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1 Introduction

The research conducted in this project concerns the geometry of extremal surfaces, embedded minimal surfaces in particular. The methods include geometric analysis, computational simulation, mathematical visualization and software development. This research is closely tied to applications, specifically to the understanding of the geometry of compound polymers and the determination and prediction of intermaterial dividing surfaces.

2 Natural equilibrium interfaces

Minimal surface research stands at the intersection of partial differential equations, calculus of variations, complex function theory and topology. Advances in this area are often—as is the case with our research—tied to the development and implementation of computational methods and tools of mathematical visualization. Understanding the structure of the space of minimal surfaces has been important in applications from cosmology to structural engineering, as well as other applied areas including polymer physics.

One of the basic goals of polymer physics is to predict how macromolecules assemble themselves. Our interest is in systems composed of di- and triblock copolymers. Understanding the achievable structures is equivalent to understanding the “intermaterial dividing surfaces” (IMDS) formed when such systems are in stable equilibrium. These structures will be (in their ideal form) periodic; since the material is made up of bound strings of monomers of different substances, they cannot segregate. Because the differing materials are naturally repelling, there is a natural tendency to reduce, as much as possible, the area of the interface between the differing substances. This naturally leads to the mathematical problem of area minimization subject to volume constraint. The constraint comes from the “volume fraction” of the composite materials. Solutions to this problem are surfaces of *constant mean curvature*. When there is an equal amount of two substances (50-50 volume fraction), one expects no

volume constraint and the formation of surfaces with zero mean curvature: ***minimal surfaces***. Since these minimal surfaces are the boundary of solid regions of space, they are, of course, free of self-intersections. Such surfaces are sometimes referred to as “*natural*.” In mathematical parlance, they are called “***embedded***” surfaces.

The study of embedded minimal surfaces, either singly, doubly, or triply periodic, is one that requires abstract methods of analysis, usually global in nature, together with computational techniques that allow the testing of conjectures. The subject has benefited from the discovery of new examples by the use of computation, examples far beyond the range current theoretical construction techniques. Not only are these surfaces important for the understanding of equilibrium morphology via inter-material dividing surfaces, they arise in the study of grain boundaries and dislocations. These same examples are in turn signposts for the further theoretical development in mathematics.

This research project has made fundamental advances in the study of equilibrium interfaces. Carrying on the parent project that was based at the University of Massachusetts, we have

- *Proved the existence of large families of periodic minimal surfaces that serve as models for compound polymers.*
- *Developed software to simulate the transmission electron microscopy of the nanostructure of block copolymers. The laboratory research was done in collaboration with scientists at UMASS and MIT . Our software acts as a virtual transmission electron microscope (TEM). It allows the user to select different geometries as models of the IMDS and view projections of them using simulated microscope controls and other parameters. The user can search through different model libraries and parameter spaces to find projections that match experimental TEM images.*
- *Pioneered the use of numerical approximation and image simulation for minimal and CMC surfaces in the theoretical investigation of these variationally define equilibrium interfaces*
- *Developed and maintained an archival site and model libraries, SGP (for Scientific Graphics Project), reachable at www.msri.org for the description of embedded minimal and CMC surfaces. This website was one of the first such sites and has served as a model for others*
- *Ported to Java for some fifty minimal surfaces a display and manipulation version of the MESH software package developed under this package. The original package is written in C++ is also available. This package allows the skilled*

expert to program and explore new examples. The Java version, JMESH, has precomputed surfaces and can be used on Windows or Linux platforms.

Some representative publications are listed here. (Note: James Hoffman (no relation) was the scientific graphics programmer on the project.)

- “A new turn for Archimedes,” NATURE, Nov. 8, 1996 28-29
- “Ordered Bicontinuous Nanoporous and Nanorelief Ceramic Films from Self Assembling Polymer Precursors” SCIENCE, November 26, 1999 (Vanessa Z.-H. Chan, James Hoffman, Victor Y. Lee, Hermis Latrou, Apostolos Avgeropoulos, Nikos Hadjichristidis, Robert D. Miller, and Edwin L. Thomas)
- “Triply Periodic Bicontinuous Cubic Microdomain Morphologies by Symmetries,” in Macromolecules (2004) (Meinhard Wohlgenuth, Nataliya Yufa, James Hoffman, and Edwin L. Thomas)
- “A family of genus-one, screw-motion invariant minimal surface with helicoidal ends,” Experimental Mathematics, (December, 2002) Experimental Mathematics (with F. Wei).
- “Constant mean curvature surfaces in $M^2 \times \mathbf{R}$,” Trans. Amer. Math. Soc. 358 (2006), 491-507.(with Jorge H. S. de Lira and Harold Rosenberg)

Embedded minimal surfaces—surfaces that minimize area locally and are part of the boundary of a solid region in space—have been studied for over two hundred years not only because of their physical significance but also because they lie at the intersection of so many important areas of mathematics and applied science: nonlinear elliptic PDE; complex function theory; differential geometry; differential topology of three manifolds; surface modelling for manufacture and computer graphics; complex manifolds and algebraic geometry; inter-material dividing surfaces; scientific visualization). The theory of minimal surfaces is often the first place where new general phenomena are discovered and understood.

