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Estimating Terrorist Risk with Possibility Theory

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Estimating Terrorist Risk with Possibility Theory

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1 Introduction

This report summarizes techniques that use possibility theory to estimate the risk of terrorist acts. These techniques were developed under the sponsorship of the Department of Homeland Security (DHS) as part of the National Infrastructure Simulation Analysis Center (NISAC) project. The techniques have been used to estimate the risk of various terrorist scenarios to support NISAC analyses during 2004.

The techniques are based on the Logic Evolved Decision (LED) methodology developed over the past few years by Terry Bott and Steve Eisenhower at LANL. [LED] The LED methodology involves the use of fuzzy sets, possibility theory, and approximate reasoning. LED captures the uncertainty due to vagueness and imprecision that is inherent in the fidelity of the information available for terrorist acts; probability theory cannot capture these uncertainties.

This report does not address the philosophy supporting the development of non-probabilistic approaches, and it does not discuss possibility theory in detail. The references provide a detailed discussion of these subjects. [Shafer] [Klir and Yuan] [Dubois and Prade] Suffice to say that these approaches were developed to address types of uncertainty that cannot be addressed by a probability measure. An earlier report discussed in detail the problems with using a probability measure to evaluate terrorist risk. [Darby Methodology]

Two related techniques are discussed in this report:

1. a numerical technique, and
2. a linguistic technique.

The numerical technique uses traditional possibility theory applied to crisp sets, while the linguistic technique applies possibility theory to fuzzy sets. Both of these techniques as applied to terrorist risk for NISAC applications are implemented in software called PossibleRisk.

The techniques implemented in PossibleRisk were developed specifically for use in estimating terrorist risk for the NISAC program.

The LEDTools code can be used to perform the same linguistic evaluation as performed in PossibleRisk. [LEDTools] LEDTools is a general purpose linguistic evaluation tool and allows user defined universes of discourse and approximate reasoning rules, whereas PossibleRisk uses predefined universes of discourse (risk, attack, success, loss, and consequence) and rules. Also LEDTools has the capability to model a large number of threat scenarios with a graph and to integrate the scenarios (paths from the graph) into the linguistic evaluation.

Example uses of PossibleRisk and LEDTools for the possibilistic evaluation of terrorist risk are provided in this report.

2 Risk of a Terrorist Act

Risk is defined in general as a relation between likelihood and consequence; conceptually, a low likelihood/high consequence event can have the same risk as a high likelihood/low consequence event.¹ For a terrorist act, risk must consider the likelihood of a successful terrorist attack and the consequences of that attack. For a terrorist act, we will mathematically define Risk as a relation among: Threat, Vulnerability, and Consequence; that is:

$$\text{Terrorist Risk}_{i,j,k} = \text{Threat}_i \times \text{Vulnerability}_{i,j} \times \text{Consequence}_{i,j,k} \quad (\text{Eq. 1})$$

where i denotes a specific threat scenario, j a specific facility, and k a specific type of consequence.² The “x” operation represents convolution, since Threat, Vulnerability, and Consequence are not numbers but “likelihood” distributions. Vulnerability is defined as the likelihood of success of the threat, considering the protective measures in place; thus, Threat and Vulnerability together provide the likelihood of a *successful* terrorist attack.

Conceptually, Equation 1 is similar to the equation for a probabilistic evaluation of an accident:

$$\text{Safety Risk}_{i,j,k} = \text{Initiating Event}_i \times \text{System Response}_{i,j} \times \text{Consequence}_{i,j,k} \quad (\text{Eq. 2})$$

Equation 2 is evaluated by assigning probability distributions to Initiating Event, System Response, and Consequence, and then convoluting these distributions to form a probability distribution for Risk.

However, due to the greater (and different) types of uncertainty for terrorist risk compared to safety risk, the probabilistic techniques used to evaluate safety risk are not appropriate for evaluating terrorist risk, and a probabilistic approach will not capture the uncertainty inherent in the information available. The technique explained in this report evaluates Equation 1 using a possibility measure.

In the evaluation of risk, dependence must be considered. For safety risk (Eq. 2) System Response is dependent on Initiating Event, and Consequence is dependent on System Response. Similarly, for terrorist risk (Eq.1) Vulnerability is dependent on Threat, and Consequence is dependent on Vulnerability. The dependence is more complex for terrorist risk than for safety risk. Safety risk involves random events associated with “dumb” components; terrorist risk involves intentional acts by malevolent, thinking terrorists. The likelihood of a Threat depends on the terrorists’ perception of the Vulnerability of a potential target and the terrorists’ perception of the types and magnitudes of Consequence achievable.

