Quark Confinement in a Constituent Quark Model

Kurt Langfeld and Mannque Rho

Service de Physique Théorique,
C.E.A. Saclay, F-91191 Gif-sur-Yvette Cedex, France

and

Institute for Nuclear Theory, Physics/Astronomy Building, Box 351550
University of Washington, Seattle, Washington 98195-1550, USA

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Abstract

On the level of an effective quark theory, we define confinement by the absence of quark anti-quark thresholds in correlation functions. We then propose a confining Nambu-Jona-Lasinio-type model. The confinement is implemented in analogy to Anderson localization in condensed matter systems. We study the model's phase structure as well as its behavior under extreme conditions, i.e. high temperature and/or high density.
1 Introduction

The description of hadron physics starting from quantum chromodynamics (QCD) – the theory of strong interactions – is one of the most challenging problems in medium energy physics of today. The difficulty in the description of the low energy sector of QCD lies in the large effective coupling constant, which precludes a perturbative treatment.

Various non-perturbative methods shed some light on its ground state properties: the operator product expansion [1] improves the perturbative expansion by the inclusion of effects from vacuum condensates in Green's functions, QCD sum rules [2] relate these condensates to hadron masses, variational methods [3] attempt to evolve a qualitative picture of the gluonic vacuum, the semi-classical evolution of the QCD functional integral, leading to instanton physics [4], and the reformulation of QCD in terms of the field strength [5, 6, 7, 8] provides some insight into the vacuum properties of the quark sector. These analytical approaches can be contrasted to numerical investigations of lattice QCD [9, 10]. The latter approach is not bounded by any approximation, but is restricted by the capacities of computers.

One of the most striking features of low energy QCD is the absence of quarks in the asymptotic states. This quark confinement is explained in lattice QCD by a linear rising potential between two static quarks. This behavior is also confirmed by considerations within the $1/N_c$-expansion, $N_c$ being the number of colors [11]. A natural explanation of the linear confining potential is provided by the dual superconductor picture [12]. As pointed out by 't Hooft, in a certain gauge, the non-abelian Yang-Mills theories possess monopole configurations [13]. If these monopoles condense, a dual Meissner effect occurs, expelling electric field strength out of the vacuum. This implies that the electric field between two static quarks is squeezed into a flux tube, which subsequently gives rise to the linear confining potential. A recent SUSY model of Seiberg and Witten suggests that monopole condensation could be the mechanism of confinement in certain Yang-Mills theories [14, 15]. There have been suggestions that the same phenomenon occurs in the Yang-Mills sector of QCD proper [16]. In this development, it is found that the confinement of the quarks is intimately related to the spontaneous breakdown of chiral symmetry [15].

Despite the recent progress in understanding the ground state properties of QCD, the description of hadron properties is still feasible only with effective models. These models include aspects of QCD by incorporating its symmetries. The most important symmetry constraining the variety of hadron models is chiral symmetry. For QCD ($N_c = 3$) and for zero current masses, the quark sector is invariant under global $SU(N_f)_L \times SU(N_f)_R$ transformations of the left- and right-handed quarks, where $N_f$ is the number of quark flavors. The vacuum breaks this symmetry down
to the diagonal $SU(N_f)$.\footnote{For $SU(2)$ Yang-Mills theory, the chiral symmetry group is much larger, i.e. $SU(2N_f)$, since the $SU(2)$ gauge group is pseudoreal [17]. More on this later.}

Most effective theories of mesons and nucleons contain many parameters restricting the predictive power of these models. It is therefore desirable to "derive" the effective hadron theory from an underlying effective quark model in order to obtain constraints on the parameter range. One of these quark models is the model of Nambu and Jona-Lasinio (NJL) [18]. The NJL model is one of the most economical models that possess the essence of dynamical symmetry breaking that is the hall-mark of modern field theories and has enjoyed an impressive phenomenological success in hadron physics [19]. Now the NJL-model reflects the principal low-energy QCD symmetry properties of the quark sector, but does not include the confinement of quarks. This implies, in particular, that the mesons of this model – quark-anti-quark bound states – can decay into free quark-anti-quark pairs, if such a process is allowed by kinematics. This manifestly unphysical threshold puts severe constraints on the applicability of the NJL-model [21].

In this paper, we propose an NJL-type model that possesses the confinement property in the sense that quark-anti-quark thresholds are absent in (mesonic) Green's functions. Our reasoning, albeit exploratory, will be guided by a close analogy to a phenomenon in condensed-matter physics known as Anderson localization [22]. In Anderson localization, it is observed that freely moving electrons get localized when the strength of a random distributed potential stemming from the impurities of the conducting solid exceed a certain strength. We shall argue in analogy that the quarks feel randomly distributed background fields generated by the gluon sector.

The paper is organized as follows. In the next section, a motivation and a detailed description of the model is presented. The gap equation describing quark ground-state properties is derived. In section 3, we first describe the particular ansatz which produces a remarkable confinement property. The absence of quark-anti-quark thresholds is explicitly demonstrated for the scalar correlation function. We then show that this particular ansatz is indeed a solution of the gap equation. The phase structure of the model is discussed in some detail. In section 4, we describe the implication of the model on chiral properties. There is no surprise here as the model is designed to reproduce the chiral structure of QCD (section 2). We reproduce the corresponding low-energy theorems by first solving the Bethe-Salpeter equation for the pion field, normalize the amplitude by the electromagnetic form factor and extract the pion decay constant from its definition. We then establish that the Gell-Mann-Oakes-Renner (GMOR) relation is valid in our model. Finally, we compute the pseudoscalar correlation function. The sections 5 and 6 are devoted to the temperature and density dependence of the vacuum properties of the model. We will
show that a deconfinement phase transition occurs at some large temperature and/or density. We will find that the deconfinement phase transition is accompanied by the restoration of the spontaneously broken chiral symmetry. The model predicts the two transitions at the same critical point. The final section contains some concluding remarks.

2 The Model

2.1 Motivation

It is known for a long time that the lattice formulation of Yang-Mills theory provides information on the confinement of quarks [10]. Subsequently quark liberation due to temperature was studied with the lattice version of QCD. It turned out that quantities which are dominated by the properties of the gluonic sector, e.g., the gluon condensate, vary smoothly throughout the deconfinement phase transition [25, 26]. This suggests that quark liberation may be described solely by a dynamical effect of the quark sector. If so, the description of this phase transition should be feasible by an effective quark model. We hope to gain qualitative insight into the nature of quark liberation by including temperature effects in the quark loops, while the temperature dependence of the quark interaction, mediated by the gluonic sector, is kept temperature-independent.

The properties of the light pseudoscalar mesons, in particular those of the scalar meson and the pion, are protected by chiral symmetry of QCD and might therefore serve as a convenient test ground to explore other features of low energy QCD. We have learned in the past years that at low energies, the pion physics is phenomenologically well described in terms of an effective quark model, in which the quarks interact via a local current-current interaction of the NJL type [18, 19]. However, the open question is: What is the signature of confinement at the level of an effective quark theory?

In order to answer this question, we can be guided by a phenomenon in solid state physics: the localization of electrons in random potentials [22], known as Anderson localization. In his pioneering work, Anderson showed that the key idea of localization can be traced back to the Hamiltonian

\[ H = \sum_{\langle ik \rangle} (-V) a_i^\dagger a_k + \sum_i \epsilon_i a_i^\dagger a_i, \]  

(1)

where \( a_i^\dagger \) is the creation operator of a spin at site \( i \) of a lattice. The sum in the first term of (1) extends only over the nearest neighbors, and the constant \( V \) measures the strength of the nearest-neighbor hopping. This term is responsible for the spin
diffusion on the lattice and is the analogue of a kinetic term in continuum quantum field theory. The energy $\epsilon_i$ corresponds to a potential at site $i$. In the Anderson model, the $\epsilon_i$'s are random variables distributed over a range $-W/2 < \epsilon < W/2$. A continuum version of this model is described by the Hamiltonian

$$ H = p^2 + w \sum_\alpha \delta(r - R_\alpha), $$

which describes electrons elastically scattered off impurities which are randomly distributed at points $R_\alpha$. It turns out that the precise form of the distribution of the potential $\epsilon_i$ is not crucial. It has been found that if the random potential exceeds a certain strength, i.e.

$$ W > W_c = 4KV, $$

where the connective constant $K$ is characteristic of the lattice type, then the spins are localized at their sites, whereas for $W < W_c$ the spins are liberated, and the spin diffusion takes place.

The localization of spins is intimately related to the presence of random potentials. Is there a random interaction of quarks which stems from the gluonic sector and which survives the low-energy limit? The answer to this question is not known. There is, however, a first hint from the field-strength formulation of QCD [5, 6]. When QCD ($N_c = 3$) is formulated in terms of the field strength, the resulting quark interaction takes the form

$$ Z[j] = \int DT^{a}_{\mu\nu} e^{-S[T]} \exp \left\{ \int d^4x \left[ j^a\gamma^\mu V^a_{\mu} - i\frac{g}{2} j^a\left(\tilde{T}^{-1}\right)^{ab}_{\mu\nu} \partial_\rho T^{b}_\rho \right] \right\}, $$

where $j^a = \bar{q}t^a\gamma_\mu q$ is the color octet current of the quarks, $\tilde{T}^{ab}_{\mu\nu} = f^{abc}T^c_{\mu\nu}$ is defined with the help of the $SU(3)$ structure functions $f^{abc}$, and $V^a_{\mu} = (\tilde{T}^{-1})^{ab}_{\mu\nu}\partial_\rho T^{b}_\rho$ is the gauge potential induced by the conjugate field strength $T^{a}_{\mu\nu}$. The action of the field strength $S[T]$ need not to be specified here and can be found in [5, 6]. Within the strong coupling limit, one observes that the gluonic vacuum decays into domains of constant field strength $iT^a_{\mu\nu} [6]$ giving rise to a Nambu-Jona-Lasinio type quark interaction in (4). Due to gauge covariance, all orientations of the constant background field contribute to the quark interaction. Subsequently, the spontaneous breakdown of chiral symmetry was observed in the strong coupling limit [7, 8]. The specific form of the quark interaction due to the gluon background fields gives rise to a splitting of the strange- and up-quark condensates as is predicted by QCD sum rules [8].

We conclude from these observations that the low energy-effective quark interaction of the NJL type is in fact mediated by the gluon background field. Our key assumption is that in strongly fluctuating gluonic background fields (with an average scale given by the gluon condensate), we will be endowed with a random quark
interaction which induces quark confinement by a mechanism similar to Anderson localization in solid state physics. In this paper, we shall construct a simple toy model which has the expected qualitative feature mentioned above of low-energy QCD.

2.2 Description of the model

In order to investigate the implications of strong random colored interactions of quarks, we study a model in which the quark fields are doublets of a global $SU(2)$ color symmetry. We write the generating functional for mesonic Green's functions in Euclidean space as

\[
Z[\phi] = \left\langle \int \mathcal{D}q \mathcal{D}\bar{q} \exp \left\{ -\int d^4x [\mathcal{L} - \bar{q}(x)\phi(x)q(x)] \right\} \right\rangle_0,
\]

\[
\mathcal{L} = \bar{q}(x)(i\partial + im)q(x) + \frac{G_0}{2} [\bar{q}q(x)\bar{q}q(x) - \bar{q}\gamma_5q(x)\bar{q}\gamma_5q(x)]
\]

\[
+ \frac{1}{2} [\bar{q}\tau^\alpha q(x)G^{\alpha\beta}\bar{q}\tau^\beta q(x) - \bar{q}\gamma_5\tau^\alpha q(x)G^{\alpha\beta}\bar{q}\gamma_5\tau^\beta q(x)],
\]

where $m$ is the current quark mass. We assume that the quark interaction is given by a color-singlet four-fermion interaction of strength $G_0$ and a color-triplet interaction mediated by a positive definite matrix $G^{\alpha\beta}$, which represents gluonic background fields. An average of all orientations $O$ of the background field $G^{\alpha\beta}$, transforming as $G' = O^TGO$ with $O$ being a $3 \times 3$ orthogonal matrix, is understood in (5) to restore global $SU(2)$ color symmetry. Our basic assumption as motivated above is that this averaging amounts to assuring confinement.

