CONF-9507165--1

LA-UR- 95-2268

TITLE: DIFFEOMORPHISM GROUPS AND ANYON FIELDS

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SUBMITTED TO: 13TH ANNUAL WORKSHOP ON GEOMETRIC METHODS IN PHYSICS, BIALOWIEZA, POLAND

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DIFFEOMORPHISM GROUPS AND ANYON FIELDS

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Abstract

We make use of unitary representations of the group of diffeomorphisms of the plane to construct an explicit field theory of anyons. The resulting anyon fields satisfy q-commutators, where q is the well-known phase shift associated with a single counterclockwise exchange of a pair of anyons. Our method uses a realization of the braid group by means of paths in the plane, that transform naturally under diffeomorphisms of $\mathbf{R}^2$

1. DIFFEOMORPHISM GROUP REPRESENTATIONS AND ANYONS

The intrinsic structure of standard quantum mechanics includes representations of an infinite-dimensional group, whose infinitesimal generators are the mass density operator $\rho(\mathbf{x},t)$ and the momentum density operator $\mathbf{J}(\mathbf{x},t)$. By examining the commutation relations and other properties of ρ and $\mathbf{J}$, one determines that the corresponding group is the natural semidirect product $G = \mathcal{S}(M) \times \mathit{Diff}(M)$, where the manifold M is physical space (typically $\mathbf{R}^3$), $\mathcal{S}(M)$ is the additive group of smooth, real-valued scalar functions on M that together with all derivatives vanish rapidly at infinity (Schwartz' space), and $\mathit{Diff}(M)$ is the group of diffeomorphisms of M under composition that, together with all derivatives, become rapidly trivial at infinity[1].

Quantum-mechanical systems are described by the continuous unitary representations (CURs) of G, or in certain cases (such as an ideal, incompressible fluid) particular subgroups of G (e.g., the volume-preserving diffeomorphisms). This fact has been established and used in our previous work to obtain a unified description of an astonishing variety of quantum systems, ranging from extended objects such as vortex configurations[2] to point particles obeying boson, fermion, and (in two space dimensions) intermediate, or "anyon", statistics[3]. The latter possibility had already been conjectured from the topology of two-particle configuration space in the plane[4]; the diffeomorphism group approach provided a rigorous prediction even without the assumed exclusion of configurations where the particle coordinates coincide. From diffeomorphism group representations there also followed many of the fundamental physical properties of anyons–the shifts in angular momentum and energy spectra, the connection with configuration space topology, the relation to charged particles circling regions of magnetic flux, and the mathematical role of the braid group[3,5,6]. Anyon statistics find application in physics to surface phenomena, particularly the fractional quantum Hall effect[7].

Based on the diversity of known quantum systems associated with CURs of the diffeomorphism group or its semidirect product, we believe that G can be regarded as a universal, or generic, group of local symmetries describing non-relativistic quantum theory[6]. In the formulation of quantum mechanics based on diffeomorphism group representations, canonical fields ψ and ψ^* do not play a fundamental role. While the infinitesimal generators ρ and $\mathbf{J}$ can be constructed from canonical fields in specific models (see below), is not necessary to use this fact to obtain representations of $Diff(M)$, or to establish the physical interpretations of these representations.

It is nevertheless useful to reintroduce the field operators when that is possible: for example, annihilation and creation operators provide a way to construct states with specified numbers of particles, and fields are a starting point for many computational methods. It is thus worth asking how canonical fields can be constructed, taking as a starting point a collection of CURs of G.

We show here how creation and annihilation fields can be constructed, uniquely up to unitary equivalence, as operators *intertwining a hierarchy* of representations of G. So defined, these operators create or annihilate *configurations*, where the type of object created is defined by the representations from which one starts. We take as a specific example the construction of anyon fields from diffeomorphism group representations. Then we are able to determine the algebra that the field operators satisfy. Thus we obtain q-commutation relations for anyon fields *not* as a starting assumption, nor by introducting a Chern-Simons potential into a canonical field theory, but strictly as a *consequence* of the unitary group representations that describe anyons together with the intertwining property defining the fields. In the course of doing this, we shall also make clear how elements of $Diff(\mathbf{R}^2)$ act on the braid group.

First we provide some basic facts about the infinite-dimensional Lie algebra of mass and momentum density operators, and the corresponding Lie group of unitary operators, in canonical nonrelativistic theories. One has formally,

$$\begin{aligned} \rho(\mathbf{x},t) &= m\psi^*(\mathbf{x},t)\psi(\mathbf{x},t), \\ \mathbf{J}(\mathbf{x},t) &= (\hbar/2i)\{\psi^*(\mathbf{x},t)[\nabla\psi(\mathbf{x},t)] - [\nabla\psi^*(\mathbf{x},t)]\psi(\mathbf{x},t)\}\,. \end{aligned} \tag{1.1}$$

where the fields in (1.1) obey, at equal times t, for all $\mathbf{x},\mathbf{y}$,

$$\begin{aligned} [\psi(\mathbf{x},t),\psi(\mathbf{y},t)]_\pm &= [\psi^*(\mathbf{x},t),\psi^*(\mathbf{y},t)]_\pm = 0, \\ [\psi(\mathbf{x},t),\psi^*(\mathbf{y},t)]_\pm &= \delta(\mathbf{x}-\mathbf{y}). \end{aligned} \tag{1.2}$$