To correctly account for dependence, an evaluation of terrorist risk must be performed for each Threat of concern; Vulnerability is evaluated as Vulnerability|Threat (vulnerability

¹ With apologies to Bayesians, when risk is discussed in this report the term likelihood means a fuzzy measure. The term likelihood is used instead of fuzzy measure to avoid confusion with the term fuzzy set by those not versed in the nomenclature of the theory of evidence.

Fuzzy measures are discussed in Section 3.1.

² In this report, the term “Threat” means “Threat Scenario”; that is, the Threat includes the knowledge, attributes, and attack plans of the terrorists.

conditional on threat), and Consequence is evaluated as $\text{Consequence}|\text{Vulnerability}$. [For example, the vulnerability of a nuclear power plant to a threat restricted to hand-held tools is much lower than the vulnerability to a threat involving crew-served weapons, that is, small weapons (e.g., stinger missiles, bazookas) that can be handled by 1 to 3 people or so.]

In the context of Equation 1, Threat is sometimes called Attack and Vulnerability is sometimes called Success (for the adversary), because Threat can be represented as frequency of attack (number/year), and Vulnerability can be represented as the likelihood that the attack is successful. The techniques discussed in this report represent Threat as an Attack frequency and Vulnerability as a Success likelihood, so in the remainder of this report, Threat and Vulnerability are called Attack and Success, respectively.

One type of Consequence is used in this report; specifically, equivalent cost [in dollars (\$)] to repair. All consequences of concern are converted to equivalent dollars to repair; for example, consequences involving fatalities and injuries are converted into a dollar equivalent.

Attack, Success, and Consequence are represented by the following discrete sets.
Attack = $\{0, 10^{-5}, 10^{-4}, 10^{-3}, 10^{-2}, 10^{-1}, 1, 10\}$ where each element has units of per year.
Success = $\{0, 10^{-3}, 10^{-2}, 10^{-1}, 1\}$ where each element is dimensionless.
Consequence = $\{1, 10^1, 10^2, 10^3, 10^4, 10^5, 10^6, 10^7, 10^8, 10^9, 10^{10}, 10^{11}, 10^{12}\}$ where each element has units of dollars.

3 Fuzzy Measures and the Numerical Technique

Traditional sets are called crisp sets to distinguish them from fuzzy sets. Each element in a crisp set is completely in that set with a degree of membership of 1.0. Fuzzy sets contain elements with partial membership indicated by degrees of membership less than 1.0.

Uncertainty can be segregated into two types: Ambiguity and Vagueness. Ambiguity occurs when it is not clear which of several crisp events will contain the outcome. Vagueness is due to the lack of sharpness for a fuzzy event; that is, the outcomes that are part of a given fuzzy event cannot be precisely identified. Therefore, ambiguity is associated with a fuzzy measure on crisp subsets (events) and vagueness is associated with fuzzy subsets (events). The numerical technique addresses ambiguity, but does not address vagueness, so the fuzzy measures discussed in this section are associated with crisp sets. The linguistic technique presented in Section 4 addresses vagueness as well as ambiguity through the use of fuzzy sets.

Section 3.1 summarizes the fuzzy measures of interest for crisp sets. Then, the numerical technique, which uses a possibility measure on crisp sets, is presented.

3.1 Fuzzy Measures

This section provides a brief summary of different fuzzy measures. The references provide a detailed description of these measures. [Shafer] [Klir and Yuan] [Dubois and Prade] [Darby Methodology]

The different fuzzy measures can be explained using set theory. The power set of a set is defined as the set of all subsets of that set including the null or empty set; for example, for the set $\{a, b, c\}$ the power set is $\{\text{null}, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$. Each element of the power set that has a single element is called a singleton; $\{a\}$, $\{b\}$, and $\{c\}$ are the singletons of the power set of $\{a, b, c\}$.

The set containing all the elements that are possible for the problem of interest is called the Universe of Discourse (UOD). Assume that a Universe of Discourse can be represented by a finite set of elements: $X = \{x_1, x_2, x_3, \dots, x_n\}$ ³. [Dougherty] A set cannot contain duplicate elements and the order of the elements does not matter. The elements of X are mutually exclusive outcomes of an experiment of interest. The problem is to predict the outcome of an experiment yet to be performed; the experiment will have one specific outcome, but there is uncertainty as to exactly which outcome. So the solution to the problem is an estimate of the likelihood for the various elements of X . An event is defined as a collection of outcomes. An event is an element of the power set

³ Finite sets are used in our model. All further discussions related to sets assume finite sets unless stated otherwise.

of X ; a singleton of the power set is an outcome (an event with one element). The term event and (sub)set are used interchangeably.