Our model is defined in Euclidean space for two reasons. First, the motivation of the model is provided by the Euclidean formulation of QCD. It seems reasonable to assume that classical Euclidean configurations such as instantons or monopoles might provide the random background. Second, the averaging procedure in (5) is better defined in Euclidean space, since a superposition of weight functions is again a weight function (with a correct normalization), whereas the superposition of phases (as the integrand of the Minkowskian functional integral is) does not give a phase. The theory in Minkowski space is defined by the standard Wick rotation. We will have more discussions on this later.

Instead of the current-current interaction (4), we shall use its pseudoscalar-scalar part that results from a Fierz transformation. The reason is as follows. As mentioned, it is known that QCD in $SU(2)$ (being pseudoreal) undergoes a symmetry breakdown which is quite different from what one expects in three-color QCD. To simulate what happens in QCD with three colors, we choose interactions so that we would have the correct symmetry-breaking pattern. Specifically whereas the interaction of two color triplet currents $j_\mu^a$ which presumably follows from QCD exhibits
the full $SU(2N_f)$ chiral symmetry group, the reduced interaction in (5) is, however, only invariant under $SU(N_f) \times SU(N_f)$ transformation, as QCD is. Although we are dealing with an $SU(2)$ color group, we assume that the basic idea of the confinement mechanism, developed below, does not depend qualitatively on the color group under investigations. In order to keep contact with QCD as closely as possible, we therefore choose our low-energy effective quark theory to exhibit the chiral patterns of QCD in four dimensions.

In order to make contact with more familiar formulations of effective quark models, we first study the limit where the color-triplet interaction $G^{\alpha\beta}$ is weak. In this limit we can perform the average over the background orientation $O$ using a cumulant expansion. The colored part of the quark interaction becomes

$$
\int d\lambda \ f(\lambda) \exp \left\{ -\frac{\lambda}{3} \int d^4x \ [\bar{q} \tau^\alpha q(x) \bar{q} \tau^\alpha q(x) - \bar{q} \gamma_5 \tau^\alpha q(x) \bar{q} \gamma_5 \tau^\alpha q(x)] + O(\lambda^2) \right\},
$$

where $\lambda$ is an eigenvalue of the matrix $G^{\alpha\beta}$, and $f(\lambda)$ is the corresponding eigenvalue density. At lowest order of the cumulant expansion, we obtain the familiar Nambu-Jona-Lasinio model with a global color symmetry. Terms multiplied by $\lambda^2$ are eight-quark interactions. The average over the background field $G^{\alpha\beta}$ in (5) obviously incorporates the interaction of more than four quarks. If these interactions are not small or equivalently if the background fields in (5) are not weak, one then has to abandon the cumulant expansion, and instead study the quark theory of (5) with fixed background field in a certain approximation, which we will specify below, and average over the background fields afterwards. If the approximation used is not bound to weak couplings, this approach can be applied even if the cumulant approximation fails.

Specifically, the approximation under investigation is to introduce meson fields on top of the scalar condensate of the vacuum and to treat their interactions perturbatively. This approximation does not resort to small couplings and is expected to give good results if the number of mesonic degrees of freedom is large. This approximation is the usual one applied to study the physics of light hadrons in the context of the Nambu-Jona-Lasinio model. We will not further question the validity of the approximation, but investigate the ground state and the properties of the light mesons within this scheme. The quark interaction in (5) is linearized by means of color-triplet ($\sigma^\alpha, \pi^\alpha$) and color-singlet mesons ($\sigma, \pi$). Integrating out the quark fields in the Hubbard-Stratonovich formalism, the resulting effective meson theory

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\(^2\)We do not expect the answer to the question as to whether the quarks are confined or not to depend on the color group used, whereas the actual value of e.g. the pion decay constant might depend on whether we have the $SU(3)$ or the $SU(2)$ gauge group.
is

\[ Z[\phi] = \left\langle \int D\sigma^\alpha D\pi^\alpha D\sigma D\pi \exp \left\{ -\int d^4x \mathcal{L}_M \right\} \right\rangle_0 , \]  \hspace{1cm} (8)

\[ \mathcal{L}_M = -\ln \left( i\bar{\phi} + iM_0 + i\sigma + i\bar{\tilde{M}}^\alpha \tau^\alpha + i\sigma^\alpha \tau^\alpha + \pi \gamma_5 + \pi^\alpha \tau^\alpha \gamma_5 \right) \]  \hspace{1cm} (9)

\[ + \frac{1}{2} \left( \sigma^\alpha + \tilde{M}^\alpha \right) \left( G^{-1} \right)^{\alpha\beta} \left( \sigma^\beta + \tilde{M}^\beta \right) + \frac{1}{2\pi^\alpha} \left( G^{-1} \right)^{\alpha\beta} \pi^\beta \]

\[ + \frac{1}{2G_0} \left( \left( \sigma + M_0 - m - i\phi_0 \right)^2 + \left( \pi + \phi_5 \right)^2 \right) , \]

where \( \tilde{M}^\alpha \) and \( M_0 \) are, respectively, the color-triplet and color-singlet constituent quark masses, and we have decomposed the external source \( \phi \) into a scalar and pseudoscalar parts, i.e. \( \phi = \phi_0 + \phi_5 \gamma_5 \). The approximation mentioned above consists of truncating the expansion of (8) in terms of the meson fields. The zeroth-order approximation provides access to ground state properties, in particular, the quark condensate. From

\[ \frac{\delta \ln Z[0]}{\delta \sigma^\alpha} = 0, \quad \frac{\delta \ln Z[0]}{\delta \sigma} = 0 \]  \hspace{1cm} (10)

one obtains

\[ -\frac{1}{V_4} \text{Tr} \left\{ \frac{i}{i\bar{\phi} + iM_0 + i\tilde{M}^\alpha \tau^\alpha} \right\} + \frac{1}{G_0} (M_0 - m) = 0 , \]  \hspace{1cm} (11)

\[ -\frac{1}{V_4} \text{Tr} \left\{ \frac{i}{i\bar{\phi} + iM_0 + i\tilde{M}^\alpha \tau^\alpha} \right\} + \left( G^{-1} \right)^{\alpha\gamma} \tilde{M}^\gamma = 0 , \]  \hspace{1cm} (12)

where \( V_4 \) is the Euclidean space-time volume, and the trace extends over internal degrees of freedom as well as over space-time. In the latter case, a regularization is required, which we will specify when needed. Different solutions of the equations (11-12) correspond to different phases of the system.

3 The Confining Phase

Here we will show that a particular solution of the gap equations (11)-(12) with an imaginary color-triplet constituent mass exists, i.e. \( \tilde{M}^\alpha = -iM^\alpha \) with \( M^\alpha \) real. Before discussing in detail the existence of such a solution, we first illustrate its remarkable physical consequence.
3.1 The scalar correlation function

Consider the color-singlet scalar correlation function

\[ \Delta_s(p) = \int d^4x \, e^{-ipx} \langle \bar{q}q(x) \bar{q}q(0) \rangle , \]  

within the effective meson theory (9). Its connected part is given by

\[ \Delta_s^c(p) = \int d^4x \, e^{-ipx} \frac{\delta^2 \ln Z[\phi]}{\delta \phi_0(x) \delta \phi_0(0)}|_{\phi=0} . \]  

To compute this, one has to consider the fluctuations of the scalar color-singlet field \( \sigma_0 \) as well as those of the fields \( \sigma^a \), since, for a fixed interaction \( G^{\alpha\beta} \), the colored mesons couple to \( \sigma_0 \) by a quark loop (see the first term on the right-hand side of (9)). Fluctuations of the pion fields \( \pi^a, \pi \) do not contribute to the scalar correlation function, since the pion fields have the wrong quantum numbers. Expanding (9) up to second order in the meson fields, the action, in momentum space representation, is

\[ S^{(2)} = \int \frac{d^4p}{(2\pi)^4} \left\{ \frac{1}{2} \sigma(p) \Pi_s^0(p^2) \sigma(-p) + \frac{1}{2} \sigma^\alpha(p) \Pi_s^\alpha(p^2) \sigma^\alpha(-p) \right\} , \]

\[ + \frac{i}{2G_0} \phi_0(p) \phi_0(-p) - \frac{i}{G_0} \phi_0(p) \sigma(-p) \right\} , \]  

where

\[ \Pi_s^0(p^2) = \frac{1}{G_0} - \int \frac{d^4k}{(2\pi)^4} \text{tr} \{ S(k + p) S(k) \} , \]

\[ \Pi_s^{\alpha\beta}(p^2) = (G^{-1})^{\alpha\beta} - \int \frac{d^4k}{(2\pi)^4} \text{tr} \{ \tau^\alpha S(k + p) \tau^\beta S(k) \} , \]

\[ K^\alpha(p^2) = -i \int \frac{d^4k}{(2\pi)^4} \text{tr} \{ \tau^\alpha S(k + p) S(k) \} . \]

The quark propagator \( S(k) \), in momentum space, is

\[ S(k) := \frac{1}{k^2 + i(M_0 + iM^\alpha \tau^\alpha)} . \]

Our model (5) is defined with an average over all local \( SU(2) \) orientations of the interaction matrix \( G^{\alpha\beta} \). In order to render the averaging procedure easy, we integrate out the colored mesons in (15), i.e.

\[ S^{(2)}_{\text{eff}} = \int \frac{d^4p}{(2\pi)^4} \left\{ \frac{1}{2} \sigma(p) \left[ \Pi_s^0 + K^\alpha (\Pi_s^{-1})^{\alpha\beta} K^\beta \right] \sigma(-p) \right\} \]

\[ - \frac{1}{2G_0} \phi_0(p) \phi_0(-p) - \frac{i}{G_0} \phi_0(p) \sigma(-p) \right\} . \]
In fact, we will observe that $S_{\text{eff}}^{(2)}$ no longer depends on the orientation of the interaction matrix, so the averaging procedure is trivial.

In order to study the effective theory of the singlet meson $\sigma$, we explicitly calculate the polarization functions $\Pi_0^0$, $\Pi^\beta_\gamma$ in (16) and (17), respectively, as well as the mixing $K^\alpha$ (18). For this purpose, it is convenient to introduce the eigenvectors of the matrix $M^\alpha\tau^\alpha$, i.e.

\[ M^\alpha\tau^\alpha |\pm\rangle = \pm M |\pm\rangle, \quad M = \sqrt{M^\alpha M^\alpha}. \] (21)

They possess the property

\[ \langle \pm |\tau^\alpha|\pm\rangle = \pm \frac{M^\alpha}{M}, \] (22)

which will be extensively used later. The detail of the calculation is left to Appendix A. It turns out that the quantities (16) and (18) can be expressed in terms of two functions $H_0(p^2)$, $H_\nu(p^2)$, i.e.

\[ \Pi_0^0(p^2) = \frac{1}{G_0} - H_0(p^2), \quad K^\alpha(p^2) = \frac{M^\alpha}{M} H_\nu(p^2). \] (23)

For later convenience, we explicitly write them down here:

\begin{align*}
H_0(p^2) &= 4 \int_0^1 d\alpha \int \frac{d^4q}{(2\pi)^4} \left\{ \frac{q^2 - Q}{[q^2 + Q]^2} + (M \rightarrow -M) \right\}, \quad (24) \\
H_\nu(p^2) &= -4i \int_0^1 d\alpha \int \frac{d^4q}{(2\pi)^4} \left\{ \frac{q^2 - Q}{[q^2 + Q]^2} - (M \rightarrow -M) \right\} \quad (25)
\end{align*}

where

\[ Q = \alpha(1 - \alpha)p^2 + (M_0 + iM)^2. \] (26)

We also find (see Appendix A) that $M^\alpha$ is an eigenvector of the polarization matrix $\Pi_\gamma^\beta$, i.e.

\[ \Pi_\gamma^\beta \frac{M^\beta}{M} = \left( \frac{1}{G_c} - H_0(p^2) \right) \frac{M^\alpha}{M}, \] (27)

where we have used the property (of which we will have more to say in the next subsection) of the solution of the Dyson-Schwinger equations (11,12), namely, that $M^\alpha$ is an eigenvector of the symmetric matrix $G^\alpha\beta$ with eigenvalue $G_c$. It is now straightforward to calculate $S_{\text{eff}}^{(2)}$, (20). Since $K^\alpha$ is proportional to $M^\alpha$ and using that the eigenvectors of the symmetric matrix $G^\alpha\beta$ are orthogonal, one obtains

\begin{align*}
S_{\text{eff}}^{(2)} &= \int \frac{d^4p}{(2\pi)^4} \left\{ \frac{1}{2} \sigma(p) \left[ \frac{1}{G_0} - H_0(p^2) + \frac{H_\nu^2(p^2)}{G_c} - H_0(p^2) \right] \sigma(-p) \right. \quad (28) \\
&\quad - \frac{1}{2G_0} \phi_0(p) \phi_0(-p) - \frac{i}{G_0} \phi_0(p) \sigma(-p) \left\}.
\end{align*}
Note that $S_{eff}^{(2)}$ depends only on $M_0$ and $M$, which are invariant under $SU(2)$ rotations. This is the desired result, since the average over the $SU(2)$ orientations can now be trivially performed.