The subscript "-" denotes the commutator, and "+" the anticommutator bracket. To interpret ρ and $\mathbf{J}$ as operator-valued distributions on the spatial manifold M, define $\rho(f) = \int_M \rho(\mathbf{x})\,f(\mathbf{x})\,d\mathbf{x}$ and $J(\mathbf{g}) = \int_M \mathbf{J}(\mathbf{x})\cdot\mathbf{g}(\mathbf{x})\,d\mathbf{x}$; where $f \in \mathcal{S}(M)$, and $\mathbf{g}$ is a vector field whose components (together with all derivatives) vanish rapidly at infinity. We shall call the set of such vector fields $Vect(M)$. We then obtain the same infinite-dimensional semidirect product Lie algebra (a nonrelativistic local current algebra) *independent* of which bracket is chosen in (1.1), namely:

$$\begin{aligned} [\rho(f_1),\rho(f_2)] &= 0\,, \qquad [\rho(f),J(\mathbf{g})] = i\hbar\rho(\mathbf{g}\cdot\nabla f)\,, \\ [J(\mathbf{g}_1),J(\mathbf{g}_2)] &= -i\hbar J([\mathbf{g}_1,\mathbf{g}_2])\,, \end{aligned} \tag{1.3}$$

where $\mathbf{g}\cdot\nabla f = L_{\mathbf{g}}f$ is the Lie derivative, and $[\mathbf{g}_1,\mathbf{g}_2] = \mathbf{g}_1\cdot\nabla\mathbf{g}_2 - \mathbf{g}_2\cdot\nabla\mathbf{g}_1$ is the Lie bracket of the vector fields. The fact that the Lie algebra (1.3) is the same for fermions and bosons means that the information about the particle statistics, formerly encoded in the algebra of fields, will now be contained in the choice of a *representation* satisfying (1.3). We have shown that inequivalent representations, in spaces of dimension greater than one, describe the different statistics[3].

It is a standard result that the C^∞ vector field $\mathbf{g}$ generates a unique one-parameter group of C^∞ diffeomorphisms, $\phi_t^{\mathbf{g}} : M \to M \, (t \in \mathbf{R})$ with $\phi_{t_1}^{\mathbf{g}} \circ \phi_{t_2}^{\mathbf{g}} = \phi_{t_1+t_2}^{\mathbf{g}}$; satisfying the ordinary differential equation $(\partial/\partial t)\phi_t^{\mathbf{g}}(\mathbf{x}) = \mathbf{g}(\phi_t(\mathbf{x}))$, together with the initial condition $\phi_{t=0}^{\mathbf{g}}(\mathbf{x}) = \mathbf{x}$. (The conditions on $\mathbf{g}$ at infinity are important to the global existence of ϕ_t.) Then, defining the unitary operators $U(f) = \exp[(i/m)\rho(f)]$ and $V(\phi_t^{\mathbf{g}}) = \exp[(it/\hbar)J(\mathbf{g})]$, we have the semidirect product group law,

$$U(f_1)V(\phi_1)U(f_2)V(\phi_2) = U(f_1 + f_2 \circ \phi_1)V(\phi_1\phi_2), \tag{1.4}$$

where $\phi_1\phi_2 = \phi_2 \circ \phi_1$ is the composition of the diffeomorphisms.

The simplest representations of (1.3) and (1.4) are the N-particle Bose and Fermi representations. For specificity let $M = \mathbf{R}^2$ or $\mathbf{R}^3$ and let the wave function $\Phi_N^{s,a}$ belong to the Hilbert space $\mathcal{H}_N^{s,a}$ of symmetric (s) or antisymmetric (a) functions of N variables $(\mathbf{x}_1, \ldots \mathbf{x}_N) \in M$, square integrable with respect to Lebesgue measure μ. Then these representations are given by

$$\rho_N(f)\Phi_N^{s,a}(\mathbf{x}_1, \ldots, \mathbf{x}_N) = m \sum_{j=1}^{N} f(\mathbf{x}_j)\, \Phi_N^{s,a}(\mathbf{x}_1, \ldots, \mathbf{x}_N),$$

$$J_N(\mathbf{g})\Phi_N^{s,a}(\mathbf{x}_1, \ldots, \mathbf{x}_N) = \frac{\hbar}{2i} \sum_{j=1}^{N} \{\mathbf{g}(\mathbf{x}_j) \cdot \nabla_j \Phi_N^{s,a}(\mathbf{x}_1, \ldots, \mathbf{x}_N) + \nabla_j \cdot [\mathbf{g}(\mathbf{x}_j)\Phi_N^{s,a}(\mathbf{x}_1, \ldots, \mathbf{x}_N)]\}, \tag{1.5}$$

and correspondingly

$$U_N(f)\Phi_N^{s,a}(\mathbf{x}_1, \ldots, \mathbf{x}_N) = \exp[i \sum_{j=1}^{N} f(\mathbf{x}_j)]\, \Phi_N^{s,a}(\mathbf{x}_1, \ldots, \mathbf{x}_N),$$

$$V_N(\phi)\Phi_N^{s,a}(\mathbf{x}_1, \ldots, \mathbf{x}_N) = \Phi_N^{s,a}(\phi(\mathbf{x}_1), \ldots, \phi(\mathbf{x}_N)) [\prod_{j=1}^{N} \mathcal{J}_\phi(\mathbf{x}_j)]^{\frac{1}{2}}, \tag{1.6}$$

where $\mathcal{J}_\phi(\mathbf{x})$ denotes the Jacobian of the diffeomorphism ϕ at $\mathbf{x}$.

Note that the operators in (1.5) are self-adjoint and those in (1.6) are unitary. They preserve the particle number N, and are also manifestly symmetric with respect to exchange of particle coordinates $\mathbf{x}_j$; thus they also preserve the wave function symmetry. For $N = 0, 1, 2, \ldots$, the N-particle Bose (respectively, Fermi) representations constitute a *hierarchy*, in an obvious physical sense that we make precise below.

2. HIERARCHIES OF REPRESENTATIONS AND THEIR INTERTWINING FIELD OPERATORS

The first step in our development is to establish the conditions that have to be satisfied for creation and annihilation field operators to intertwine representations of the diffeomorphism group. This allows us to specify a well-defined sense, satisfied by the above examples, in which a collection of continuous unitary representations of the group *Diff*(M) forms a hierarchy. The representations in the hierarchy are labeled by the number N of elementary configurations; thus we establish the bracket that an intertwining field must obey with the operators in the N-configuration Hilbert space.

