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ON A SMALL BODY

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Inverse geophysical and potential scattering on a small body

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ABSTRACT

Simple and numerically stable approaches to approximate solution of inverse geophysical and potential scattering problems are described. The method we propose consists of two steps. Let $v(z)$ be the inhomogeneity (potential), and let D be its support. First, we find approximations to the zeroth moment (total intensity) $\int_D v(z)dz$ and the first moment (center of gravity) $\int_D zv(z)dz / \int_D v(z)dz$ of the function $v(z)$. We call this step “inhomogeneity localization”, because in many cases the center of gravity lies inside D or is located close to it. Second, we refine the above moments and find the tensor of the second central moments of $v(z)$. Using this information, we find an ellipsoid $\tilde{D}$ and a real constant $\tilde{v}$, such that the inhomogeneity (potential) $\tilde{v}(z) = \tilde{v}, z \in \tilde{D}$, and $\tilde{v}(z) = 0, z \notin \tilde{D}$, fits best the scattering data and has the same zeroth, first, and second moments. We call this step “approximate inversion”. The proposed method does not require any intensive computations, it is very simple to implement and it is relatively stable towards noise in the data.

Keywords: inverse problem, approximate solution, geophysical scattering, potential scattering.

1. INTRODUCTION

Inverse problem of geophysics consists of finding the inhomogeneity from the surface data. Inverse problem of potential scattering consists of finding the potential knowing the scattering amplitude. There exist many different approaches to solving the above inverse problems, e.g. parameter fitting^{7,11}, linearized inversion based on low-frequency approximation^{6,9}, inversion based on weak scatterer assumption³, exact fixed frequency inversion^{8,10}. However, the majority of the known algorithms are computationally intensive and do not work stably, because the inverse problems are severely ill-posed.

The purpose of this paper is to describe computationally simple and numerically more stable approaches to approximate solution of inverse geophysical and potential scattering problems, which were proposed in^{4,5}. First, consider the geophysical inverse problem. Let an unknown inhomogeneity D be characterized by an unknown real-valued function $v(z), z \in D$. The method we propose consists of two steps. First, we find approximations to the zeroth moment (total intensity) $v_D = \int_D v(z)dz$ and the first moment (center of gravity) $z^{(0)} = \int_D zv(z)dz / \int_D v(z)dz$ of the function $v(z)$. We call this step “inhomogeneity localization”, because in many cases the center of gravity lies inside the inhomogeneity D , or, at least, is located close to it. Second, we refine the above moments and find the tensor of the second central moments of $v(z)$. Using this information, we find an ellipsoid $\tilde{D}$ and a real constant $\tilde{v}$, such that the inhomogeneity $\tilde{v}(z) = \tilde{v}, z \in \tilde{D}$, and $\tilde{v}(z) = 0, z \notin \tilde{D}$, fits best the surface data and has the same zero, first, and second moments. We call this step “approximate inversion”. In case of potential scattering, we, similarly, find the first three moments of the unknown potential $q(x), x \in D$. The accuracy of the above procedure has been analyzed. This preliminary information about the inhomogeneity (potential) is of practical interest by itself and can be used as an initial approximation in solving the full inverse problem. It is important to emphasize that the above methods do not require any intensive computations, they are very simple to implement and they are relatively stable towards noise in the data.

The basic assumption which allows us to construct the desired algorithm is that the domain D , the support of inhomogeneity (potential) shrinks to some point, i.e. $d := \text{diam}(D) \rightarrow 0$. More specifically, the limit “ $d \rightarrow 0$ ” means that we consider a sequence of domains $\{D_d\}$ and functions $\{v_d\}$ satisfying the following conditions:

- (a) $D_{d_1} \subset D_{d_2}$ if $d_1 < d_2$, and $\exists c_1, c_2 > 0$ such that $c_1 d^3 \leq \text{Vol}(D_d) \leq c_2 d^3$ as $d \rightarrow 0$;
- (b) $v_d(z) = v(z)\chi(D_d)$, where $v(z)$ is a real-valued piecewise-continuous function which does not depend on d , $\chi(D_d)$ is a characteristic function of D_d ; and
- (c) $\int_{D_d} |v(z)| dz / |\int_{D_d} v(z) dz| < r$ for all $d, 0 < d \leq d_0$, and $\sup_{z \in \mathbb{R}^3} |v(z)| < \infty$.

In what follows, we drop the subscript “ d ” in the notation D_d and q_d . In case of potential scattering, we assume that the potential q satisfies assumptions (a) - (c).

Note that the approximation based on the assumption $\text{diam}(\text{supp}(v)) := d \rightarrow 0$, used in this paper, differs considerably from the standard Born (low-frequency) approximation which is used in the geophysical setting when $k \rightarrow 0$ ($kd \ll 1$). Using our approach, we are able to find approximately moments of the inhomogeneity $v(z)$ *itself*, whereas considering the limit $k \rightarrow 0$ leads either to a very ill-posed integral equation with respect to v ^{6,9} or to finding the moments of the scattering solution², but not to the moments of the inhomogeneity. In the potential scattering case, the limiting static inverse problem at $k = 0$ is even more complicated than the similar problem for the geophysical scattering. In contrast, the approach taken in this paper yields an inversion procedure which is not nearly as ill-posed as the procedures mentioned above. The reason is simple: we find only a finite-parametric family of the inhomogeneities (potentials). On the other hand, an inhomogeneity (potential) from this family generates the scattering data which fit well the observed scattering data. From this point of view our solution may be quite satisfactory in many cases. Also, it is important to mention that the emphasis in this paper is on a new method for finding the first few moments of the unknown inhomogeneity (potential) from the scattering data. Thus the problem we consider is different from the standard problem of finding an unknown function given its moments.

In Section 2 we give mathematical description of the inverse problems. Geophysical inverse problem is considered in Section 3. Since factors in the terms of the type $O(d^\gamma)$, $\gamma = \text{const}$, which give the criteria for the smallness of the parameter d are quite complicated, we do not give them here. The interested reader might find them in⁴, where also the proofs of all theorems from Section 3 are given and results of numerical experiments are presented. The case of potential scattering is considered in Section 4. Again, all theorems there are presented without proofs, and these proofs can be found in⁵.

