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This document was prepared in conjunction with work accomplished under Contract No. DE-AC09-96SR18500 with the U. S. Department of Energy.

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TITLE OF PAPER:

Application of the Two-Parameter J-A² Description to Ductile Crack Growth

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**ABSTRACT:** Typical ASTM fracture testing determines J-integral resistance (J-R) curve or fracture toughness ($J_{IC}$) based on specimens with high constraint geometry such as those specified in ASTM E1737-96. A three-term asymptotic solution with two parameters J and $A_2$ (a constraint parameter) has been developed for characterizing the constraint effect of various geometries. The present paper extends the $J-A_2$ characterization of a stationary crack tip to the regime of stable crack growth. Similar to the concept of J-controlled crack growth, the $J-A_2$ description can be approximately used to characterize ductile crack growth under certain amount of crack extension. The region of $J-A_2$ controlled crack growth is much larger than that controlled by J-integral alone. From the relationships between $A_2$ and the test data, $J_{IC}$ and tearing modulus ($T_R$), the coefficients used to define a J-R curve can be determined. For non-standard specimens or actual structures, once the constraint parameter $A_2$ is determined, the J-R curves appropriate for these geometries can then be obtained. A procedure of transferring J-R curves determined from the standard ASTM procedure to non-standard specimens or flawed structures is outlined in the paper.

**KEYWORDS:** constraint effect, fracture toughness, tearing modulus, J-R curve, $J-A_2$, asymptotic solution, J-controlled crack growth
The ductile failure of engineering materials is characterized by the fracture initiation toughness $J_{IC}$ and the subsequent fracture resistance (J-R) curve. The standard J-R curve shows that J-integral is a function of crack extension and is size-independent within a J-controlled growth region. However, for non-standard, low constraint specimens or flawed structures, the J-R curve may be size and geometry dependent when J-dominance is lost.

The American Society for Testing Materials (ASTM) standardized the specimen geometries for measuring $J_{IC}$ and J-R curves. These specimens are typically high crack-tip constrained, such as the deeply cracked three-point bend (3PB) and the compact tension (CT) specimens which are specified in the ASTM Standard Test Method for J-Integral Characterization of Fracture toughness (E 1737).

Using the ASTM 710 Grade A steel, Hancock et al. [1] measured the fracture toughness $J_{IC}$ and J-R curves for specimens of 3PB, 1CT, center-cracked panel (CCP), and surface cracked panel (SCP). Joyce and Link [2,3] presented the experiment data of ductile crack extension in A533B, HY-100 and HY-80 steels with specimens of 3PB, 1CT, single edge-notched bend (SENB), single edge-notched tensile (SENT), and double edge-cracked plate (DECP). A variety of crack tip constraints were achieved by using these specimens with different crack depths. The results did not show noticeable constraint effects on the crack growth initiation $J_{IC}$, but a significant difference in the slopes of the J-R curves was observed after some amount of crack growth. Similar results were obtained from ductile crack extension experiments of large-size fracture specimens by Marschall et al. [4], Eisele et al. [5] and Roos et al. [6,7], Kordisch et al. [8], Kelmm et al. [9], Henry et al. [10], and Haynes and Gangloff [11].

Roos et al. [6] showed the dependence of the J-R curve on specimen geometry and specimen thickness for KS01 (22NiMoCr37) which is typically used in German nuclear industry. Their
result indicated that thicker specimen tends to lower the J-R curve. Of particular interest is the J-R curve from the SENT specimen. It is substantially lower than that from the standard CT which is widely used in fracture testing. This implies that the standard ASTM CT specimen may not always form the lower bound J-R curve among various specimen geometries. Therefore, the fracture properties that determined from the standard CT specimens might not safely represent the tearing resistance of an actual structural component. They also showed that in the case of DENT specimens, deeper crack resulted in a lower J-R curve. The trend of all their test data is consistent with a general rule in the constraint effect of fracture, that is, specimens with higher constraint result in lower J-R curves.

In some nuclear applications such as the fracture testing for irradiated materials, the specimen size is limited due to the test facility or the material availability. Miniature or sub-sized disk compact tension (DCT) specimens are often used (Alexander [12]). Similar sub-sized fracture specimens were also used, for instance, by Elliot et al., [13] and Yoon et al. [14], and in ceramics materials used by Zhang and Ardell [15] and Gilbert et al [16]. The J-R curves of this kind of specimens differed significantly from those of standard specimens because of the effect of crack-tip constraints, while the values of initiation toughness $J_{IC}$ remained similar [12].

For a stationary crack in elastic-plastic materials, the effect of constraint on crack-tip fields has been investigated extensively by Betegon and Hancock [17], O’Dowd and Shih [18,19], Yang et al. [20,21] and Chao et al. [22], etc. A review can be found in Chao and Zhu [23]. This paper is focused on the J-A$_2$ approach [20-22] which was derived from a rigorous asymptotic solution and has been developed for a two-parameter fracture toughness testing. With J being the driving force and A$_2$ a constraint parameter, this approach has been successfully used to quantify the constraints of crack-tip fields for various geometry and loading configurations [22-
Note that the parameter $A_2$ is almost independent of its position near the crack tip (Nikishkov et al., [27]).

Similar to the concept of J-controlled crack growth, it is expected that $J-A_2$ description can approximately characterize the effect of crack-tip constraints on ductile crack growth, at least for certain amount of crack extension. As shown in Figure 1, the amount of $J-A_2$ controlled crack growth, $\Delta a$, should be much larger than that controlled by J alone [23].

The objective of this paper is to extend the $J-A_2$ characterization of crack tip fields to the stable crack growth regime. A procedure is outlined for transferring the J-R curves determined from ASTM standard specimens to non-standard specimens or to flawed structures. Based on a set of test data, a constraint modified J-R curve can be developed. Using literature data [3], the predicted J-R curve is demonstrated and is compared to the experimental J-R curve.