The required conditions follow from the structure of the commmutation relations between the fields ψ and ψ^* and the operators ρ and $\mathbf{J}$. For bosons and fermions (where we already know ρ, $\mathbf{J}$, ψ, and ψ^*), these commutation relations can be obtained by direct formal computation starting from (1.1) and (1.2). To facilitate the calculation, and in anticipation of the results of Section 3.2, we shall generalize this procedure from the outset and start directly from the q-commutation relations for the fields. These are based on the q-deformed bracket $[A, B]_q = AB - qBA$, where q is assumed to be a complex number of modulus one. When $q = 1$, we recover the commutator brackets "-" in (1.2), and when $q = -1$, we have the anticommutator brackets "+". We write

$$[\psi(\mathbf{x},t), \psi(\mathbf{y},t)]_q = [\psi^*(\mathbf{x},t), \psi^*(\mathbf{y},t)]_q = 0\,,$$

$$[\psi(\mathbf{y},t), \psi^*(\mathbf{x},t)]_q = \delta(\mathbf{x}-\mathbf{y})\,. \tag{2.1}$$

Note that for the first two equations of (2.1) to be consistent when $q \neq \pm 1$, they cannot be interpreted as holding for all ordered pairs $(\mathbf{x},\mathbf{y})$, but only in a half-space H of $M \times M$. In the complementary half-space $\bar{H} = M \times M - H$, we have instead the $(1/q)$-bracket. Then the equation for $[\psi(\mathbf{x}), \psi(\mathbf{y})]_q$ is consistent with the equation for $[\psi^*(\mathbf{x}), \psi^*(\mathbf{y})]_q$, since we are assuming that $|q| = 1$. The third equation of (2.1) is written as indicated for $(\mathbf{x},\mathbf{y}) \in H$; it may be written equivalently (using $\bar{q} = 1/q$) as

$$[\psi(\mathbf{x},t), \psi^*(\mathbf{y},t)]_{1/q} = \delta(\mathbf{x}-\mathbf{y})\,. \tag{2.2}$$

Now we are ready to obtain brackets for ψ and ψ^* with ρ and $\mathbf{J}$. We shall use the algebraic identity,

$$[AB, C]_- = A[B, C]_q + q\,[A, C]_{1/q}B\,, \tag{2.3}$$

that relates the ordinary commutator to the q-commutator. Then we can calculate the commutators of the field operators with the generators of the infinite-dimensional group. We obtain, for field operators obeying (2.1) and (2.2) for any fixed value of q having modulus one,

$$[\rho(\mathbf{y},t), \psi^*(\mathbf{x},t)] = m\psi^*(\mathbf{y},t)\delta(\mathbf{x}-\mathbf{y}),$$

$$[\rho(\mathbf{y},t), \psi(\mathbf{x},t)] = -m\psi(\mathbf{y},t)\delta(\mathbf{x}-\mathbf{y}), \tag{2.4}$$

$$[\mathbf{J}(\mathbf{y},t), \psi^*(\mathbf{x},t)] = \frac{\hbar}{2i}\{\psi^*(\mathbf{y},t)\nabla_{\mathbf{y}}\delta(\mathbf{x}-\mathbf{y}) - \delta(\mathbf{x}-\mathbf{y})\nabla_{\mathbf{y}}\psi^*(\mathbf{y},t)\}\,,$$

$$[\mathbf{J}(\mathbf{y},t), \psi(\mathbf{x},t)] = -\frac{\hbar}{2i}\{\psi(\mathbf{y},t)\nabla_{\mathbf{y}}\delta(\mathbf{x}-\mathbf{y}) - \delta(\mathbf{x}-\mathbf{y})\nabla_{\mathbf{y}}\psi(\mathbf{y},t)\}. \tag{2.5}$$

The justification of these equations for all values of $\mathbf{x}$ and $\mathbf{y}$ involves performing the calculation in each half-space separately, and noting that the answer is the same.

Next we multiply (2.4) by the test functions $h(\mathbf{x})$ and $f(\mathbf{y})$, and (2.5) by the test function $h(\mathbf{x})$ and vector field $\mathbf{g}(\mathbf{y})$, and integrate over $\mathbf{x}$ and $\mathbf{y}$. We thus obtain,

$$[\rho(f), \psi^*(h)] = \psi^*(mfh)\,, \quad [\rho(f), \psi(h)] = -\psi(mfh)\,, \tag{2.6}$$

$$[J(\mathbf{g}), \psi^*(h)] = \psi^*(\frac{\hbar}{2i}\{\mathbf{g}\cdot\nabla h + \nabla\cdot(\mathbf{g}h)\})\,,$$

$$[J(\mathbf{g}), \psi(h)] = -\psi(\frac{\hbar}{2i}\{\mathbf{g}\cdot\nabla h + \nabla\cdot(\mathbf{g}h)\})\,. \tag{2.7}$$

The essential point is that we find the same *commutator* brackets independent of whether we begin the calculation with Bose fields, Fermi fields, or even fields satisfying q-commutators; that is, equations (2.6) and (2.7) are representation-independent!

It is also straightforward to verify that (1.3) together with (2.6) and (2.7) satisfy the Jacobi identity, as long as we do not include brackets of fields with each other in the identity. However, ψ and ψ^* satisfy different relations with each other in the Bose and Fermi cases, and (as we shall shortly see) in the anyon case. Only in the Bose case do we have an actual Lie algebra of fields together with densities and currents.