2. GENERAL DESCRIPTION OF THE MODEL

In many applications, the geophysical inverse problem can be formulated as follows. In the known medium, there is an unknown compactly supported inhomogeneity D , which is characterized by a real-valued compactly supported function $v(z)$, $z \in D := \text{supp}(v)$. The inhomogeneity is excited by a point source situated at a point y on the Earth's surface $P := \{x : x_3 = 0\}$ (the x_3 -axis is directed downward) and, for various positions x of the receiver on the plane P , one knows the total field $U(x, y, k)$, which satisfies the equation

$$(L + k^2 + k^2 v(x))U(x, y, k) = -\delta(x - y), \quad x \in \mathbb{R}^3, \quad (2.1)$$

where L is a second order elliptic operator with coefficients which are smooth in a neighborhood of D , v is a piecewise-continuous function, $k > 0$ is a wavenumber. The field U satisfies the outgoing radiation condition

$$|x \cdot \nabla_x U(x, y, k) - ik|x|U(x, y, k)| \rightarrow 0 \text{ as } |x| \rightarrow \infty. \quad (2.2)$$

It is also assumed that

- (A1) The operator $L + k^2$ has a Green function $G(x, y, k)$, which satisfies the radiation condition for all k at which $U(x, y, k)$ is known;
- (A2) Fix an arbitrary compact domain $K \subset \mathbb{R}_+^3$. There exists $C > 0$, such that $|G(x, y, k)| \leq C|x - y|^{-1}$, $\forall x, y \in K$, and for all k at which $U(x, y, k)$ is known, where C may depend on K , but is independent of x, y , and k .

The problem is to find D and to determine the function $v(z)$, $z \in D$, from the knowledge of $U(x, y, k)$ for x and y in some finite subset of $P \times P$ and for one (or several) fixed $k > 0$.

In the potential scattering case, the governing equation is

$$[\nabla^2 + k^2 - q(x)]U(x, \alpha) = 0, \quad x \in \mathbb{R}^3, \quad k = \text{const} > 0, \quad (2.3)$$

$$U(x, \alpha) = e^{ik\alpha \cdot x} + A(\alpha', \alpha, k) \frac{\exp(ik|x|)}{|x|} + o(|x|^{-1}), \quad \alpha' = x/|x|, \quad (2.4)$$

where $\alpha, \alpha' \in S^2$, the unit sphere in $\mathbb{R}^3$, α is the direction of the incident plane wave, and $q(x)$ is a compactly supported real-valued potential. The solution $U(x, \alpha)$ to the problem (2.3) and (2.4) is called the scattering solution, the coefficient $A(\alpha', \alpha)$ is called the scattering amplitude.

The problem is to find D and to determine the function $q(x)$, $x \in D$, from the knowledge of $A(\alpha', \alpha) := A(\alpha', \alpha, k)$ for α and α' in some finite subset of $S^2 \times S^2$ and for one (or several) fixed $k > 0$.

It is known that under some suitable assumptions about the given data, the solutions to the above two inverse problems are unique¹⁰.

3. GEOPHYSICAL INVERSE PROBLEM

3.1. Localization of inhomogeneity.

For convenience, we introduce the function

$$f_k(x, y) := k^{-2}(U(x, y, k) - G(x, y, k)), \quad (3.1)$$

which can be computed for $\xi := (x, y) \in P \times P$ where U is known. Let us assume that the domain D is not large, that is the Green function $G(x, z, k)$ does not change much in D if $x \in P$ is fixed:

$$\max_{z \in D} |\nabla G(x, z, k)| d \ll \min_{z \in D} |G(x, z, k)|, \quad d := \text{diam}(D) = \sup_{z, z' \in D} |z - z'|. \quad (3.2)$$

We have

$$U(x, y, k) = G(x, y, k) + k^2 \int_D G(x, z, k) v(z) U(z, y, k) dz, \quad x \in D, \quad y \in P. \quad (3.3)$$

Using (3.3) in (3.1) and dropping k from the arguments for notational convenience, we get

$$f_k(x, y) = \int_D G(x, z) v(z) G(z, y) dz + \int_D G(x, z) v(z) [k^2 \int_D G(z, w) v(w) U(w, y) dw] dz, \quad x, y \in P. \quad (3.4)$$

If $\int_D v(z) dz \neq 0$, using (3.2) and neglecting the contribution of the second integral in (3.4), we can rewrite the last equation

$$f_k(x, y) \approx G(x, \tilde{z}^{(0)}) G(\tilde{z}^{(0)}, y) \int_D v(z) dz, \quad x, y \in P, \quad (3.5)$$

for some $\tilde{z}^{(0)}$ in a neighborhood of D . Let us denote $\xi := (x, y)$, $\hat{G}_k(\xi, z) := G(x, z) G(z, y)$, and let ξ_i , $i \in I$, be the set of data points for which the total field is known, I is a finite set of indices. One sees from (3.5) that the functions $f_k(\xi)$ and $\hat{G}_k(\xi, \tilde{z}^{(0)})$ are approximately proportional for some $\tilde{z}^{(0)}$ inside (or close to) D . Thus we can expect that the position $\tilde{z}^{(0)}$ of a global maximum of the correlation coefficient

$$r_k(z) := \frac{|\sum_{i \in I} f_k(\xi_i) \overline{\hat{G}_k(\xi_i, z)}|^2}{\sum_{i \in I} |f_k(\xi_i)|^2 \sum_{i \in I} |\hat{G}_k(\xi_i, z)|^2} \quad (3.6)$$

determines a point in a neighborhood of D . In (3.6) and below, the bar stands for complex conjugate. Let v_D and $z^{(0)}$ denote the total intensity (the zeroth moment) and the center of gravity (the vector of the first moments) of the inhomogeneity v , respectively:

$$v_D = \int_D v(z) dz, \quad z^{(0)} := \frac{\int_D z v(z) dz}{\int_D v(z) dz}.$$

It follows from Assumption (c) in Section 1 that $\int_D v(z)dz \neq 0$, hence $z^{(0)}$ is well defined. Using Assumption (A2), we estimate the second term in (3.4) as follows

$$\left| \int_D G(x, z)v(z) \left[k^2 \int_D G(z, w)v(w)U(w, y)dw \right] dz \right| = v_D O(d^2) \text{ as } d \rightarrow 0, \quad x, y \in P. \quad (3.7)$$

Since $\int_D (z - z^{(0)})v(z)dz = 0$, we obtain using (3.4) and (3.7):

$$f_k(\xi) = v_D (\hat{G}_k(\xi, z^{(0)}) + O(d^2)), \quad \xi \in P \times P. \quad (3.8)$$

Substituting (3.8) into (3.6), we get after some transformations

$$1 - r_k(z) = (B(z - z^{(0)}), z - z^{(0)}) + O(d^3), \quad (3.9)$$

where B is a positive-definite matrix. In case of the noisy data, the only difference is that the noise in U : $U_n(\xi_i, k) := U(\xi_i, k) + n_i$ results in the following noise in f_k (see (3.1)): $f_k^{(n)}(\xi_i) := f_k(\xi_i) + n_i/k^2$. Using (3.9), we prove