**Theoretical Background**

A Mode-I crack under plane strain condition is considered. The elastic-plastic material behavior is described by the Ramberg-Osgood power-law strain hardening curve where the uniaxial strain $\varepsilon$ is related to the uniaxial stress $\sigma$ in simple tension by

$$\frac{\varepsilon}{\varepsilon_0} = \frac{\sigma}{\sigma_0} + \alpha \left( \frac{\sigma}{\sigma_0} \right)^n$$

(1)

where $\sigma_0$ is a reference stress, $\varepsilon_0 = \sigma_0 / E$ is a reference strain with $E$ as the Young’s modulus, (for actual elastic-plastic solids, $\sigma_0$ and $\varepsilon_0$ may be taken as the yield stress and the yield strain of the material, respectively), $\alpha$ is a material constant and $n$ is a strain hardening exponent. By using the $J_2$ deformation theory of plasticity, the uniaxial stress-strain relation (1) can be generalized to
\[ \frac{\varepsilon_{ij}}{\varepsilon_0} = (1 + \nu) \frac{\sigma_{ij}}{\sigma_0} - \nu \frac{\sigma_{kk}}{\sigma_0} \delta_{ij} + \frac{3}{2} \alpha \left( \frac{\sigma_e}{\sigma_0} \right)^{n-1} s_{ij} \]  

(2)

where \( \nu \) is the Poisson’s ratio, \( \delta_{ij} \) is the Kronecker delta, \( s_{ij} \) is the deviatoric stress and \( \sigma_e \) is the von Mises effective stress defined as \( \sigma_e = \sqrt{3s_{ij}s_{ij}} / 2 \). The tensor summation convention has been used.

**J-A_2 three-term asymptotic solution**

Equation 2 is rewritten with respect to a polar coordinate system \((r, \theta)\) with centered at the crack tip. The \( \theta = 0 \) is along the uncracked ligament. Yang [28], Yang et al. [20,21] and Chao et al. [22] developed a three-term asymptotic crack-tip solution with only two parameters \( J \) and \( A_2 \), in which \( J \)-integral can be used to quantify the magnitude of applied loading and \( A_2 \) describes the crack tip constraints. The asymptotic fields of stress \((\sigma_{ij})\), strain \((\varepsilon_{ij})\) and displacement \((u_i)\) can be expressed as

\[
\frac{\sigma_{ij}}{\sigma_0} = A_1 \left[ \left( \frac{r}{L} \right)^{n_1} \tilde{\sigma}_{ij}^{(1)}(\theta,n) + A_2 \left( \frac{r}{L} \right)^{n_2} \tilde{\sigma}_{ij}^{(2)}(\theta,n) + A_2^2 \left( \frac{r}{L} \right)^{n_3} \tilde{\sigma}_{ij}^{(3)}(\theta,n) \right] 
\]

(3)

\[
\frac{\varepsilon_{ij}}{\alpha \varepsilon_0} = A_1^n \left[ \left( \frac{r}{L} \right)^{n_1} \tilde{\varepsilon}_{ij}^{(1)}(\theta,n) + A_2 \left( \frac{r}{L} \right)^{(n-1)n_1 + n_2} \tilde{\varepsilon}_{ij}^{(2)}(\theta,n) + A_2^2 \left( \frac{r}{L} \right)^{(n-1)n_1 + n_3} \tilde{\varepsilon}_{ij}^{(3)}(\theta,n) \right] 
\]

(4)

\[
\frac{u_i}{\alpha \varepsilon_0 L} = A_1^{n_1+1} \left[ \left( \frac{r}{L} \right)^{n_1+1} \tilde{u}_i^{(1)}(\theta,n) + A_2 \left( \frac{r}{L} \right)^{(n-1)n_1 + n_2 + 1} \tilde{u}_i^{(2)}(\theta,n) + A_2^2 \left( \frac{r}{L} \right)^{(n-1)n_1 + n_3 + 1} \tilde{u}_i^{(3)}(\theta,n) \right] 
\]

(5)
where the angular functions \( \tilde{\sigma}_q^{(k)} \), \( \tilde{\varepsilon}_q^{(k)} \) and \( \tilde{u}_i^{(k)} \), the stress power exponents \( s_k \), and the dimensionless integration constant \( I_n \) are only dependent of the hardening exponent \( n \) and independent of the other material constants (i.e. \( \alpha, \varepsilon_0, \sigma_0 \)) and applied load. The characteristic length, \( L \), can be the crack length \( a \), the specimen width \( W \), the thickness \( B \) or an unity 1 cm.

The parameters \( A_1 \) and \( s_1 \) are given by the Hutchinson-Rice-Rosengren (HRR) field \[29-31\],

\[
A_1 = \left( \frac{J}{\alpha \varepsilon_0 \sigma_0 I_n L} \right)^{-s_1}, \quad s_1 = -\frac{1}{n+1}
\]

(6)

and \( s_3 = 2s_2 - s_1 \) for \( n \geq 3 \). The parameter \( A_2 \) is undetermined and is related to the loading and geometry of specimen. Plane strain Mode I dimensionless functions \( \tilde{\sigma}_q^{(k)} \), \( \tilde{\varepsilon}_q^{(k)} \), \( \tilde{u}_i^{(k)} \), \( s_k \) and \( I_n \) have been calculated and tabulated by Chao and Zhang [32]. When \( A_2 = 0 \), the three-term asymptotic solutions (3) - (5) coincide with the HRR singular field.