We have proved the existence of an embedded minimal surface of genus one with one helicoidal end. This is the first properly embedded minimal surface with infinite total curvature and finite topology to be found since the helicoid was shown to be minimal by Muesnier in the 1770’s. The significance of this advance is reflected in the following publications and papers.

- *The Global Theory of Minimal Surfaces, Proceedings of the Clay Mathematics Institute 2001 Summer School at the Mathematical Sciences Research Institute, Berkeley California*, David Hoffman, editor. Clay Mathematics Proceedings, (2005) Vol.2, 800 pages.

- “An embedded genus-one helicoid,” submitted for publication to the *Annals of Mathematics*. 112 pages.
- “An embedded genus-one helicoid,” *Proceedings of the National Academy of Sciences*, (November 15, 2005), vol. 102, no.46, 16566-16568 (with M.Weber and M.Wolf).
- “Surface story” *Science News*, (December 17, 2005) Vol. 168, 25, 393-394 (by Ivars Peterson).

(The last two items are appended to this report.)

It is a landmark result in the subject, achieved by a combination of methodologies that are themselves of interest. The surface was proved to exist by showing it to be the limit of a family of screw-motion-invariant embedded minimal surfaces. That such a technique could possibly work was established by careful computational experiments using MESH, software developed by this project, as well as ideas based on previous theoretical results. To make it mathematically rigorous required creation of a theory of “flat structures” and a careful analysis of limiting behavior of the associated “cone metrics.” This work has opened up the subject. We hope to be able to show that any such example must have a symmetry and that higher-genus examples exist (known at this time computationally up to genus four).

In the last year of the project, Hoffman has been working with Brian White of Stanford on an approach to this subject that uses degree theory and the minmax principle instead of the Weierstrass representation. Based upon geometric information that could be gleaned from the simulations, we have developed a strikingly simple approach. This is now available in manuscript form (copy on request) and will be submitted in the next months. The advantages of this approach are that it well suited for generalization to higher genera (although there is now a developing theory of such surfaces, there is only one example—not counting the helicoid discovered 230 years ago!—and that example has genus one), that it gives qualitative and quantitative information not available from the previous methods—information that confirms what we observed in the computer simulations—and that it explains in some fundamental sense why these surfaces should exist in the first place.

3 The space of minimal surfaces at the boundary: the limit as $k \rightarrow 1/2$

We include this section of the final report as an example of the many unexpected offshoots of the project. In the investigation of $\mathcal{H}e_1$, it was discovered that, numerically, this surface could be approximated by a sequence of screw-motion-invariant

surfaces, $\mathcal{H}_k$, which are asymptotic to the helicoid outside of a vertical cylinder. In the construction of the family $\mathcal{H}_k$, there is no reason not to let the twist angle $2\pi k$ decrease below 2π . Computationally, there are solutions for $k < 1$ and there seemed to be no theoretical barrier to letting k approach zero. The expectation was that the handles would accumulate around the vertical axis. That is, as $k \rightarrow 0$, the handles would shrink in the limit to a dense sequence of points on the z -axis, while away from that axis, the family would smoothly converge to a helicoid.

However, there seemed to be a barrier at $k = 1/2$. The computation failed for $k \leq 1/2$ and behaved erratically for values of k very near to but larger than $1/2$. This was a mystery until recently.

The method of proving that the screw-motion-invariant surfaces exist involves a construction that naturally has a barrier at $k = 1/2$. For $k > 1$, the construction requires one to "sew a cone of angle $2\pi(k - 1)$ " into the plane minus the real $[-1, 1]$, with its vertex on the positive imaginary axis. For $k < 1$, it is required to remove a cone with angle $2\pi(k - 1)$. However, when $k \leq 1/2$, removing a the cone angle is impossible because of the missing interval $[-1, 1]$.

With this insight, we have been able to see that there is in fact a natural barrier here. Using an adapted theta-function representation and MESH, an adapted grid generation program developed by our project, ¹ Matthias Weber (now a Professor of Mathematics at the University of Indiana), who was a research scientist working with the project, and M. Traizet, who visited our group at MSRI in September, 2000, were able to compute very close to the putative limit of $k = 1/2$. Traizet had developed a way of constructing families of minimal surfaces by looking at degenerate examples, considered as limits of these imagined families, and using implicit-function-theoretic arguments to "open nodes." That is, he was able to perturb away from the degenerate limit to nearby, nondegenerate examples. The hope was that the same thing could be done in this case. The idea is, roughly, that the nodes open to catenoids.