Theorem 3.1. *Suppose that (3.2) holds, and one knows a priori a bounded domain V containing D , $D \subset V$. Given the exact data $U(\xi_i, k)$, $i \in I$, the position $\tilde{z}^{(0)}$ of the global maximum of $r_k(z)$ in V is*

$$\tilde{z}^{(0)} = z^{(0)} + O(d^{1.5}) \text{ as } d \rightarrow 0. \quad (3.10)$$

Given the noisy data $U_n(\xi_i, k) := U(\xi_i, k) + n_i$, $i \in I$, where n_i are independent random variables with zero mean value and variance σ^2 , the position $\tilde{z}^{(0)}$ of the global maximum of $r_k(z)$ in V is

$$\tilde{z}^{(0)} = z^{(0)} + O\left(\sqrt{d^3 + 2|\nu| \|f_k\|^{-1} + |\nu|^2 \|f_k\|^{-2}}\right) \text{ as } d \rightarrow 0, |I| \rightarrow \infty, \quad (3.11)$$

where $|I|$ is the number of indices in the set I , that is the number of data points, ν is a random variable with zero mean value and variance $\sigma^2/(k^4|I|)$, and $\|f_k\| := (|I|^{-1} \sum_{i \in I} |f_k(\xi_i)|^2)^{1/2}$.

Let us denote $\varphi(d, |I|) := \sqrt{d^3 + 2|\nu| \|f_k\|^{-1} + |\nu|^2 \|f_k\|^{-2}}$. In (3.11), $O(\varphi(d, |I|))$ means a random variable, such that there exist $c_1, c_2 > 0$ with $|E[O(\varphi(d, |I|))]| \leq c_1 E[\varphi(d, |I|)]$ and $\sigma^2[O(\varphi(d, |I|))] \leq c_2 \sigma^2[\varphi(d, |I|)]$ as $d \rightarrow 0$ and $|I| \rightarrow \infty$. Here E and σ^2 stand for mathematical expectation and variance, respectively.

Although Theorem 3.1 asserts that the position of the global maximum of the correlation coefficient $r(z)$ is attained in a neighborhood of the center of gravity of $v(z)$, this does not guarantee that $\tilde{z}^{(0)} \in D$. In the Corollary below, we list some important cases when $\tilde{z}^{(0)}$ belongs to D (or its convex hull) provided that d is sufficiently small.

Corollary.

- (1) If $v(z)$ does not change sign in D , then $\tilde{z}^{(0)}$ belongs to the convex hull of D ;
- (2) If $v(z)$ does not change sign in D and D is convex, then $\tilde{z}^{(0)} \in D$; and
- (3) If $v(z)$ possesses central symmetry and D is connected, then $\tilde{z}^{(0)} \in D$ and $\tilde{z}^{(0)}$ coincides with the center of symmetry of D (up to the terms of order $O(d^{1.5})$).

Using Theorem 3.1 and its Corollary, we obtain the following

Algorithm for localization of the inhomogeneity. One calculates $r(z)$ at a discrete set of points in a lower half-space (below the Earth's surface). The position $\tilde{z}^{(0)}$ of the global maximum of $r(z)$ determines a point inside (or close to) the inhomogeneity.

After the point $\tilde{z}^{(0)}$ of the global maximum of $r(z)$ have been found, it is possible to determine the total intensity of the inhomogeneity. We see from (3.5), that v_D is the coefficient of proportionality between $f_k(\xi)$ and $\hat{G}_k(\xi, \tilde{z}^{(0)})$. Therefore, the formula to compute v_D approximately is

$$\tilde{v}_D := \frac{\text{Re}[\sum_{i \in I} f_k(\xi_i) \overline{\hat{G}_k(\xi_i, \tilde{z}^{(0)})}]}{\sum_{i \in I} |\hat{G}_k(\xi_i, \tilde{z}^{(0)})|^2}.$$

Theorem 3.2. Under the assumptions of Theorem 3.1, one has

$$\tilde{v}_D = v_D(1 + O(d^{1.5})) \quad \text{as } d \rightarrow 0, \quad (3.12)$$

given the exact data $U(\xi_i, k)$, $i \in I$; and

$$\tilde{v}_D = v_D(1 + O(\sqrt{d^3 + 2|\nu|}\|f_k\|^{-1} + |\nu|^2\|f_k\|^{-2})) \quad \text{as } d \rightarrow 0, |I| \rightarrow \infty,$$

given the noisy data $U_n(\xi_i, k)$, $i \in I$.

3.2. Approximate inversion.

In this section, we develop an algorithm for determining an ellipsoid-shaped inhomogeneity $\tilde{D}$ and a constant function $v(z) = \tilde{v}$, $z \in \tilde{D}$, which best fit the data. This algorithm is based on finding the tensor of the second central moments of $v(z)$. Using smallness of D and the equation $|G(z, y) - U(z, y)| = O(d^2)$, $z \in D, y \in P$, let us simplify the second integral in (3.4).

$$\int_D G(x, z)v(z)[k^2 \int_D G(z, w)v(w)U(w, y)dw]dz = k^2(v_D)^2 \hat{G}_k(\xi, z^{(0)})R + O(d^6), \quad x, y \in P, \quad d \rightarrow 0, \quad (3.13)$$

where

$$R := \int_D \int_D v(z)G(z, w)v(w)dw dz / \left(\int_D v(z)dz \right)^2. \quad (3.14)$$

Using Assumption (A2), we get $(v_D)^2 R = O(d^5)$. From (3.4) and (3.13), we obtain

$$f_k(\xi) = v_D \left(\hat{G}_k(\xi, z^{(0)}) + \sum_{\substack{j,l=1 \\ j \leq l}}^3 A_{jl}(\xi, z^{(0)}) \overset{\circ}{J}_{jl} + k^2 v_D \hat{G}_k(\xi, z^{(0)})R \right) + O(d^6), \quad \xi \in P \times P, \\ A_{jl}(\xi, z) := (1 - \frac{\delta_{jl}}{2}) \frac{\partial^2 \hat{G}_k(\xi, z)}{\partial z_j \partial z_l}, \quad \overset{\circ}{J}_{jl} := \frac{\int_D (z_j - z_j^{(0)})(z_l - z_l^{(0)})v(z)dz}{\int_D v(z)dz}, \quad j, l = 1, 2, 3, \quad (3.15)$$

where $\delta_{jl} = 1$ if $j = l$ and $\delta_{jl} = 0$ if $j \neq l$, and $\overset{\circ}{J} = (\overset{\circ}{J}_{jl})_{j,l=1}^3$ is the symmetric tensor of the second central moments of the inhomogeneity v . Let us introduce "the apparent total intensity"

$$v_D^* := v_D + k^2(v_D)^2 R. \quad (3.16)$$

Using (3.16), we transform (3.15) as follows:

$$f_k(\xi) = v_D^* \left(\hat{G}_k(\xi, z^{(0)}) + \sum_{\substack{j,l=1 \\ j \leq l}}^3 A_{jl}(\xi, z^{(0)}) \overset{\circ}{J}_{jl} \right) + O(d^6), \quad \xi \in P \times P. \quad (3.17)$$