Yang et al. [20] showed that the above three-term asymptotic solutions are the fully plastic or the pure power-law solutions when the material is moderate or low strain hardening, \( n \geq 3 \). Comparing with the finite element results, it can be shown that the three-term solution can be used to characterize the crack tip region well beyond \( r / (J / \sigma_0) = 5 \) [20,21,23,26]. This solution is valid under both small scale yielding and large scale yielding conditions, and is independent of the crack tip constraint in any strain hardening materials.

**Relationship between \( J \) and \( A_2 \) under fully plastic conditions**

For a pure power-law or fully plastic material, Ilyushin [33] showed that under monotonically increasing load a solution to the boundary value problem has a simple form, that
is, the stress is linearly related to the applied load $P$, the strain and the displacement are proportional to $P^n$. Therefore,

$$\frac{\sigma_{ij}}{\sigma_0} = \left[ \frac{P}{P_0} \right] \hat{\sigma}_{ij}(x_i, n)$$  \hspace{1cm} (7)$$

$$\frac{\varepsilon_{ij}}{\alpha\varepsilon_0} = \left[ \frac{P}{P_0} \right]^n \hat{\varepsilon}_{ij}(x_i, n)$$  \hspace{1cm} (8)$$

$$\frac{u_i}{\alpha\varepsilon_0 L} = \left[ \frac{P}{P_0} \right]^n \hat{u}_i(x_i, n)$$  \hspace{1cm} (9)$$

where the dimensionless $\hat{\sigma}_{ij}$, $\hat{\varepsilon}_{ij}$ and $\hat{u}_i$ are functions of spatial coordinates $x_i$ and strain hardening exponent $n$, and are independent of the applied load $P$. The reference load, $P_0$, can be the limit load. These expressions are the direct result of the homogenous nature of the equations of equilibrium, compatibility, and the constitutive relation (2).

Since the integrand of the $J$-integral involves products of stress and displacement gradients, from (7) - (9), Goldman and Hutchinson [34] showed that the fully plastic $J$-integral is proportional to $P^{n+1}$:

$$\frac{J}{\alpha\varepsilon_0\sigma_0 a} = \left[ \frac{P}{P_0} \right]^{n+1} \hat{J}(a/W, n)$$  \hspace{1cm} (10)$$

where $a$ is a crack length, $W$ is a specimen width. The dimensionless function $\hat{J}$ depends only on the geometry size ratio $a/W$ and strain hardening exponent $n$.

Substituting (10) into (7) - (9) and eliminating load term $P/P_0$, one can obtain the relationship between the field quantities and $J$-integral
\[
\frac{\sigma_{\bar{y}}}{\sigma_0} = \left[ \frac{J}{\alpha \varepsilon_0 \sigma_0 a \tilde{J}(a/W, n)} \right]^{\frac{1}{n+1}} \tilde{\sigma}_{\bar{y}}(x_1, n) \tag{11}
\]

\[
\frac{\varepsilon_{\bar{y}}}{\alpha \varepsilon_0} = \left[ \frac{J}{\alpha \varepsilon_0 \sigma_0 a \tilde{J}(a/W, n)} \right]^{\frac{n}{n+1}} \hat{\varepsilon}_{\bar{y}}(x_1, n) \tag{12}
\]

\[
\frac{u_j}{\alpha \varepsilon_0 L} = \left[ \frac{J}{\alpha \varepsilon_0 \sigma_0 a \tilde{J}(a/W, n)} \right]^{\frac{n}{n+1}} \hat{u}_i(x_1, n) \tag{13}
\]

The above expressions are the unique forms of solutions for plane crack problems under fully plastic deformation. Therefore, the three-term asymptotic solutions (3) - (5) must meet the functional forms (11) - (13), or

\[
\left( \frac{r}{L} \right)^{s_1} \tilde{\sigma}_{\bar{y}}^{(1)}(\theta) + A_2 \left( \frac{r}{L} \right)^{s_2} \tilde{\sigma}_{\bar{y}}^{(2)}(\theta) + A_2^2 \left( \frac{r}{L} \right)^{s_3} \tilde{\sigma}_{\bar{y}}^{(3)}(\theta) = \left[ \frac{I_n L}{a \tilde{J}(a/W, n)} \right]^{\frac{1}{n+1}} \tilde{\sigma}_{\bar{y}}(r, \theta, n) \tag{14}
\]

\[
\left( \frac{r}{L} \right)^{s_1'} \hat{\varepsilon}_{\bar{y}}^{(1)}(\theta) + A_2 \left( \frac{r}{L} \right)^{s_2'} \hat{\varepsilon}_{\bar{y}}^{(2)}(\theta) + A_2^2 \left( \frac{r}{L} \right)^{s_3'} \hat{\varepsilon}_{\bar{y}}^{(3)}(\theta) = \left[ \frac{I_n L}{a \tilde{J}(a/W, n)} \right]^{\frac{n}{n+1}} \hat{\varepsilon}_{\bar{y}}(r, \theta, n) \tag{15}
\]

\[
\left( \frac{r}{L} \right)^{s_1''} \tilde{u}_i^{(1)}(\theta) + A_2 \left( \frac{r}{L} \right)^{s_2''} \tilde{u}_i^{(2)}(\theta) + A_2^2 \left( \frac{r}{L} \right)^{s_3''} \tilde{u}_i^{(3)}(\theta) = \left[ \frac{I_n L}{a \tilde{J}(a/W, n)} \right]^{\frac{n}{n+1}} \tilde{u}_i(r, \theta, n) \tag{16}
\]

In equations (14) - (16), the exponents \( s_k \) and the constant \( I_n \) are only dependent of the hardening exponent \( n \). The angular functions \( \tilde{\sigma}_{\bar{y}}^{(k)}(\theta) \), \( \hat{\varepsilon}_{\bar{y}}^{(k)}(\theta) \) and \( \tilde{u}_i^{(k)}(\theta) \), the dimensionless quantities \( \tilde{\sigma}_{\bar{y}} \), \( \hat{\varepsilon}_{\bar{y}} \) and \( \tilde{u}_i \) are the functions of polar coordinates \( (r, \theta) \) and hardening exponent \( n \). The
dimensionless function \( \hat{J} \) depends upon geometry size ratio \( a/W \) and hardening exponent \( n \).

