COMPUTER IMPLEMENTATION OF AN ANALYTICAL SOLUTION TO THE NEUTRAL PARTICLE BOLTZMANN
TRANSPORT EQUATION FOR HETEROGENEOUS SLABS,
A BENCHMARK POSTER

by

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Computer Implementation of an Analytical Solution to the Neutral Particle Boltzmann Transport Equation for Heterogeneous Slabs, A Benchmark Poster

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Abstract

An analytical solution to the neutral particle, isotropic down scatter Boltzmann transport equation is implemented for heterogeneous slabs and critical slabs using the $F_N$ method. The results of this implementation are compared to the ANISN/PC code, an $S_N$ solution method, in regions where $S_N$ codes fail. The flux distribution determined by ANISN/PC compare well to $F_N$ results for heterogeneous slabs even when ANISN/PC is being stressed. However, critical widths and fluxes do not compare well for critical slabs until ANISN/PC is used with a large angular quadrature and the number of secondaries is far away from 1.0.

I. Introduction

To solve problems for dilute sub-atomic particle transport in a dense medium, three major solution techniques have been widely used: deterministic (discrete ordinates or $S_N$), stochastic (Monte Carlo), and analytic solutions (closed-form or numerically derived). The discrete ordinates and Monte Carlo solution techniques have been used to solve most of the real engineering and physics problems encountered in reactor and shield design, well logging, stellar dynamics, etc. . The closed-form analytic solution technique has mainly been used to solve simple problems in the academic setting and to help benchmark the large numerical codes modeling very simple physical situations. A fourth solution technique has become more popular as the availability of faster and larger capacity computers has increased. This solution technique can solve more complicated physical models by calculating numbers for an analytical solution that is not necessarily in a closed-form: convergent infinite series, numerically evaluated real and complex integrals, etc. . These types of solutions can be controlled so that four to six place accuracy can be achieved in a relatively short computer code which executes in a reasonable amount of time on a workstation or personal class of computer.

This paper investigates a solution technique that involves an infinite series expansion solution to the Boltzmann transport equation in heterogeneous slabs. The method was original published by C.E. Siewert, et. al. as the $F_N$ method; however, a generalised program appears to have never been released. A synopsis of the derivation has been included in this paper with some minor additions and clarifications to the original $F_N$ method. The resultant equations have been programmed on a computer in ANSI FORTRAN 77 and can be obtained from the author. The code package contains three programs: a heterogeneous slab solver (MGSLAB), a semi-infinite slab solver (MGSEMI), and a critical slab dimension search with resultant steady state flux (FNCRIT). MGSEMI is not discussed further because of the inability of the $S_N$ method to model semi-infinite slabs effectively.

MGSLAB and FNCRIT are compared to an $S_N$ code in various situations where it begins to fail. The $F_N$ method is very robust and in some circumstances could be faster and more accurate than the discrete ordinates method. The total group scalar flux is presented as the comparison measure.
II. NEUTRAL PARTICLE TRANSPORT MODEL

An analytical solution to the one dimensional, multiple energy group, isotropic down scatter, neutral particle, linearized Boltzmann transport equation is developed and solved for heterogeneous, non-critical slabs. A solution for critical slabs is also developed for a single slab and one energy group. These solutions are then compared with $S_N$ solutions to validate and to show the advantages of the $F_N$ method in various situations where the $S_N$ codes fail.

A. Heterogeneous Slab Media

The $F_N$ method uses an angular variable expansion technique to solve an inhomogeneous Fredholm type integral equation that represents the transport equation of interest:

$$\left[\mu \frac{\partial}{\partial z} + \sigma^i_g\right] \phi_g(x, \mu) = \frac{1}{2} \sum_{g' = 1}^{g} \sigma^i_{g' \rightarrow g} \int_{-1}^{+1} d\mu' \phi_{g'}(x, \mu') + \frac{1}{2} S^i_g(x),$$

with the boundary conditions of

$$\phi_g(x_{i-1}, \mu) = F^i_{L}(\mu) \quad \mu > 0, \quad \text{and} \quad \phi_g(x_i, -\mu) = F^i_{R}(\mu) \quad \mu > 0,$$

where, $i$ represents the slab number; $g$ represents the energy group number; $\sigma^i_g$ and $\sigma^i_{g' \rightarrow g}$ are the total and isotropic scattering cross sections for slab $i$; $S^i_g(x)$ is a general isotropic distributed source in slab $i$; $\phi_g(x, \mu)$ is the angular neutron flux in the media; and $F^i_{L}(\mu)$ and $F^i_{R}(\mu)$ are known general functions.