For the cases of bosons and fermions, we can now look again at the N-particle representations (1.5) and (1.6), and interpret ψ^* and ψ as creation and annihilation operators respectively *intertwining* these representations. In the Bose or Fermi Fock representations of the usual second-quantized nonrelativistic field operators, we have Hilbert space vectors $\Phi^{s,a} = (\Phi_N^{s,a})$, $N = 0, 1, 2, \ldots$, with $\Phi_N^{s,a} \in \mathcal{H}_N^{s,a}$. For bosons ($s$),

$$[\psi(\mathbf{x})\Phi^s]_N(\mathbf{x}_1 \ldots \mathbf{x}_N) = \sqrt{N+1}\,\Phi^s_{N+1}(\mathbf{x}_1 \ldots \mathbf{x}_N, \mathbf{x}),$$

$$[\psi^*(\mathbf{x})\Phi^s]_N(\mathbf{x}_1 \ldots \mathbf{x}_N) = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \delta(\mathbf{x} - \mathbf{x}_j)\,\Phi^s_{N-1}(\mathbf{x}_1 \ldots \hat{\mathbf{x}}_j \ldots \mathbf{x}_N). \qquad (2.8)$$

where $\hat{\mathbf{x}}_j$ means that $\mathbf{x}_j$ is to be omitted; for fermions (a),

$$[\psi(\mathbf{x})\Phi^a]_N(\mathbf{x}_1 \ldots \mathbf{x}_N) = \sqrt{N+1}\,\Phi^a_{N+1}(\mathbf{x}_1 \ldots \mathbf{x}_N, \mathbf{x}),$$

$$[\psi^*(\mathbf{x})\Phi^a]_N(\mathbf{x}_1 \ldots \mathbf{x}_N) = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} (-1)^{N-j} \delta(\mathbf{x} - \mathbf{x}_j)\,\Phi^a_{N-1}(\mathbf{x}_1 \ldots \hat{\mathbf{x}}_j \ldots \mathbf{x}_N). \qquad (2.9)$$

When all but one of the N-particle components of Φ vanish, we can see from (2.8) or (2.9) that $\psi^* : \mathcal{H}_N^{s,a} \to \mathcal{H}_{N+1}^{s,a}$, while $\psi : \mathcal{H}_{N+1}^{s,a} \to \mathcal{H}_N^{s,a}$. It is straightforward to verify explicitly that both (2.8) and (2.9) satisfy (2.6) and (2.7).

Note next that the expressions mfh and $(\hbar/2i)\{\mathbf{g}\cdot\nabla h + \nabla\cdot(\mathbf{g}h)\}$, which occur on the right-hand sides of (2.6) and (2.7), are just the one-particle representations of $\rho(f)$ and $J(\mathbf{g})$ respectively, applied to h (if we regard h as an element of the Hilbert space $\mathcal{H}_1$). Then we can rewrite (2.6) and (2.7) in the form

$$[\rho(f), \psi^*(h)] = \psi^*(\rho_{N=1}(f)h), \quad [J(\mathbf{g}), \psi^*(h)] = \psi^*(J_{N=1}(\mathbf{g})h),$$

$$[\rho(f), \psi(h)] = -\psi(\rho_{N=1}(f)h), \quad [J(\mathbf{g}), \psi(h)] = -\psi(J_{N=1}(\mathbf{g})h). \qquad (2.10)$$

Finally we exponentiate $\rho(f)$ and $J(\mathbf{g})$ in (2.10), and obtain

$$U(f)\psi^*(h)U^{-1}(f) = \psi^*(U_{N=1}(f)h), \quad V(\phi)\psi^*(h)V^{-1}(\phi) = \psi^*(V_{N=1}(\phi)h),$$

$$U(f)\psi(h)U^{-1}(f) = \psi(U_{N=1}(-f)h), \quad V(\phi)\psi(h)V^{-1}(\phi) = \psi((V_{N=1}(\phi^{-1})h). \qquad (2.11)$$

When we make the dependence on N explicit in (2.11), we have

$$U_{N+1}(f)\psi^*(h) = \psi^*(U_{N=1}(f)h)U_N(f), \quad V_{N+1}(\phi)\psi^*(h) = \psi^*(V_{N=1}(\phi)h)V_N(\phi),$$

$$U_N(f)\psi(h) = \psi(U_{N=1}(-f)h)U_{N+1}(f), \quad V_N(\phi)\psi(h) = \psi((V_{N=1}(\phi^{-1})h)V_{N+1}(\phi). \quad (2.12)$$

The preceding calculations for the case of canonical fields motivate the following general perspective. For a collection of CURs of the diffeomorphism group (or its semidirect product) to form a hierarchy labeled by N, a necessary and sufficient condition is that ψ^* and ψ can be constructed obeying (2.12). Especially noteworthy is the fact that the argument of the fields ψ^* and ψ in these equations is a Hilbert space vector from the $N = 1$ space in the hierarchy. This fact *defines* the $N = 1$ space, and establishes the nature of the configuration that ψ^* creates and ψ annihilates.

We expect this general structure to occur not only for point-like configurations, but also for configurations of extended objects such as vortex filaments or ribbons. With vortex configurations, the argument of ψ^* and ψ is a one-vortex Hilbert space vector, so that the unsmeared creation and annihilation fields no longer depend on a single point in space but on a spatially extended configuration. Only the currents, in unsmeared form, have as their arguments individual points in space. In the case of quantum vortex configurations, we also have additional complications associated with the possibility of overlapping or knotted vortices. This is a topic of our current research.

In the next section we use the above results to construct explicit fields for anyons that obey (2.12), anticipating that the fields will satisfy different commutation relations from those satisfied by Bose and Fermi fields. It turns out that these are the q-commutators written above.

3. CONSTRUCTING ANYON FIELDS

In this section, we construct anyon fields from a hierarchy of continuous unitary representations of the group $Diff(\mathbf{R}^2)$, using the N-anyon representations[3]. We display these fields explicitly, using a convenient diagrammatic representation of the elements of the braid group. It then emerges that the fields so obtained satisfy a q-commutator algebra. We stress that the q-commutator is not put in by hand, but is one of the consequences of the diffeomorphism group approach, just as anyons themselves are a consequence of the representation theory of the diffeomorphism group.

We construct anyon fields obeying the commutation relations (2.12) in the following steps: First, we write the N-anyon representation of $Diff(\mathbf{R}^2)$ and its semidirect product, in the Hilbert space $\mathcal{H}_N^{eq}$ of *equivariant* wave functions, defined on the universal covering space of N-particle configuration space in $\mathbf{R}^2$. The equivariance is for a one-dimensional unitary representation (a character) of the braid group B_N. Second, we make this representation of $Diff(\mathbf{R}^2)$ concrete by introducing a way of labeling an element in the covering space by a set of N paths in the plane. Third, we make use of this to define ψ^* as a creation operator mapping $\mathcal{H}_N^{eq}$ to $\mathcal{H}_{N+1}^{eq}$. Finally we state the results about ψ^* and ψ that are obtained in this framework.