Moreover, all these quantities are independent of the other material properties \( (\alpha, \varepsilon_0, \sigma_0) \) and the level of the applied load or \( J \)-integral. If the characteristic length \( L \) is independent of \( J \), for instance, \( L = a, W, B \) or 1 cm, one can conclude immediately from (14)-(16) that \( A_2 \) is only the function of strain hardening exponent \( n \) and geometry dimension \( (a, W, a/W) \), namely,

\[
A_2 \bigg|_{\text{fully plastic}} = f(n, \text{geometry})
\]

As a result, under fully plastic deformation conditions the constraint parameter \( A_2 \) in the three-term solutions (3) - (5) are independent of the applied \( J \) for a given specimen and material.

Using a 3PB and SEN specimens, the finite element results [22,28] showed that \( A_2 \) becomes a constant value as the applied load (characterized by \( J \)) increases beyond about 1.2 times limit load with the hardening exponent \( n = 4, 7 \) and 12. Therefore, for a specimen fracturing at large-scale yielding or near fully plastic condition, the value of \( A_2 \) determined at \( J = J_{IC} \) can be used for loads \( J \geq J_{IC} \), as long as the growing crack tip is still within the \( J-A_2 \) field and the elastic unloading behind the crack tip does not significantly alter the crack tip field.

### Determination of the constraint parameter \( A_2 \)

A point matching technique was used by Yang et al. [20,21] and Chao et al. [22] to determined the value of \( A_2 \). The stress obtained by finite element analysis at a point \((r, \theta)\) near the crack tip is set equal to the three-term analytical solution (3) to solve for \( A_2 \). In particular, these authors used \( \sigma_r \) and \( \sigma_\theta \) at \( r = 2(J/\sigma_0) , \theta = 0^\circ \) or \( 45^\circ \). Chao and Zhu [23] also used this approach to determine \( A_2 \), but \( r \) is chosen from \( J/\sigma_0 \) to \( 5(J/\sigma_0) \).
To reduce the finite element mesh size sensitivity on the value of $A_2$, Nikishkov et al. [27] developed another technique, the *least square procedure*, for fitting the finite element data in the region $1 < r / (J / \sigma_0) < 5$. They found that $A_2$ is almost independent of its location within the region of interest.

A *simple weight average technique* is developed for the determination of $A_2$. It is assumed that the resultant force due to the crack opening stress $\sigma_{00}(r, 0)$ on the remaining ligament in the region of $1 < r / (J / \sigma_0) < 5$ has the same magnitude from a finite element result and from the three-term solution (3). Mathematically it is expressed as

$$\int_{r_1}^{r_2} \frac{\sigma_{00}^{FEA}(r, 0)}{\sigma_0} d\bar{r} = \left( \frac{J}{\alpha \varepsilon_0 \sigma_0 J_n L} \right)^{-1} \left[ J_{\bar{r}} \sigma_{ij}^{(1)}(0) + A_2 \left( \frac{J_{\bar{r}}}{\sigma_0 L} \right)^2 \sigma_{ij}^{(2)}(0) + A_2^2 \left( \frac{J_{\bar{r}}}{\sigma_0 L} \right)^3 \sigma_{ij}^{(3)}(0) \right] d\bar{r}$$

where $\bar{r} = r / (J / \sigma_0)$ and $\sigma_{00}^{FEA}$ is the crack opening stress for points on the $(r, \theta) = (r, 0)$ line determined from finite element calculations. Upon integrating the above expression, the value of $A_2$ can be determined simply by solving the following second-order algebraic equation:

$$aA_2^2 + bA_2 + c = 0 \quad (18)$$

where

$$a = \frac{5^{s_1+1} - 1}{(s_1 + 1)(L\sigma_0 / J)^{s_1}} \bar{\sigma}_{00}^{(3)}(0) \quad (19a)$$

$$b = \frac{5^{s_2+1} - 1}{(s_2 + 1)(L\sigma_0 / J)^{s_2}} \bar{\sigma}_{00}^{(2)}(0) \quad (19b)$$

$$c = \frac{5^{s_1+1} - 1}{(s_1 + 1)(L\sigma_0 / J)^{s_1}} \bar{\sigma}_{00}^{(1)}(0) - \left( \frac{J}{\alpha \varepsilon_0 \sigma_0 J_n L} \right)^{s_1} \int_{r_1}^{r_2} \frac{\sigma_{00}^{FEA}(\bar{r}, 0)}{\sigma_0} d\bar{r} \quad (19c)$$
The integration in (19c) can be determined approximately by numerical summation of \( \sigma_{60}^{FEA}(\bar{r}, 0) \) on the finite element nodes in \( 1 < \bar{r} < 5 \). Note that the weight averaging in (18) and (19) determines the averaged value for \( A_2 \) within the range \( 1 < \bar{r} < 5 \) along the \((r, 0)\) line. Other ranges of \( \bar{r} \) for averaging \( A_2 \) may be used.

**J-A \(_2\) description of ductile crack growth**

The two-parameter, J-A \(_2\) three-term solution has been successfully used to quantify the effect of constraint for stationary crack-tip fields with various geometries and loading configurations [20-22,26]. For a growing crack, similar to the concept of \( J \)-controlled crack growth, it is assumed that that, within certain amount of crack extension, the \( J-A_2 \) description can approximately characterize the effect of geometry constraint on ductile crack growth with \( J \) being the driving force and \( A_2 \) the constraint parameter. It is expected that the amount of \( J-A_2 \) controlled crack growth is much larger than that of the \( J \)-controlled since the crack tip zone dominated by the \( J-A_2 \) is much larger than that controlled by \( J \) alone [23].