We are now going to study the occurrence of an imaginary part of the scalar correlation function signaling a quark-anti-quark threshold. The crucial observation will be that whenever the trace of the color indices is required, the contributions with $M^\alpha$ occur in conjugate complex pairs. In particular, this will erase imaginary parts of the correlations function, and no quark-anti-quark threshold will occur.

In order to work out this phenomena in some detail for the scalar correlation function (13), it is sufficient to study the functions $H_0(p^2)$, (24) and $H_v(p^2)$, (25), since they provide the complete correlator with the help of (28). We rewrite, for instance, $H_0(p^2)$ as

$$H_0(p^2) = \frac{1}{4\pi^2} \int_0^1 d\alpha \int_0^\Lambda^2 du \left\{ 1 - \frac{3Q}{u + Q} + \frac{2Q^2}{[u + Q]^2} \right\} + (M \rightarrow -M). \quad (29)$$

To illustrate the disappearance of the quark-anti-quark threshold for $M \neq 0$, we first study its occurrence for $M = 0$. In this case our model describes the scalar correlation function in the usual constituent quark model with a constituent quark mass $M_0$. The term of interest is the second one in the curly bracket in (29). After integration over $u$, this term becomes essentially $\ln Q$. This implies that whenever $Q$ becomes negative, the function $H_0$, (29), acquires an imaginary part. In order for $Q$ to become negative, the Euclidean momentum $p^2$ must satisfy,

$$-p^2 < 4M_0^2,$$ \quad (30)

implying that the quark-anti-quark threshold occurs at a (Minkowskian) momentum $p_M = 2M_0$, which is the familiar result. For $M = 0$ the function $H_v(p^2)$ does no harm, since it is identically zero by definition, (25).

We are now going to show that for $M \neq 0$ no threshold will occur at all. Adding up the contributions from $M$ and $-M$ in (29), the crucial term becomes

$$H_0^{\text{crit}} = -\frac{3}{2\pi^2} \int_0^1 d\alpha \int_0^\Lambda^2 du \frac{(u + W)W + 4M_0^2M^2}{[u + W]^2 + 4M_0^2M^2}, \quad (31)$$

where $W = \alpha(1 - \alpha)p^2 + M_0^2 - M^2$. An analogous result holds for the function $H_v(p^2)$. One finds

$$H_v^{\text{crit}} = -\frac{3}{2\pi^2} \int_0^1 d\alpha \int_0^\Lambda^2 du \frac{2M_0^2M^2}{[u + W]^2 + 4M_0^2M^2}. \quad (32)$$

We find that the logarithmic divergence is screened if $MM_0 \neq 0$. No imaginary part occurs for $M_0 \neq 0$ and $M \neq 0$. This is our main observation. For non-vanishing
current mass $m$, one always expects at least a small constituent quark mass $M_0$. Therefore, the main ingredient in avoiding the decay of the scalar meson into a quark-anti-quark pair is the non-vanishing value of $M$. In the following, we refer to the phase of the constituent quark model (5) with $M \neq 0$ as confining phase.

In the chiral limit, the chiral symmetric phase ($M_0 = 0$) needs further discussion. In section (5) we will find that temperature induces a deconfinement phase transition, and that chiral symmetry is restored at the same time. Here we find by an inspection of (31,32) that the restoration of chiral symmetry ($M_0 = 0$) is accompanied by the occurrence of quark-anti-quark thresholds.

### 3.2 Phase structure

Here we will search for solutions of the gap equations (11-12) with a non-vanishing, imaginary constituent quark mass in the color-triplet channel, which implies the remarkable consequences discussed above. We will discuss the dependence of the phase structure, and, in particular, the phase transition from the confining phase ($M \neq 0$) to a non-confining phase ($M = 0$), on the parameters of the model, i.e. $G_0$, $G^{a\beta}$ and $m$. For this purpose, we have to analyze the solution of the gap
equations (11-12). In order to solve (12), we assume \( M^\alpha \) to be an eigenvector of the matrix \( G^\alpha^\beta \), i.e.
\[
G^\alpha^\beta M^\beta = G_c M^\alpha .
\] (33)
This reduces eqs. (11) and (12) to (see Appendix B)
\[
\frac{1}{G_0}(M_0 - m) = -M_0 I_0(M_0, M) + M I_v(M_0, M) ,
\] (34)
\[
\frac{1}{G_c} M = -M I_0(M_0, M) - M_0 I_v(M_0, M) ,
\] (35)
where
\[
I_0(M_0, M) = -\frac{1}{2\pi^2} \int_0^\Lambda^2 du \, u \, \frac{u + M_0^2 - M^2}{u^2 + 2u(M_0^2 - M^2) + (M_0^2 + M^2)^2} ,
\] (36)
\[
I_v(M_0, M) = \frac{1}{\pi^2} \int_0^\Lambda^2 du \, \frac{M_0 M}{u^2 + 2u(M_0^2 - M^2) + (M_0^2 + M^2)^2} ,
\] (37)
where a sharp \( O(4) \) cutoff \( \Lambda \) was introduced. Since \( I_v \) is proportional to \( M_0 M \), \( M = 0 \) is always a solution of eq. (35). In this case, eq. (34) is reduced to the gap equation of the standard Nambu-Jona-Lasinio model with an additional \( SU(2) \) degree of freedom. This implies that the standard Nambu-Jona-Lasinio model is contained in the extended model (5) as a special case, and we expect the theory (5) to be phenomenologically as successful as the Nambu-Jona-Lasinio model.

The solution of the system of eqs. (34,35) was investigated numerically. Figure 1 shows the color-singlet and color-triplet constituent quark masses, \( M_0 \) and \( M \), as function of the color-singlet coupling \( G_0 \). For sufficiently small color-singlet coupling strength a confining phase with \( M \neq 0 \) exists. Also shown is the non-confining phase with \( M = 0 \) for all values of \( G_0 \). In order to decide which solution forms the vacuum, one has to compare the classical action, i.e.
\[
A_c = \int \frac{d^4k}{(2\pi)^4} \left\{ -2\ln[(k^2 - M_0^2 - M^2)^2 + 4M_0^2k^2] - \frac{1}{2G_c} M^2 + \frac{1}{2G_0} (M_0 - m)^2 \right\} ,
\] (38)
of both solutions. One finds that the confining solution has a lower action and forms the vacuum.

It is observed that the color-triplet coupling strength \( G_c \) must exceed a critical value in order to allow for the desired imaginary constituent quark mass in the color-triplet channel. Figure 2 shows the critical coupling as function of the color-singlet coupling \( G_0 \) for different values of the current mass \( m \). A strong color-singlet constituent mass \( M_0 \) seems to suppress the occurrence of the confining phase. The situation can be compared with that of free electrons in a solid. It was discovered by Anderson that the electrons get localized if the density of impurities exceeds a critical limit [22].
Finally, we present the result for the scalar correlation function $\Delta_s(p^2)$, (14) which we have calculated in the last subsection. The final result (two integrations are left to a numerical calculation) is given by

$$\Delta_s(p^2) = \frac{1}{G_0} - \frac{1}{G_0} \frac{1}{G_0 - H_0(p^2)} + \frac{\frac{H_0(p^2)}{G_0}}{\frac{1}{G_0} - H_0(p^2)}.$$  \hspace{1cm} (39)

Figure 3 shows the correlation function $\Delta_s(p^2)$ as function of the Euclidean momentum transfer $p^2$ for different values for the current quark mass $m$. From Figure 2, we conclude that increasing the current mass $m$ at fixed values of $G_0$ and $G_c$ drives the system towards the deconfinement phase transition. For large $m$ (close to the critical value of $m$ where the transition to the deconfined phase occurs) one observes a resonance-like peak at a negative momentum squared, which is reminiscent of the quark anti-quark threshold of the deconfined phase ($M = 0$). In the latter case, our model is identical to the Nambu-Jona-Lasinio model. It is well known that in the standard Nambu-Jona-Lasinio model, a weakly bound scalar particle occurs in the spectrum. It has a mass of two times the constituent quark mass implying that the particle pole occurs at the threshold position. The physics of the scalar meson is therefore beyond the scope of the standard Nambu-Jona-Lasinio model. This defect of the NJL-model is cured in our model (5): the threshold is avoided.
Figure 3: The scalar correlation function as function of the Euclidean momentum transfer for different values of the current quark mass $m$ for $\frac{G_0 \Lambda^2}{2\pi^2} = 1$ and $\frac{G_1 \Lambda^2}{2\pi^2} = 2$.

in the confining phase ($M \neq 0$). At the same time, no scalar particle is present in the spectrum. There appears a “resonance” near the threshold position with a width proportional to $M$. This result is in agreement with nature, since no scalar bound state is observed in the meson spectrum at the relevant energy scale. The scalar Green’s function develops a structure around the threshold position without allowing the meson to decay into two quarks. This might be a precursor to a parton structure of hadrons at high energy. Unfortunately we cannot push this conjecture further, since our model is limited to low energies.

One might raise the question as to whether this result is in not disagreement with dispersion relations, since the correlation function we have is a real function in the entire momentum space. It is interesting to note that our model does not allow a scalar particle to appear, although a scalar field is needed in the bosonization approach to represent the quark interaction. The scalar correlation function at hand is, therefore, not appropriate for discussing dispersion relations. On the other hand, as we will see soon, the pion-pion correlation function exhibits the pion pole and is otherwise real, so one might worry about the dispersion relation in this case. Note, however, that in order to arrive at this result, we have expanded the effective meson theory (9) up to second order in the meson fields ignoring mesonic interactions. If these interactions are included, we expect to recover the physical thresholds of an interacting meson theory satisfying dispersion relations.
Although the correlation function is real, a non-trivial structure appears at the would-be threshold position. A close inspection of the correlator (39) (see Appendix C) shows that the function possesses cuts in the complex $p$-plane. One might then question whether these complex structures violate fundamental principles of quantum field theory (e.g. stability). We note that our approach – the bosonization procedure of section 2.2 and the related approximations – is supposed not to violate any of the analyticity requirements. In comparison to other NJL-type models, our ground state (vacuum) is simply more complicated, with the standard NJL model corresponding to a particular case ($M = 0$). It may be that some of the fundamental requirements of QCD are not correctly implemented. If so, then our detailed calculation may be providing mechanisms for the cancellation of unwanted features. In this case, we could expect our model to give some insight in the analyticity structure of Green's functions of a confining theory. This question is of particular interest, but is beyond the scope of the present paper. See [23] for a discussion on this matter.