However, the image of the surface close to the limiting value of $k = 1/2$ did not look like small degenerating catenoids. As inconceivable as it may be, what seems to be happening is that there is a opening of nodes to *helicoids*.² This visual information is counter to intuition. However, seeing this in the computed image has forced the realization that one can, in fact, open a node to be a helicoid. In fact, not only does this procedure seem to work for the family in question, it can be generalized to allow the construction of several classical families of surfaces, including some known only by computation. These include: the Hoffman-Karcher singly periodic versions of

¹(www.msri.org/publications/sgp/jim/software/mesh/indexc.html)

²The computation and the underlying theoretical reasoning is described in the paper "A family of genus-one, screw-motion invariant minimal surface with helicoidal ends," Experimental Mathematics, December 2002 Experimental Mathematics (with F. Wei).

the Fischer-Koch crystallographic minimal surfaces and the Scherk Towers of higher genus. Perhaps more important, they can be used to give a construction of screw-motion-invariant examples of arbitrary genus. Of course, the existence proof only works near the limit value. But this is the first such general construction principle to be found for examples of this complexity. This has appeared in

- M. Traizet and M. Weber, “Hermite polynomials and helicoidal minimal surfaces,” *Invent. Math.*, Vol 161, 2005 113–149

It is a subject of current investigation to extend the existence of these families to get to, say, $k = 1$, which would correspond to translation-invariant examples. While, at first glance, it may seem to be straightforward to be able to do this, the very existence of the singular barrier at $k = 1/2$ in the breakthrough case does indicate that families can terminate in unexpected ways.

Sewing together half-helicoids (“spiral staircases”) to form “parking garages” fits extremely well with the theory of limit laminations developed by Colding (MIT) and Minicozzi (Johns Hopkins). It is I think fair to say that the theoretical conception and computational simulation of this surface family was key to this development. It was not possible to interpret the output of the computation until other theoretical advances were made, and it was also impossible to make use of these advances without a detailed knowledge of the manner in which these surfaces were first approximated.

4 Scientific organization and structural support for collaborative research

During the period that this project was carried out at MSRI, we provided support and consultation for mathematicians who had interests in using our graphical software and expertise in their research. MSRI has upwards of 1200 visitors per year staying for periods of a few days to 9 months. In addition, Hoffman organized, initiated or acted as MSRI coordinator for a variety of scientific meetings and workshops on topics ranging from visualization of polymer structures and the mathematics of CAD, to mathematical communication. The following is a partial list of such meetings and workshops related to this project that were organized and or initiated by the PI at MSRI. A record of most or all of them can be found on the MSRI website: www.msri.org

- The Future of Mathematical Communication, 1999

This was a follow-up meeting to the December, 1994 conclave of the same name, of which Hoffman was the principal organizer. (This meeting preceded

the installation of the project at MSRI by several months. At this meeting, the following new developments were presented to the mathematical community for the first time: Oak and WebRunner, the precursor to Java; the Multicast Backbone (MBONE) precursor to streaming video; Adobe's early version of PDF. In addition (I cannot say this was a first for the mathematical community but it may have been) the LANL X-server for physics were described, and the first steps were taken to develop a similar server for mathematics.

The 1999 meeting took place in a world where these early possibilities had become commonplace realities for mathematical scientists. One of the foci was a hands-on workshop on the use (both for consumers and producers) of the streaming video system that has been a model for several other centers. It has over 4000 hours of advanced lectures on demand, together with accompanying notes and graphics. This system was conceived by Hoffman and implemented by a team under his leadership.

- Self-Assembling Geometric Structures in Material Science: The Geometry of Interfaces in Mesoscopic Materials, 1999 Organized with Andrew Canning of (NERSC) Michel A. Van Hove (Materials Sciences Division LBNL), and Edwin Thomas (Materials Science and Engineering, MIT)
- Mathematical Foundations of Computer Aided Design AD, 1999. Organized by: Hyeong In Choi, Herbert Edelsbrunner, Rida Farouki, David Ferguson, David Hoffman, Helmut Pottmann
- Analysis Models and Methods: A conference on Applied Mathematics and Computational Methods in memory of Fred Howes. Organized by: A. Chorin, I. Singer and M. Wright. Hoffman was involved as the MSRI coordinator, and as a member of the initial committee that conceived the meeting.
- Clay Mathematics Institute Summer School on the Global Theory of Minimal Surfaces June 25,2001 to July 27,2001. Hoffman was the principal organizer of this month-long comprehensive research meeting, the proceedings of which were cited earlier in this document. It was the largest and most extensive meeting on this subject in its 250-year history.
- Genetics of Complex Disease Organized by: Jun Liu, Mary Sara McPeck, Richard Olshen (chair), David O. Siegmund, and Wing Wong. Hoffman originated the idea for this meeting, which focussed on the use of applied mathematical techniques to deal with the problem of extracting information from very large data sets, for example those that come from microchip arrays, in the medical sciences.