To write the N-anyon representation, recall that an N-particle configuration is an *unordered* set γ of N distinct points in the plane: $\gamma = \{\mathbf{x}_1 \dots \mathbf{x}_N\} \subset \mathbf{R}^2$; the indexing of the points is arbitrary. Let the configuration space Δ_N be the set of all such configurations γ, and let μ be a normalized measure on Δ_N locally equivalent to Lebesgue measure. The fundamental group $\pi_1(\Delta_N)$ is the braid group B_N. An element $\tilde{\gamma}$ of the universal covering space $\tilde{\Delta}_N$ can be labeled by a configuration γ, together with a braid $b \in B_N$ that specifies the *sheet* in $\tilde{\Delta}_N$ to which the element belongs; we shall write $\tilde{\gamma} = (\gamma, b)$. This labeling is not unique, but conventional: the sheet associated with the identity element $e \in B_N$ may be selected arbitrarily. We denote by p the projection mapping, $p(\tilde{\gamma}) = \gamma$. The braid group also *acts* on $\tilde{\Delta}_N$: for $b' \in B_N$, we have $b'(\gamma, b) = (\gamma, bb')$. An equivariant wave function $\tilde{\Psi}$ is a complex-valued function on $\tilde{\Delta}_N$ that transforms in accordance with a character T of B_N; that is, $\tilde{\Psi}(\gamma, bb') = T(b')\tilde{\Psi}(\gamma, b)$. Because $\tilde{\Psi}$ is equivariant, the product $\overline{\tilde{\Phi}(\gamma, b)}\tilde{\Psi}(\gamma, b)$ is independent of the particular choice of b. Thus we can use the measure μ on Δ_N to define square-integrable wave functions and to introduce an inner product: $(\tilde{\Phi}, \tilde{\Psi}) = \int_{\Delta_N} \overline{\tilde{\Phi}(\gamma, b)}\tilde{\Psi}(\gamma, b)\, d\mu(\gamma)$. The result is the Hilbert space $\mathcal{H}_N^{eq}$.

As we have emphasized strongly in our earlier work[8], these ideas are not restricted to complex-valued functions and one-dimensional representations of B_N; quantum theories based on higher-dimensional, irreducible representations are equally possible (braid parastatistics). However, we limit ourselves here to discussing the usual anyon case where, when b is the braid for a single, counterclockwise exchange of two particles, $T(b) = \exp i\theta$.

Now the action of diffeomorphisms in the base space, which is given by $\phi\gamma = \{\phi(\mathbf{x}_1), \ldots \phi(\mathbf{x}_N)\}$ for $\phi \in Diff(\mathbf{R}^2)$, lifts to the covering space in such a way that if $p(\tilde{\gamma}) = \gamma$, then $p(\phi\tilde{\gamma}) = \phi\gamma$. Denote by K_γ the stability subgroup of γ. Diffeomorphisms in K_γ act, as do braids, on the points $(\gamma, b) \in p^{-1}(\gamma)$ belonging to the different sheets in the covering space. There is thus a natural homomorphism from K_γ onto B_N; and T determines a CUR of K_γ in which distinct components are represented by (in general) distinct complex numbers. The N-anyon representation of the semidirect product group G may be written on $\mathcal{H}_N^{eq}$ as:

$$U_N(f)\tilde{\Psi}(\tilde{\gamma}) = \exp[i\langle\tilde{\gamma}, f\rangle]\tilde{\Psi}(\tilde{\gamma}), \quad V_N(\phi)\tilde{\Psi}(\tilde{\gamma}) = \tilde{\Psi}(\phi\tilde{\gamma})\sqrt{\frac{d\mu_\phi}{d\mu}(\gamma)}, \qquad (3.1)$$

where $\langle\tilde{\gamma}, f\rangle = \sum_j f(\mathbf{x}_j)$ when $\gamma = \{\mathbf{x}_1 \ldots \mathbf{x}_N\}$, and where μ_ϕ is the transformed measure given by $d\mu_\phi(\gamma) = d\mu(\phi\gamma)$.

Next we introduce a concrete way to label points in $\tilde{\Delta}_N$ that assists in understanding the action of diffeomorphisms in this space. For $\mathbf{x} \in \mathbf{R}^2$, write $\mathbf{x}$ in Cartesian coordinates as (x^1, x^2). Choose a typical configuration $\gamma = \{\mathbf{x}_j \,|\, j = 1, \ldots, N\}$ in which (for now) all the $\mathbf{x}_j$ have distinct values of their first coordinates: i.e., $x_j^1 \neq x_k^1$ for $j \neq k$. For such a choice of γ, we consider a set Γ of N continuous, non-self-intersecting and non-mutually-intersecting paths $\{\Gamma_j \,|\, j = 1, \ldots, N\}$, coming in from infinity and terminating at the $\mathbf{x}_j$. For specificity we shall take all the Γ_j at infinity to be parallel to the x^2-axis, and to extend in the direction of the negative x^2-axis. For a fixed configuration γ, consider two such sets of paths, Γ^1 and Γ^2, terminating at γ. They are said to be homotopic if the individual paths Γ_j^1 can be continuously deformed into the paths Γ_j^2, without moving the terminal points $\mathbf{x}_j \in \gamma$, without changing the direction of the paths at infinity (though they may be translated), and of course without any of the paths intersecting each other. Denote the homotopy class containing Γ by $[\Gamma]$. An element $\tilde{\gamma}$ of the covering space, with $p(\tilde{\gamma}) = \gamma$, can now be identified with a class $[\Gamma]$ whose set of terminal points is γ.

Given a configuration γ as above, we can make a canonical choice for an element $\tilde{\gamma}$ of the covering space by letting all the Γ_j be straight half-lines parallel to the x^2-axis. We call this particular set of paths $\Gamma_0^{\{\mathbf{x}_1,\ldots,\mathbf{x}_N\}}$, or Γ_0^γ (see Fig. 1). Since the indexing of the $\mathbf{x}_j$ is to this point arbitrary, we can also label the paths and their terminal points in accordance with the order of their x^1 coordinates. Thus we have $x_1^1 < x_2^1 < \ldots < x_N^1$, with Γ_j terminating at $\mathbf{x}_j$. The homotopy class $[\Gamma_0^\gamma]$ is the element of $p^{-1}(\gamma)$ that we shall conventionally associate with the identity element in the braid group.