To check that nothing is amiss with our model, we have studied the analytic structure of the scalar correlation function in Appendix C and explicitly verified that the stability criterion is indeed satisfied thanks to a cancellation mechanism mentioned above.

4 Chiral Properties

In this section we study the properties of the pion that should emerge as a Goldstone boson of the spontaneously broken chiral symmetry. We will show that our model is in agreement with the low-energy theorems, and will verify by an explicit calculation that the Gell-Mann-Oakes-Renner relation holds.

4.1 The pion Bethe-Salpeter equation

One way to extract the properties of the pion is to study the pseudoscalar correlation function in the color-singlet channel which can be obtained in the same way as for the scalar correlator discussed in section 3.1. Here we prefer to use a different approach which is to calculate the pion Bethe-Salpeter amplitude. This method will illustrate the role of the hidden color components of the pion, and will readily provide an access to such observables as the pion decay constant and the pion electromagnetic form factor.

The Bethe-Salpeter equation for the pion amplitude $(P_0, P^a)$ is directly obtained
from the effective meson Lagrangian $\mathcal{L}_M$ (9) by

$$\sum_{b=0,1,2,3} \frac{\delta^2 \mathcal{L}_M}{\delta \pi^a(-p) \delta \pi^b(p)} P^b(p)|_{p^2=-m^2_\pi} = 0 \quad (40)$$

For a fixed orientation of the interaction matrix $G^{\alpha\beta}$, the left-hand side of this equation becomes

$$\left( \frac{\delta}{\delta \mathcal{G}_0} + \text{Tr}\{\gamma_5 S(k+p)\gamma_5 S(k)\} \right) \left( \frac{\delta}{\delta \mathcal{G}_0} + \text{Tr}\{\tau^\alpha \gamma_5 S(k+p)\gamma_5 S(k)\} \right) \left( \begin{array}{c} P_0 \\ P^\alpha \end{array} \right),$$

where the trace $\text{Tr}$ extends over the momentum space $(k)$ as well as Lorentz- and color-space. One observes that the ansatz

$$P^\alpha = i \frac{M^\alpha}{M} P_1$$

for the color-components of the pion\(^3\) reduces (41) to (see Appendix D)

$$\left( \frac{\delta}{\delta \mathcal{G}_0} + \mathcal{I}_0(p^2) - \mathcal{I}_v(p^2) \right) \left( \begin{array}{c} P_0(p^2) \\ P^\alpha(p^2) \end{array} \right) = 0 \quad (43)$$

where the functions $\mathcal{I}_0$ and $\mathcal{I}_v$ are defined by

$$\mathcal{I}_0(p^2) = -\frac{1}{4\pi^2} \int_0^1 d\alpha \int_0^{\Lambda^2} du \frac{u - \alpha(1-\alpha)p^2 + A^2}{[u + \alpha(1-\alpha)p^2 + A^2]^2} + (M \rightarrow -M), \quad (44)$$

$$\mathcal{I}_v(p^2) = \frac{i}{4\pi^2} \int_0^1 d\alpha \int_0^{\Lambda^2} du \frac{u - \alpha(1-\alpha)p^2 + A^2}{[u + \alpha(1-\alpha)p^2 + A^2]^2} - (M \rightarrow -M), \quad (45)$$

with $A = M_0 + iM$. An inspection of (45) yields that both $\mathcal{I}_0$ and $\mathcal{I}_v$ are real. Demanding (43) to have a non-trivial solution for $P^2 = -m^2_\pi$ leads to a nonlinear equation to determine $m_\pi$. Instead of solving this equation numerically, it is more instructive to solve it analytically for pion masses small compared with the constituent quark mass $M_0$. For this purpose we expand the functions $\mathcal{I}_0$ and $\mathcal{I}_v$ up to linear order in $p^2$, i.e.

$$\mathcal{I}_0(p^2) = I_0 + F_0 p^2 + O(p^4), \quad \mathcal{I}_v(p^2) = I_v + F_v p^2 + O(p^4), \quad (46)$$

where $I_0$ and $I_v$ depend, respectively, on $M_0$ and $M$ (see (36) and (37)), and the quantities $F_0, F_v$ are defined by

$$F_0 = \frac{1}{8\pi^2} \int_0^{\Lambda^2} dk^2 \frac{d(k^2) k^2}{[k^2 + (M_0 + iM)^2]^2} + (M \rightarrow -M), \quad (47)$$

$$F_v = -\frac{i}{8\pi^2} \int_0^{\Lambda^2} d(k^2) k^2 \frac{d(k^2) k^2}{[k^2 + (M_0 + iM)^2]^2} - (M \rightarrow -M), \quad (48)$$

\(^3\)Once the average over all orientations of the interaction matrix is performed, these parts become the hidden-color components (compare section 4.3).
A non-trivial solution for the Bethe-Salpeter amplitudes $P_0, P_1$ exists provided
\[
\text{det} \left( \begin{array}{cc}
\frac{1}{G_0} + I_0 + F_0 p^2 & -I_v - F_v p^2 \\
I_v + F_v p^2 & \frac{1}{G_c} + I_0 + F_0 p^2
\end{array} \right) = 0 .
\]
Equation (49) can be rewritten as (setting $p^2 = -m_n^2$)
\[
m_n^2 f_{\pi c}^2 = 4m \langle \bar{q}q \rangle + O(m_n^4) + O(m^2) ,
\]
where we have used that the quark condensate $\langle \bar{q}q \rangle$ is given by $M_0/G_0$, and where $f_{\pi c}^2$ is defined by
\[
f_{\pi c}^2 = 4 \left\{ M_0^2 F_0 - M^2 F_0 - 2M M_0 F_v \right\} .
\]
Equation (51) is the Gell-Mann-Oakes-Renner relation. It tells us that in the chiral limit ($m = 0$) the pion is massless if chiral symmetry is spontaneously broken ($\langle \bar{q}q \rangle \neq 0$). Below, we will show that $f_{\pi c}^2$, defined in (52), coincides with the pion decay constant in the chiral limit ($m = 0$).

## 4.2 The electromagnetic form factor

Since the Bethe-Salpeter equation provides only the relative weights of the amplitudes $P_0, P^\alpha$, an additional physical input is needed for the normalization of the amplitudes. One may use the electromagnetic form factor of the pion. The electromagnetic form factor provides the desired additional constraint. The form factor $F(q^2)$ is defined via the electromagnetic vertex [27], i.e.
\[
\langle \pi(p')|J_\mu(0)|\pi(p) \rangle = ie (p'_\mu + p_\mu) F(q^2) ,
\]
where $J_\mu$ is the electromagnetic current and $q = p' - p$. In order to normalize the pion charge to unity, we demand that
\[
F(0) = 1 .
\]
Once the form factor is calculated, the (mean-square) charge radius of the pion is simply given by
\[
r_{ms}^2 = (r^2)_\pi = -6 \frac{\partial F(q^2)}{\partial q^2}|_{q^2=0} ,
\]
where $q$ is the momentum in Minkowski space. Writing the electromagnetic current in terms of the quark fields,

$$ J_\mu(x) = \bar{q}(x) \gamma_\mu q(x) , $$

we can study the matrix element (53) most efficiently by using the Bethe-Salpeter amplitude of the pion. Its graphical representation is given in Fig.4\footnote{Since the diagrams (in particular, (a)) are divergent, the result depends on the actual choice of the loop momentum. A more sophisticated cutoff procedure (e.g. Schwinger’s proper-time regularization) would remove this ambiguity.}. For $p'=0$, the matrix element is

$$ -\int \frac{d^4k}{(2\pi)^4} \text{tr}\left\{ \gamma_5 \left( P_0 + i \frac{M^\alpha}{M} \tau^\alpha P_1 \right) S(k - \frac{p}{2}) \gamma_\mu S(k + \frac{p}{2}) \right\} . $$

The evaluation of this matrix element is left to Appendix E. With (E.7), the form factor, for small momentum, reads

$$ F(p^2) = \left( P_0^2 - P_1^2 \right) \left( F_0 - p^2 R_0 \right) - 2P_0 P_1 \left( F_v - p^2 R_v \right) . $$

Figure 4: Graphical representation of the electromagnetic form factor (a) and the pion decay constant (b).
From this expression follow the desired normalization (54) of the pion Bethe-Salpeter amplitude and the pion charge radius (55):

$$1 = \left( P_0^2 - P_1^2 \right) F_0 - 2P_0P_1 F_v ,$$  \hspace{1cm} (59)

$$r_{ms}^2 = 6 \left[ \left( P_0^2 - P_1^2 \right) R_0 - 2P_0P_1 R_v \right] .$$  \hspace{1cm} (60)

We can now discuss the full momentum dependence of the form factor $F(p^2)$. The expression for $F(p^2)$ is derived in Appendix E. The integration over the angle variable and the radial component of the momentum space was performed numerically. The numerical result is presented in Figure 5. There exists no vector-meson poles at negative momentum transfer, but instead a peak, its width strongly depending on the parameters of the model. Presumably the resonance-like structure is an artifact of the model, unrelated to any physical processes. It probably reflects the suppression of the quark-anti-quark pole present in the standard NJL model without confinement. As mentioned, the physics of the vector mesons is beyond the scope of a local four-quark interaction\(^5\). We expect that a more realistic model with quantum loops will produce a vector-meson (i.e., $\rho$) pole instead of the resonance. In

\(^5\)In particular, we do not have any vector and axial-vector channels in the quark interaction (6), which are known to be important for the physics of vector mesons.
fact, we observe a pion pole and no further structure in the pseudoscalar correlation function in our model, which is supposed to give good results for the pion physics (see below).

It is quite remarkable that we obtain a real form factor for arbitrary large (Minkowskian) momentum manifesting the absence of quark and anti-quark thresholds. This cures an outstanding problem of the standard non-confining NJL-model. This may have an important consequence in heavy-meson physics. For instance, the non-perturbative description of the electroweak currents of a heavy and a light quark is beyond the reach of the usual NJL-model due to the occurrence of quark anti-quark threshold effects. Electroweak currents such as the one considered here enter into the decay rate of a B-meson into a pion and an electron. In the limit of the electron being very fast, one should be able to extract the element $V_{bu}$ of the Kobayashi-Maskawa mixing matrix from the differential cross section. However, it has been shown that in the limit of a fast electron, non-perturbative contributions to the electroweak currents become important. We believe that a more realistic formulation ($SU(3)$ color) of our model might be able to provide the desired results.

The normalization (59) completes the calculation of the pion Bethe-Salpeter amplitude. This is the desired result, since we are now able to calculate matrix elements involving pions. In particular, we will obtain the pion decay constant in the next chapter by a direct calculation.

### 4.3 The pion decay constant

An expression of pion decay constant was already deduced in section 4.1 by demanding that the Gell-Mann-Renner-Oakes relation (51) be quantitatively satisfied. The aim of this subsection is to show that the decay constant (52), obtained there, is identical to that derived from its definition.

