} n \\ r \end{smallmatrix} \right] = n \left[\begin{smallmatrix} n-1 \\ r \end{smallmatrix} \right] + (n-1) \left[\begin{smallmatrix} n-2 \\ r-1 \end{smallmatrix} \right] + \cdots + (n-r) \left[\begin{smallmatrix} n-(r+1) \\ 0 \end{smallmatrix} \right].$$

Example 1.

$$\begin{bmatrix} 5 \\ 2 \end{bmatrix} = 5 \begin{bmatrix} 4 \\ 2 \end{bmatrix} + 4 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

Analogous Property and Example:

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-2}{r-1} + \binom{n-3}{r-2} + \cdots + \binom{n-(r+1)}{0}.$$

$$\binom{5}{2} = \binom{4}{2} + \binom{3}{1} + \binom{2}{0}.$$

Property 2.

$$\begin{bmatrix} n \\ r \end{bmatrix} = \begin{bmatrix} n-1 \\ r-1 \end{bmatrix} + n \begin{bmatrix} n-2 \\ r-1 \end{bmatrix} + n(n-1) \begin{bmatrix} n-3 \\ r-1 \end{bmatrix} + \cdots + n(n-1)\cdots(r+1) \begin{bmatrix} r-1 \\ r-1 \end{bmatrix}.$$

Example 2.

$$\begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} + 5 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + 5 \cdot 4 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 5 \cdot 4 \cdot 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Analogous Property and Example:

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-2}{r-1} + \binom{n-3}{r-1} + \cdots + \binom{r-1}{r-1}.$$

$$\binom{5}{2} = \binom{4}{1} + \binom{3}{1} + \binom{2}{1} + \binom{1}{1}.$$

Where x is a real number, define $[x] \equiv$ the greatest integer less than or equal to x . Property 3 is a *symmetry* property.

Property 3.

$$\sum_{r=0}^{[n/2]} \begin{bmatrix} n \\ 2r \end{bmatrix} = \sum_{r=0}^{[n/2]} \begin{bmatrix} n \\ 2r+1 \end{bmatrix}.$$

Example 3.

$$\begin{bmatrix} 5 \\ 0 \end{bmatrix} + \begin{bmatrix} 5 \\ 2 \end{bmatrix} + \begin{bmatrix} 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix} + \begin{bmatrix} 5 \\ 3 \end{bmatrix} + \begin{bmatrix} 5 \\ 5 \end{bmatrix}.$$

Analogous Property and Example:

$$\sum_{r=0}^{[n/2]} \binom{n}{2r} = \sum_{r=0}^{[n/2]} \binom{n}{2r+1}.$$

$$\binom{5}{0} + \binom{5}{2} + \binom{5}{4} = \binom{5}{1} + \binom{5}{3} + \binom{5}{5}.$$

Property 4.

$$\sum_{r=0}^n \begin{bmatrix} n \\ r \end{bmatrix} = (n+1) \sum_{r=0}^{n-1} \begin{bmatrix} n-1 \\ r \end{bmatrix}.$$

Example 4.

$$\begin{bmatrix} 4 \\ 0 \end{bmatrix} + \begin{bmatrix} 4 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \end{bmatrix} + \begin{bmatrix} 4 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 \\ 4 \end{bmatrix} = 5 \left\{ \begin{bmatrix} 3 \\ 0 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 3 \end{bmatrix} \right\}.$$

Analogous Property and Example:

$$\sum_{r=0}^n \binom{n}{r} = 2 \sum_{r=0}^{n-1} \binom{n-1}{r}.$$

$$\binom{4}{0} + \binom{4}{1} + \binom{4}{2} + \binom{4}{3} + \binom{4}{4} = 2 \left\{ \binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3} \right\}.$$

Property 5 follows from Property 4.