Now the important observation is that diffeomorphisms of the plane act not only on the configurations γ but on the sets of paths Γ, since these also lie in the plane. It is also evident that a diffeomorphism that becomes trivial at infinity respects homotopy equivalence as we have defined it, so that it actually acts on $[\Gamma]$. Thus, for fixed γ, diffeomorphisms in the stability subgroup K_γ map the classes $[\Gamma]$ of paths terminating at γ into each other.

Suppose, for specificity, that we have a fixed pair of points $\{\mathbf{x}_1, \mathbf{x}_2\}$ in the plane, and consider the canonical paths $\Gamma_0^{\{\mathbf{x}_1,\mathbf{x}_2\}}$ constructed in accordance with Fig. 1, terminating at $\{\mathbf{x}_1, \mathbf{x}_2\}$. Let ϕ be a diffeomorphism, trivial at infinity, that exchanges the points: i.e., $\mathbf{x}_2 = \phi(\mathbf{x}_1)$ and $\mathbf{x}_1 = \phi(\mathbf{x}_2)$. One way in which ϕ might act on the pair of paths $\Gamma_0^{\{\mathbf{x}_1,\mathbf{x}_2\}}$ is to map them to a pair of paths as in Fig. 2 (imagine ϕ to have support in the shaded region of the plane). Then we may regard ϕ as implementing a single *counterclockwise* exchange of $\mathbf{x}_1$ and $\mathbf{x}_2$, and associate with this diffeomorphism the corresponding generator b_{12} in the braid group. Alternatively, a diffeomorphism may implement a clockwise exchange of the points, as in Fig. 3. With such a diffeomorphism, we associate the inverse braid group generator b_{12}^{-1}. Clearly a group homomorphism is defined in this manner, from the stability subgroup K_γ of $Diff(\mathbf{R}^2)$ onto the braid group. The example generalizes in the obvious way to N-particle configurations in $\mathbf{R}^2$. We denote the homomorphism by $h_\gamma : K_\gamma \to B_N$, and write $h_\gamma(\phi) = b$ for the braid associated with ϕ.

Figure 1. For $\gamma = \{\mathbf{x}_1, \ldots, \mathbf{x}_N\}$, a canonical choice Γ_0^γ of paths $\{\Gamma_j\}$ terminating at $\{\mathbf{x}_j\}$.

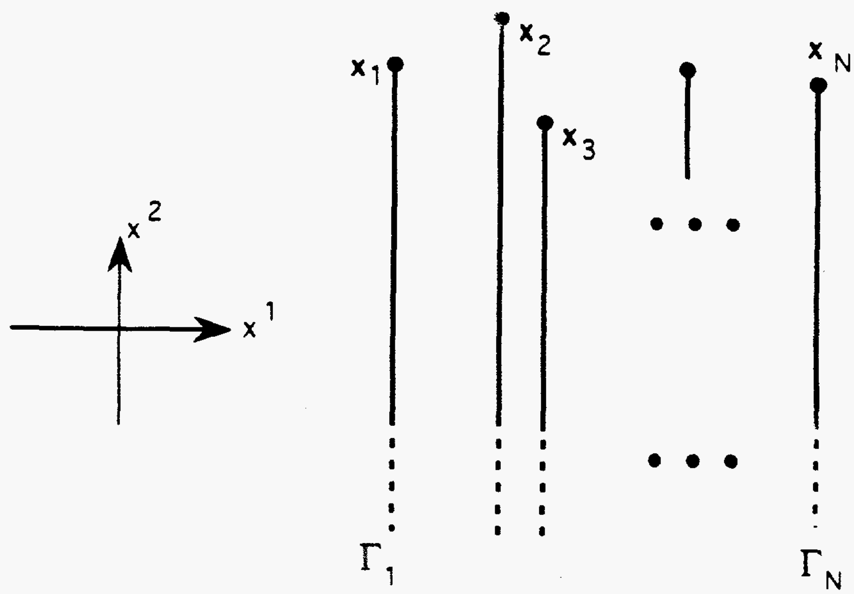


Figure 2. A diffeomorphism with support inside the indicated region moves the paths to a different homotopy class, implementing a single counterclockwise exchange of two points labeled originally as in Fig. 1.

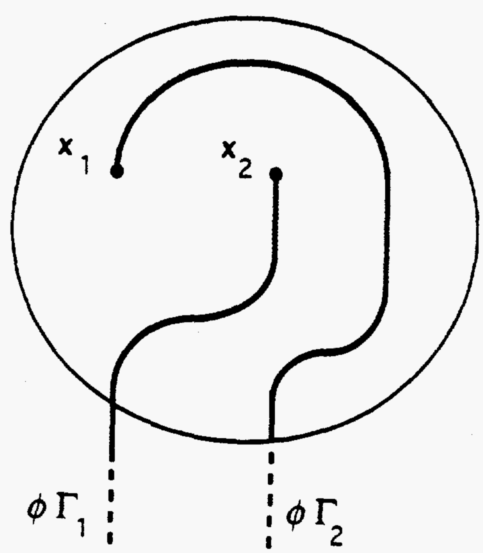
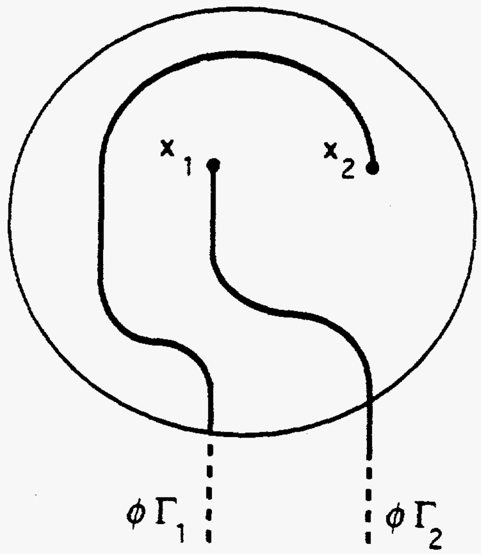


Figure 3. A diffeomorphism implementing a single clockwise exchange, as distinct from the counterclockwise exchange of Fig. 2.