The pion decay constant $f_\pi$ is defined by the matrix element

$$
\langle 0 | A_\mu(x) | \pi(p) \rangle |_{p \to 0} = i e^{ipx} p_\mu f_\pi , \tag{61}
$$

which describes the coupling of the pion to the vacuum via the axial vector current $A_\mu(x) = \bar{q}(x) \gamma_5 \gamma_\mu q(x)$. The matrix element (61), also shown in Figure 4, is

$$
- \int \frac{d^4k}{(2\pi)^4} \text{tr} \left\{ \gamma_5 \gamma_\mu S(k+p) \left( P_0 + i \frac{M^\alpha}{M} P_1 \right) \gamma_5 S(k) \right\} . \tag{62}
$$

An average over all color orientations of the interaction matrix $G^{\alpha\beta}$ – which is equivalent to an average over all color directions of $M^\alpha$ – is understood in (62). A straightforward calculation of (62) for a given vector $M^\alpha$ yields

$$
4i p_\mu \left\{ \int \frac{d^4k}{(2\pi)^4} \left[ \frac{A}{[k^2 + A^2]^2} + (M \rightarrow -M) \right] P_0 \right. \tag{63}
$$
\[- (i) \int \frac{d^4 k}{(2\pi)^4} \left\{ \frac{A}{[k^2 + A]^2} - (M \rightarrow -M) P_1 \right\}.\]

This result can be further simplified by introducing the functions $F_0$ and $F_v$ from (47,48), i.e.

\[2i p_{\mu} \left\{ (M_0 F_0 - M F_v) P_0 - (M_0 F_v + M F_0) P_1 \right\}. \tag{64}\]

The ratio $\epsilon$ of $P_0$ over $P_1$ is provided by the Bethe-Salpeter equation (43), i.e.

\[\epsilon := \frac{P_1}{P_0} = -\frac{\mathcal{I}_v(p^2 = -m^2_\pi)}{\frac{1}{\alpha_c} + \mathcal{I}_0(p^2 = -m^2_\pi)}, \tag{65}\]

whereas the overall normalization is constrained by the electromagnetic form factor (59). Expressing $P_1$ in terms of $\epsilon$ and $P_0$ and eliminating $P_0$ with the constraint (59), we obtain the final result

\[f^2_\pi = 4 \frac{\left[ M_0 F_0 - M F_v - (M_0 F_v + M F_0) \epsilon \right]^2}{(1 - \epsilon^2) F_0 - 2\epsilon F_v}, \tag{66}\]

where the functions $F_0$, $F_v$ are given by (47) and (48). This is our result for the pion decay constant, valid for all values of the current mass. This decay constant needs to agree with the one extracted from the Gell-Mann-Renner-Oakes relation, (52), only in the chiral limit ($m = 0$). Before we show that this is indeed the case, we would like to comment on the result (66). First note that the average of all orientations has still to be performed. Since our result (66), however, depends only on the invariant quantity $M$ rather than on $M^\alpha$, this average is trivial. Note further that a term proportional to $P_1$ enters into the physical decay constant. This is the contribution from the hidden-color components of the pion.

In order to calculate $f_\pi$ in the chiral limit, we proceed with (63). From the Bethe-Salpeter equation (43) in the chiral limit $p^2 = -m^2_\pi = 0$, one finds

\[P_0 = \frac{M_0}{M} P_1. \tag{67}\]

With this result, the decay constant in the chiral limit becomes

\[f_{\pi c} = 2 \left\{ M^2 F_0 - 2M M_0 F_v - M^2 F_0 \right\} \frac{P_1}{M}. \tag{68}\]

Using (67) in the normalization of the Bethe-Salpeter amplitude (59), one may eliminate $P_1$ in (68) to obtain

\[f^2_{\pi c} = 4 \left\{ (M^2 - M^2) F_0 - 2M M_0 F_v \right\}. \tag{69}\]

This expression is identical to (52). This establishes that the Gell-Mann-Oakes-Renner relation is indeed valid in our model.
Figure 6: Mass $m_\pi$ and decay constant $f_\pi$ of the pion as function of the current mass $m$ for $\frac{G_0 \Lambda^2}{2\pi^2} = 1$ and $\frac{G_e \Lambda^2}{2\pi^2} = 2$. Also shown are the constituent quark masses $M_0$ and $M_e$.

We finally present the numerical data for the pion mass and the pion decay constant. We have numerically solved the Bethe-Salpeter equation (43) to obtain the mass of the pion and the ratio $\epsilon = P_1/P_0$. The decay constant $f_\pi$ was then calculated from (66). The result for $m_\pi$ and $f_\pi$ as a function of the current mass is shown in Figure 6. One observes that $m_\pi^2$ depends almost linearly on the current mass with the slope given by the Gell-Mann-Oakes-Renner relation. The decay constant $f_\pi$ decreases with increasing current mass.

4.4 The pseudoscalar correlation function

It is instructive to compare the result of the scalar correlation function with that for the pseudoscalar correlator where one expects a pion pole. The pseudoscalar correlation function is defined by

$$
\Delta_\pi^\phi(p) = \int d^4x \ e^{-ipx} \frac{\delta^2 \ln Z[\phi]}{\delta \phi_\pi(x) \delta \phi_\pi(0)}|_{\phi=0}.
$$

In order to derive $\Delta_\pi^\phi$, we expand the effective meson theory (8) up to second order in pion fields. Since an average over all orientations of the interaction matrix $G^{ab}$ is
Figure 7: The pion dispersion relation as function of the Euclidean momentum transfer for zero current quark mass $m = 0$ and for $\frac{G_\pi A^2}{2\pi^2} = 1$ and $\frac{G_\pi A^2}{2\pi^2} = 2$.

required, it is convenient to integrate out the colored pion fields. One observes that the resulting theory for the color-singlet pion does not depend on this orientation, so the averaging is trivial. Since the explicit calculation parallels closely that of the scalar correlator in section 3.1, we shall simply present the final result:

$$S_{\text{eff}}^{(2)} = \int \frac{d^4p}{(2\pi)^4} \left\{ \frac{1}{2} \pi(p) \Pi_\pi(p^2) \pi(-p) + \frac{1}{2G_0} \phi_\pi(p) \phi_\pi(-p) + \frac{1}{G_0} \phi_\pi(p) \pi(-p) \right\},$$

where the pion dispersion formula $\Pi_\pi(p^2)$ is given by

$$\Pi_\pi(p^2) = \frac{1}{G_0} + \mathcal{I}_0(p^2) + \frac{\mathcal{I}_\pi^2(p^2)}{\mathcal{I}_0(p^2)} + \frac{1}{G_0}. \quad (72)$$

The functions $\mathcal{I}_0$ and $\mathcal{I}_\pi$ were defined in (44) and (45), respectively. The dispersion formula $\Pi_\pi$ is shown in Figure 7 as function of the Euclidean momentum transfer in the chiral limit ($m = 0$). Note that the function $\Pi_\pi(p^2)$ is zero at $p^2 = 0$. This zero gives rise to a pole of the correlation function at zero momentum transfer confirming the pion as Goldstone boson. The most striking feature of Figure 7 is a singularity at positive Euclidean momentum. It arises from the second term in (72), which stems from the integration over the colored pions. The singularity occurs at the momentum where the colored pions go on-shell. The occurrence of the singularity
therefore is quite natural and does not depend on the details of the model. In our model, the colored pions go on-shell at positive Euclidean momentum $p^2$, which might indicate that the colored pions condense. In this case our approach, expanding the Lagrangian up to second order in the colored pion fields and integrating them out, is no longer appropriate. We will leave this issue to future investigations and here study the pion pole in some detail.

One can deduce the pion mass $m^2_\pi$ from the position of the pion pole in the correlation function $\Delta^c_\pi(p^2)$. It is worthwhile to check whether this result for the pion mass agrees with that obtained from solving the Bethe-Salpeter equation (see section 4.2). One observes that both masses are indeed identical, because the constraint for the dispersion relation to be satisfied coincides with the condition (49) which guarantees a non-zero Bethe-Salpeter amplitude. Finally the numerical result for the correlator (70) is presented in Figure 8. The correlation function is entirely dominated by the pion pole. This is compared with the result for the scalar correlator. In the latter case, no pole occurs at all, but the scalar correlation function is significantly influenced by a peak structure.

A second pole in the correlation function occurs at a positive value of the Euclidean momentum $p^2$. This pole is due to the fact that the occurrence of the colored pion pole in the dispersion relation $\Pi_5$ gives rise to a further zero of $\Pi_5(p^2)$. The pole of $\Delta^c_\pi$ at a positive momentum squared is intimately related to the properties

Figure 8: The pseudoscalar correlation function as function of the Euclidean momentum transfer for a current quark mass $m = 0.01\Lambda$ and for $\frac{G_L\Lambda^2}{2\pi^2} = 1$ and $\frac{G_T\Lambda^2}{2\pi^2} = 2$. 
of the hidden-color pions and will be the subject of a future work.

5 Temperature Effects

The model (5) offers a confining phase and a deconfining phase, both emerging from the classical equations of motion (11-12). In the section given above, we investigated the phase structure in the model's parameter space. We found that the ground state is in the confining phase if the color-triplet coupling strength is strong enough. In this section, we will investigate the influence of temperature on the phase structure. In particular, we will start from the system with the vacuum in the confining phase and will study the type of phase transitions, if any, to the deconfining phase.

For this purpose, we must generalize the gap equations (11-12) to finite temperature. The trace in the first terms on the left-hand side stems from the integration over the quark fields. In order to introduce temperature, we adopt the usual imaginary-time formalism [24] and confine the configuration space of the fermionic fields to the configurations which are anti-periodic in the Euclidean time direction with a periodic length $1/T$ with $T$ the temperature. At finite temperature, the integration over the zeroth component of the Euclidean momentum in the terms of (11-12) is replaced by a discrete sum over Matsubara frequencies. In order to illustrate the evaluation of such trace terms at finite temperature, we explicitly work out a term, which arises in (11-12) once the color trace is performed, i.e.

$$F := \text{Tr} \left\{ \frac{i}{\not{k} + i(M_0 + iM)} \right\}.$$  \hfill (73)

Performing the Dirac trace as well as the trace over space-time, one obtains

$$F = 4V \sum_{n=-\infty}^{\infty} \int \frac{d^3k}{(2\pi)^3} \frac{M_0 + iM}{\pi^2 T^2 (2n+1)^2 + (a + ib)^2},$$  \hfill (74)

where $V$ is the space volume, and the quantities

$$a = \sqrt{\frac{1}{2} \left[ \tilde{k}^2 + M_0^2 - M^2 + \sqrt{(\tilde{k}^2 + M_0^2 - M^2)^2 + 4M_0^2 M^2} \right]}$$  \hfill (75)

$$b = \frac{M_0 M}{a},$$  \hfill (76)

were introduced as an abbreviation. In order to split the zero temperature part, which is cutoff-dependent, from the temperature-dependent contributions, which
are finite, we perform a Poisson resummation of the Matsubara sum in (74), i.e.

\[ F = 4V \int d\mathbf{k} \frac{d^3k}{(2\pi)^3} \frac{M_0 + iM}{\pi^2T^2(2n + 1)^2 + \vec{k}^2 + (M_0 + iM)^2} + F(\nu \neq 0). \]  
(77)

The first term is the term with zero conjugate Matsubara frequency ($\nu = 0$), which yields the zero temperature result if we substitute $k_0 = \pi T(2n + 1)$ for $n$ in the integral in (77). It is this term which needs regularization. It will be performed by introducing a sharp $O(4)$ cutoff as it was done in the zero-temperature case. The part of the non-zero conjugate Matsubara indices is finite and needs no regularization. It is

\[ F(\nu \neq 0) = -\frac{4V M_0 + iM}{T} \frac{a + ib}{a^2 + b^2 + 2 \cos(b/T)e^{a/T} + 1} \]  
(78)

Evaluating the trace terms in (11-12) as sketched above, one finds that the gap equations acquire an additional term that contains a temperature dependence. Introducing

\[ \Sigma_R = \frac{ae^{a/T} \cos(b/T) + a - be^{a/T} \sin(b/T)}{(a^2 + b^2)(e^{2a/T} + 2 \cos(b/T)e^{a/T} + 1)}, \]  
(79)

\[ \Sigma_I = \frac{ae^{a/T} \sin(b/T) + b + be^{a/T} \cos(b/T)}{(a^2 + b^2)(e^{2a/T} + 2 \cos(b/T)e^{a/T} + 1)}, \]  
(80)
the modified gap equations are

\[ G_0^{-1}(M_0 - m) = M_0 I_+(M_0, M) - \frac{4}{\pi^2} \int dk \, k^2 (M_0 \Sigma_R + M \Sigma_I) \] \tag{81} \\
\[ G_c^{-1} M = M I_-(M_0, M) - \frac{4}{\pi^2} \int dk \, k^2 (M \Sigma_R - M_0 \Sigma_I) \] \tag{82}

where the functions \( I_\pm \) are defined with the help of (36,37) by

\[ I_+ = -I_0 + \frac{M}{M_0} I_v, \quad I_- = -I_0 - \frac{M_0}{M} I_v. \] \tag{83}

This is the main result of this section; the temperature-dependent color-singlet and color-triplet constituent quark masses will emerge from these equations.