A fuzzy measure, g , is a mapping from $P(X)$ to $[0,1]$ where $P(X)$ is the power set of the crisp set X :

$$g : P(X) \rightarrow [0,1] \quad (\text{Eq. 3})$$

It is noted that a fuzzy measure maps the power set of X , not X .

A fuzzy measure satisfies the following three axioms:

- (1) $g(\text{null}) = 0$ and $g(X) = 1$ where null is the set with no elements
- (2) for all (crisp) subsets A and B of X , if A is a subset of B then $g(A) \leq g(B)$
- (3) For every monotonic sequence of subsets of X (the subsets A_i i from 1 to n where either: A_1 is a subset of A_2 is a subset of A_3 ... is a subset of A_n , or: A_n is a subset of A_{n-1} is a subset of A_{n-2} ... is a subset of A_1)

$$\lim_{i \rightarrow \infty} g(A_i) = g(\lim_{i \rightarrow \infty} A_i) . \quad (\text{Eq. 4})$$

From axiom (2), since

$$\begin{aligned} A \subseteq A \cup B \quad \text{and} \quad B \subseteq A \cup B \\ A \cap B \subseteq A \quad \text{and} \quad A \cap B \subseteq B \end{aligned} \quad (\text{Eq. 5})$$

we can state that

$$\begin{aligned} g(A \cup B) &\geq \max[g(A), g(B)] \\ g(A \cap B) &\leq \min[g(A), g(B)] . \end{aligned} \quad (\text{Eq. 6})$$

A fuzzy measure considers uncertainty due to Ambiguity which can be segregated into Nonspecificity and Discord (also called Strife).

A fuzzy measure involves nonspecificity when a degree of evidence is associated with a crisp subset that is not a singleton; that is, when a degree of evidence is applied to a crisp subset that has more than one outcome there is nonspecificity.

Discord is associated with conflict in evidence; the greater the difference in the elements (outcomes) among the crisp sets to which evidence is given, the greater the conflict in the evidence.

3.1.1 Belief and Plausibility

A Belief measure is a type of fuzzy measure; it has a dual measure, Plausibility. Under certain conditions Belief/Plausibility both become Probability; under certain different conditions Belief/Plausibility become Necessity/Possibility, respectively. Figure 3.1 summarizes this hierarchy.

Figure 3.1. Hierarchy of Fuzzy Measures

Note that Probability is *not* a special case of Necessity/Possibility.

A *Belief* measure, Bel , is a fuzzy measure that satisfies an additional axiom:

$$\begin{aligned}
 Bel(A_1 \cup A_2 \cup \dots \cup A_n) \geq & \sum_i Bel(A_i) - \sum_{i < j} Bel(A_i \cap A_j) + \dots \\
 & + (-1)^{n+1} Bel(A_1 \cap A_2 \cap \dots \cap A_n)
 \end{aligned}
 \quad (\text{Eq. 7})$$

Associated with Belief is a complementary measure called *Plausibility*, Pl , defined by

$$Pl(A) = 1 - Bel(\bar{A}) \quad (\text{Eq. 8})$$

where $\bar{A}$ is the complement of A ($X - A$).

$Bel(A)$ is the *degree of belief* that a given element of X belongs to the crisp subset A of X . It can be shown that

$$\begin{aligned}
 Bel(A) + Bel(\bar{A}) & \leq 1 \\
 Pl(A) + Pl(\bar{A}) & \geq 1
 \end{aligned}
 \quad (\text{Eq. 9})$$

Let $m(A)$ represent the *degree of evidence* that the outcome will be a member of the set (event) A , and **not** a member of any subset of A . Let $P(X)$ represent the power set of X ; the set whose elements are all the crisp subsets of X . $P(X)$ contains all the crisp events possible. Those members of $P(X)$ for which the degrees of evidence are greater than zero are called the *focal elements* of X ; that is, the focal elements are those sets (events) upon which the evidence focuses.

Whereas $m(A)$ is the degree of evidence that an element of X belongs exactly to A , the Belief that an element belongs to A , $Bel(A)$, is the total evidence that the element belongs to A as well as to any subset of A . $Pl(A)$ is the total evidence that the element belongs to A as well as to any subset of A , plus additional evidence for sets that overlap with A .