A procedure for transferring the \( J-R \) curve determined from a standard ASTM procedure to that for a non-standard specimen or to a flawed structure is described in the following. It is then illustrated by using the test data \((J_{IC} \text{ and } T_R)\) of Joyce and Link [3]. A constraint based \( J-R \) curve is established for the case of stable crack growth which is characterized by the parameter \( A_2 \).
General framework to construct constraint-modified J-R curves

The ASTM E 1737 specifies test procedures for the determination of fracture toughness as characterized by the $J$-Integral under plane strain conditions. Based on the amount of crack extension, three toughness properties are identified as: (a) instability without significant prior crack extension ($J_C$); (b) onset of stable crack extension ($J_{IC}$); (c) stable crack growth resistance curve ($J$-$R$) in the region of $J$-controlled growth. The present investigation is focused on the last two fracture properties, (b) and (c).

It is specified in ASTM E 1737 that the fracture toughness $J_{IC}$ is defined as the value of $J$ at crack extension $\Delta a = \Delta a_Q = \frac{J_{IC}}{2\sigma_f} + 0.2$ (mm) or

$$J_{IC}(\Delta a) = J(\Delta a_Q)\big|_{\Delta a_Q = \frac{J_{IC}}{2\sigma_f} + 0.2}$$

(20)

where $\sigma_f$ is the flow stress or effective yield stress. In addition, the $J$-resistance curve can be approximated by a best-fit power-law relationship,

$$J(\Delta a) = C_1 \left( \frac{\Delta a}{k} \right)^{C_2}$$

(21)

where the coefficients $C_1$ and $C_2$ are constants, and $k = 1$ mm or 1 inch depending on the unit of $\Delta a$. Note that equation (21) assumes $J = 0$ at $\Delta a = 0$.

Since $J_{IC}$ corresponds to crack initiation, the constraint parameter $A_2$ can be solved from equation (18) by matching the three-term solution (3) to the finite element results at load level $J_{IC}$. If the specimens are nonstandard, the corresponding value of $A_2$ can be obtained in a similar manner, as long as the $J_{IC}$ value is measured.
As described earlier, under large scale yielding or near fully plastic deformation, the constraint parameter $A_2$ determined at $J = J_{IC}$ remains a constant for $J \geq J_{IC}$ in the case of a stationary crack. If the crack extension incurs and the location of the current crack tip is within the $J-A_2$ dominant region, the value of $A_2$ is approximately invariant for the specific specimen. Incorporating the constraint effect into the $J$-$R$ curve, a curve of $J$ versus crack extension $\Delta a$ under $J$-$A_2$ controlled growth can be expressed as

$$J(\Delta a, A_2) = C_0(A_2) + C_1(A_2) \left( \frac{\Delta a}{k} \right)^{C_2(A_2)} \quad (22)$$

The coefficients $C_0(A_2), C_1(A_2), C_2(A_2)$ are unknown constants and depend upon the constraint $A_2$ at the crack tip for a specific material and specimen. The coefficient $C_0(A_2)$ is added to include the fact that $J$ is nonzero at zero crack extension ($\Delta a$). That is, if the J value is known prior to crack initiation, then

$$C_0(A_2) = J(\Delta a = 0, A_2) = J_0(A_2) \quad (23)$$

Equation (22) extends the current ASTM $J$-resistance curve concept of $J(\Delta a)$ to a constraint modified $J$-resistance curve $J(\Delta a, A_2)$. Once the functional forms of $C_0(A_2), C_1(A_2)$ and $C_2(A_2)$ are known, the constraint modified $J$-resistance curve (or function), (22), is completely determined. The determination of $C_0(A_2), C_1(A_2)$ and $C_2(A_2)$ is described in the following steps:

Step 1: $J$-$R$ curves are determined experimentally based on ASTM procedures, but for different crack sizes or specimen types, as shown in Figure 2(a). The test specimens should be chosen to include several low constraint to high constraint specimens. As shown in Figure 2(a), each $J$-$R$
curve has a constant $A_2$ value, which is determined at the crack initiation load using finite element results and equation (18).

Step 2: Using three points on each $J$-$R$ curve as shown in Figure 2(a) the three coefficients $C_0(A_2), C_1(A_2)$ and $C_2(A_2)$ can be determined for each $J$-$R$ curve or $A_2$. That is

$$C_1(A_2) \left( \frac{\int \Delta a_i}{k} \right)^{C_2(A_2)} = J_i(\Delta a_i, A_2) - C_0(A_2); \quad i = 1, 2, 3 \quad (24)$$

If $\Delta a = \Delta a_0 = \frac{J_{ic}}{2\sigma_f} + 0.2$ (mm), $k = 1$mm and the International System of Units (SI) units are used, one has

$$C_1(A_2) \left( \frac{J_{ic}}{2k\sigma_f} + 0.2 \right)^{C_2(A_2)} = J_{ic}(A_2) - C_0(A_2) \quad (25)$$

Step 3: The step 2 is repeated for several $J$-$R$ curves, see Figure 2(a). Through curve fitting, the functional forms of $C_0(A_2), C_1(A_2)$ and $C_2(A_2)$ are determined as shown in Figure 2(b).