We have studied numerically the solutions \( M \) and \( M_0 \) of (81-82) as function of temperature. The coupling strengths \( G_0 \) and \( G \) were chosen in order for the system to be in the confining phase at zero temperature. The result is shown in Figure (9). If the temperature exceeds a critical value, a first order phase transition from the confining phase \( (M \neq 0) \) to the deconfined phase \( (M = 0) \) takes place. The deconfining phase transition is accompanied by a sudden drop of the color-singlet constituent mass \( M_0 \) indicating at the same time the restoration of chiral symmetry. The small residual constituent quark mass is due to the current mass \( m = 0.02 \Lambda \) which explicitly breaks chiral symmetry.
For phenomenological applications, the dependence of the deconfining phase transition on the current quark mass is of particular interest. Figure 10 shows the critical color-triplet coupling strength $G_c$ as function of the temperature for some values of the current mass. The result suggests that quark liberation of all quark flavors occurs approximately at the same temperature (assuming that the system was in the confining phase at zero temperature).

### 6 Finite-Density Effects

In the last section, we observed a phase transition from the confined phase to the deconfined phase at high temperature. For phenomenological reasons [28], one also expects a phase transition to occur at high baryonic density. Here we investigate the existence of this phase transition in the random background quark model (5).

In order to study the system at a non-zero baryonic density, we introduce a chemical potential $\mu$ into the Lagrangian (6):

$$
\mathcal{L}_D = \bar{q}(x)(i\partial + im + i\mu_\gamma)q(x) + G_0[\bar{q}q(x)\bar{q}q(x) - \bar{q}\gamma_5q(x)\bar{q}\gamma_5q(x)] (84)
+ \left[\bar{q}\gamma^\alpha q(x)G^\alpha_\beta\bar{q}\gamma^\beta q(x) - \bar{q}\gamma_5\tau^\alpha q(x)G_\beta^\alpha\bar{q}\gamma_5\tau^\beta q(x)\right],
$$

The bosonization procedure — introducing scalar and pion fields — is unchanged by the presence of the chemical potential. The gap equations (11,12) acquire an additional part due to finite density, i.e.

$$
-\frac{1}{V_4}\text{Tr}\left\{\frac{i}{i\partial + i\mu_\gamma + iM_0 + i\bar{M}^\alpha_\tau}\right\} - I_F(M_0, M) + \frac{1}{G_0} (M_0 - m) = 0, (85)
$$

$$
-\frac{1}{V_4}\text{Tr}\left\{\frac{i}{i\partial + i\mu_\gamma + iM_0 + i\bar{M}^\alpha_\tau}\right\} - I_F^\beta(M_0, M) + (G^{-1})^\beta_\gamma \bar{M}^\gamma = 0, (86)
$$

where

$$
I_F = \frac{1}{V_4}\text{Tr}\left\{\frac{i}{i\partial + i\mu_\gamma + iM_0 + i\bar{M}^\alpha_\tau}\right\} - \frac{1}{V_4}\text{Tr}\left\{\frac{i}{i\partial + iM_0 + i\bar{M}^\alpha_\tau}\right\}, (87)
$$

$$
I_F^\beta = \frac{1}{V_4}\text{Tr}\left\{\frac{i}{i\partial + i\mu_\gamma + iM_0 + i\bar{M}^\alpha_\tau}\right\} - \frac{1}{V_4}\text{Tr}\left\{\frac{i}{i\partial + iM_0 + i\bar{M}^\alpha_\tau}\right\}. (88)
$$

One easily verifies that both functions $I_F$ and $I_F^\beta$ are finite and need no regularization. In order to calculate these functions, we first perform the color trace by introducing the eigenvectors of the matrix $M^\alpha_\tau$ defined in (21). After taking the trace over Dirac indices, the function $I_F$ becomes

$$
I_F(M_0, M) = \int \frac{d^4k}{(2\pi)^4} \left[\frac{4(M_0 + iM)}{k_0 + i\mu + (a + ib)^2} - \frac{4(M_0 + iM)}{k_0^2 + (a + ib)^2}\right] + (M \rightarrow -M), (89)
$$
Figure 11: The constituent quark masses $M_0$ and $M$ as a function of the Fermi momentum $k_f$ for $\frac{G_\alpha \Lambda^2}{2\pi^2} = 1$ and $\frac{G_\beta \Lambda^2}{2\pi^2} = 2$. $M_0$, $M$ and $k_f$ in units of the cutoff $\Lambda$.

where the functions $a$ and $b$ depend on the momentum $\vec{k}$ as defined in (75,76). It is now straightforward to evaluate the $k_0$ integration in (89):

\begin{equation}
I_F(M_0, M) = -i \int_{a(\vec{k}) < \mu} \frac{d^3k}{(2\pi)^3} \frac{2(M_0 + iM)}{ia - b} + (M \rightarrow -M). (90)
\end{equation}

One observes that the color trace again enforces that the function $I_F$ be real. Analogous considerations hold for the function $I_F^\beta$. The final result for the gap equations (85, 86) is

\begin{align*}
G_0^{-1}(M_0 - m) &= M_0 I_+(M_0, M) - \frac{2}{\pi^2} \int_0^{k_f} dk \ k^2 \frac{M_0 a + M b}{a^2 + b^2}, \quad (91) \\
G_c^{-1} M &= M I_-(M_0, M) - \frac{2}{\pi^2} \int_0^{k_f} dk \ k^2 \frac{M a - M_0 b}{a^2 + b^2}, \quad (92)
\end{align*}

where the functions $I_\pm$ are defined in (83) and $a(\vec{k}^2)$ and $b(\vec{k}^2)$ are defined in (75) and (76) respectively. The upper bound of the integration over the three-momentum $\vec{k}$ is provided by the Fermi sphere of radius $k_f$ where $k_f$ is defined by $a(k_f^2) = \mu$.

The solutions of the coupled system (91,92) provide the constituent quark mass $M_0$ and the mass $M$ in the colored channel as a function of the chemical potential $\mu$. For physical applications, it is convenient to express the chemical potential in terms
Figure 12: The color-triplet critical coupling strength $G_c$ as a function of the Fermi momentum $k_f$ for different values of the current mass $m$ in units of the cutoff.

of the baryonic density defined by

$$\rho_B := -i \left( \frac{\partial \ln Z}{\partial \mu} (\mu) - \frac{\partial \ln Z}{\partial \mu} (\mu = 0) \right).$$

From (5) and the Lagrangian $\mathcal{L}_D$ at finite chemical potential $\mu$ in (84), one obtains

$$\frac{\partial \ln Z}{\partial \mu} (\mu) = \text{Tr} \left\{ \frac{-1}{(k_0 + i\mu)\gamma_0 + \vec{k}\gamma + i(M_0 + iM)\gamma_0} \right\}. \quad (94)$$

A calculation along the line sketched above yields the simple result

$$\rho_B = 2 \int_{\vec{k}^2 < \mu} \frac{d^3k}{(2\pi)^3}. \quad (95)$$

This provides the familiar relation $k_f^3 = 3\pi^2\rho_B$ verifying that this relation also holds in the context of the augmented model (5).

We have studied the solutions $M_0$ and $M$ of the gap equations (91,92) as a function of the Fermi momentum $k_f$. The result is shown in Figure 11. The coupling strengths $G_0$ and $G_c$ were chosen in order for the system to be in the confined phase ($M \neq 0$) at zero density. For increasing Fermi momentum $k_f$ one observes
an increase in $M$ up to a critical momentum $k_f^*$, where $M$ rapidly drops to zero implying that the deconfined phase is realized at high density. The behavior of the color-singlet constituent quark mass $M_0$ is different. It smoothly decreases and vanishes at $k_f^*$. This behavior is different from that found at finite temperature (i.e. section 6) where the phase transition causes a discontinuity of $M_0$ at the critical temperature. The deconfining phase transition at finite density is accompanied by the restoration of chiral symmetry as in the temperature case.

Figure 12 shows the critical color-triplet coupling $G_c^{(\text{crit})}$ as a function of the Fermi momentum $k_f$. The dependence of $G_c^{(\text{crit})}$ on the Fermi momentum $k_f$ is qualitatively the same as the dependence on the temperature (compare Figure 10). The critical density is nearly independent of the current mass of the quarks.

### 7 Conclusions

This paper describes how the NJL model can be modified minimally so as to take into account quark confinement. Confinement is understood in the sense that quark-anti-quark thresholds which plague the standard NJL model are screened by a random color background field: physical quantities are free of unwanted colored excitations. The mechanism that confines the quarks is analogous to Anderson localization in electronic systems and deconfinement can be induced by temperature and/or density in a way that seems to be consistent with QCD. In this model, deconfinement and chiral symmetry restoration occur at the same critical point. We have used, for simplicity, color $SU(2)$ group but we have no reason to expect that qualitative features would be modified if we were to consider the realistic $SU(3)$ gauge group. For applications to phenomenology, $SU(3)$ gauge group will have to be treated. The extension to color $SU(3)$ remains to be worked out.

The problem that we hope to be able to resolve with the confining NJL model is to describe how hadrons – both mesons and baryons – behave in medium at finite temperature and density. Since confinement is suitably implemented in the model, we can treat the excitations that are not amenable to the conventional NJL model, such as scalar and vector mesons. For instance, we should be able to “derive” the BR scaling predicted in mean field of effective chiral Lagrangians [32] and address the properties of hadrons in hot and dense medium created in heavy-ion collisions or compact star matter [28]. A microscopic model of the kind presented here will have certain advantage over the “macroscopic” treatment of [32].
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Appendices

A Some ingredients for the scalar correlation function

Here we calculate the functions $\Pi_0^S(p^2)$, $K^\alpha(p^2)$ and $\Pi^2\sigma(p^2)$, defined in (16,17,18) that enter in the scalar correlation function (13). For this purpose, we need to evaluate the trace, which extends over Lorentz- and color-space, of the functions

$$\text{tr} \{ S(k+p)S(k) \} , \quad \text{tr} \{ \tau^\alpha S(k+p)\tau^\beta S(k) \} , \quad \text{tr} \{ \tau^\alpha S(k+p)S(k) \} , \quad (A.1)$$

where the fermion propagator $S(k)$, (19),

$$S(k) := \frac{1}{\not{k} + i(M_0 + iM^\alpha \tau^\alpha)} \quad (A.2)$$

possesses a non-trivial color structure. The Lorentz trace is straightforward. The color trace is most conveniently evaluated by introducing the eigenvectors $|\pm\rangle$ of the matrix $M^\alpha \tau^\alpha$ which form a complete set in color space giving rise to a specific representation of the unit element, i.e.

$$1 = \sum_{i=\pm} |i\rangle\langle i| . \quad (A.3)$$

For instance, the color trace of the first expression in (A.1) is

$$\sum_{i,l=\pm} \text{tr}_L \{ (i|S(k+p)|l)\langle l|S(k)|i\rangle \} , \quad (A.4)$$

where $\text{tr}_L$ indicates the trace over Lorentz indices only. Using

$$S(k)|\pm\rangle = \frac{1}{\not{k} + i(M_0 \pm iM^\alpha)} |\pm\rangle , \quad (A.5)$$
one obtains for (A.4)
\[ \text{tr}_L \{ s(k + p)s(k) \} + (M \rightarrow -M) , \]  

where
\[ s(k) = \frac{1}{\not k + iA} \quad \text{with} \quad A = M_0 + iM . \]  

It is now straightforward to calculate \( \Pi^0_s(p^2) \) in (16):
\[ \Pi^0_s(p^2) = \frac{1}{G_0} - 4 \int_0^1 d\alpha \int \frac{d^4 k}{(2\pi)^4} \left\{ \frac{\text{tr}_L (\not k + \not p - iA)(\not k - iA)}{[|(k + p)^2 + A^2|][k^2 + A^2]} + (M \rightarrow -M) \right\} . \]  

The Lorentz trace can be easily performed. Introducing Feynman’s parametrization, one has
\[ \Pi^0_s(p^2) = \frac{1}{G_0} - 4 \int_0^1 d\alpha \int \frac{d^4 q}{(2\pi)^4} \left\{ \frac{k^2 + kp - A^2}{[|(k + \alpha p)^2 + Q^2|][q^2 + Q^2]} + (M \rightarrow -M) \right\} , \]  

where \( Q = \alpha(1 - \alpha)p^2 + A^2 \). After the substitution \( q = k + \alpha p \), we obtain the final result
\[ \Pi^0_s(p^2) = \frac{1}{G_0} - 4 \int_0^1 d\alpha \int \frac{d^4 q}{(2\pi)^4} \left\{ \frac{q^2 - Q}{[q^2 + Q^2]} + (M \rightarrow -M) \right\} , \]  

with \( H_0(p^2) \) defined in (24). Since the integral in (A.10) is divergent, we cut off the \( q \)-integral by a sharp \( O(4) \) cutoff \( \Lambda \). This regularization procedure is included into the definition of our low-energy effective quark theory. We have ensured (see section 4) that chiral symmetry is not violated by the cutoff procedure.