Two singular integral equations in the half range of $\mu$ are created from the transport equation by extending it into the complex plane and integrating on $x$ for each slab and over the positive half range of $\mu$. The complex variable is then restricted to the positive real axis. A second equation is created by changing the sign of $\mu$ and the complex variable and then integrating as before. All singular integrals are evaluated using the Plemelj relations. A set of constraint equations (like boundary conditions for a differential equation) is constructed by substituting the positive root of the infinite medium dispersion relation, $\nu^i_0$, into the integral equations. The dispersion relation is derived to be:

$$\Lambda_{g'g}(\nu) = \delta_{g'g} + \nu \frac{\sigma^i_{g' \rightarrow g}}{2 \sigma^i_g} \int_{-1}^{+1} d\eta \frac{1}{\eta - \nu}.$$  

Since this is monotonic, a single positive root can be found from:

$$\Lambda_{g'g}(\nu^i_0) = 0.$$  

An angular variable expansion method is used to solve the neutron flux. The expansion coefficients are found using a collocation scheme. The particular expansions at the slab boundary are:

$$\phi_g(x_{i-1}, \mu) = F^i_{L}(\mu) e^{-\frac{\int_{-1}^{x_{i-1}} \sigma^i_{g' \rightarrow g} \phi_{g'}(x, \mu')} - \frac{1}{2} \sum_{\alpha=0}^{N-1} a^i_{\alpha} \phi(\mu),$$

$$\phi_g(x_i, -\mu) = F^i_{R}(\mu) e^{-\frac{\int_{-1}^{x_{i-1}} \sigma^i_{g' \rightarrow g} \phi_{g'}(x, \mu')} - \frac{1}{2} \sum_{\alpha=0}^{N-1} b^i_{\alpha} \phi(\mu),$$

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where, $a_\alpha^i$ and $b_\alpha^i$ are the expansion coefficients and $\psi_\alpha(\mu)$ are the basis functions. The representation becomes exact as $N$ tends to infinity. To determine the coefficients, these expansions are substituted into the integral equations and evaluated at the collocation points and the constraint point, $\nu_\beta < 1$ for $\beta = 1, \ldots, N-1$ and $\nu_0 > 1$. The resultant set of matrix equations is:

\begin{align*}
\sum_{\alpha=0}^{N-1} \left[ a_\alpha^i B_\alpha^i(\nu_\beta) + b_\alpha^i A_\alpha^i(\nu_\beta) e^{-\frac{\sigma_\alpha^i(x_i-x_{i-1})}{\nu_\beta}} \right] &= R_1^i(\nu_\beta, x_{i-1}) + \\
&+ \frac{1}{\sigma_{g'-g}^i} T_1^i(\nu_\beta, x_{i-1}, x_i) + \frac{1}{\sigma_{g'-g}^i} S_1^i(\nu_\beta, x_{i-1}, x_i),
\end{align*}

(4a)

\begin{align*}
\sum_{\alpha=0}^{N-1} \left[ b_\alpha^i B_\alpha^i(\nu_\beta) + a_\alpha^i A_\alpha^i(\nu_\beta) e^{-\frac{\sigma_\alpha^i(x_i-x_{i-1})}{\nu_\beta}} \right] &= R_2^i(\nu_\beta, x_i) + \\
&+ \frac{1}{\sigma_{g'-g}^i} T_2^i(\nu_\beta, x_{i-1}, x_i) + \frac{1}{\sigma_{g'-g}^i} S_2^i(\nu_\beta, x_{i-1}, x_i),
\end{align*}