Note that

$$Pl(A) \geq Bel(A). \quad (\text{Eq. 10})$$

The degrees of evidence satisfy:

$$\begin{aligned} m : P(X) &\rightarrow [0,1] \\ \text{such that} \\ \sum_{A \in P(X)} m(A) &= 1 \quad \text{and} \\ m(\text{null}) &= 0. \end{aligned} \quad (\text{Eq. 11})$$

The fuzzy measures Bel and Pl for any subset A can be calculated from the focal elements as follows:

$$\begin{aligned} Bel(A) &= \sum_{B \subseteq A} m(B) \\ Pl(A) &= \sum_{B \cap A \neq \text{null}} m(B) \end{aligned} \quad (\text{Eq. 12})$$

where B ranges over all the elements of the power set of the UOD.

3.1.2 Possibility and Necessity

A family of proper subsets of X is nested if these subsets can be ordered such that each is a subset of another; i.e. $A_1 \subset A_2 \subset \dots A_n$. If the focal elements of a body of evidence are nested, the degrees of evidence assigned to the focal elements do not conflict with each other; the focal elements are free from *dissonance* in evidence if they are nested. In this case, the Belief/Plausibility measures are called *consonant*. Another way of stating the requirement for nesting is as follows. If $m(A)$ and $m(B)$ are both greater than zero; then, either A is a subset of B, or B is a subset of A.

Note that it is possible to have a situation where $m(A_1) > m(A_2)$ for A_1 a proper subset of A_2 ; this corresponds to evidence suggesting that an element is in a particular subset, but there is a “possibility” that the element is in another subset within which the first subset is nested.

For a consonant body of evidence, the following hold:

$$\begin{aligned} Bel(A \cap B) &= \min[Bel(A), Bel(B)] \quad \text{for all } A, B \in P(X) \\ Pl(A \cup B) &= \max[Pl(A), Pl(B)] \quad \text{for all } A, B \in P(X). \end{aligned} \quad (\text{Eq. 13})$$

Consonant Belief/Plausibility measures are referred to as *Necessity/Possibility* measures, respectively. Necessity will be denoted by N and Possibility will be denoted as Π .

Therefore:

$$\begin{aligned} N(A \cap B) &= \min[N(A), N(B)] \quad \text{for all } A, B \in P(X) \\ \Pi(A \cup B) &= \max[\Pi(A), \Pi(B)] \quad \text{for all } A, B \in P(X). \end{aligned} \quad (\text{Eq. 14})$$

Necessity and Possibility are complementary measures related by:

$$N(A) = 1 - \Pi(\bar{A}) \text{ .(Eq. 15)}$$

Also,

$$\begin{aligned} N(A) > 0 &\Rightarrow \Pi(A) = 1 \\ \Pi(A) < 1 &\Rightarrow N(A) = 0 \text{ .} \end{aligned} \quad (\text{Eq. 16})$$

Possibility measures can be represented by a *possibility distribution function*:

$$\begin{aligned} \pi : X &\rightarrow [0,1] \text{ where} \\ \Pi(A) &= \max \{ \pi(x) \mid x \in A \} \\ N(A) &= \min \{ [1 - \pi(x)] \mid x \notin A \} \text{ .} \end{aligned} \quad (\text{Eq. 17})$$

Note that π is a mapping from X , not a mapping from $P(X)$; for an element x_i of X , $\pi(x_i)$ can be related to m as follows:

$$\pi(x_i) = \sum_{k=i}^n m(A_k) \quad (\text{Eq. 18})$$

where k ranges over nested focal elements; Equation 18 assumes the focal elements have been ordered and renumbered such that A_i is a subset of A_j if $i < j$, and the x_i have been ordered and renumbered such that $\pi(x_i) \geq \pi(x_j)$ if $i < j$. Since the sum over all m is 1, π_1 is always 1.

3.1.3 Probability

When the following holds:

$$Bel(A \cup B) = Bel(A) + Bel(B) \text{ if } A \cap B = \text{null} \quad (\text{Eq. 19})$$

both Belief and Plausibility become the classical *Probability* measure.

Let Pr denote a probability measure. Since a Probability measure is a special case of a fuzzy measure, it is required that $Pr(X) = 1$ and $Pr(\text{null}) = 0$.

Let x denote an element (outcome) of X . A Belief Measure is a Probability Measure if and only if the function m is given by:

$$m(\{x\}) = Bel(\{x\}) \text{ and } m(A) = 0 \text{ for all } A \subseteq X, A \text{ not a Singleton} \quad (\text{Eq. 20})$$

That is, the degrees of evidence are sufficiently known so that they can be assigned to singleton events (instead of a less precise assignment to events that are not singletons). A probability measure can be formed from a probability density function:

$$p : X \rightarrow [0,1] \text{ such that } p(x) = m(\{x\}) \text{ . (Eq. 21)}$$

For a Probability Measure, both Belief and Plausibility are equivalent to Probability:

$$Pr(A) = Bel(A) = Pl(A) = \sum_{x \in A} m(\{x\}) = \sum_{x \in A} p(x) \text{ .} \quad (\text{Eq. 22})$$

$Pr(A)$ is the algebraic sum of all the $p(x)$ since the outcomes are mutually exclusive.