Equations (3.10) and (3.12) yield

$$\Delta_z^{\circ} := z^{(0)} - \tilde{z}^{(0)} = O(d^{1.5}), \quad \Delta \overset{\circ}{v}_D := v_D^* - \tilde{v}_D = (v_D - \tilde{v}_D) + k^2(v_D)^2 R = O(d^{4.5}) + O(d^5) = O(d^{4.5}). \quad (3.18)$$

The Taylor formula and (3.18) imply

$$\hat{G}_k(\xi, z^{(0)}) = \hat{G}_k(\xi, \tilde{z}^{(0)}) + \nabla \hat{G}_k(\xi, \tilde{z}^{(0)}) \cdot \Delta_z^{\circ} + O(d^3), \quad \xi \in P \times P, \quad d \rightarrow 0.$$

Substituting the last equation and (3.18) into (3.17), we get

$$\psi_k := \tilde{v}_D \hat{G}_k(\xi, \tilde{z}^{(0)}) + \Delta \overset{\circ}{v}_D \hat{G}_k(\xi, \tilde{z}^{(0)}) + \tilde{v}_D \nabla \hat{G}_k(\xi, \tilde{z}^{(0)}) \cdot \Delta_z^{\circ} + \tilde{v}_D \sum_{\substack{j,l=1 \\ j \leq l}}^3 A_{jl}(\xi, \tilde{z}^{(0)}) \overset{\circ}{J}_{jl}, \quad (3.19)$$

where

$$|\psi_k(\xi) - f_k(\xi)| = O(d^6), \quad \xi \in P \times P. \quad (3.20)$$

Note that the function ψ_k is linear with respect to the unknown parameters $\Delta \overset{\circ}{v}_D$, $\Delta \overset{\circ}{z}$, and $\overset{\circ}{J}_{jl}$. Using (3.19) and (3.20), we obtain the error functional (in the L_2 - norm)

$$\begin{aligned} & \Phi(\Delta v_D, \Delta z, J) \\ & := \sum_{i \in I} |f_k(\xi_i) - \tilde{v}_D \hat{G}_k(\xi_i, \tilde{z}^{(0)}) - \Delta v_D \hat{G}_k(\xi_i, \tilde{z}^{(0)}) - \tilde{v}_D \nabla_z \hat{G}_k(\xi_i, \tilde{z}^{(0)}) \cdot \Delta z - \tilde{v}_D \sum_{\substack{j,l=1 \\ j \leq l}}^3 A_{jl}(\xi_i, \tilde{z}^{(0)}) J_{jl}|^2 \rightarrow \min. \end{aligned} \quad (3.21)$$

From (3.19)-(3.21) it follows that we have to find ten scalar variables Δv_D , $\Delta z := (\Delta z_1, \Delta z_2, \Delta z_3)$, and J_{jl} , $j, l = 1, 2, 3, j \leq l$ (recall that $J_{jl} = J_{lj}$), what can be done by solving the minimization problem (3.21). Let $\Delta \tilde{v}_D$, $\Delta \tilde{z}$, and $\tilde{J}_{jl}$ be the optimal values of the parameters which minimize the functional Φ . Since Φ is quadratic, these values satisfy the system of normal equations, which can be written as

$$(\Psi^T \Psi)(\Delta \tilde{v}_D, \Delta \tilde{z}_j, \tilde{J}_{jl})^T = \Psi^T (f_k(\xi_1) - \tilde{v}_D \hat{G}_k(\xi_1, \tilde{z}^{(0)}), \dots, f_k(\xi_{|I|}) - \tilde{v}_D \hat{G}_k(\xi_{|I|}, \tilde{z}^{(0)}))^T, \quad (3.22)$$

where Ψ is a rectangular matrix, and $(\Psi^T \Psi)$ is a nondegenerate symmetric 10×10 matrix. Therefore equations (3.19) - (3.22) together with the relation $\tilde{v}_D = O(d^3)$ yield

$$\Delta \tilde{v}_D = \Delta \overset{\circ}{v}_D + O(d^6), \quad \Delta \tilde{z}_j = \Delta \overset{\circ}{z}_j + O(d^3), \quad \tilde{J}_{jl} = \overset{\circ}{J}_{jl} + O(d^3).$$

Theorem 3.3. *Let the assumptions be the same as that of Theorem 3.1. One has*

$$\tilde{v}_D^* = v_D^*(1 + O(d^3)), \quad \tilde{z}_j^{(0)} = z_j^{(0)} + O(d^3), \quad \tilde{J}_{jl} = \overset{\circ}{J}_{jl} + O(d^3), \quad j, l = 1, 2, 3,$$

where

$$\tilde{v}_D^* := \tilde{v}_D + \Delta \tilde{v}_D, \quad \tilde{z}_j^{(0)} := \tilde{z}_j^{(0)} + \Delta \tilde{z}_j, \quad (3.23)$$

given the exact data; and

$$\tilde{v}_D^* = v_D^*(1 + O(d^3 + \hat{\nu})), \quad \tilde{z}_j^{(0)} = z_j^{(0)} + O(d^3 + \hat{\nu}), \quad \tilde{J}_{jl} = \overset{\circ}{J}_{jl} + O(d^3 + \hat{\nu}) \quad j, l = 1, 2, 3,$$

given the noisy data, where $\hat{\nu} := 2|\nu| \|f_k\|^{-1} + |\nu|^2 \|f_k\|^{-2}$.

Suppose that the apparent total intensity $\tilde{v}_D^*$, the vector of the first moments $\tilde{z}^{(0)}$, and the tensor of the second central moments $\tilde{J} := (\tilde{J}_{jl})$ of the function $v(z)$ have been found from (3.22) and (3.23). Then one can determine an ellipsoid $\tilde{D}$ and a real number $\tilde{v}$ such that a function $\tilde{v}(z) = \tilde{v}\chi(\tilde{D})$ has exactly the same first three moments. This can be done as follows.