For the evaluation of \( K^\alpha(p^2) \), we proceed along the same line as sketched above. We first perform the color trace in the last expression of (A.1):
\[ \sum_{i,j=\pm} \text{tr}_L \{ (i|\tau^\alpha S(k + p)|l\rangle\langle l|S(k)|i\} = \sum_{i,j=\pm} \text{tr}_L \{ (i|\tau^\alpha|l\rangle s(k + p)s(k)\langle l|i\} . \]  

From (22) and the orthogonality of the eigenvectors \( |\pm\rangle \), we find
\[ K^\alpha(p^2) = -i\frac{M^\alpha}{M} \int \frac{d^4 k}{(2\pi)^4} \text{tr}_L \{ s(k + p)s(k) \} - (M \rightarrow -M) . \]  

For the further calculation of \( K^\alpha \), one might employ directly the above results to finally obtain
\[ K^\alpha(p^2) = -4i\frac{M^\alpha}{M} \int_0^1 d\alpha \int \frac{d^4 q}{(2\pi)^4} \left\{ \frac{q^2 - Q}{[q^2 + Q^2]} - (M \rightarrow -M) \right\} , \]  

34
and therefore $K^\alpha(p^2) = \frac{M^2}{M} H_v(p^2)$ with $H_v(p)$ defined in (25).

It remains to show that $M^\alpha$ is an eigenvector of the polarization matrix $\Pi^\alpha_\beta$. To do this, we first evaluate the color trace of

$$\text{tr}\left\{ \tau^\alpha S(k + p) \tau^\beta \frac{M^\beta}{M} S(k) \right\} = \sum_{i, l = \pm} \text{tr}_L \left\{ \langle i|\tau^\alpha|l\rangle s(k + p)s(k)\langle l|\tau^\beta \frac{M^\beta}{M} |i\rangle \right\} .$$

(A.15)

Since $|\pm\rangle$ are eigenvectors to $\tau^\beta M^\beta$, the expression (A.15) is

$$\frac{M^\alpha}{M} \text{tr}_L \{ s(k + p)s(k) \} + (M \rightarrow -M) .$$

(A.16)

This expression was already calculated to obtain $\Pi^\alpha_\beta(p^2)$. Since $M^\beta$ is also an eigenvector of the interaction matrix $G^\alpha_\beta$ (33), the final result is

$$\Pi^\alpha_\beta(p^2) \frac{M^\beta}{M} = \left( \frac{1}{G_c} - H_0(p^2) \right) \frac{M^\alpha}{M} .$$

(A.17)

B The equations of motion

In this section we derive an explicit expression for the equations of motion (11) and (12). To perform the color trace of the quark propagator $S(k)$, we introduce the eigenstates $|\pm\rangle$ of the color matrix $M^\alpha \tau^\alpha$ (compare Appendix A):

$$\text{tr} S(k) = \text{tr}_L \sum_{i = \pm} \langle i| S(k) |i\rangle ,$$

(B.1)

where $\text{tr}_L$ is the trace over Lorenz-indices only. The equation of motion (11) becomes

$$\frac{1}{G_0}(M_0 - m) - i \int \frac{d^4k}{(2\pi)^4} \left\{ \frac{k - iA}{k^2 + A^2} + (M \rightarrow -M) \right\} = 0 ,$$

(B.2)

where $A$ was defined in (A.7). It is straightforward to evaluate (B.2). One obtains

$$\frac{1}{G_0}(M_0 - m) - 8M_0 \int \frac{d^4k}{(2\pi)^4} \frac{k^2 + M_0^2 + M^2}{k^4 + 2(M_0^2 - M^2)k^2 + (M_0^2 + M^2)^2} = 0 .$$

(B.3)

Setting $\tilde{M}^\alpha = iM^\alpha$, we derive the explicit form of the remaining equation of motion (12) in a similar fashion. The color-triplet part of the quark propagator, $\text{tr} S(k) \tau^\alpha$, is

$$\text{tr}_L \sum_{i = \pm} \langle i| S(k) \tau^\alpha |i\rangle = \text{tr}_L \sum s(k) \langle i| \tau^\alpha |i\rangle = \text{tr}_L s(k) \frac{M^\alpha}{M} - (M \rightarrow -M) ,$$

(B.4)
with $s(k)$ from (A.7). If $M^\alpha$ is an eigenstate of the interaction matrix, i.e. $G^\alpha \beta M^\beta = G_c M^\alpha$, the equation of motion (12) reduces to

$$\frac{i}{G_c} M - i(-i) \int \frac{d^4 k}{(2\pi)^4} \left\{ \frac{k - iA}{k^2 + A^2} - (M \rightarrow -M) \right\} = 0. \quad (B.5)$$

A direct calculation yields

$$\frac{1}{G_c} M - 8M \int \frac{d^4 k}{(2\pi)^4} \frac{k^2 - M_0^2 - M^2}{k^4 + 2(M_0^2 - M^2)k^2 + (M_0^2 + M^2)^2}. \quad (B.6)$$

Both integrals in (B.3) and (B.6) can be expressed with the help of the functions $I_0$ and $I_1$, defined respectively, in (36) and (37), to obtain the desired result presented in (34) and (35).

C Analytic structure of the scalar correlation function

As argued in section 3.2, we believe that the approximations made to derive the scalar correlation function do not violate any fundamental requirements of quantum field theory. Nevertheless we observe a non-trivial analytic structure of the scalar correlation function which might make one suspect that some constraints may be incorrectly implemented. In this subsection, we show by an explicit calculation that one particular constraint, namely, the stability criterion, is indeed satisfied. Of course there are many more constraints. Here we shall offer one evidence that our model is compatible with a general axiom of quantum field theory. The investigation of further constraints which might provide insight into the possible analytic structure of Green's functions of a confining theory seems very interesting to us and will be relegated to a future work.

For completeness, we rederive the stability criterion. Following the standard procedure [31], the correlation function of a local operator $j(x)$ is written in momentum space as

$$G(q) = \int d^4 x \ e^{iqx} \langle 0 | T j(x) j(0) | 0 \rangle, \quad (C.1)$$

where $T$ is the time-ordering operator. Rewriting the time ordering, one has

$$G(q) = \int d^4 x \ e^{iqx} \theta(x_0) \langle 0 | [j(x), j(0)] | 0 \rangle + \int d^4 x \ e^{iqx} \langle 0 | j(0) j(x) | 0 \rangle \quad (C.2)$$

Inserting a complete set of eigenstates, the last expression becomes

$$\sum_n \int d^4 x \ e^{i(q+p_n)x} \langle 0 | j(0) | n \rangle \langle n | j(0) | 0 \rangle \quad (C.3)$$

$$= \sum_n (2\pi)^4 \delta^4(q + p_n) \langle 0 | j(0) | n \rangle \langle n | j(0) | 0 \rangle.$$
In the lab frame ($q^0 > 0$), there is no contribution from the term (C.3) to the correlator (C.1), if we are dealing with a stable theory, since there are no states with negative energy ($E_n = p_n^0 > 0$). We therefore observe that the correlation function vanishes for $x_0$ less than zero, i.e.

\[ \langle 0| T j(x) j(0) |0 \rangle = f(x) \theta(x_0) . \]  

(C.4)

In the standard case of a constituent quark model, the stability criterion is satisfied for the following reason: the correlation function in momentum space is analytic in the upper $q_0$ complex-plane and possesses cuts at the real axes due to quark anti-quark thresholds (see Figure 13 (a)). For $x_0 < 0$ one might calculate the Fourier transform by closing the path by a semi-circle in the upper half plane. Since there are no poles or cuts, we conclude that the Fourier transformation yields zero.

In order to check the stability criterion in our model, we first study the analytic structure of the scalar correlation function (39). To this aim, we have explicitly calculated the functions $H_0(p^2)$ in (24) and (25). The result is

\[ H_0(p^2) = h(p^2; M_0, M) + h(p^2; M_0, -M) , \]
\[ H_\nu(p^2) = -i[h(p^2; M_0, M) - h(p^2; M_0, -M)] \]  

(C.5)
\[
\frac{4\pi^2}{p^2} h(p^2; M_0, M) = \frac{\Lambda^2}{p^2} + \frac{1}{2} \left( \frac{4\Lambda^2}{p^2} + 1 \right)^{3/2} \ln \frac{\sqrt{\frac{4\Lambda^2}{p^2} + 1} + 1}{\sqrt{\frac{4\Lambda^2}{p^2} + 1} - 1} \\
- \frac{1}{2} \left( \frac{6\Lambda^2}{p^2} + 1 \right) \ln \left( 1 + \frac{\Lambda^2}{A^2} \right) \\
- \frac{1}{2} \left( \frac{4\Lambda^2}{p^2} + 1 \right) \frac{\left( \frac{2\Lambda^2 + 4\Lambda^2}{p^2} + 1 \right)}{\left( \frac{4\Lambda^2 + 4\Lambda^2}{p^2} + 1 \right) - 1},
\]

where \( A = M_0 + iM \). We confine the momentum to \( |p|^2 \leq \Lambda^2 \). In this case the only cuts which occurs in the upper half \( q_0 \) complex-plane are shown in Figure 13(b). Their orientation is given by the phase \( \phi \) of the complex number \( M_0 + iM \). In order to perform the Fourier transform to the coordinate space, we close the path in the upper half plane by the contour depicted in Figure 13(b). It seems that the Fourier transform does not yield zero as it would be required by the stability criterion, since there are now the contributions from the cut. Our main observation is, however, that these contributions exactly cancel. In order to see this, we first note that the correlation function \( \Delta(q_0) \) possesses reflection positivity, i.e.

\[
\Delta(z^*) = \Delta^*(z),
\]

as the functions \( H_{0/\nu} \) do. This can be shown either by a direct inspection of (C.5) or, which is more instructive, by tracing it back to the fact that we consider color singlet correlation functions. In the latter case, the appropriate superposition of terms with \( M \) and \( -M \) provides (C.7). Equation (C.7) immediately implies that the complex part of the integration over the contour in the half-plane vanishes, since this contour is chosen to be symmetric with respect to the imaginary axis. In order to prove that the contributions from the cuts cancel, it is sufficient to show that the contribution from either cut is purely imaginary. That this is indeed the case holds on very general grounds. Although this may be well-known to the readers, we nevertheless feel that we should give some arguments on this matter. We have to prove that integrals of the type

\[
I := \int_C dx \ K(\ln(x))f(x)
\]

produce purely imaginary results, where \( C \) is the contour surrounding the cut (like the contour \( ACB \) in Figure 13(b)). In fact, one has to study the more general integrand \( K[\ln(g(x))]f(x) \), the integral of which, however, can be traced back to the one in (C.8) by a change of variable \( x \to y = g(x) \) and a redefinition of the function \( f(x) \). This change of variables provides us with a linear cut in the complex plane. If we define the cut of the logarithm to coincide with the negative real-axis,
Figure 14: The Wick rotation to relate the scalar correlation function in Euclidean space to that in Minkowski space.