Since a Probability measure has degrees of evidence for singletons only, a Probability measure has no Nonspecificity. The type of Ambiguity addressed by a Probability measure is Discord only.

A Probability measure involves discord since the degrees of evidence are assigned to numerous singletons, each of which is totally distinct (each singleton set has no elements in common with any other singleton set). For example, with no evidence, the Probability measure assigns an equal probability density to each x subject to the constraint they all add to 1.0; this leads to maximum discord (strife).

3.2 The Numerical Technique

An earlier report discussed in detail the problems with using a probability measure to evaluate terrorist risk. [Darby Methodology] For example, consider the set $\text{Attack} = \{0 \text{ per year}, 10^{-5} \text{ per year}, 10^{-4} \text{ per year}, 10^{-3} \text{ per year}, 10^{-2} \text{ per year}, 10^{-1} \text{ per year}, 1 \text{ per year}, 10 \text{ per year}\}$. For a given target from among a large number of targets, the likelihood that the attack is 0 per year is very high, so nonzero likelihood must be assigned to the value 0 per year. If the likelihood of Attack is estimated as a probability, then whatever likelihood is assigned to 0 per year affects the likelihoods assigned to the other values (10^{-5} per year, 10^{-4} per year, etc.) because all the likelihoods are constrained to sum to 1.0 using probability. We have insufficient information to assign a probability to 0 per year yet we are forced to do so if likelihood is assigned using probability. Using a possibility measure for likelihood, a possibility of 1.0 can be assigned to the frequency 0 per year, and that does not constrain the possibilities for the other values.

3.2.1 Convolution of Possibility Distributions

The purpose of the numerical technique is to solve Equation 1 by convoluting possibility distributions. This section summarizes how that convolution can be performed.

3.2.1.1 Possibility Distributions

Let X denote a discrete Universe of Discourse (UOD) to which degrees of evidence are assigned. If the focal elements are nested, a Necessity and a Possibility can be calculated for each element x of X , and therefore a necessity/possibility distribution can be created for X .

Furthermore, assume that X is a fuzzy quantity. A fuzzy quantity is defined as a UOD consisting of real numbers.⁴ [Dubois and Prade]

⁴ Most texts discuss a Fuzzy Number which is a convex fuzzy set on real numbers. A Fuzzy Number is a special case of a Fuzzy Interval which is a special case of a Fuzzy Quantity. [Dubois and Prade Section 2.1] A Fuzzy Quantity is a fuzzy set on the reals, but is *not* required to be convex. The extension principle provides the process by which to extend functions on real numbers to functions on Fuzzy Quantities; most texts address the extension principle for Fuzzy Numbers, but it also applies to Fuzzy Quantities. [Dubois and Prade Section 2.1.2]

Figure 3.2 illustrates a necessity/possibility distribution for a fuzzy quantity $X = \{1, 2, 3, 4\}$.

Figure 3.2. Example Necessity/Possibility Distribution

Note that the necessity/possibility distribution is similar to, yet different from a probability distribution, as evident from comparing the distribution of Figure 3.2 to the probability distribution of Figure 3.3. This is due to the consideration of Nonspecificity with the possibility measure.

Figure 3.3. Example Probability Distribution

3.2.1.2 Expected Value

For a probability distribution, the expected value (mean) is: [Dougherty]

$$E(x) = \sum_{x \in X} x * p(x) \quad (\text{Eq. 23})$$

where $p(x)$ is the probability of value x .

A probability distribution has a single value, a probability, assigned to each element in the UOD, and expectation for a probability distribution is a value, the mean.

A necessity/possibility distribution has a range (necessity to possibility) assigned to each element in the UOD. Therefore it is not surprising that expectation for a belief/plausibility distribution should be an interval.

The expected value interval is denoted as:

$$[E_*(x), E^*(x)] \quad (\text{Eq. 24})$$

Both necessity/possibility and probability are special cases of belief/plausibility, and for belief/plausibility the expected value interval can be calculated as: [Advances Dempster-Shafer]

$$\begin{aligned}
E_*(x) &= \sum_{A_i \subseteq X} \inf(A_i) * m(A_i) \\
E^*(x) &= \sum_{A_i \subseteq X} \sup(A_i) * m(A_i)
\end{aligned}
\tag{Eq. 25}$$

where m is a degree of evidence. For a probability distribution each m is a p and each A_i has one element, so $E_*(x)$ and $E^*(x)$ are both equal to the mean value.