(4b)

where, the integral terms are:

\begin{align*}
A_\alpha^i(\nu_\beta) &= \frac{\sigma_{g'-g}^i}{2\sigma_g^i} \int_0^1 d\eta \frac{\eta}{\eta + \nu_\beta} \psi_\alpha(\eta),
\end{align*}

(5a)

\begin{align*}
B_\alpha^i(\nu_\beta) &= \lambda_{g'}(\nu_\beta) \psi_\alpha(\nu_\beta) - \frac{\sigma_{g'-g}^i}{2\sigma_g^i} \int_0^1 d\eta \frac{\eta}{\eta - \nu_\beta} \psi_\alpha(\eta),
\end{align*}

(5b)

\begin{align*}
\lambda_{g'}(\nu_\beta) &= 1 + \nu_\beta \frac{\sigma_{g'-g}^i}{2\sigma_g^i} \int_{-1}^{+1} d\eta \frac{1}{\eta - \nu_\beta},
\end{align*}

(5c)

the inhomogeneous terms are:

\begin{align*}
R_1^i(\nu_\beta, x_{i-1}) &= \int_0^1 d\eta \left[ F_{R}^{g,i}(\eta) C(\sigma_g^i(x_i-x_{i-1}), \eta, \nu_\beta) + F_{L}^{g,i}(\eta) S(\sigma_g^i(x_i-x_{i-1}), \eta, \nu_\beta) \right],
\end{align*}

(6a)

\begin{align*}
R_2^i(\nu_\beta, x_i) &= \int_0^1 d\eta \left[ F_{L}^{g,i}(\eta) C(\sigma_g^i(x_i-x_{i-1}), \eta, \nu_\beta) + F_{R}^{g,i}(\eta) S(\sigma_g^i(x_i-x_{i-1}), \eta, \nu_\beta) \right],
\end{align*}

(6b)

where,

\begin{align*}
C(\xi, \eta, \nu) &= \begin{cases} 
\xi e^{-\xi/\nu} & \eta = \nu \\
\frac{\nu^2 e^{-\xi/\nu} - e^{-\xi/\nu}}{\eta - \nu} & \eta \neq \nu
\end{cases},
\end{align*}

(6c)

\begin{align*}
S(\xi, \eta, \nu) &= \frac{1 - e^{-\xi/\nu}}{\eta + \nu},
\end{align*}

(6d)
the distributed source terms are:

\[
S_{1g}^i(\nu_\beta, x_{i-1}, x_i) = \begin{cases} \frac{\sigma^i_{x}}{\nu_\beta} \int_{x_{i-1}}^{x_i} dx e^{-\frac{\sigma^i_{x}}{\nu_\beta}(x-x_{i-1})} S_{g}^i(x) & \nu_\beta \neq 0 \\ S_{g}^i(x) & \nu_\beta = 0 \end{cases}
\] (7a)

\[
S_{2g}^i(\nu_\beta, x_{i-1}, x_i) = \begin{cases} \frac{\sigma^i_{x}}{\nu_\beta} \int_{x_{i-1}}^{x_i} dx e^{-\frac{\sigma^i_{x}}{\nu_\beta}(x-x_{i-1})} S_{g}^i(x) & \nu_\beta \neq 0 \\ S_{g}^i(x) & \nu_\beta = 0 \end{cases}
\] (7b)

and, the down scatter terms are:

\[
T_{1g}^i(\nu_\beta, x_{i-1}, x_i) = \sigma_g^i \sum_{g' = 1}^{g-1} \sigma_{g' \rightarrow g}^i T_{g'}^i(\nu_\beta, x_{i-1}, x_i),
\] (8a)

\[
T_{2g}^i(\nu_\beta, x_{i-1}, x_i) = \sigma_g^i \sum_{g' = 1}^{g-1} \sigma_{g' \rightarrow g}^i J_{g'}^i(\nu_\beta, x_{i-1}, x_i).
\] (8b)