4.1 Visitors and Postdoctoral students

During the period of the project there were research visitors Michael Wolf (Rice), William Minicozzi (Johns Hopkins), Mohamed Ghomi (Johns Hopkins and UC Santa Cruz), William Meeks III (U. Massachusetts), Henry Wente (U. Toledo), Karsten Grosse Brauckmann (Bonn), Len Radzilowski (MIT and Ratheon), Helen Moore (Stanford), Matthias Weber (Bonn), Marc Soret (Tours, Johns Hopkins), Edwin Thomas (MIT), Martin Traizet (Tours), Dick Palais (Brandeis), Konrad Polthier (Berlin), Alexander Bobenko (Berlin), Hermann Karcher (Bonn), Harold Rosenberg (U. Paris, VII), E.L. Thomas (MIT), Joel Hass (UC Davis), E. Thayer (private industry), Wayne Rossman (Kobe)

Postdoctoral and research associates included Peter Topping (Cornell, U. Warwick), John McCuan (now at Georgia Tech.), Fusheng Wei (Virginia Tech, now a private consultant), Michael Callahan (Oxford, now founder and CFO of Polyserve, Inc.), Matthias Weber (Bonn, Indiana), Pengzi Miao (Stanford, UC Santa Barbara)

An embedded genus-one helicoid

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There exists a properly embedded minimal surface of genus one with a single end asymptotic to the end of the helicoid. This genus-one helicoid is constructed as the limit of a continuous one-parameter family of screw-motion invariant minimal surfaces, also asymptotic to the helicoid, that have genus equal to one in the quotient.

cone metrics | global theory of minimal surfaces | flat structures

Complete Embedded Minimal Surfaces of Finite Topology and Infinite Total Curvature

We prove the existence of a properly embedded minimal surface in $\mathbf{R}^3$ with finite topology and infinite total curvature.[¶] In refs. 1 and 2, an immersed surface with these properties was proved to exist, and strong mathematical and graphical evidence was given to show that the constructed surface was embedded (i.e., free of self intersections). However, a proof of embeddedness was not found, even though this earlier existence result has a considerable influence of the subsequent development of the global theory of minimal surfaces. We give a full theoretical proof of the existence of an embedded example with finite topology and infinite total curvature. Such a surface has not been reported since 1776, when Meusnier (3) showed that the helicoid was a minimal surface. Our surface has genus one and is asymptotic to the helicoid.

We exhibit this minimal surface as a geometric limit of periodic embedded minimal surfaces. The periodic surfaces, $\mathcal{H}_k$, indexed by a real number $k \geq 1$, are invariant under a cyclic group of screw motions generated by σ_k : rotation by $2\pi k$ about the vertical axis, followed by a vertical translation by $2\pi k$. Thus, for fixed k , the quotient surface has two topological ends and genus one. The limit is taken as $k \rightarrow \infty$; a compact set in the limit surface $\mathcal{H}_1$ is arbitrarily well-approximated (for k sufficiently large) by corresponding pieces of fundamental domains of $\mathcal{H}_k/\sigma_k$ (see Fig. 1).

The requirement to prove embeddedness was the main motivation of our work. We prove that embeddedness is inherited from the embeddedness of the approximating simpler (periodic) surfaces, using that in this particular minimal surface setting, the condition of being embedded is both open and closed on families. This method of proving embeddedness for surfaces defined using the Weierstrass representation contrasts with previous methods: here, the characteristic of being embedded follows naturally from the property holding for simpler surfaces, whereas previously one proved embeddedness by ad hoc methods, for instance by cutting the surface into graphs.

A second important feature is that we approximate a surface of finite topology (and finite symmetry group) by surfaces of infinite topology (and infinite symmetry group). We are unaware of this method being previously applied to produce a nonclassical example.

The third feature is that we construct the Weierstrass data of these approximating minimal surfaces in terms of flat singular structures on the tori corresponding to the quotients. The salient feature to note is that the defining flat structures have singularities corresponding to the two ends^{||} with cone angles of $\pm 2\pi k$. Thus, as the size of the twist tends to infinity, the cone angles also tend to infinity, with the limit surface, our genus-one helicoid, represented in terms of flat cone metrics with an infinite cone

angle. The flat geometry of this end corresponds to the Weierstrass data for a helicoid, whose Gauss map has an essential singularity at the end. Establishing this correspondence required the development of a theory of singular flat structures that admits infinite cone angles.