For necessity/possibility, the expected value interval can also be calculated based on the formulation summarized in Dubois and Prade for the expected value interval of a real function f defined on the fuzzy quantity X : [Dubois and Prade]:⁵

$$\begin{aligned}
E_*(f) &= \sum_i f(x_i) [\Pi(\{x_j \mid f(x_j) \leq x_i\}) - \Pi(\{x_j \mid f(x_j) \leq x_{i-1}\})] \\
E^*(f) &= \sum_i f(x_i) [\Pi(\{x_j \mid f(x_j) > x_{i-1}\}) - \Pi(\{x_j \mid f(x_j) > x_i\})] .
\end{aligned}
\tag{Eq. 26}$$

In Equation 26 the x 's have been ordered such that $x_{i-1} < x_i$. By specifying $f(x) = x$, Equation 26 can be used to calculate the expected value interval for x .

In application, both Equations 25 and 26 yield the same result for the expected value interval for a necessity/possibility distribution.

Since a possibility distribution can be considered to define membership for a fuzzy set, a point estimate for a necessity/possibility distribution can be based on the centroid measure of a fuzzy set.⁶ Following Klir and Yuan: [Klir and Yuan]

$$\text{Centroid}_x = \frac{\sum_{x \in X} x * \pi(x)}{\sum_{x \in X} \pi(x)} . \tag{Eq. 27}$$

Note that both the expected value interval and the centroid have the units of the elements of X . For example, if X is a set of frequencies with units of per year, then both $[E_*(x), E^*(x)]$ and Centroid_x have units of per year.

Example 1

Consider the situation presented in Figure 3.4 for the fuzzy quantity $X = \{0, 10^{-5}, 10^{-4}, 10^{-3}, 10^{-2}, 10^{-1}, 1, 10\}$.

⁵ Equation 26 was derived by the author based on Equations 1.31 and 1.32 (the Lebesgue-Stieljes integrals) provided in Dubois and Prade. [Dubois and Prade]

⁶ Every possibility distribution defines a normal fuzzy set with degrees of membership given by the possibilities of the singletons. [Dubois and Prade] Thus, although the numerical technique considers possibilities for a crisp set, the *mathematics* of an equivalent fuzzy set to represent the possibility distribution can be used.

Figure 3.4. Degrees of Evidence for X in Example 1

Using Equation 25, the expected value interval for X is:

$$E^*(x) = (0.3 \times 10^{-2}) + (0.3 \times 10^{-4}) + (0.2 \times 0) + (0.2 \times 0) = 3.03 \times 10^{-3} \text{ and}$$

$$E^*(x) = (0.3 \times 10^{-2}) + (0.3 \times 10^{-2}) + (0.2 \times 10^{-2}) + (0.2 \times 10^{-1}) = 2.8 \times 10^{-2}.$$

Table 3.1 summarizes the calculation of the expected value interval for X using Equation 26; Table 3.1 also provides the possibility distribution, π , for X based on the degrees of evidence given in Figure 3.4.

Table 3.1. Expected Value Interval for X in Example 1

x_i $\pi(x_i)$	$\Pi(\{x_j \mid x_j \leq x_i\})$ (Term A)	$\Pi(\{x_j \mid x_j \leq x_{i-1}\})$ (Term B)	$\Pi(\{x_j \mid x_j > x_{i-1}\})$ (Term C)	$\Pi(\{x_j \mid x_j > x_i\})$ (Term D)	A – B	C – D	$x_i^* (A - B)$	$x_i^* (C - D)$
0 0.4	0.4	0	1	1	0.4	0	0	0
10^{-5} 0.2	0.4	0.4	1	1	0	0	0	0
10^{-4} 0.7	0.7	0.4	1	1	0.3	0	3×10^{-5}	0
10^{-3} 0.7	0.7	0.7	1	1	0	0	0	0
10^{-2} 1	1	0.7	1	0.2	0.3	0.8	3×10^{-3}	8×10^{-3}
10^{-1} 0.2	1	1	0.2	0	0	0.2	0	2×10^{-2}
1 0	1	1	0	0	0	0	0	0
10 0	1	1	0	0	0	0	0	0
							$E_*(x) =$ 3.03×10^{-3} (Sum)	$E^*(x) =$ 2.8×10^{-2} (Sum)

The point estimate using the centroid measure of Equation 27 is:

$$[(0.2 \times 10^{-5}) + (0.7 \times 10^{-4}) + (0.7 \times 10^{-3}) + (1 \times 10^{-2}) + (0.2 \times 10^{-1})] /$$

$$[0.4 + 0.2 + 0.7 + 0.7 + 1 + 0.2] = 9.6 \times 10^{-3}$$

3.2.1.3 Convolution

Just as probability distributions can be combined (convoluted), necessity/possibility distributions can be combined.