1. Since a function $\tilde{v}(z) = \tilde{v}\chi(\tilde{D})$, where $\tilde{D}$ is an ellipsoid, has central symmetry, the center of $\tilde{D}$ is chosen to be $\tilde{z}^{(0)}$ (see Theorem 3.1);
2. One finds the eigenvalue decomposition $\tilde{U}\tilde{\Sigma}\tilde{U}^T$, $\tilde{\Sigma} = \text{diag}(\tilde{\sigma}_1, \tilde{\sigma}_2, \tilde{\sigma}_3)$, $\tilde{U}\tilde{U}^T = I$, of the tensor $\tilde{J}$. Assuming in (3.15) that D is an ellipsoid and $v(z) = \text{const}$, $z \in D$, we get that the tensor $\tilde{J}$ is diagonal with the j -th diagonal element $h_j^2/5$, where h_j is the j -th half-axis of the ellipsoid. Therefore, the half-axes of the ellipsoid $\tilde{D}$ are $\tilde{h}_j = \sqrt{5\tilde{\sigma}_j}$, $j = 1, 2, 3$, and it is defined to be

$$\tilde{D} := \{z : z = \tilde{U}z', \sum_{j=1}^3 (z'_j - \tilde{z}_j^{(0)})^2 / \tilde{h}_j^2 \leq 1\};$$

3. Using (3.14) and (3.16), one computes

$$\tilde{R} = \frac{1}{\text{Vol}^2(\tilde{D})} \int_{\tilde{D}} \int_{\tilde{D}} G(z, w) dz dw, \quad \tilde{v}_D := \frac{-1 + \sqrt{1 + 4k^2 \tilde{R} \tilde{v}_D^*}}{2k^2 \tilde{R}} \approx \tilde{v}_D^* (1 - k^2 \tilde{R} \tilde{v}_D^*);$$

4. Finally, the parameter $\tilde{v}$ equals

$$\tilde{v} = \tilde{v}_D / \text{Vol}(\tilde{D}) = \tilde{v}_D / \left(\frac{4\pi}{3} \tilde{h}_1 \tilde{h}_2 \tilde{h}_3 \right).$$

Suppose now that the original inhomogeneity $v(z)$ is indeed of the type $v(z) = v, z \in D, v(z) = 0, z \notin D$, where D is an ellipsoid. Then the above algorithm is consistent if it recovers v and D with increasing accuracy as $\text{diam}(D) \rightarrow 0$. The accuracy (and consistency) of the above inversion procedure is described in Theorem 3.4.

Theorem 3.4. *Let $\gamma_j > 0, j = 1, 2, 3$, be fixed, and consider ellipsoids*

$$D_d := \{z : z = Uz', \sum_{j=1}^3 (z'_j - z_j^{(0)})^2 / (\gamma_j d)^2 \leq 1\}, \quad d \rightarrow 0,$$

where U is a fixed orthogonal matrix, and $z^{(0)} \in \mathbb{R}_+^3$ is a fixed point. Let a number $v \in \mathbb{R}$ be also fixed. Then the algorithm for the approximate inversion recovers D_d and v with the accuracy

$$\frac{\text{Vol}(D_d \Delta \tilde{D}_d)}{\text{Vol}(D_d)} = O(d), \quad \tilde{v} = v(1 + O(d)),$$

where $D_d \Delta \tilde{D}_d := (D_d \cup \tilde{D}_d) \setminus (D_d \cap \tilde{D}_d)$ is the symmetric difference, given the exact data; and

$$\frac{\text{Vol}(D_d \Delta \tilde{D}_d)}{\text{Vol}(D_d)} = O(d + \frac{\hat{v}}{d^2}), \quad \tilde{v} = v(1 + O(d + \frac{\hat{v}}{d^2})),$$

given the noisy data.

4. INVERSE POTENTIAL SCATTERING

4.1. Localization of the potential.

As in Section 3, we assume that the frequency $k > 0$ is fixed. We start with a pair of well-known equations

$$A(\alpha', \alpha) = -\frac{1}{4\pi} \int_D e^{-ik\alpha' \cdot x} q(x) U(x, \alpha) dx, \quad (4.1)$$

$$U(x, \alpha) = e^{ik\alpha \cdot x} - \int_D \frac{e^{ik|x-y|}}{4\pi|x-y|} q(y) U(y, \alpha) dy, \quad \alpha, \alpha' \in S^2, \quad (4.2)$$

where S^2 is the unit sphere in $\mathbb{R}^3$. Here $e^{ik\alpha \cdot x}$ is the incident wave, and $A(\alpha', \alpha)$ is the scattering amplitude. Substitution of (4.2) into (4.1) yields

$$-4\pi A(\alpha', \alpha) = \int_D e^{-ik\alpha' \cdot x} q(x) e^{ik\alpha \cdot x} dx - \int_D e^{-ik\alpha' \cdot x} q(x) \left\{ \int_D \frac{e^{ik|x-y|}}{4\pi|x-y|} q(y) U(y, \alpha) dy \right\} dx. \quad (4.3)$$

Let q_D and $x^{(0)}$ denote the total intensity and the center of gravity of the potential q , respectively (cf. Section 3.1). Expanding exponentials in the Taylor series around $x^{(0)}$ in (4.3), we obtain

$$\begin{aligned} -4\pi A(\alpha', \alpha) = e^{ik(\alpha - \alpha') \cdot x^{(0)}} & \left(\int_D (1 + ik(\alpha - \alpha') \cdot \Delta x - \frac{1}{2}(k(\alpha - \alpha') \cdot \Delta x)^2 + O(k^3 d^3)) q(x) dx \right. \\ & \left. + \int_D (1 + O(kd)) q(x) \left[\int_D \frac{e^{ik|x-y|}}{4\pi|x-y|} q(y) (1 + O(Qd^2)) dy \right] dx \right), \end{aligned} \quad (4.4)$$

where $Q := \max_{x \in D} |q(x)|$, $\Delta x := x - x^{(0)}$, and we have used the property $U(y, \alpha) = e^{ik\alpha \cdot y} + O(Qd^2)$, $d \rightarrow 0$. Similarly to Section 3.1, we estimate the double integral in (4.4) (cf. (3.7)):

$$\left| \int_D (1 + O(kd))q(x) \left[\int_D \frac{e^{ik|x-y|}}{4\pi|x-y|} q(y)(1 + O(Qd^2))dy \right] dx \right| \leq q_D O(Qd^2), \quad (4.5)$$

where we have used Assumptions (a) – (c) from Section 1. Since $\int_D q(x)\Delta x dx = \int_D q(x)(x - x^{(0)})dx = 0$ and

$$\int_D \left(-\frac{1}{2}(k(\alpha - \alpha') \cdot \Delta x)^2 + O(k^3 d^3) \right) q(x) dx = q_D O(k^2 d^2),$$

substitution of (4.5) into (4.4) yields

$$-4\pi A(\alpha', \alpha) = e^{ik(\alpha - \alpha') \cdot x^{(0)}} q_D (1 + O(\xi d^2)), \quad \xi := k^2 + Q. \quad (4.6)$$