Recently, Meeks and Rosenberg (4) showed that the helicoid is the unique simply connected, properly embedded (nonplanar) minimal surface $\mathbf{R}^3$ with one end. The method of proof uses in an essential manner the work of Colding and Minicozzi (5–8) concerning curvature estimates for embedded minimal disks and geometric limits of those disks. [Colding and Minicozzi recently showed that a complete and embedded minimal surface with finite topology in $\mathbf{R}^3$ must be proper (9).] These results together with the work we describe here (M. Weber, D.H., and M. Wolf, unpublished results) and numerical work of Traizet (M. Traizet, unpublished data) and Bobenko (10) suggest that there may be a substantial theory of complete embedded minimal surfaces with one end, infinite total curvature and finite topology (see also ref. 11). For complete embedded surfaces of finite total curvature, the theory is surveyed in ref. 1.

$\mathcal{H}_1$, an Embedded Genus-One Helicoid

In 1993, Hoffman, Karcher, and Wei (12) constructed a surface, $\mathcal{H}_1 \subset \mathbf{R}^3$, which they called the genus-one helicoid.

It has the following properties:

- (i) $\mathcal{H}_1$ is a properly immersed minimal surface;
- (ii) $\mathcal{H}_1$ has genus one and one end asymptotic to the helicoid;
- (iii) $\mathcal{H}_1$ contains a single vertical line (the axis) and a single horizontal line.

Condition 1

Note that *Condition 1.ii* implies that any $\mathcal{H}_1$ has finite topology and infinite total curvature.**

We will refer to any surface with the properties in *Condition 1* as a genus-one helicoid and denote such a surface by $\mathcal{H}_1$. It is known that any $\mathcal{H}_1$ must be embedded outside of a compact set (12, 13). However, as mentioned above, even though computer simulations and computational estimates strongly suggested that this $\mathcal{H}_1$ was embedded, an independent proof has been elusive.

We prove the following.

Theorem 1. *There exists an embedded $\mathcal{H}_1$.*

We believe that the surface we have found is the same one constructed by Hoffman, Karcher, and Wei (12). In fact, we believe

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[¶]A surface is said to have “finite topology” if it is homeomorphic to a compact surface with a finite number of points removed.

^{||}There is an additional cone point (with cone angle 6π) in these structures at a vertical point.

**The helicoid, a surface swept out by a horizontal line rotating at a constant rate as it moves up a vertical axis at a constant rate, is clearly properly embedded and has finite topology (in fact it is simply connected). Because it is singly periodic and evidently not flat, it has infinite total curvature. Any periodic surface asymptotic to the helicoid must also have infinite total curvature.

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Fig. 3. The dh structure. The vertical period condition is satisfied when V_1 and V_2 lie in a single vertical line. The images in *Right* and *Center* depict the dh structure for k fixed, but d near zero (*Center*) and d very large (*Right*). Notice that in *Center*, V_1 lies to left of V_2 , while in *Right*, V_1 lies to the right of V_2 . The argument that for each fixed k there exists a $d = d(k)$ satisfying the vertical period condition is by the intermediate-value theorem, because as d increases, the point V_1 must pass from being on the left of V_2 to being on the right of V_2 .

parameter $d \in (0, \infty)$ for which the vertical period problem is also solved; here this solution value d should depend continuously on the screw-motion parameter k .

To find such a family of $d = d(k)$, we show that the period of the height function over a given cycle naturally defines a real analytic function, $h = h(k, d)$, and the desired $d(k)$ would then satisfy $h(k, d(k)) = 0$. If $h = 0$ for some (k, d) , then the corresponding (k, d) -flat geometric structure defines a properly immersed minimal surface $\mathcal{H}_k$ satisfying *Condition 3*. We show that the zero set of h contains a piecewise-smooth curve C that begins at the point $(1, d_1)$ corresponding to $\mathcal{H}_1$, and crosses each vertical line $\{k = \text{const.}\}$ in the (k, d) rectangle. Each point, (k, d) , on this curve defines a properly immersed minimal surface satisfying the *Condition 3*.

To find this curve C , we first compactify the “moduli space” of (k, d) -structures, adding degenerate structures for the loci $k = \infty$, $d = 0$, and $d = \infty$ in a manner compatible with the topology of $[1, \infty] \times [0, \infty]$. Then, we show that the height function h is continuous on the full compact rectangle $[1, \infty] \times [0, \infty]$ (i.e., h extends continuously to $d = 0, \infty$ and $k = \infty$). This method of compactifying has the crucial advantage that the signs of h on the degenerate surfaces $(k, 0)$ and (k, ∞) are evident and opposite, and so the intermediate-value theorem provides for a solution $(k_0, d) \in C$ for each choice of $k_0 \in (1, \infty)$. More precisely, there must be a curve C on which $h = 0$ because this curve separates the neighborhoods of the boundary components $\{d = 0\}$ and $\{d = \infty\}$ on which h has opposite signs. Finally, a maximum-principle argument then shows that the embeddedness of $\mathcal{H}_1$ implies that all of the $\mathcal{H}_k$, $k > 1$, on this curve C are embedded.