Consider two UODS X and Y . The cartesian product

$X \times Y = \{ \langle x, y \rangle \mid x \text{ in } X \text{ and } y \text{ in } Y \}$ where $\langle \rangle$ denotes a tuple.⁷ A relation is a subset of $X \times Y$.

The following discussion is based on Dubois and Prade, Section 2.1.2. [Dubois and Prade] Assume X and Y are fuzzy quantities (real numbers). Let f be a real function such that $f: X \times Y \rightarrow \mathbb{R}$, $\mathbb{R}$ being the set of real numbers. Let $f(x, y) = u$ and let $\pi_f(u)$ be the possibility for u . Assume that X and Y are separable; that is, $\pi_f(u) = \min[\pi_X(x), \pi_Y(y)]$. Using the extension principle, the possibility distribution for u is:

$$\pi_f(u) = \sup \{ \min[\pi_X(x), \pi_Y(y)] \mid f(x, y) = u \} \quad (\text{Eq. 28})$$

Since a possibility distribution defines a normal fuzzy set (see footnote 6), Equation 28 defines a fuzzy set $f(X, Y)$ with membership values $\mu_{f(X, Y)} = \pi_f$.

Example 2

Consider the following example. $X = \{0, 10^{-3}, 10^{-1}\}$ and $Y = \{10^2, 10^4, 10^6\}$ with the degrees of evidence as indicated in Figures 3.5 and 3.6. Take $f(x, y)$ as $x \cdot y$.

⁷ Whereas a set cannot contain duplicate elements and the order of the elements does not matter, the order of elements does matter in a tuple and the elements can be duplicated.

Figure 3.5. Degrees of Evidence for X

Figure 3.6. Degrees of Evidence for Y

Using Equation 25, the expected value intervals for X and Y are:

$$E^*(x) = (0.2 \times 10^{-3}) + (0.7 \times 10^{-3}) = 9.0 \times 10^{-4},$$
$$E^*(x) = (0.2 \times 10^{-3}) + (0.7 \times 10^{-1}) + (0.1 \times 10^{-1}) = 8.02 \times 10^{-2},$$

$$E^*(y) = (0.3 \times 10^6) + (0.7 \times 10^2) = 3.001 \times 10^5, \text{ and}$$
$$E^*(y) = (0.3 \times 10^6) + (0.7 \times 10^6) = 1.0 \times 10^6.$$

Table 3.2 summarizes the use of Equation 26 to calculate the expected value intervals for X and Y.

Table 3.2. Expected Value Intervals for X and Y in Example 2

x_i $\pi(x_i)$	$\Pi(\{x_j x_j \leq x_i\})$ (Term A)	$\Pi(\{x_j x_j \leq x_{i-1}\})$ (Term B)	$\Pi(\{x_j x_j > x_{i-1}\})$ (Term C)	$\Pi(\{x_j x_j > x_i\})$ (Term D)	A – B	C – D	$x_i^* (A - B)$	$x_i^* (C - D)$
0 0.1	0.1	0	1	1	0.1	0	0	0
10^{-3} 1	1	0.1	1	0.8	0.9	0.2	9×10^{-4}	2×10^{-4}
10^{-1} 0.8	1	1	0.8	0	0	0.8	0	8×10^{-2}
							$E_*(x) = 9.0 \times 10^{-4}$ (Sum)	$E^*(x) = 8.02 \times 10^{-2}$ (Sum)
y_i $\pi(y_i)$	$\Pi(\{y_j y_j \leq y_i\})$ (Term A)	$\Pi(\{y_j y_j \leq y_{i-1}\})$ (Term B)	$\Pi(\{y_j y_j > y_{i-1}\})$ (Term C)	$\Pi(\{y_j y_j > y_i\})$ (Term D)	A – B	C – D	$y_i^* (A - B)$	$y_i^* (C - D)$
10^2 0.7	0.7	0	1	1	0.7	0	7×10^1	0
10^4 0.7	0.7	0.7	1	1	0	0	0	0
10^6 1	1	0.7	1	0	0.3	1	3×10^5	1×10^6
							$E_*(y) = 3.001 \times 10^5$ (Sum)	$E^*(y) = 1.0 \times 10^6$ (Sum)

Using Equation 27, the centroid values for X and Y are:

Centroid of x = $[(10^{-3}) + (8 \times 10^{-2})] / 1.9 = 4.3 \times 10^{-2}$ and

Centroid of y = $[(0.7 \times 10^2) + (0.7 \times 10^4) + (10^6)] / 2.4 = 4.2 \times 10^5$.