The derivation of the group-to-group particle transfer terms, \(I_{g'}^i(\nu_\beta, x_{i-1}, x_i)\) and \(J_{g'}^i(\nu_\beta, x_{i-1}, x_i)\), is based on an analysis of the transport equation in group \(g'\). When \(\sigma_{g'}^i\nu_\beta\) is smaller than \(\sigma_{g}^i\xi\), the integral equation kernel is still singular so the analysis is of the same form as the original. After the \(P_N\) approximations are substituted into the integral equations they are solved for \(I_{g}^i(\xi, x_{i-1}, x_i)\) and \(J_{g}^i(\xi, x_{i-1}, x_i)\). If \(s_{gg'} = \frac{\sigma_{g'}^i}{\sigma_g^i}\), then for \(\xi \in [0, \frac{1}{s_{gg'}}]\):

\[
\sigma_{g' \rightarrow g}^i I_{gg'}^{ii}(\xi, x_{i-1}, x_i) = \sigma_{g' \rightarrow g}^i \sum_{\alpha = 0}^{N-1} \sigma_g^{ii} \psi_g(s_{gg'}\xi) - \frac{1}{\sigma_g^i} S_{1g}^i(s_{gg'}\xi, x_{i-1}, x_i) - \sum_{g'' = 1}^{g'-1} \sigma_{g'' \rightarrow g}^i I_{gg''}^{ii}(\xi, x_{i-1}, x_i),
\] (9a)

\[
\sigma_{g' \rightarrow g}^i J_{gg'}^{ii}(\xi, x_{i-1}, x_i) = \sigma_{g' \rightarrow g}^i \sum_{\alpha = 0}^{N-1} \sigma_g^{ii} b_g^{ii} \psi_g(s_{gg'}\xi) - \frac{1}{\sigma_g^i} S_{2g}^i(s_{gg'}\xi, x_{i-1}, x_i) - \sum_{g'' = 1}^{g'-1} \sigma_{g'' \rightarrow g}^i J_{gg''}^{ii}(\xi, x_{i-1}, x_i).
\] (9b)

When \(\sigma_{g'}^i\nu_\beta\) is larger than \(\sigma_{g}^i\xi\), the integral equation kernel is no longer singular. Therefore, the original analysis is performed without using the Plemelj relations. For \(\xi \notin [0, \frac{1}{s_{gg'}}]\):

\[
\sigma_g^i \Lambda_{g'g'}(s_{gg'}\xi) I_{gg'}^{ii}(\xi, x_{i-1}, x_i) = \sum_{\alpha = 0}^{N-1} \left[ a_{g}^{ii} A_{g'}^{\alpha} (-s_{gg'}\xi) - b_{g}^{ii} e^{-\frac{s_{gg'}(x_{i-1}-x_{i-1})}{s_{gg'}}} A_{g}^{\alpha}(s_{gg'}\xi) \right] + \frac{R_{1g'}^i(s_{gg'}\xi, x_{i-1}) - (s_{gg'}\xi) L_{g'}^i(s_{gg'}\xi) S_{1g}^i(s_{gg'}\xi, x_{i-1}, x_i) - \frac{1}{\sigma_{g'}} \sum_{g'' = 1}^{g'-1} A_{g'g''}(s_{gg'}\xi) I_{gg''}^{ii}(\xi, x_{i-1}, x_i),
\] (10a)
The FN approximations can be used directly to solve for the angular neutron flux; however, they become unstable as $\alpha$ increases and $\mu$ approaches zero. This instability can be overcome by regularizing the integral term $B^a_\alpha(\mu)$ as follows:

\[
B^a_\alpha(\mu) = \left(1 + \frac{\sigma_g^a - \mu}{\sigma_g^a} \int_{-1}^{1} \frac{d\eta}{\eta - \mu} \right) \psi_\alpha(\mu) - \frac{\sigma_g^a}{\sigma_g^a} \int_{0}^{1} \frac{d\eta}{\eta - \mu} \psi_\alpha(\eta)
\]

Using this new formulation and the matrix equations, the FN approximations, equations (3a) and (3b), can be rewritten. This new formulation is called the post processor:

\[
\phi_g^i(x_{i-1}, -\mu) = \frac{F_{Rg}^i(\mu)e^{-\frac{\epsilon^i_{g}(x_{i-1})}{\mu}}}{\lambda_{gg}(\mu)} + \frac{1}{2\sigma_g^i} \lambda_{gg}(\mu) \left[\sigma_{g-g}^i S_{rg}^i(\mu, x_{i-1}) + \sigma_{g-g}^i N_{\alpha=0}^{N-1} \left\{ a_{\alpha}^g b_{\alpha}^g(\mu) - b_{\alpha}^g \epsilon_{g}(x_{i-1}) \right\} \right]
\]

These equations are solved using a computer and numerical methods as described in Section III.