For $k = \infty$, the (k, d) -structures are defined by sewing in a cone with an infinite cone angle (see below.) Thus, any structure (∞, d) corresponds to a potential $\mathcal{H}_{e_1}$, which will exist provided $h(\infty, d) = 0$; the endpoint of C on the locus $\{k = \infty\}$ is then such a point. Thus, we obtain a limit flat structure that defines an $\mathcal{H}_{e_1}$, and we argue that, as a limit of a family $\{\mathcal{H}_k\}$ of embedded surfaces, the surface $\mathcal{H}_{e_1}$ is also embedded.

Existence and Uniqueness of Cone Metrics with Infinite Cone Angles

The approximating minimal surfaces $\mathcal{H}_k/\sigma_k$ each have meromorphic Weierstrass data, whereas the limit surface $\mathcal{H}_{e_1}$ has Weierstrass data with an essential singularity at its end. This phenomenon of meromorphic data limiting on transcendental data corresponds on the level of the developed flat structures to taking limits of cone metrics whose cone angles grow without bound: the limit cone metric has a cone point with infinite cone angle. To accommodate the taking of limits of metrics with cone points of arbitrarily large cone angles, we introduce the definition of a cone point of simple exponential type and the idea of asymptotic isometry. In particular, an exponential cone of simple type is a cone isometric to the disk D with the metric

$$\left| e^{1/w} \frac{dw}{w^2} \right|$$

(the origin is the cone point), and two such cones with representations

$$\left| e^{1/w} \frac{dw}{w^2} \right| \quad \text{and} \quad \left| e^{1/z} \frac{dz}{z^2} \right|$$

are asymptotically isometric provided $\frac{dw}{dz}(0) = 1$.

We show that cone metrics are essentially determined by their cone points, with a natural Gauss–Bonnet-type restriction being the only obstruction to existence. These results extend the work of Troyanov (18) on cone metrics with positive and finite cone angles to the cases where the cone angles may be negative, zero, positive, or of simple exponential type.

We thank Hermann Karcher, Harold Rosenberg, Bill Meeks, and Fusheng Wei for useful conversations over the past years. Wei’s ingenious ideas about how to represent the family $\mathcal{H}_k$ in a computationally tractable manner was key to establishing the belief that the family not only existed but also converged to a genus-one helicoid. The idea that embeddedness was inherited from $\mathcal{H}_1$ by the family $\mathcal{H}_k$ came from discussions that one of us had with Meeks about the same phenomena for finite-total-curvature surfaces. Rosenberg’s idea to sew a helicoid into a genus-one surface to create a genus-one helicoid, conveyed to Karcher in a conversation reported in ref. 12, was pivotal to the discovery of $\mathcal{H}_{e_1}$. Karcher’s deep understanding of the Weierstrass data of the surfaces $\mathcal{H}_1$ and $\mathcal{H}_{e_1}$ was of crucial help at numerous instances, and his intelligence and spirit continue to be fundamental to our work. This work was partly supported by National Science Foundation Grants DMS-0139476 (to M. Weber), DMS-9971563 (to M. Wolf), and DMS-0139887 (to M. Wolf). D.H. was partly supported by U.S. Department of Energy Office of Energy Research Applied Mathematical Science Subprogram Research Grant DE-FG03-95ER25250/A007 and National Science Foundation, Division of Mathematical Sciences Research Grant DMS-0139410.

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Surface Story

Inspired by spiral soap films, mathematicians zero in on a novel, economical, and infinite helix

Ivars Peterson

Dip a flat wire ring into a basin of soapy water. The ring comes out spanned by a taut, iridescent soap film in the form of a thin disk. Its area is smaller than it would be if the surface had peaks and valleys, or even small wrinkles. A clinging soap film invariably settles into the shape that mathematicians call a minimal surface. They can also imagine minimal surfaces that don't exist in nature.

SLIPPERY SLOPE. A spiral slide, as in this photo of a soap film, makes up the central core of half of an infinite minimal surface called a helicoid. The computer-generated image (right) shows a central portion of a helicoid. The distinct red and blue surfaces demonstrate that this is a double spiral. The sheets extend laterally to infinity.

J. Oprea; Weber

Consider a perfectly flat disk of soap film, for example, that extends so far over the horizon that its boundary can't be seen. This two-dimensional plane is the simplest example of a minimal surface that is infinite in extent and not an endless repetition of some basic shape.