Taking modulus and argument on both sides of (4.6), we obtain

$$4\pi |A(\alpha', \alpha)| = |q_D| (1 + O(\xi d^2)), \quad (4.7)$$

$$\arg(-A(\alpha', \alpha)) = c(\alpha', \alpha), \quad 0 \leq c(\alpha', \alpha) < 2\pi, \quad (4.8)$$

and

$$c(\alpha', \alpha) = k(\alpha - \alpha') \cdot x^{(0)} + 2\pi n(\alpha', \alpha) + O(\xi d^2), \quad (4.9)$$

where $n(\alpha', \alpha)$ is an integer-valued function. Denote $\beta := \alpha - \alpha'$. If $\alpha, \alpha' \in S^2$, then, clearly, $|\beta| \leq 2$. Here S^2 is the unit sphere in $\mathbb{R}^3$. Conversely, given any β , $|\beta| \leq 2$, one can find uniquely $\alpha, \alpha' \in S^2$ so that $\beta = \alpha - \alpha'$. Therefore equation (4.9) can be written as

$$c(\beta) = k\beta \cdot x^{(0)} + 2\pi n(\beta) + O(\xi d^2), \quad \beta \in \mathbb{R}^3, |\beta| \leq 2. \quad (4.10)$$

According to equation (4.8), the function $c(\beta)$ is discontinuous for β only where $c(\beta) = 0$ or $c(\beta) = 2\pi - 0$. Suppose that the origin is located inside V , where V is the domain which contains the potential, $D \subset V$. Suppose also that the constant r from Assumption (c) in Section 1 is not too large, so that $x^{(0)} \in V$. For example, if $q(x) \geq 0$, then $r = 1$ and $x^{(0)} \in V$ if V is convex (cf. also the Corollary to Theorem 3.1). Define

$$\Delta := \pi / (\text{diam}(V)k). \quad (4.11)$$

Clearly,

$$|k(\beta_2 - \beta_1) \cdot x^{(0)}| < \pi \text{ if } |\beta_2 - \beta_1| < \Delta. \quad (4.12)$$

Fix now any β_1, β_2 such that $|\beta_2 - \beta_1| < \Delta$, $\beta_1 \neq \beta_2$, and consider three possible cases:

(1) One has

$$c(\beta_1) - (\pi - O(\xi d^2)) < c(\beta_2) < c(\beta_1) + (\pi - O(\xi d^2)). \quad (4.13)$$

Using (4.10) and (4.12), we obtain

$$(\beta_2 - \beta_1) \cdot x^{(0)} = \frac{c(\beta_2) - c(\beta_1) + O(\xi d^2)}{k}. \quad (4.14)$$

(2) One has

$$c(\beta_2) > c(\beta_1) + (\pi + O(\xi d^2)). \quad (4.15)$$

In view of (4.12), equation (4.15) may be valid if and only if the function $c(\beta)$ decreases from $c(\beta_1)$ to 0, jumps up to 2π , and then decreases to $c(\beta_2)$ as β moves along a line segment joining β_1 and β_2 in the direction from β_1 to β_2 . Therefore equation (4.10) yields

$$(\beta_2 - \beta_1) \cdot x^{(0)} = \frac{(c(\beta_2) - 2\pi) - c(\beta_1) + O(\xi d^2)}{k}. \quad (4.16)$$

(3) One has

$$c(\beta_2) < c(\beta_1) - (\pi + O(\xi d^2)). \quad (4.17)$$

In view of (4.12), equation (4.17) may be valid if and only if the function $c(\beta)$ increases from $c(\beta_1)$ to 2π , jumps down to 0, and then increases to $c(\beta_2)$ as β moves along a line segment joining β_1 and β_2 in the direction from β_1 to β_2 . Therefore equation (4.10) yields

$$(\beta_2 - \beta_1) \cdot x^{(0)} = \frac{(c(\beta_2) + 2\pi) - c(\beta_1) + O(\xi d^2)}{k}. \quad (4.18)$$

Using equations (4.13)-(4.18), we propose the following algorithm for locating the potential:

1. Let $\{\beta_j, c(\beta_j)\}$ be the available data. For all possible pairs j_1, j_2 , such that $|\beta_{j_1} - \beta_{j_2}| < \Delta$, $\beta_{j_1} \neq \beta_{j_2}$, form the equations

$$e_l \cdot x = \frac{1}{k} c_l, \quad l = 1, \dots, L, \quad (4.19)$$

where $e_l = \beta_{j_2} - \beta_{j_1}$, $c_l = c(\beta_{j_2}) - c(\beta_{j_1}) + 2\pi m$, and $m = 0, -1$, or 1 , depending on whether the first, second, or the third case has occurred, respectively;

2. Find the solution to (4.19) by solving the corresponding 3×3 system

$$(E^T E) \tilde{x}^{(0)} = \frac{1}{k} E^T C, \quad E := \begin{pmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ \dots & \dots & \dots \\ e_{L1} & e_{L2} & e_{L3} \end{pmatrix}, \quad C := (c_1, \dots, c_L)^T. \quad (4.20)$$

Analysing the proposed algorithm, we see that an accurate and stable recovery of $x^{(0)}$ is possible if

- (1) directions $\beta/|\beta|$ of the observation points $\beta = \alpha - \alpha'$ are distributed sufficiently uniformly over S^2 ; and
- (2) distribution of observation points $\beta = \alpha - \alpha'$ inside $B_2 := \{x : |x| \leq 2\} \subset \mathbb{R}^3$ is sufficiently dense, so that there are many pairs β_{j_1}, β_{j_2} , $|\beta_{j_1} - \beta_{j_2}| < \Delta$, $\beta_{j_1} \neq \beta_{j_2}$.

If the above two conditions are met, we easily prove using equations (4.15)-(4.20):

Theorem 4.1. Let $\tilde{x}^{(0)} = (\tilde{x}_1^{(0)}, \tilde{x}_2^{(0)}, \tilde{x}_3^{(0)})$ be the estimated value of $x^{(0)}$. Under Assumptions (a) - (c) from Section 1 one has $|\tilde{x}^{(0)} - x^{(0)}| = O\left[\left(k + \frac{Q}{k}\right) d^2\right]$, $d \rightarrow 0$.

To estimate q_D , fix arbitrary $\alpha', \alpha \in S^2$ and define $\tilde{q}_D = -4\pi A(\alpha', \alpha) e^{-ik(\alpha - \alpha') \cdot \tilde{x}^{(0)}}$. Using Theorem 4.1 and equation (4.6), one immediately proves

Theorem 4.2. Under Assumptions (a) - (c) from Section 1 one has $\tilde{q}_D = q_D(1 + O((k^2 + Q)d^2))$, $d \rightarrow 0$.