Property 5.

$$\sum_{r=0}^n \begin{bmatrix} n \\ r \end{bmatrix} = (n+1)!.$$

Example 5.

$$\begin{bmatrix} 4 \\ 0 \end{bmatrix} + \begin{bmatrix} 4 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \end{bmatrix} + \begin{bmatrix} 4 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 \\ 4 \end{bmatrix} = 5!.$$

Analogous Property and Example:

$$\sum_{r=0}^n \binom{n}{r} = 2^n.$$

$$\binom{4}{0} + \binom{4}{1} + \binom{4}{2} + \binom{4}{3} + \binom{4}{4} = 2^4.$$

For the nonnegative integers x and y where $x \leq y$, define P_x^y to be

$$P_x^y \equiv \frac{y!}{(y-x)!}. \quad (4)$$

Property 6.

$$\sum_{m=0}^n P_{n-m}^{n+1} \sum_{r=0}^m \begin{bmatrix} m \\ r \end{bmatrix} = \sum_{m=0}^n P_{n-m}^{n+1} (m+1)! = (n+1)(n+1)!.$$

Example 6. For $n = 3$,

$$\begin{aligned} P_3^4 \begin{bmatrix} 0 \\ 0 \end{bmatrix} + P_2^4 \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} + P_1^4 \left\{ \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right\} + P_0^4 \left\{ \begin{bmatrix} 3 \\ 0 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 3 \end{bmatrix} \right\} \\ = 4 \cdot 3 \cdot 2(1!) + 4 \cdot 3(2!) + 4(3!) + (4!) = 4(4!). \end{aligned}$$

Analogous Property and Example:

$$\sum_{m=0}^n \sum_{r=0}^m \binom{m}{r} = \sum_{m=0}^n 2^m = 2^{n+1} - 1.$$

For $n = 3$,

$$\binom{0}{0} + \left\{ \binom{1}{0} + \binom{1}{1} \right\} + \left\{ \binom{2}{0} + \binom{2}{1} + \binom{2}{2} \right\} + \left\{ \binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3} \right\} = 2^4 - 1.$$

Property 7.

$$\sum_{i=0}^r \begin{bmatrix} n \\ i \end{bmatrix} = (n+1) \sum_{i=0}^{r-1} \begin{bmatrix} n-1 \\ i \end{bmatrix} + n \begin{bmatrix} n-1 \\ r \end{bmatrix}.$$

Example 7. For $n = 6$ and $r = 4$,

$$\begin{bmatrix} 6 \\ 0 \end{bmatrix} + \begin{bmatrix} 6 \\ 1 \end{bmatrix} + \begin{bmatrix} 6 \\ 2 \end{bmatrix} + \begin{bmatrix} 6 \\ 3 \end{bmatrix} + \begin{bmatrix} 6 \\ 4 \end{bmatrix} = (6+1) \left\{ \begin{bmatrix} 5 \\ 0 \end{bmatrix} + \begin{bmatrix} 5 \\ 1 \end{bmatrix} + \begin{bmatrix} 5 \\ 2 \end{bmatrix} + \begin{bmatrix} 5 \\ 3 \end{bmatrix} \right\} + 6 \begin{bmatrix} 5 \\ 4 \end{bmatrix}.$$

Analogous Property and Example:

$$\sum_{i=0}^r \binom{n}{i} = 2 \sum_{i=0}^{r-1} \binom{n-1}{i} + \binom{n-1}{r}.$$

For $n = 6$ and $r = 4$,

$$\binom{6}{0} + \binom{6}{1} + \binom{6}{2} + \binom{6}{3} + \binom{6}{4} = 2 \left\{ \binom{5}{0} + \binom{5}{1} + \binom{5}{2} + \binom{5}{3} \right\} + \binom{5}{4}.$$

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REFERENCES

1. A. W. F. Edwards, *Pascal's Arithmetical Triangle*, Charles Griffin Books Series, Oxford University Press, New York, (1987).
2. R. L. Graham, D. E. Knuth, and O. Patashnik, *Concrete Mathematics*, Addison-Wesley, Reading, (1989).
3. D. E. Knuth, *The Art of Computer Programming, Vol I, Seminumerical Algorithms*, 2nd, Addison-Wesley, Reading, (1981).
4. B. Pascal, *Traite du Triangle Arithmetique*, Desprez(1665).
5. S. M. Ross, *Introduction to Probability Models*, 4th Ed., Academic Press, San Diego, (1989).
6. H. S. Wilf, *Combinatorial Algorithms: An Update*, # 55 in *CBMS-NSF Regional Conference Series in Applied Mathematics*, Society for Industrial and Applied Mathematics, Philadelphia, (1989).
7. T. Wright, A note on Pascal's triangle and simple random sampling, *College Mathematics Journal*, 20(1989)59-66.
8. T. Wright, *Exact Confidence Bounds When Sampling From Small Finite Universes*, Springer-Verlag, New York, (1991).
9. T. Wright, "Pascal's triangle gets its genes from Stirling numbers of the first kind", *The College Mathematics Journal*, (in press).

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