Alternatively, these coefficients can be determined by the values of the initiation fracture toughness $J_{ic}$ and the corresponding tearing modulus $T_R$. Note that the current ASTM E 1737 specifies detailed procedures for the determination of $J_{ic}$ ( $J$ at $\Delta a = \Delta a_0 = \frac{J_{ic}}{2\sigma_f} + 0.2$ mm) and

$$T_R = \frac{E}{\sigma_\infty} \frac{dJ}{da}$$

at $\Delta a = 1$ mm. The fracture toughness at $\Delta a = 0$ is often close to zero, as assumed in (21). The constraint modified $J$-$R$ curve, equation (22), has therefore only two coefficients $C_1(A_2)$ and $C_2(A_2)$ for each curve or $A_2$, which can then be determined by $J_{ic}$ and $T_R$. Using
(22) and (25) with SI units, one obtains the following two equations to determine the unknown coefficients $C_1$ and $C_2$:

$$
C_1(A_2) \left( \frac{J_{IC}}{2k\sigma_F} + 0.2 \right)^{C_1(A_2)} = J_{IC}(A_2)
$$

$$
C_1(A_2)C_2(A_2) = \frac{\partial J(\Delta a, A_2)}{\partial a} \bigg|_{\Delta a=1\text{mm}} = T_R(A_2) \frac{\sigma_0^2}{E}
$$

(26)

Note that both the initiation fracture toughness $J_{IC}$ and the tearing modulus $T_R = \frac{E}{\sigma_0^2} \frac{dJ}{da}$ are now functions of the constraint parameter $A_2$. Using the J-R curves obtained for several specimens of various constraint levels, the functional relationship on the right hand sides of (26) can be established as shown in Figures 3(a) and 3(b). For a given $A_2$ value (e.g. $A_2$ from 0 to –1.0), solving equation (26) gives the solutions for $C_1$ and $C_2$. Consequently, the functional forms of $C_1(A_2)$ and $C_2(A_2)$ are obtained by curve fitting as shown in Figure 3(c).

Once the constraint modified J-R curve, equation (22), is completely determined for a material of interest, the J-R curves appropriate for any test specimen or for any structural component can then be obtained, provided that the value of the constraint parameter $A_2$ is known.

*Experimental results of Joyce and Link (1997)*

Following ASTM E 1737, Joyce and Link [3] tested a series of SENB specimens with $a/W$ ratios ranging from 0.13 to 0.83 for the HY80 steel. The material properties\(^1\) are: 0.2% yield

\(^1\) Joyce, J. A., Private communication, 1998.
strength $\sigma_0 = 610\ MPa$, ultimate strength $\sigma_{us} = 726\ MPa$, Young’s modulus $E = 199\ GPa$, Poisson ratio $\nu = 0.29$, and the strain hardening exponent $n = 10$.

The initiation fracture toughness $J_{IC}$ and the material tearing resistance $T_R$ at $\Delta a = 1\ mm$, obtained with ASTM E 1737 are presented by Joyce and Link [3] for all specimens and reproduced in Table 1. Because there is a simple relation between the Q-stress [18] and the $A_2$ parameter, the $A_2$ parameter in the current analysis was obtained by converting the Q-stress reported by Joyce and Link [3].

Comparing (3) to the definition of $Q$, that is, $Q\sigma_0 = \sigma_{00} - \sigma_{00}^{HRR}$ at $\theta = 0$ and $r = 2J/\sigma_0$, the relationship between $Q$ and $A_2$ is

$$A_2 \left( \frac{2J}{\sigma_0 L} \right)^{\gamma_2} \bar{\sigma}_{00}^{(2)}(0) + A_2 \left( \frac{2J}{\sigma_0 L} \right)^{\gamma_2} \bar{\sigma}_{00}^{(2)}(0) = Q \left( \frac{J}{\alpha\sigma_0 I_n L} \right)^{\gamma_6}$$  \hspace{1cm} (27a)

For strain hardening exponent $n = 10$, the following constants were determined: $s_1 = -0.09091$, $s_2 = 0.06977$, $s_3 = 0.23044$, $I_n = 4.53985$, $\bar{\sigma}_{00}^{(2)} = 0.3130$, and $\bar{\sigma}_{00}^{(3)}(0) = -6.4127$. Letting $\alpha = 1$ and $L = 10\ mm$, (27a) becomes

$$0.313 \left( \frac{J}{3050} \right)^{0.06977} A_2 - 6.4127 \left( \frac{J}{3050} \right)^{0.23044} A_2^2 = \left( \frac{84.882}{J} \right)^{0.09091} Q$$  \hspace{1cm} (27b)

Using (27b) and the Q-stresses which were reported by Joyce and Link [3], the $A_2$ values were determined and are listed in Table 1.
**Prediction of J-R curves using the J-A$_2$ description**

With the data given in Table 1 and following the procedures described earlier, a constraint-modified J-R curve can be constructed. Figures 4(a) and 4(b) show plots of $J_{IC}$ and $T_R$ versus $A_2$, respectively. Note that the relationship of $J_{IC}$ and $A_2$ can be fitted linearly by

$$J_{IC} = -119.79 \ A_2 + 161.86 \ (kJ/m^2) \quad (28)$$

From Figure 4(a), it can be seen that $J_{IC}$ may be approximated by

$$J_{IC} = 194 \ (kJ/m^2) \quad (29)$$

In Figure 4(b), the relationship of $T_R$ and $A_2$ is fitted by a straight line

$$T_R = -187.33 \ A_2 + 36.425 \quad (30)$$

or by a quadratic equation

$$T_R = -164.77 \ A_2^2 - 277.38 \ A_2 + 25.717 \quad (31)$$

Figure 4(b) shows that (30) and (31) are almost indistinguishable. Therefore, the linear equation (30) is used in this analysis.

