A NOTE ON
SEVEN ANALOGOUS PROPERTIES BETWEEN
STIRLING NUMBERS OF THE FIRST KIND
AND BINOMIAL COEFFICIENTS

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A NOTE ON
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ABSTRACT

This notes gives seven analogous properties between Stirling numbers of the first kind and binomial coefficients.
1. INTRODUCTION

If $n$ and $r$ are both nonnegative integers where $r \leq n$, the binomial coefficient $\binom{n}{r}$ is given by

$$\binom{n}{r} \equiv \frac{n!}{r!(n-r)!}.$$  \hspace{1cm} (1)

The symbol $\binom{n}{r}$ is called a binomial coefficient because it is the coefficient of the $(r+1)^{th}$ term in the expansion of $(1+x)^n$ by the binomial theorem. Furthermore, these coefficients are the entries in Pascal's triangle. For a recent historical treatment of Pascal's arithmetic triangle's roots, which stretch backward before Christ, see Edwards (1987). The binomial coefficient plays a fundamental role in several areas including combinatorics, applied probability, and probability sampling (Knuth, 1981; Graham, Knuth, and Patashnik, 1989; Ross, 1989; Wilf, 1989; and Wright, 1989, 1991).

If $n$ and $r$ are both nonnegative integers where $r \leq n$, the Stirling Number of the First Kind $\left[\begin{array}{c} n \\ r \end{array}\right]$ is defined as

$$\left[\begin{array}{c} n \\ r \end{array}\right] \equiv \text{the sum of all possible products of } n-r \text{ integers taken from the first } n \text{ positive integers.} \hspace{1cm} (2)$$

For $r = n$, we define $\left[\begin{array}{c} n \\ n \end{array}\right] = 1$. Note that $\left[\begin{array}{c} n \\ 0 \end{array}\right] = n!$. Thus if $n = 4$ and $r = 2$, $\left[\begin{array}{c} 4 \\ 2 \end{array}\right] = 1 \cdot 2 + 1 \cdot 3 + 1 \cdot 4 + 2 \cdot 3 + 2 \cdot 4 + 3 \cdot 4 = 35$. Also $\left[\begin{array}{c} 4 \\ 0 \end{array}\right] = 4! = 24$ and $\left[\begin{array}{c} 4 \\ 4 \end{array}\right] = 1$. In general, the number of terms in the sum $\left[\begin{array}{c} n \\ r \end{array}\right]$ is $\binom{n}{r}$.

In a result analogous to the binomial theorem, it can be shown that

$$\prod_{i=1}^{k} (i+x) = \sum_{r=0}^{k} \left[\begin{array}{c} k \\ r \end{array}\right] x^r.$$  \hspace{1cm} (3)

The quantities $\left[\begin{array}{c} n \\ r \end{array}\right]$ have a triangular arrangement which is similar to Pascal's triangle for the binomial coefficients. (Graham, Knuth, and Patashnik (1989); and Wright (in press))

2. SOME ANALOGOUS PROPERTIES OF THE COEFFICIENTS $\left[\begin{array}{c} n \\ r \end{array}\right]$ AND $\binom{n}{r}$

In this section, we list several properties of the coefficients $\left[\begin{array}{c} n \\ r \end{array}\right]$. For each property, an analogous result is noted for Pascal's triangle. The proofs of these properties are straightforward.

Property 1.

$$\left[\begin{array}{c} n \\ r \end{array}\right] = n \left[\begin{array}{c} n-1 \\ r \end{array}\right] + (n-1) \left[\begin{array}{c} n-2 \\ r-1 \end{array}\right] + \cdots + (n-r) \left[\begin{array}{c} n-(r+1) \\ 0 \end{array}\right].$$
Example 1.

\[ \begin{bmatrix} 5 \\ 2 \end{bmatrix} = 5 \begin{bmatrix} 4 \\ 2 \end{bmatrix} + 4 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ 0 \end{bmatrix}. \]

Analogous Property and Example:

\[
\binom{n}{r} = \binom{n-1}{r} + \binom{n-2}{r-1} + \binom{n-3}{r-2} + \cdots + \binom{n-(r+1)}{0}.
\]

\[
\binom{5}{2} = \binom{4}{2} + \binom{3}{1} + \binom{2}{0}.
\]

Property 2.

\[
\begin{bmatrix} n \\ r \end{bmatrix} = \begin{bmatrix} n-1 \\ r-1 \end{bmatrix} + n \begin{bmatrix} n-2 \\ r-1 \end{bmatrix} + n(n-1) \begin{bmatrix} n-3 \\ r-1 \end{bmatrix} + \cdots + n(n-1) \cdots (r+1) \begin{bmatrix} r-1 \\ r-1 \end{bmatrix}.
\]

Example 2.

\[ \begin{bmatrix} 5 \\ 2 \end{bmatrix} = 5 \begin{bmatrix} 4 \\ 1 \end{bmatrix} + 5 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + 5 \cdot 4 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 5 \cdot 4 \cdot 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \]

Analogous Property and Example:

\[
\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-2}{r-1} + \binom{n-3}{r-1} + \cdots + \binom{r-1}{r-1}.
\]

\[
\binom{5}{2} = \binom{4}{1} + \binom{3}{1} + \binom{2}{1} + \binom{1}{1}.
\]

Where \( x \) is a real number, define \( [x] \equiv \) the greatest integer less than or equal to \( x \). Property 3 is a symmetry property.