Mathematicians can also imagine twisting that plane to produce another infinite shape called the helicoid. Also a minimal surface, it looks like a pair of intertwined spiral slides—a double helix—at its core and it stretches away to infinity as a stack of sheets.

For centuries, the plane and the helicoid were the only known examples of infinite, unbounded minimal surfaces that don't fold back to intersect themselves. Then, in the early 1990s, mathematicians discovered a new minimal surface that seemed to have the same basic properties that the helicoid has, but with a crucial difference: Through one of its sheets, it has a tunnel (SN: 10/24/92, p. 276). Topologists refer to such an opening as a handle, in deference to a favorite everyday shape, a coffee mug.

Although computer images and other evidence strongly suggested that the new surface met the criteria for placing it, alongside the helicoid and the plane, in the minimal-surface hall

of fame, that wasn't enough for mathematicians. "No matter how good the computer approximations are, you can never be sure just by computer," says Frank Morgan of Williams College in Williamstown, Mass.

So, mathematicians sought an airtight proof that the infinite surface doesn't somehow, somewhere, twist around enough to intersect itself.

Matthias Weber of Indiana University in Bloomington, David Hoffman of Stanford University, and Michael Wolf of Rice University in Houston now offer a proof that nearly settles the question. Their work, published in the Nov. 15 *Proceedings of the National Academy of Sciences*, establishes that a particular shape—a helicoid with a handle—doesn't intersect itself. The only remaining subtlety is to show that this surface and the helicoid with a handle discovered in the 1990s are one and the same.

Disk covery

At every point, a minimal surface is either flat, like a disk, or has a saddle shape. In the latter case, its curvature resembles that of a potato chip, which typically starts out as a flat, thin slice of moist potato. As a chip loses water during frying, it shrinks. Minimizing its area, it curls into a saddle shape.

Twisting the ordinary two-dimensional plane into a helicoid converts the plane's flatness into saddle-based curviness. Topologists classify the helicoid as a "complete embedded minimal surface of finite topology with infinite total curvature." The word *embedded* indicates that the surface doesn't fold back on itself. *Complete* means that it extends indefinitely and has no boundary.

In the early 1990s, David Hoffman and Fusheng Wei, then at the University of Massachusetts at Amherst, and Hermann Karcher of the University of Bonn in Germany discovered complicated equations that seemed to represent a surface just like the helicoid but with a tunnel penetrating one of the levels. Computer-generated images provided tantalizing glimpses of this novel surface.

Hoffman shows animations of the new surface to audiences by starting at a spot far above the hole and sliding downward.

Morgan says, "After a while, the audience is sure it's just the helicoid, when suddenly the unexpected hole appears."

That wasn't the only surprise. The team had actually identified an infinite family of minimal surfaces, each

A soap film stretching across a wire loop is a real-world example of a bounded minimal surface. Mathematicians study minimal surfaces that may also be infinite, such as the computer-generated, double-spiral form shown here. Called a helicoid, this shape played an important role in a recent mathematical proof.

Matthias Weber

HOLE IN ONE. A tunnel penetrates the central portion of this infinite helicoidal surface, which mathematicians consider a helicoid with a handle.

D. Hoffman and J. Hoffman, MSRI

characterized by a different number of handles. It appeared that the new sort of helicoid could have any number of tunnels and still qualify as the same sort of minimal surface as the basic helicoid and the plane. The mathematicians, however, had great difficulty proving that the helicoid with a handle and all its cousins don't somehow intersect themselves.

For the vexing problem of whether a surface folds back on itself, the complex equations characterizing minimal surfaces mask more than they reveal. Computer visualizations of the equations help but can't provide a complete answer.

"The problem in much of the research that we are doing lies in determining the interplay of the algebraic nature of the formulas with the geometry and topology of the shapes these formulas represent," Weber says.

Constructing such a surface is like piecing together local maps in an atlas to create a detailed map of the world. "You know how the map on page 27 fits with the maps on pages 25, 26, and 28, but the maps on pages 27 and 62 might just cross each other," Wolf says. "This makes it quite difficult to detect global properties of the shape."

Genus one

In the rubbery world of topology, it's possible to imagine creating a helicoid by carefully deforming and stretching the surface of a punctured sphere rather than by expanding and twirling a flat soap film. You'd stretch the rim of the sphere's puncture hole to infinity to create a plane and then twist the plane into the double spiral characteristic of the helicoid.

Putting a tunnel in the helicoid is equivalent to adding a handle—just like the one that sprouts from a coffee mug—to a punctured sphere. Mathematicians call the result a genus-one helicoid.