Based on (30) and the definition of material tearing resistance $T_R = \frac{E}{\sigma_0^2} \ \frac{\partial J}{\partial a}|_{\Delta a=1mm}$, substitution of material properties ($E$ and $\sigma_0$) yields the slope of the J-R curve at $\Delta a = 1 \ mm$:

$$\left. \frac{\partial J}{\partial a} \right|_{\Delta a=1mm} = -350.31 A_2 + 68.109 \ (N/mm^2) \quad (32)$$

**J-R curves with $J_{IC}$ independent of constraint** – The material flow stress of HY80 is 668 MPa and is defined as $\sigma_F = \frac{1}{2} (\sigma_0 + \sigma_u)$. Substituting (29) and (32) into (26) gives

$$\begin{cases} C_1 (0.3452)^{C_1} = 194 \\ C_1 C_2 = -350.31 A_2 + 68.109 \end{cases} \quad (33)$$
For a specific value of $A_2$, the coefficients $C_1$ and $C_2$ can be solved from (33) with a non-linear Newton iteration method. Within the range $-1.0 \leq A_2 \leq 0$, the values of $C_1$ and $C_2$ are plotted in Figures 5(a) and 5(b), respectively. They are expressed as

\[
C_1(A_2) = -226.35A_2 + 264.63 \\
C_2(A_2) = -0.5813A_2 + 0.3182
\]

By substituting (34) into (22), the constraint modified $J$-$R$ curve for HY-80 steel is found to be

\[
J(\Delta a, A_2) = (-226.35A_2 + 264.63) \left( \frac{\Delta a}{1 mm} \right)^{-0.5813A_2 + 0.3182}
\]

**J-R curves with $J_{IC}$ linearly related to constraint $A_2$** - It is not uncommon that the fracture toughness $J_{IC}$ is weakly dependent of the specimen geometry. Therefore, equation (28) is used to investigate the effect of $J_{IC}$-$A_2$ relationship on the $J$-$R$ curves. Following the same procedure and by substituting (28) and (32) into (26), it can be shown that

\[
\begin{align*}
C_1(0.3212 - 0.0897A_2)C_2^- \quad & = -119.79A_2 + 161.86 \\
C_1C_2 & = -350.31A_2 + 68.109
\end{align*}
\]

Solving (36) for a given value of $A_2$, the coefficients $C_1$ and $C_2$ are obtained and are plotted in Figures 6(a) and 6(b), respectively. For $-1.0 \leq A_2 \leq 0$, the expressions for $C_1$ and $C_2$ are,

\[
C_1(A_2) = -323.15A_2 + 237.78 \\
C_2(A_2) = -0.4365A_2 + 0.3551
\]

Substituting (37) into (22), the constraint modified $J$-$R$ curve for HY-80 steel is obtained:
\[ J(\Delta a, A_2) = (-323.15A_2 + 237.78) \left( \frac{\Delta a}{1mm} \right)^{(-0.4365A_2+0.3551)} \] (38)

Figure 7 shows the J-R curves predicted by (35), solid curves, and by (38), dashed curves, for several \( A_2 \) values (0.0, -0.168, -0.2, -0.3, and -0.4). It indicates that the predicted J-R curves are insensitive to the functional forms of \( J_{IC} \). Therefore, \( J_{IC} \) can be approximately considered as a material constant and is independent of the level of constraint. This agrees with the experimental observations of Hancock et al.

Some experimental data of Joyce and Link [3] are also included in Figure 7. The predicted J-R curves (\( A_2 = -0.168 \) and -0.393) are compared with the test data of a shallow crack specimen (\( a/W=0.13 \)) and a deep crack specimen (\( a/W=0.55 \)). Note that the present analysis used only two data points from each of the specimen of Joyce and Link, namely, \( J_{IC} \) at \( \Delta a = 0.35mm \) and \( T_R \) at \( \Delta a = 1 \) mm, and yet the predicted J-R curves match very well with the experimental data up to \( \Delta a = 7mm \). Therefore, the \( J-A_2 \) description can successfully predict the J-resistance curves even considerable amount of crack growth has occurred.

Conclusions

To quantity the crack tip constraint effect on J-R curves in ductile crack growth, this paper extends the concept of J-controlled crack growth to J-\( A_2 \) controlled crack growth, in which the J-integral represents the load level and the \( A_2 \) parameter indicates the level of constraint. The main results are summarized as follows:

(1) Under a fully plastic condition, it is proved by the deformation theory of plasticity (Ilyushin [33]) that the constraint parameter \( A_2 \) is independent of the applied load. As a result,
the constraint parameter $A_2$, determined at the crack initiation ($J_{IC}$), remains invariant as long as
the growing crack tip is still within a $J-A_2$ controlled regime.

(2) A simple weight average technique is developed to evaluate the value of $A_2$ by matching
the finite element result with the three-term asymptotic solution [20-22]. The value of $A_2$
determined in this manner is almost independent of the distance from the crack tip within
$1 \leq r/(J/\sigma_o) \leq 5$, where the fracture event normally takes place.

(3) Using $A_2$ as a constraint parameter, the concept of a constraint modified $J-R$ curve is
proposed. A procedure is outlined for transferring the $J-R$ curves determined from the ASTM
standards to non-standard specimens or to structures with flaws. Once the constraint parameter
$A_2$ is determined, the $J-R$ curve for any specimen can be readily predicted.