4.3. Algorithm for the approximate inversion.

Assumptions (a) - (c) imply that $\sup_{x \in D} |\int_D (4\pi|x - y|)^{-1} q(y) dy| = O(Qd^2)$. Using this, keeping the second order terms in equation (4.4), and letting $e^{ik|x - y|} = 1 + O(k|x - y|)$, $x, y \in D$, $d \rightarrow 0$, we obtain

$$\begin{aligned} -4\pi A(\alpha', \alpha) = e^{ik(\alpha - \alpha') \cdot x^{(0)}} & \left[\int_D q(x) dx + \int_D \int_D \frac{q(x)q(y)}{4\pi|x - y|} dx dy \right. \\ & \left. - \frac{k^2}{2} \int_D q(x) ((\alpha - \alpha') \cdot \Delta x)^2 dx + \int_D q(x) O(\mu d^3) dx \right], \end{aligned} \quad (4.21)$$

where $\mu := Q(k + Qd)$. Let $\Delta x = (\Delta x_1, \Delta x_2, \Delta x_3)$. The third integral in (4.21) can be written as

$$\int_D q(x) ((\alpha - \alpha') \cdot \Delta x)^2 dx = \sum_{1 \leq j, l \leq 3} c_{jl}(\alpha', \alpha) \int_D q(x) \Delta x_j \Delta x_l dx = q_D \sum_{1 \leq j, l \leq 3} c_{jl}(\alpha', \alpha) J_{jl}^0, \quad (4.22)$$

where $\overset{\circ}{J} = (\overset{\circ}{J}_{jl})_{j,l=1}^3$ is the symmetric tensor of the second central moments of the potential q (cf. (3.15)), $c_{jl}(\alpha', \alpha) := \nu(j, l)(\alpha'_j - \alpha_j)(\alpha'_l - \alpha_l)$, and $\nu(j, l) = 2$ if $j \neq l$ and $\nu(j, l) = 1$ if $j = l$. Using Assumptions (a) – (c) and (4.22), equation (4.21) becomes

$$-4\pi A(\alpha', \alpha) = e^{ik(\alpha - \alpha') \cdot \tilde{x}^{(0)}} q_D \left(1 + R - \frac{k^2}{2} \sum_{1 \leq j \leq l \leq 3} c_{jl}(\alpha', \alpha) \overset{\circ}{J}_{jl} + O(\mu d^3) \right), \quad (4.23)$$

where

$$R = \frac{1}{q_D} \int_D \int_D \frac{q(x)q(y)}{4\pi|x-y|} dx dy. \quad (4.24)$$

Let us denote

$$\Delta \overset{\circ}{q}_D := 1 - \frac{\tilde{q}_D}{q_D}, \quad \Delta \overset{\circ}{x} := x^{(0)} - \tilde{x}^{(0)}. \quad (4.25)$$

Substituting (4.25) into (4.23), using Theorems 4.1, 4.2, and Assumptions (a) – (c) from Section 1 (which imply that $R = O(Qd^2)$), we find after some transformations

$$-\left(\frac{4\pi A(\alpha', \alpha) e^{-ik(\alpha - \alpha') \cdot \tilde{x}^{(0)}}}{\tilde{q}_D} + 1 \right) = (\Delta \overset{\circ}{q}_D + R) - \frac{k^2}{2} \sum_{1 \leq j \leq l \leq 3} c_{jl}(\alpha', \alpha) \overset{\circ}{J}_{jl} + ik(\alpha - \alpha') \cdot \Delta \overset{\circ}{x} + O(\mu d^3).$$

Denoting

$$\hat{A}(\alpha', \alpha) := -\left(\frac{4\pi A(\alpha', \alpha) e^{-ik(\alpha - \alpha') \cdot \tilde{x}^{(0)}}}{\tilde{q}_D} + 1 \right),$$

we get

$$\text{Re}[\hat{A}(\alpha', \alpha)] = (\Delta \overset{\circ}{q}_D + R) - \frac{k^2}{2} \sum_{1 \leq j \leq l \leq 3} c_{jl}(\alpha', \alpha) \overset{\circ}{J}_{jl} + O(\mu d^3), \quad (4.26a)$$

$$\text{Im}[\hat{A}(\alpha', \alpha)] = k(\alpha - \alpha') \cdot \Delta \overset{\circ}{x} + O(\mu d^3). \quad (4.26b)$$

Let us introduce the following linear functional:

$$\Psi(\Delta q_D, J) := \Delta q_D - \frac{k^2}{2} \sum_{1 \leq j \leq l \leq 3} c_{jl}(\alpha', \alpha) J_{jl}. \quad (4.27)$$

Then, clearly,

$$\text{Re}[\hat{A}(\alpha', \alpha)] - \Psi(\Delta \overset{\circ}{q}_D + R, J) = O(\mu d^3). \quad (4.28)$$

Thus, approximations to $(\Delta \overset{\circ}{q}_D + R)$ and $\overset{\circ}{J}_{jl}$ can be found by minimizing the error functional

$$\Phi(\Delta q_D, J) := \sum_{\alpha', \alpha} (\text{Re}[\hat{A}(\alpha', \alpha)] - \Psi(\Delta q_D, J))^2 d\alpha d\alpha' \quad (4.29)$$

with respect to the seven scalar variables $\Delta q_D, J_{jl}, 1 \leq j \leq l \leq 3$. In (4.29) and below, $\sum_{\alpha', \alpha}$ denotes the summation over α and α' for which the scattering amplitude $A(\alpha', \alpha)$ is known. Since the functional in (4.29) is quadratic, the solution to the minimization problem $\Phi \rightarrow \min$ satisfies the system of normal equations which can be written explicitly. An approximation $\Delta \tilde{x}$ to $\Delta \overset{\circ}{x}$ can be obtained from (4.26b) by minimizing the quadratic functional

$$\sum_{\alpha', \alpha} \left(\text{Im}[\hat{A}(\alpha', \alpha)] - k(\alpha - \alpha') \cdot \Delta x \right)^2. \quad (4.30)$$

Let us denote

$$q_D^* := q_D(1 + R), \quad \tilde{q}_D^* := \tilde{q}_D(1 + \Delta \tilde{q}_D), \quad \tilde{x}^{(0)} := \tilde{x}^{(0)} + \Delta \tilde{x}, \quad (4.31)$$

where $(\Delta \tilde{q}_D, \tilde{J})$ and $\Delta \tilde{x}$ are optimal values of parameters which minimize (4.29) and (4.30), respectively. Similarly to Section 3.2, we call q_D^* “the apparent total intensity”. Equations (4.26) – (4.31) prove

Theorem 4.3. *Let the assumptions be the same as in Theorem 4.1. Then one has*

$$\tilde{q}_D^* = q_D^*(1 + O(\mu d^3)), \quad \tilde{x}^{(0)} = x^{(0)} + O\left(\frac{\mu}{k} d^3\right), \quad \tilde{J}_{jl} = J_{jl} + O\left(\frac{\mu}{k^2} d^3\right), \quad 1 \leq j \leq l \leq 3,$$

where $\mu = Q(k + Qd)$.