Several years ago, William H. Meeks III of the University of Massachusetts and Harold Rosenberg of the Université Denis Diderot in Paris proved that a complete, embedded minimal surface that was topologically a punctured sphere with no handles had to be either the basic helicoid or the plane. In other words, "there's only one helicoid with no handles," Wolf says. Meeks and Rosenberg built their proof on earlier investigations of the structure of minimal surfaces done by Tobias Colding of New York University and William Minicozzi of Johns Hopkins University in Baltimore.

The new proof by Weber, Hoffman, and Wolf establishes that a helicoid with one handle doesn't fold back on itself. Originally, in one special case, Hoffman, Wei, and Karcher had proved that a helicoid with an infinite number of handles—a hole at each level of an infinite tower—exists and doesn't intersect itself. Weber and Wolf later developed a new way of thinking about minimal surfaces that enabled Weber to provide an alternative, much simpler and shorter, proof that this surface with infinitely many handles is embedded.

By manipulating equations, Weber, Hoffman, and Wolf then carefully deformed this particular surface so that it always stayed free of intersections, even as the number of handles decreased. To do so, they took advantage of a particular property of the helicoid.

One can construct a plane by lifting a horizontal line upward with a constant speed, Weber says. If one simultaneously rotates the line with a constant speed around a fixed vertical axis, one obtains the basic helicoid. Decreasing the rotational speed of the line while still moving the horizontal line upward corresponds to stretching out, or "untwisting," the ordinary helicoid. The process of untwisting pushes the handles vertically farther and farther apart, until finally, at the limit, there's a helicoid with just one handle. In effect, all the other handles are twisted away. (See <http://www.msri.org/about/sgp/jim/geom/minimal/library/helicoidg1p/index.html>.)

In this way, Weber, Hoffman, and Wolf reached the genus-one helicoid, establishing that it, too, is embedded. The proof shows that helicoids with more than one handle are also embedded.

INFINITE TOWER. The central portion of this towering minimal surface features an infinite number of holes, one at each level.

D. Hoffman and J. Hoffman, MSRI

The entire proof, which runs to more than 100 manuscript pages, contains just mathematical logic and prose. None of the argument requires computer calculations or visualizations.

Nonetheless, "our understanding of what must be going on was aided by pictures, animations, hand drawings, and all manner of geometric and visual thinking," Hoffman says. "The process here is to pin down what really is correct without reference to unverifiable intuition and common belief."

There's still the delicate matter of the relationship between the surface characterized by the proof and the one-handle surface that Hoffman, Wei, and Karcher constructed more than a decade ago. "Whether it is the same surface is not yet known," Hoffman says. "More generally, we do not know, but do believe, that there is only one embedded surface with [the relevant] properties."

Future flexibility

Recent advances related to the structure of minimal surfaces and new ways of representing them promise many more developments in the realm of soap-film mathematics. Weber, for one, is interested in what happens when handles on a helicoid are squeezed together instead of being twisted away.

"Sheer curiosity forces us mathematicians to figure out what's going on in this limit as well," Weber says.

Weber and Martin Traizet of the Université François Rabelais in Tours, France, have recently produced intriguing images of compressed helicoids. "The resulting limit is quite surprising and can be described as a 'parking-garage structure,' a term we coined for this phenomenon," Weber says.

A typical real-life parking garage can be thought of as a giant helicoid, with two spiral ramps, one for driving up and the other for coming down. These ramps can involve a sequence of either right turns or left turns. The new minimal structure found by Weber and his team is like a three-ramp parking garage, with two spirals turning left and one spiral turning right. To further complicate the picture, the parking levels are close to each other.

PLENTY OF SPACES. This unusual structure, called a parking garage, represents the center of a minimal surface constructed by gluing together three helicoids. Weber

Inspired by computer-generated images, Weber and Traizet have proved that it's possible to construct families of minimal surfaces using such parking-garage structures—in effect, by gluing together helicoids laterally. These surfaces, in turn, may be related to helicoids with some finite number of handles.

The various directions of current research on minimal surfaces suggest a rich future full of amazing forms. Mathematicians will need new insights to organize and classify the members of this burgeoning zoo.

Wolf says, "The times are exciting because we are partly there, but still at the stage where we don't know if we are nearly there or only a few steps down the path."

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Matthias Weber's images of minimal surfaces can be found at <http://www.indiana.edu/~minimal/gallery/index.html>.

An introduction to soap films and minimal surfaces is available at <http://www.zib.de/polthier/booklet/intro.html>.

You can learn more about helicoids as minimal surfaces at <http://www.indiana.edu/~minimal/essays/helicoid/index.html> and <http://www.indiana.edu/~minimal/essays/helicoidal/index.html>.

To see a movie of how a helicoid can be untwisted, go to <http://www.msri.org/about/sgp/jim/geom/minimal/library/helicoidg1p/index.html>.

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