(4) The methodology of the constraint modified $J-R$ curve has been applied to a set of SENB
test data obtained by Joyce and Link [3]. Using the initiation fracture toughness $J_{IC}$ and the
tearing modulus $T_R$, the constraint modified $J-R$ curves are constructed for the test material. The
predicted $J-R$ curves agree very well with the experimental data up to a crack extension of 7 mm.
Note that only two test data points ($J_{IC}$ and $T_R$ measured at $\Delta a < 1$ mm) were needed to
successfully construct the constraint modified $J-R$ curve. These results also indicate that (a) a
nearly linear relationship exists between the slope of the material $J$-resistance curve at 1 mm of
crack extension and the constraint parameter $A_2$; and (b) the predicted $J-R$ curves are insensitive
to the value of $J_{IC}$ which may be weakly dependent of the crack tip constraint. Therefore, it can
be concluded that the initiation fracture toughness $J_{IC}$ may be regarded as a constant and is
independent of the crack tip constraint. This was proposed earlier by Hancock et al. [1] and
Joyce and Link [3].
References


### TABLE 1 -- Fracture toughness and constraint quantities for all SENB specimens

<table>
<thead>
<tr>
<th>Specimen I.D.</th>
<th>a/W</th>
<th>a/b (mm)</th>
<th>$J_{IC}$ (KJ/m$^2$)</th>
<th>$T_R(\Delta a=1mm)$</th>
<th>Q</th>
<th>$A_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>94A</td>
<td>0.29</td>
<td>14.5/35.5</td>
<td>211.8</td>
<td>95.8</td>
<td>-0.36</td>
<td>-0.274</td>
</tr>
<tr>
<td>94B</td>
<td>0.26</td>
<td>13.0/37.0</td>
<td>225.6</td>
<td>99.1</td>
<td>-0.43</td>
<td>-0.299</td>
</tr>
<tr>
<td>94D</td>
<td>0.19</td>
<td>9.5/40.5</td>
<td>217.2</td>
<td>104.0</td>
<td>-0.60</td>
<td>-0.362</td>
</tr>
<tr>
<td>94E</td>
<td>0.39</td>
<td>19.5/30.5</td>
<td>216.0</td>
<td>77.9</td>
<td>-0.24</td>
<td>-0.217</td>
</tr>
<tr>
<td>94G</td>
<td>0.55</td>
<td>27.5/22.5</td>
<td>195.2</td>
<td>72.1</td>
<td>-0.15</td>
<td>-0.168</td>
</tr>
<tr>
<td>94H</td>
<td>0.55</td>
<td>27.5/22.5</td>
<td>169.2</td>
<td>71.1</td>
<td>-0.10</td>
<td>-0.134</td>
</tr>
<tr>
<td>94J</td>
<td>0.13</td>
<td>6.5/43.5</td>
<td>219.3</td>
<td>109.4</td>
<td>-0.70</td>
<td>-0.393</td>
</tr>
<tr>
<td>94K</td>
<td>0.14</td>
<td>7.0/43.0</td>
<td>215.1</td>
<td>117.4</td>
<td>-0.70</td>
<td>-0.394</td>
</tr>
<tr>
<td>94K</td>
<td>0.14</td>
<td>7.0/43.0</td>
<td>183.0</td>
<td>100.0</td>
<td>-0.67</td>
<td>-0.395</td>
</tr>
<tr>
<td>94J</td>
<td>0.13</td>
<td>6.5/43.5</td>
<td>196.5</td>
<td>108.7</td>
<td>-0.68</td>
<td>-0.394</td>
</tr>
<tr>
<td>FYB507</td>
<td>0.61</td>
<td>30.5/19.5</td>
<td>189.5</td>
<td>55.0</td>
<td>-0.10</td>
<td>-0.132</td>
</tr>
<tr>
<td>95H</td>
<td>0.83</td>
<td>41.5/8.5</td>
<td>162.9</td>
<td>73.7</td>
<td>-0.25</td>
<td>-0.232</td>
</tr>
<tr>
<td>95G</td>
<td>0.78</td>
<td>49.0/11.0</td>
<td>145.6</td>
<td>78.7</td>
<td>-0.22</td>
<td>-0.220</td>
</tr>
<tr>
<td>95X</td>
<td>0.70</td>
<td>45.0/15.0</td>
<td>172.6</td>
<td>56.1</td>
<td>-0.15</td>
<td>-0.171</td>
</tr>
</tbody>
</table>

Note: the specimen length $l = 203mm$, $l/W = 4$, $B/W = 0.5$; $W = 50mm$, $B = 25mm$. Side groove 20%.
Figure Captions

FIG. 1 – Schematic Regimes for J and J-A2 Controlled Crack Growth.

FIG. 2-- Analysis procedure for constructing the constraint-modified J-R curves, equation (24).

(a) Experimental J-R curves from various specimens
(b) Determination of the functional relationship \( C_0(A_2), C_1(A_2), C_2(A_2) \)

FIG. 3 -- Analysis procedure for constructing the constraint-modified J-R curves, equation (26).

(a) Determination of the functional relationship between \( J_{IC} \) and \( A_2 \)
(b) Determination of the functional relationship between \( T_R \) and \( A_2 \)
(c) Determination of the functional relationship \( C_1(A_2), C_2(A_2) \)

FIG. 4 -- Experimental data and fitted curves for SENB specimens. Data points are from Joyce and Link (1997), and the lines are the best-fit curves.

(a) Initiation toughness \( J_{IC} \) versus \( A_2 \)
(b) Tearing toughness \( T_R \) versus \( A_2 \)

FIG. 5 -- Fitted curves for parameters in constraint-modified J-R curves with \( J_{IC} = 194 \text{ KJ/m}^2 \)

(the dots are calculated from equation (33); the lines are the best-fit curves).

(a) Variation of \( C_1 \) versus \( A_2 \); (b) Variation of \( C_2 \) versus \( A_2 \)

FIG. 6 -- Fitted curves for parameters in constraint-modified J-R curves with

\[
J_{IC} = -119.79 A_2 + 161.86 \text{ (KJ/m}^2 \text{)}
\]

(the dots are calculated from equation (36); the lines are the best-fit curves).

(a) Variation of \( C_1 \) versus \( A_2 \); (b) Variation of \( C_2 \) versus \( A_2 \)

FIG. 7 -- Comparisons of predicted J-R curves (Equations 35 and 38) and SENB J-R curves of Joyce and Link (1997).
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