B. Dimensional Criticality Search and Resultant Steady State Flux

From the FN formulation, the critical slab width and flux can be determined for a source-free problem with a single slab and energy group. For the current formulation, the critical problem is defined by using the imaginary root of the infinite medium dispersion relation, equation (2a). Criticality is defined when the resultant collocation matrix determinant is equal to zero. The parameter used to change the matrix values is the slab width.

The boundary matrix equations with one group, one slab, no source, and flux symmetry ($a_\alpha = b_\alpha$) reduce to

\[
\sum_{\alpha=0}^{N-1} a_\alpha \left[ B_\alpha(\nu_g) + A_\alpha(\nu_g)e^{-\Delta_\alpha/\nu_g} \right] = 0. \tag{13}
\]
To determine $\nu_0$, the imaginary root of the infinite medium dispersion relation is used

$$
\Delta(\nu_0) = 0 = 1 - \nu_0 \frac{\sigma_0}{\sigma} \tan^{-1} \left( \frac{1}{\nu_0} \right).
$$

(14)

Since $\nu_0$ is purely imaginary, the first row of the expansion coefficient matrix is complex. To find the critical width, the real and imaginary parts of the determinant of the matrix must be zero at the same value of $\Delta_c$, the critical dimensionless width. The determinant equations are

$$
\begin{align*}
\Re_{\alpha 0} & = 0 \quad \text{and} \quad \Im_{\alpha 0} = 0, \\
B_0(\nu_0) + A_0(\nu_0)e^{-\Delta_0/\nu_0} & = 0 \quad \text{and} \quad B_0(\nu_0) + A_0(\nu_0)e^{-\Delta_0/\nu_0} = 0,
\end{align*}
$$

(15a)

where,

$$
\begin{align*}
\Re_{\alpha 0} & = \frac{\sigma_0}{2\sigma} \left[ S_{\alpha} \cos \left( \frac{\Delta_0}{z_0} - 1 \right) + z_0S_{\beta} \sin \frac{\Delta_0}{z_0} \right], \\
\Im_{\alpha 0} & = \frac{\sigma_0}{2\sigma} \left[ S_{\alpha} \sin \frac{\Delta_0}{z_0} + z_0S_{\beta} \left( \cos \frac{\Delta_0}{z_0} + 1 \right) \right],
\end{align*}
$$

(15b, 15c)

and,

$$
\nu_0 = iz_0, \quad \text{and} \quad \Delta_c = \sigma (z_c - 0),
$$

(15d)

$$
S_{\alpha} = \int_0^1 d\eta \frac{\eta^2}{\eta^2 + z_0^2} \psi_0(\eta), \quad \text{and} \quad S_{\beta} = \int_0^1 d\eta \frac{\eta}{\eta^2 + z_0^2} \psi_0(\eta).
$$

(15e)

To determine the boundary flux, the matrix equation is rewritten as

$$
\sum_{\alpha=1}^N a_{\alpha} \left[ B_0(\nu_0) + A_0(\nu_0)e^{-\Delta_0/\nu_0} \right] = -a_0 \left[ B_0(\nu_0) + A_0(\nu_0)e^{-\Delta_0/\nu_0} \right],
$$

(16)

where, $a_0$ is the flux normalization parameter. To determine the additional $a_\alpha$ values, the matrix is inverted and back-substitution is performed. The post processor, equations (12a) and (12b) with the appropriate terms set to zero, is used to determine the flux.