A straightforward calculation yields

$$ I = \int_{-R}^{0} dx \ f(x) \ \{ K(\ln(-x) + i\pi) - K(\ln(-x) - i\pi) \} , $$

where $R$ is the length of the cut under considerations. To continue we have to assume that the function $K(x)$ possesses a Laurant-expansion around $x = 0$. It is straightforward to check that this is the case in the context of the scalar correlation function of our model. We then have

$$ I = \sum_{n=-m}^{\infty} c_n \int_{-R}^{0} dx \ f(x) \ \{ (\ln(-x) + i\pi)^n - (\ln(-x) - i\pi)^n \} . $$

The even powers of $i\pi$ drop out and we end up with a purely imaginary result. This completes the proof.

To summarize our results, the scalar correlation function possesses a nontrivial analytic structure in momentum space, e.g. consisting of cuts in the upper half $q_0$ complex-plane. We have proven that the particular analytic structure is compatible with the stability criterion (C.4) due to an intrinsic cancellation mechanism of the contributions from the cuts.

We finally present the scalar correlation function in Minkowski-space. From the very beginning, our model is given in Euclidean space and the Green's functions in Minkowski-space are defined by the Wick rotation (see Figure 14). In our case, this Wick-rotation is non-trivial, since we have to take into account the contribution from the cuts in Figure 14. Suppose the scalar correlation function in Euclidean
space is part of some scattering amplitude, e.g.

$$S(p^2) = \int_{-\infty}^{\infty} dq^E \Delta(q^2) G(q^2, p^2)$$  \hspace{1cm} (C.11)$$

where $G(q^2, p^2)$ is assumed to be analytic in the first and third quadrants of the $q^E$-plane. From Figure 14 we have

$$S(p^2) = \int_{-\infty}^{\infty} dq^E \Delta(q^2) G(q^2, p^2) + 2i \int_{-\infty}^{\infty} dx I_{\text{cut}},$$  \hspace{1cm} (C.12)$$

where we have used that the contribution $I_{\text{cut}}$ from one cut is of the form (C.10) and purely imaginary ($I_{\text{cut}}$ is real). Performing the substitution $q_0^M = -iq^E$ and redefining $x = q_0^M$, we obtain

$$S(p^2) = i \int_{-\infty}^{\infty} dq_0^M \Delta(-q^2) G(-q^2, p^2) + 2i \int_{-\infty}^{\infty} dq_0^M I_{\text{cut}},$$  \hspace{1cm} (C.13)$$

and arrive at the scalar correlation function in Minkowski space

$$\Delta^M(q^2) = \Delta(-q^2) + 2I_{\text{cut}}$$  \hspace{1cm} (C.14)$$

Note that the contribution from the cuts is real. We expect $I_{\text{cut}}$ to contribute only a background to $\Delta^M(q^2)$, whereas the significant structure, such as imaginary parts from thresholds and poles from particle states, is produced by the Euclidean scalar correlation function at negative momentum squared.

\section*{D \hspace{0.5cm} Reduction of the Bethe-Salpeter amplitude}

In order to solve the Bethe-Salpeter equation (41) for the hidden color structure of the pion, one has to perform the color trace of polarizations containing two quark propagators $S(k)$. Since the calculational technique parallels very much that performed to obtain the scalar correlation function, we only sketch the derivation briefly and refer the reader to Appendix A for further details. The traces of interest are

\begin{align*}
\text{tr}\{\gamma_5 S(k+p)\gamma_5 S(k)\} &= \text{tr}_L\{\gamma_5 s(k+p)\gamma_5 s(k)\} + (M \to -M), \\
\text{tr}\{\tau^a \gamma_5 S(k+p)\gamma_5 S(k)\} &= \frac{M^a}{M} \text{tr}_L\{\gamma_5 s(k+p)\gamma_5 s(k)\} - (M \to -M), \\
\text{tr}\{\tau^a \gamma_5 S(k+p)\tau^b \gamma_5 S(k)\} \frac{M^\beta}{M} &= \frac{M^a}{M} \text{tr}_L\{\gamma_5 s(k+p)\gamma_5 s(k)\} + (M \to -M)
\end{align*}

(D.1)
with \( s(k) \) defined in (A.7). The next step is to calculate the trace over Lorentz indices \( \text{tr}_L \):

\[
\text{tr}_L \left\{ \gamma^5 \frac{k + p - iA}{(k + p)^2 + A^2} \gamma^5 \frac{k - iA}{k^2 + A^2} \right\} = -4 \frac{k^2 + A^2 + kp}{[(k + p)^2 + A^2] (k^2 + A^2)}. \tag{D.2}
\]

To obtain the desired matrix elements entering into (41), an integration over the loop momentum \( k \) is required. Introducing Feynman's parametrization and shifting the momentum integration \( q = k + \alpha p \) yields

\[
\text{Tr} \left\{ \gamma_5 s(k + p) \gamma_5 s(k) \right\} = -4 \int_0^1 d\alpha \int (q) \frac{q^2 - \alpha(1 - \alpha)p^2 + A^2}{[q^2 + \alpha(1 - \alpha)p^2 + A^2]^2}. \tag{D.3}
\]

Inserting (D.3) into (D.1-D.1), we have all the ingredients to derive the result (43).

### E Calculation of the electromagnetic form factor

Here we evaluate the matrix element (57), which is directly related to the electromagnetic form factor. For this, we first perform the color trace by the method employed several times before (for details see Appendix A):

\[
(P_0^2 - P_1^2) W^0_\mu - 2P_0 P_1 W^\nu_\mu. \tag{E.1}
\]

The functions

\[
W^0_\mu = - \int \frac{d^4k}{(2\pi)^4} \text{tr}_L \left\{ \gamma_5 s(k - \frac{p}{2}) \gamma_\mu s(k + \frac{p}{2}) \gamma_5 s(k - \frac{p}{2}) \right\} + (M \rightarrow -M), \tag{E.2}
\]

\[
W^\nu_\mu = +i \int \frac{d^4k}{(2\pi)^4} \text{tr}_L \left\{ \gamma_5 s(k - \frac{p}{2}) \gamma_\nu s(k + \frac{p}{2}) \gamma_5 s(k - \frac{p}{2}) \right\} - (M \rightarrow -M) \tag{E.3}
\]

are introduced for abbreviation. The propagator \( s(k) \) is given by (A.7). Performing the Lorentz trace of the terms under investigation in (E.2,E.3), i.e.

\[
\frac{4(k_\mu + \frac{p_\nu}{2})}{[(k - \frac{p}{2})^2 + A^2][(k + \frac{p}{2})^2 + A^2]}, \tag{E.4}
\]

one obtains

\[
W^0_\mu = \frac{1}{2} P_\mu \int \frac{d^4k}{(2\pi)^4} \frac{1}{[(k - \frac{p}{2})^2 + A^2][(k + \frac{p}{2})^2 + A^2]} + (M \rightarrow -M) \tag{E.5}
\]

\[
W^\nu_\mu = -\frac{i}{2} P_\mu \int \frac{d^4k}{(2\pi)^4} \frac{1}{[(k - \frac{p}{2})^2 + A^2][(k + \frac{p}{2})^2 + A^2]} - (M \rightarrow -M) \tag{E.6}
\]
It is of a particular interest to expand these functions for small $p^2$ because they are required to normalize the pion Bethe-Salpeter amplitude and to calculate the pion charge radius. A direct calculation yields

$$W^{0/0}_\mu = p_\mu \left( F_{0/0} - p^2 R_{0/0} + O(p^4) \right) ,$$  \hspace{1cm} (E.7)

where $F_{0/0}$ are defined in (47,48) and

$$R_0 = \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \frac{A^2 - k^2}{(k^2 + A^2)^2} + (M \to -M) ,$$  \hspace{1cm} (E.8)

$$R_\nu = -i \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \frac{A^2 - k^2}{(k^2 + A^2)^2} - (M \to -M) .$$  \hspace{1cm} (E.9)

Inserting (E.7) in (E.1) leads to the desired matrix element.

In order to obtain the electromagnetic form factor for non-vanishing momentum in Minkowski space, an analytic continuation of the Euclidean momentum squared to negative values is needed. This analytic continuation is difficult from a technical point of view, since the momentum integration in (E.5, E.6) can be performed only numerically. Introducing

$$K_\pm = (k \pm \frac{p}{2})^2 = k^2 - \frac{p^2}{4} \pm k\not{p} , \hspace{1cm} A_\pm = M_0 \pm iM ,$$  \hspace{1cm} (E.10)

a term of interest is

$$\frac{1}{[K_- + A^2_\pm] [K_+ + A^2_\pm]} + (A_+ \to A_-) .$$  \hspace{1cm} (E.11)

We first remove the complex parts that enter via $A_\pm$, namely,

$$\frac{K_+ K_- + B_- (K_- + K_+) + B^2_\pm - 4M^2_0 M^2_0}{[K^2_+ + 2K_- B_- + B^2_\pm] [K^2_+ + 2K_+ B_- + B^2_\pm]} ,$$  \hspace{1cm} (E.12)

where $B_\pm = M^2_0 \pm M^2$. Introducing the angle $\alpha$ between the Euclidean four-vectors $k$ and $p$, the crucial observation is that in (E.12), only the terms quadratic in the external momentum $p$ appear:

$$K_+ K_- = (k^2 - \frac{p^2}{4})^2 - k^2 p^2 \cos^2 \alpha ,$$  \hspace{1cm} (E.13)

$$K^2_+ + K^2_- = 2 \left[ (k^2 - \frac{p^2}{4})^2 + k^2 p^2 \cos^2 \alpha \right]$$  \hspace{1cm} (E.14)

$$K_+ + K_- = 2 \left( k^2 - \frac{p^2}{4} \right) .$$  \hspace{1cm} (E.15)
It is now a simple matter to perform the analytic continuation $p^2 \to -p^2$. We finally present the result for the functions $W^0_\mu$, $W^-_\mu$ in (E.5,E.6)

\[
W^0_\mu = \frac{1}{8\pi^3 p_\mu} \int dV \frac{\alpha_- + 2T_-\beta + T^2_- - 4M^2_0 M^2}{\alpha^2 + 4T_-\alpha_- \beta + 2T^2_- \alpha_- + 4T^2_\beta \alpha_- + 4T^- T^2_\beta + T^4_+} ,
\]

\[
W^-_\mu = -\frac{1}{8\pi^3 p_\mu} \int dV \frac{4M_0 M (\beta + T_-)}{\alpha^2 + 4T_-\alpha_- \beta + 2T^2_- \alpha_- + 4T^2_\beta \alpha_- + 4T^- T^2_\beta + T^4_+} ,
\]

(E.16)

where $dV = d\alpha \sin^2 \alpha d(k^2) k^2$, $\alpha \in [0, \pi]$ and

\[
\alpha_\pm = \left( k^2 + \frac{p^2}{4} \right)^2 \pm k^2 p^2 \cos^2 \alpha , \quad \beta = k^2 - \frac{p^2}{4} , \quad T_\pm = M_0^2 \pm M^2 . \quad (E.17)
\]

The momentum integration $k^2$ and the angle integral in (E.16,E.16) are left to a numerical calculation.

References