Property 3.

\[
\sum_{r=0}^{[n/2]} \begin{bmatrix} n \\ 2r \end{bmatrix} = \sum_{r=0}^{[n/2]} \begin{bmatrix} n \\ 2r + 1 \end{bmatrix}.
\]

Example 3.

\[ \begin{bmatrix} 5 \\ 0 \end{bmatrix} + \begin{bmatrix} 5 \\ 2 \end{bmatrix} + \begin{bmatrix} 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix} + \begin{bmatrix} 5 \\ 3 \end{bmatrix} + \begin{bmatrix} 5 \\ 5 \end{bmatrix}. \]
Analogous Property and Example:

\[
\sum_{r=0}^{\lfloor n/2 \rfloor} \binom{n}{2r} = \sum_{r=0}^{\lfloor n/2 \rfloor} \binom{n}{2r + 1}.
\]

\[
\binom{5}{0} + \binom{5}{2} + \binom{5}{4} = \binom{5}{1} + \binom{5}{3} + \binom{5}{5}.
\]

Property 4.

\[
\sum_{r=0}^{n} \binom{n}{r} = (n + 1) \sum_{r=0}^{n-1} \binom{n-1}{r}.
\]

Example 4.

\[
\binom{4}{0} + \binom{4}{1} + \binom{4}{2} + \binom{4}{3} + \binom{4}{4} = 5 \left\{ \binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3} \right\}.
\]

Analogous Property and Example:

\[
\sum_{r=0}^{n} \binom{n}{r} = 2 \sum_{r=0}^{n-1} \binom{n-1}{r}.
\]

\[
\binom{4}{0} + \binom{4}{1} + \binom{4}{2} + \binom{4}{3} + \binom{4}{4} = 2 \left\{ \binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3} \right\}.
\]

Property 5 follows from Property 4.

Property 5.

\[
\sum_{r=0}^{n} \binom{n}{r} = (n + 1)!.
\]

Example 5.

\[
\binom{4}{0} + \binom{4}{1} + \binom{4}{2} + \binom{4}{3} + \binom{4}{4} = 5!.
\]

Analogous Property and Example:

\[
\sum_{r=0}^{n} \binom{n}{r} = 2^n.
\]

\[
\binom{4}{0} + \binom{4}{1} + \binom{4}{2} + \binom{4}{3} + \binom{4}{4} = 2^4.
\]
For the nonnegative integers $x$ and $y$ where $x \leq y$, define $P^y_x$ to be

$$ P^y_x = \frac{y!}{(y-x)!}. \quad (4) $$

**Property 6.**

$$ \sum_{m=0}^{n} P^{n+1}_{n-m} \sum_{r=0}^{m} \binom{m}{r} = \sum_{m=0}^{n} P^{n+1}_{n-m}(m+1)! = (n+1)(n+1)!. $$

**Example 6.** For $n = 3$,

\[
P^3_3 \left[ \binom{0}{0} \right] + P^3_2 \left[ \binom{1}{0} + \binom{1}{1} \right] + P^3_1 \left[ \binom{2}{0} + \binom{2}{1} + \binom{2}{2} \right] + P^3_0 \left[ \binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3} \right]
\]

\[= 4 \cdot 3 \cdot 2(1!) + 4 \cdot 3(2!) + 4(3!) + (4!) = 4(4!). \]

**Analogous Property and Example:**

\[
\sum_{m=0}^{n} \sum_{r=0}^{m} \binom{m}{r} = \sum_{m=0}^{n} 2^m = 2^{n+1} - 1.
\]

For $n = 3$,

\[
\binom{0}{0} + \left\{ \binom{1}{0} \right\} + \left\{ \binom{1}{1} \right\} + \left\{ \binom{2}{0} \right\} + \left\{ \binom{2}{1} \right\} + \left\{ \binom{2}{2} \right\} + \left\{ \binom{3}{0} \right\} + \left\{ \binom{3}{1} \right\} + \left\{ \binom{3}{2} \right\} + \left\{ \binom{3}{3} \right\} = 2^4 - 1.
\]

**Property 7.**

\[
\sum_{i=0}^{r} \binom{n}{i} = (n+1) \sum_{i=0}^{r-1} \binom{n-1}{i} + n \binom{n-1}{r}.
\]

**Example 7.** For $n = 6$ and $r = 4$,

\[
\binom{6}{0} + \binom{6}{1} + \binom{6}{2} + \binom{6}{3} + \binom{6}{4} = (6+1) \left\{ \binom{5}{0} + \binom{5}{1} + \binom{5}{2} + \binom{5}{3} \right\} + 6 \binom{5}{4}.
\]

**Analogous Property and Example:**

\[
\sum_{i=0}^{r} \binom{n}{i} = 2 \sum_{i=0}^{r-1} \binom{n-1}{i} + \binom{n-1}{r}.
\]

For $n = 6$ and $r = 4$,

\[
\binom{6}{0} + \binom{6}{1} + \binom{6}{2} + \binom{6}{3} + \binom{6}{4} = 2 \left\{ \binom{5}{0} + \binom{5}{1} + \binom{5}{2} + \binom{5}{3} \right\} + \binom{5}{4}.
\]
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