To calculate the flux normalization parameter, $a_0$, so the $F_N$ method scales to the $S_N$ code, the scalar flux found at $z = 0$ from both methods is set equal to one another. The scalar flux at the boundary for the $F_N$ method is

$$
\phi_{\text{FN}}(0) = \frac{\sigma_s}{2\sigma} a_0.
$$

(17)

For a set of cross sections defined as $\nu \sigma_s + \sigma_s = c \sigma$, the flux normalization parameter used in the $F_N$ algorithm is

$$
a_0 = \frac{2\phi_{\text{FN}}(0)}{c}.
$$

(18)
III. Numerical Considerations

To solve these equations, various simple numerical techniques are used. To invert matrices and evaluate determinants, LU decomposition is used. For root-finding tasks (determination of \( \psi \) and \( \Delta \)) a simple bi-section technique is used. The Gauss-Legendre quadrature method for numerically evaluating integrals is used to evaluate the integral, inhomogeneous, and source terms. The value of the distributed source in the source term is found by fitting the point-by-point user supplied source to a cubic spline and interpolating the needed integration values. The FN method is not sensitive to the choice of the collocation points, as long as they include \( \psi \). The basis functions used in the expansion are the most sensitive part of the FN method. The closer they fit the flux, the faster the method will converge. If the basis functions do not fit the flux distribution well, the method does not converge within the limits of the computer resources and another set of basis functions are to be chosen by the user.

To determine convergence, the number of expansion terms is increased until the maximum angular flux difference between the last iteration and the current is less than a user supplied tolerance. Since the expansion series is convergent, correct flux values are always generated. Another iteration scheme allows multiple slabs to be considered. The inhomogeneous terms, equations (6a) and (6b), contain the slab boundary conditions. The boundary conditions were assumed to be known functions in the FN derivation and in reality are the out-going fluxes of the adjacent slabs. In a simple marching solution technique through the slabs, the flux of the right-hand slab is not known to the current expansion term number accuracy by the time it is used for the current slab; therefore, an iterative method of solution is chosen in the interest of computational simplicity and speed of convergence. A simultaneous method could also have been employed, but at a greater cost in computer resources and program complexity.

IV. Benchmark Results with ANISN/PC

The MGSLAB and FNCRIT results are compared with ANISN/PC results with various models that stretch the limitations of \( S_N \) theory. The cross sections for the heterogeneous slabs are eleven group aluminium cross sections at equal lethargies from \( 10^7 \) eV to \( 10^{-2} \) eV, except for the first group which is from \( 2 \times 10^7 \) eV to \( 10^7 \) eV. The cross sections are generated by NJOY from the ENDF/B-V library. For the critical slab calculations, consistent cross sections are created and do not represent any real material. Since ANISN/PC works in mean free paths, the total size of the models is manipulated in order to achieve the desired results.

The first heterogeneous slab model makes an abrupt change in mesh size in the middle of the slab. A constant distributed source with strength \( 1.0 \) neutron sec cm\(^{-3}\) is present in the first half of the slab. Figures 1 and 2 show the group scalar flux for the slab and all eleven groups. The symbols represent the flux values from ANISN/PC and the curves represent the flux values from MGSLAB. The second model uses too coarse of a grid with the same configuration as the first model. Figures 3 and 4 show the group scalar flux.

The first critical slab model is a valid physical configuration for an ANISN/PC critical width search with the number of secondaries, \( c = \frac{\psi_{\text{crit}} + \psi_o}{\sigma_{\text{total}}} \), set equal to 1.6; however, two different quadrature sets are used: \( S_8 \) (Figure 5) and \( S_{32} \) (Figure 6). The symbols represent the ANISN/PC flux values and the curves represent the flux values from FNCRIT. Figure 7 shows the same \( S_{32} \) physical situation with \( c = 1.02 \) and Figure 8 uses \( c = 1.00001 \). Table 1 shows the FNCRIT versus ANISN/PC calculated critical thicknesses for these and other examples. Exact values from Table 2 in Reference 12 are also shown.