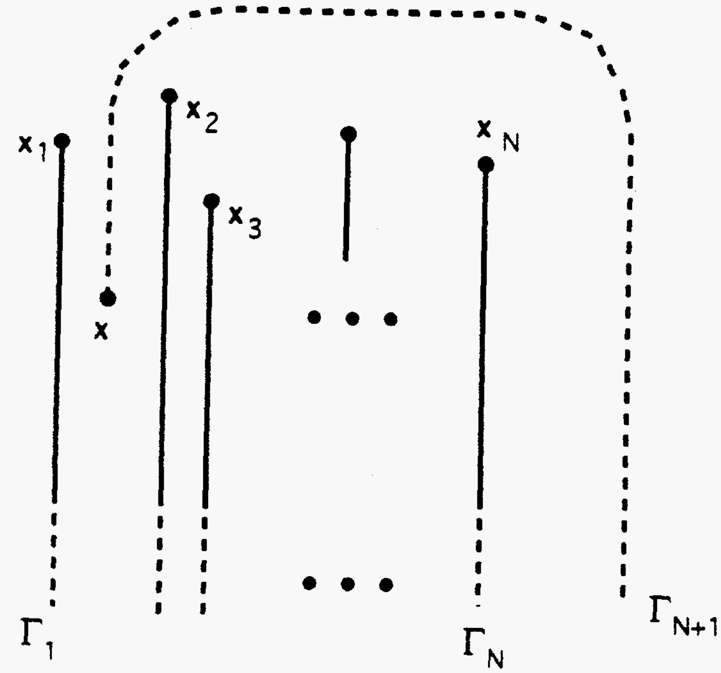
The above provides a faithful mapping from B_N to the homotopy classes $[\Gamma]$. Then we can write $T(\Gamma)$ in place of $T(b)$, when b is the braid that takes $[\Gamma_0^\gamma]$ to $[\Gamma]$.

We shall next make use of this picture to define the anyon creation field ψ^*, mapping $\mathcal{H}_N^{eq}$ to $\mathcal{H}_{N+1}^{eq}$ and satisfying the desired equations (2.12). To describe the way that such an anyon field acts, we introduce one more important convention: a way to denote the procedure for adding a single anyonic particle at $\mathbf{x}$, not merely to an N-anyon configuration, but to an element of the N-anyon covering space. Doing this, of course, will *break* the homotopy equivalence, because points on *many* different sheets of the $(N+1)$-anyon covering space correspond to the introduction of a a new anyon at a point $\mathbf{x}$. We therefore need a standard way to make the choice.

Given the homotopy class $[\Gamma_0^\gamma]$, and the additional point $\mathbf{x}$, define a new set of paths $\Gamma_{\mathbf{x}}^{\{\mathbf{x}_1,\ldots,\mathbf{x}_N\}}$ by placing the point $\mathbf{x}$ in the plane among the N paths comprising Γ_0^γ, and drawing a new path Γ_{N+1} that terminates at $\mathbf{x}$, and comes in from infinity to the *right* of the N existing paths without intersecting them (see Fig. 4). The homotopy class $[\Gamma_{\mathbf{x}}^{\{\mathbf{x}_1,\ldots,\mathbf{x}_N\}}]$ is thus defined, specifying a particular element of the $(N+1)$-anyon covering space. We stress the rather subtle point that $[\Gamma_{\mathbf{x}}^{\{\mathbf{x}_1,\ldots,\mathbf{x}_N\}}]$ is defined by this procedure as a homotopy class: but in order to define it, we needed to use not merely the class $[\Gamma_0^\gamma]$, but the actual element Γ_0^γ within that class.

We now have all we need to construct the anyon creation field acting on the Hilbert space $\mathcal{H}_N^{eq}$, in close analogy with the second-quantized, nonrelativistic Bose and Fermi creation fields discussed above. Roughly speaking, we can already see from Figs. 1-4 how the q-commutator will enter. Suppose $\{\mathbf{x}_1, \mathbf{x}_2\}$ are as in Fig. 2. If we first create an anyon at $\mathbf{x}_2$, we obtain the path $\Gamma_1 = \Gamma_0^{\{\mathbf{x}_2\}}$, which is a straight line parallel to the x^2-axis, terminating at $\mathbf{x}_2$. Creating the next anyon at $\mathbf{x}_1$ gives us the paths in the class $[\Gamma_{\mathbf{x}_1}^{\{\mathbf{x}_2\}}]$. Such a pair of paths is depicted in Fig. 2, corresponding to the braid group generator. On the other hand, if we first create the anyon at $\mathbf{x}_1$, we consider the path $\Gamma_1 = \Gamma_0^{\{\mathbf{x}_1\}}$. Creating the next anyon at $\mathbf{x}_2$ gives us the paths in the class $[\Gamma_{\mathbf{x}_2}^{\{\mathbf{x}_1\}}]$, which for this example is just the class $\Gamma_0^{\{\mathbf{x}_1,\mathbf{x}_2\}}$ associated with the identity element of the braid group. There will thus be a relative phase $q = T(b_{12})$ occurring in the two products of creation operators, where T is the one-dimensional unitary representation of the braid group characterizing the anyons in the hierarchy.

Figure 4. An anyonic particle is created at $\mathbf{x}$, defining the element $\Gamma_{\mathbf{x}}^{\{\mathbf{x}_1,\dots,\mathbf{x}_N\}}$ of the $(N+1)$-anyon covering space.