Suppose that approximate values of the apparent total intensity $\tilde{q}_D^*$, the vector of the first moments $\tilde{x}^{(0)}$, and the tensor of the second central moments $\tilde{J} := (\tilde{J}_{jl})$ of the potential $q(x)$ have been found from (4.29) – (4.31). Then one can determine an ellipsoid $\tilde{D}$ and a real number $\tilde{q}$ such that the function $\tilde{q}(x) = \tilde{q}\chi(\tilde{D})$ has exactly the same first three moments and produces the scattering amplitude which fits best the data $A(\alpha', \alpha)$. This can be done by an algorithm analogous to that from Section 3.2:

1. The center of $\tilde{D}$ is chosen to be $\tilde{x}^{(0)}$;
2. One finds the eigenvalue decomposition $\tilde{J} = \tilde{U}\tilde{\Sigma}\tilde{U}^T$, $\tilde{\Sigma} = \text{diag}(\tilde{\sigma}_1, \tilde{\sigma}_2, \tilde{\sigma}_3)$, $\tilde{U}\tilde{U}^T = I$, of the tensor $\tilde{J}$. The half-axes of the ellipsoid $\tilde{D}$ are $\tilde{h}_j = \sqrt{5\tilde{\sigma}_j}$, $j = 1, 2, 3$, and $\tilde{D}$ is defined to be

$$\tilde{D} := \{x : x = \tilde{U}x', \sum_{j=1}^3 (x'_j - \tilde{x}_j^{(0)})^2 / \tilde{h}_j^2 \leq 1\};$$

3. One computes $\tilde{R}_1 = \frac{1}{\text{Vol}(\tilde{D})} \int_{\tilde{D}} \int_{\tilde{D}} \frac{dx dy}{4\pi|x-y|}$;
4. Finally, using (4.24) and the first equation in (4.31), we see that $\tilde{q}$ satisfies the quadratic equation $\tilde{q}^2 \tilde{R}_1 + \tilde{q} - (\tilde{q}_D^* / \text{Vol}(\tilde{D})) = 0$, therefore the parameter $\tilde{q}$ equals

$$\tilde{q} = \frac{-1 + \sqrt{1 + \frac{4\tilde{q}_D^* \tilde{R}_1}{\text{Vol}(\tilde{D})}}}{2\tilde{R}_1} \approx \frac{\tilde{q}_D^*}{\text{Vol}(\tilde{D})} \left(1 - \frac{\tilde{q}_D^* \tilde{R}_1}{\text{Vol}(\tilde{D})}\right), \quad \text{Vol}(\tilde{D}) = \frac{4\pi}{3} \tilde{h}_1 \tilde{h}_2 \tilde{h}_3.$$

The accuracy (and, hence, consistency) of the above inversion procedure is analogous to that given by Theorem 3.4:

Theorem 4.4. *Consider the family of ellipsoids from Theorem 3.4. Let a number $q \in \mathbb{R}$ be fixed. Then the algorithm of the approximate inversion recovers D_d and q with the accuracy*

$$\frac{\text{Vol}(D_d \Delta \tilde{D}_d)}{\text{Vol}(D_d)} = O\left(\frac{Q(k + Qd)}{k^2} d\right), \quad \tilde{q} = q \left(1 + O\left(\frac{Q(k + Qd)}{k^2} d\right)\right). \quad (4.32)$$

4.4. What can be found from the modulus of the scattering amplitude?

Returning to equation (4.7), we see that choosing any $\alpha'_0, \alpha_0 \in S^2$, we find an estimate of the modulus of the total intensity of the potential

$$|\widetilde{q_D}| = 4\pi |A(\alpha'_0, \alpha_0)|, \quad |\widetilde{q_D}| = |q_D| (1 + O(\xi d^2)). \quad (4.33)$$

To find an estimate of the second central moments of q , we denote $\epsilon = (|\widetilde{q_D}| / |q_D|) - 1$, take modulus on both sides of (4.23), and use equation (4.33) to get

$$4\pi |A(\alpha', \alpha)| = |\widetilde{q_D}| (1 + \epsilon) \left| 1 + R - \frac{k^2}{2} \sum_{1 \leq j \leq l \leq 3} c_{jl}(\alpha', \alpha) J_{jl} + O(\mu d^3) \right|,$$

therefore

$$\frac{|A(\alpha', \alpha)|}{|A(\alpha'_0, \alpha_0)|} = 1 + (\epsilon + R) - \frac{k^2}{2} \sum_{1 \leq j \leq l \leq 3} c_{jl}(\alpha', \alpha) J_{jl} + O(\mu d^3). \quad (4.34)$$

Introducing the function $\hat{A}_1$:

$$\hat{A}_1(\alpha', \alpha) = \frac{|A(\alpha', \alpha)|}{|A(\alpha'_0, \alpha_0)|} - 1, \quad (4.35)$$

and using (4.27), we conclude that approximations to $(\epsilon + R)$ and $\overset{\circ}{J}_{jl}$ can be found by minimizing the error functional

$$\Phi_1(\Delta q_D, J) = \sum_{\alpha', \alpha} (\hat{A}_1(\alpha', \alpha) - \Psi(\Delta q_D, J))^2 d\alpha d\alpha' \quad (4.36)$$

with respect to the seven scalar variables Δq_D , J_{jl} , $1 \leq j \leq l \leq 3$. Since functional (4.36) is quadratic, the solution to the minimization problem $\Phi_1 \rightarrow \min$ satisfies the system of normal equations, which can be written explicitly.

Comparing (4.34) – (4.36) with (4.28) and (4.29), we see that the accuracy of the estimates $\tilde{J}_{jl}$, where $(\Delta \tilde{q}_D, \tilde{J}_{jl})$ minimize functional (4.36), is the same as that described in Theorem 4.3. Thus, assuming that D is an ellipsoid with the known center $x^{(0)}$, this ellipsoid can be recovered using Step 2 of the algorithm from Section 4.3 (with $\tilde{x}^{(0)} = x^{(0)}$), and the accuracy of such recovery is given by the first equation in (4.32).

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