The heterogeneous slab results are as expected. The ANISN/PC scalar flux does not match the MGSLAB scalar flux. The main reason for this is the abuse of the mesh parameters in ANISN/PC. The advantage of the FN method is that the only necessary mesh points where the flux must be calculated is at the slab boundaries; however, the \( z \) variable appears as a parameter of the equations in Section II, instead of an independent discretized variable as in ANISN/PC. In \( S_N \) theory, the mesh spacing is very important and determines the accuracy and precision of the resultant answer. If the mesh is too coarse or changes abruptly, then \( S_N \) theory begins to fail. If there are too many mesh cells (a case not shown here), errors accumulate until the answer \( S_N \) theory gives has no meaning because the accumulated error is on the same order as the answer. However, even though ANISN/PC is abused and generates
wrong answers, it still behaves like it had found the correct one.

For the critical thickness calculations, the angular quadrature used in ANISN/PC is important. This concept is also important in the FN method, because the more expansion terms included in the thickness search, the more accurate the answer will become. However, a large quadrature is needed in ANISN/PC to get a thickness within 0.1% of the exact answer and a small number of expansion terms in FNCRIT is needed to get within 0.01%. The most esoteric failing of ANISN/PC is as \( c \) approaches 1.0. The numerics in \( S_N \) theory need a lot of help in order to cope with this situation.

V. Conclusions

The \( S_N \) solution is very accurate in the regions where it applies. The problem is that in the regions were it does not apply, it generates answers. It is the user's responsibility to know that problems exist and that these problems are not obvious. A tool like the FN method can be used to benchmark the region of applicability for the \( S_N \) and other methods for simple but valuable physical configurations because when the FN method converges, it converges to the correct answer.

There are additions that can be made to the presented program set. Derivations have already been published for anisotropic scattering\(^5\)\(^-\)\(^6\) but the codes presented here do not yet incorporate it. A critical, heterogeneous slab model is being written by the author. A non-iterative upscatter technique is also being investigated by the author, but it is in the initial stages of development and may not have a clean analytical solution. Many more extensions can be made to the current FN method. On top of being robust in its region of applicability, it is a tribute to the method that it can be used in such complex models of neutral particle transport.

Table 1: Examples of Critical Thicknesses Calculated by ANISN/PC and FNCRIT

<table>
<thead>
<tr>
<th>( c )</th>
<th>( S_N )</th>
<th>( F_N )</th>
<th>( )</th>
<th>Critical Thickness and Relative Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c )</td>
<td>( S_N )</td>
<td>( F_N )</td>
<td>( )</td>
<td>( )</td>
</tr>
</tbody>
</table>
| \begin{tabular}[c]{|c|c|c|c|c|c|}
1.6 & 8 & 15 & 1.0240 & 1.04069 & -1.630 \\
1.6 & 32 & 31 & 1.0240 & 1.02476 & -0.074 \\
1.05 & 32 & 31 & 6.6004 & 6.60033 & 0.044 \\
1.02 & 32 & 31 & 11.331 & 11.3366 & -0.049 \\
1.001 & 32 & 31 & - & 56.3440 & - \\
1.0001 & 32 & 31 & - & 192.494 & - \\
1.00001 & 32 & 31 & - & 652.335 & - \\
\end{tabular} |
Figure 1: Comparison for an Abrupt Change in Mesh Size for Energy Groups 1 through 5 (Symbols are ANISN/PC output, lines are MGSLAB output)

Figure 2: Comparison for an Abrupt Change in Mesh Size for Energy Groups 6 through 11 (Symbols are ANISN/PC output, lines are MGSLAB output)

Figure 3: Comparison for too Coarse of a Grid for Energy Groups 1 through 5 (Symbols are ANISN/PC output, lines are MGSLAB output)

Figure 4: Comparison for too Coarse of a Grid for Energy Groups 6 through 11 (Symbols are ANISN/PC output, lines are MGSLAB output)
Figure 5: Comparison for a Critical Slab with $S_8$ with $c = 1.6$

Figure 6: Comparison for a Critical Slab with $S_{32}$ with $c = 1.6$

Figure 7: Comparison for Critical Slabs with $S_{32}$ and $c = 1.02$

Figure 8: Comparison for Critical Slabs with $S_{32}$ and $c = 1.0001$
REFERENCES