More precisely, consider an equivariant wave function $\tilde{\Psi}_N$ in $\mathcal{H}_N^{eq}$. We write $\tilde{\Psi}_N = \tilde{\Psi}_N(\gamma, \Gamma)$, where $\gamma = \{\mathbf{x}_1, \dots, \mathbf{x}_N\}$ and the paths in Γ terminate at the points in γ. For fixed γ it is convenient to regard $\tilde{\Psi}_N$ as defined for the *individual* sets of paths Γ, but constant on the equivalence classes $[\Gamma]$. The equivariance of $\tilde{\Psi}_N$ is put in by requiring that if Γ is obtained from Γ_0^γ by the braid $b = h_\gamma(o)$, then

$$\tilde{\Psi}_N(\gamma, \Gamma) = T(b)\tilde{\Psi}_N(\gamma, \Gamma_0^\gamma) \, . \tag{3.2}$$

or with an alternative notation, $\tilde{\Psi}_N(\gamma, \Gamma) = T(\Gamma)\tilde{\Psi}_N(\gamma, \Gamma_0^\gamma)$. That is, specifying $\tilde{\Psi}_N$ for (almost) all values on a single sheet in the covering space $\tilde{\Delta}_N$, defines its values on any other sheet. We see now that the condition we imposed earlier in defining Γ_0^γ, that all the $\mathbf{x}_j$ have distinct values of their first coordinates, can be regarded merely as the omission of an arbitrary boundary (having measure zero) associated with crossing from one sheet to the next in $\tilde{\Delta}_N$.

Now we write, in analogy with Eqs. (2.8) and (2.9), the anyon annihilation and creation fields. Let $\tilde{\Psi}$ denote the sequence $(\tilde{\Psi}_N)$, $N = 0, 1, 2, \dots$ with $\tilde{\Psi}_N \in \mathcal{H}_N^{eq}$, and $(\tilde{\Psi}, \tilde{\Psi}) = \sum_N (\tilde{\Psi}_N, \tilde{\Psi}_N) < \infty$. We define

$$[\psi(\mathbf{x})\tilde{\Psi}]_N(\{\mathbf{x}_1, \dots, \mathbf{x}_N\}, \Gamma_0^{\{\mathbf{x}_1,\dots,\mathbf{x}_N\}}) = \tilde{\Psi}_{N+1}(\{\mathbf{x}_1, \dots, \mathbf{x}_N, \mathbf{x}\}, \Gamma_{\mathbf{x}}^{\{\mathbf{x}_1,\dots,\mathbf{x}_N\}}) \, ,$$

$$[\psi^*(\mathbf{x})\tilde{\Psi}]_N(\{\mathbf{x}_1, \dots, \mathbf{x}_N\}, \Gamma_0^{\{\mathbf{x}_1,\dots,\mathbf{x}_N\}}) =$$

$$\sum_{j=1}^{N} \delta(\mathbf{x} - \mathbf{x}_j)\, \tilde{\Psi}_{N-1}(\{\mathbf{x}_1, \dots, \hat{\mathbf{x}}_j, \dots, \mathbf{x}_N\}, \Gamma_0^{\{\mathbf{x}_1,\dots,\hat{\mathbf{x}}_j,\dots,\mathbf{x}_N\}})\, T^*(\Gamma_{\mathbf{x}}^{\{\mathbf{x}_1,\dots,\hat{\mathbf{x}}_j,\dots,\mathbf{x}_N\}}) \, , \tag{3.3}$$

where as before $\hat{\mathbf{x}}_j$ means that the point $\mathbf{x}_j$ is omitted. This defines $[\psi(\mathbf{x})\Psi]_N$ and $[\psi^*(\mathbf{x})\tilde{\Psi}]_N$ on one sheet. We extend the definitions in (3.3) to the other sheets of $\tilde{\Delta}_N$ (in general, infinitely many of them) using the equivariance property (3.2).

Comparing (3.3) with (2.8) and (2.9), note that the factors $\sqrt{N+1}$ and $1/\sqrt{N}$ no longer appear. This is because, in the case of anyons, the inner product is defined with respect to integration in the base space Δ_N (or, equivalently, over just one sheet of $\tilde{\Delta}_N$). It *must* be defined so, as the number of sheets in the covering space is not $N!$ any longer, but is now infinite.

When all but one of the N-particle components of Ψ vanish, we see from (3.3) that $\psi : \mathcal{H}^{eq}_{N+1} \to \mathcal{H}^{eq}_N$, and $\psi^* : \mathcal{H}^{eq}_N \to \mathcal{H}^{eq}_{N+1}$. In fact, ψ and ψ^* defined in this way are just the intertwining fields obeying Eqs. (2.12).

Finally, we are in a position to determine, by straightforward calculation from (3.3), what commutation relations the anyon fields we have obtained satisfy with each other. The answer is just the q-commutation relations given by Eqs. (2.1), where q is the phase specified by the representation T of the braid group generator. Furthermore, we recover the operators $\rho(\mathbf{x})$ and $\mathbf{J}(\mathbf{x})$ in terms of the anyon fields as the desired expressions given by (1.1), with $\int(\tilde{\Psi}_N, \rho(\mathbf{x})\tilde{\Psi}_N)\, d^2x = Nm$.

4. CONCLUSION

To sum up we have proposed a way, beginning with a collection of diffeomorphism group representations, to classify them into hierarchies based on the existence of intertwining field operators. Our method works not only for the N-particle Bose and Fermi representations, $N = 0, 1, 2, \ldots$ (by which it is motivated), but for the N-anyon representations of $Diff(\mathbf{R}^2)$ that we previously obtained. Anyons with distinct values of the phase characterizing the intermediate statistics belong naturally to different hierarchies.

Then we obtain q-commutation relations for the resulting anyon fields as a *consequence* of our prescription. Assuming little more than the fundamental role played by CURs of the diffeomorphism group, we thus arrive by entirely natural means at a framework for treating the many-anyon system. Our approach provides an alternative to beginning with the introduction of fields obeying noncanonical commutation relations, or to beginning with particles obeying canonical fields and introducing a Chern-Simons potential to describe the anyons.

We expect this method to generalize to still other hierarchies of diffeomorphism group representations, such as those describing extended objects like quantized vortex loops and filaments.

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