$X \times Y = \{ \langle 0, 10^2 \rangle, \langle 0, 10^4 \rangle, \langle 0, 10^6 \rangle, \langle 10^{-3}, 10^2 \rangle, \langle 10^{-3}, 10^4 \rangle, \langle 10^{-3}, 10^6 \rangle, \langle 10^{-1}, 10^2 \rangle, \langle 10^{-1}, 10^4 \rangle, \langle 10^{-1}, 10^6 \rangle \}$. Using Equation 28 with $f(x,y)$ as $x*y$, the values and possibilities for the tuples are given in Table 3.3.

Table 3.3 Tuple Values and Possibilities

Tuple	Value	π
$\langle 0, 10^2 \rangle$	0	$\min[0.1, 0.7] = 0.1$
$\langle 0, 10^4 \rangle$	0	$\min[0.1, 0.7] = 0.1$
$\langle 0, 10^6 \rangle$	0	$\min[0.1, 1] = 0.1$
$\langle 10^{-3}, 10^2 \rangle$	10^{-1}	$\min[1, 0.7] = 0.7$
$\langle 10^{-3}, 10^4 \rangle$	10^1	$\min[1, 0.7] = 0.7$
$\langle 10^{-3}, 10^6 \rangle$	10^3	$\min[1, 1] = 1$
$\langle 10^{-1}, 10^2 \rangle$	10^1	$\min[0.8, 0.7] = 0.7$
$\langle 10^{-1}, 10^4 \rangle$	10^3	$\min[0.8, 0.7] = 0.7$
$\langle 10^{-1}, 10^6 \rangle$	10^5	$\min[0.8, 1] = 0.8$

The set of interest $f(X,Y)$ is $\{0, 10^{-1}, 10^1, 10^3, 10^5\}$ with possibilities given in Table 3.4.

Table 3.4. Possibilities for $f(X,Y)$

Element	π
0	$\max[0.1, 0.1, 0.1] = 0.1$
10^{-1}	0.7
10^1	$\max[0.7, 0.7] = 0.7$
10^3	$\max[1, 0.7] = 1$
10^5	0.8

Table 3.5 summarizes the use of Equation 26 to calculate the expected value interval for $f(X,Y)$.

Table 3.5. Expected Value Interval for $f(X*Y)$ in Example 2

z_i $\pi(z_i)$	$\Pi(\{z_j \mid z_j \leq z_i\})$ (Term A)	$\Pi(\{z_j \mid z_j \leq z_{i-1}\})$ (Term B)	$\Pi(\{z_j \mid z_j > z_{i-1}\})$ (Term C)	$\Pi(\{z_j \mid z_j > z_i\})$ (Term D)	A – B	C – D	$z_i^* (A - B)$	$z_i^* (C - D)$
0 0.1	0.1	0	1	1	0.1	0	0	0
10^{-1} 0.7	0.7	0.1	1	1	0.6	0	6×10^{-2}	0
10^1 0.7	0.7	0.7	1	1	0	0	0	0
10^3 1	1	0.7	1	0.8	0.3	0.2	3×10^2	2×10^2
10^5 0.8	1	1	0.8	0	0	0.8	0	8×10^4
							$E_*(z) =$ 3.001×10^2 (Sum)	$E^*(z) =$ 8.02×10^4 (Sum)

Using Equation 27 the centroid is:

$$\text{Centroid of } X*Y = [(7 \times 10^{-2}) + (7) + (10^3) + (8 \times 10^4)] / 3.3 = 2.4 \times 10^4.$$

3.2.2 Terrorist Risk using the Numerical Measure

A necessity/possibility distribution for Risk = Attack x Success x Consequence can be generated using Equation 28 to combine (convolute) the separate necessity/possibility distributions for Attack, Success, and Consequence. For this combination $f(a, s, c) = a*s*c$ in Equation 28 where a , s , and c are elements of Attack, Success, and Consequence, respectively. Once the necessity/possibility distribution for Risk has been created, an expected value interval and a centroid for Risk can be calculated using Equations 26 and 27. This procedure is the numerical technique for evaluating risk from acts of terrorism.

Risk has units of dollars/year, assuming consequences are expressed as equivalent dollars to repair. For risk so defined, its expected value interval and its centroid also have units of dollars/year. The expected value interval and the centroid for the necessity/possibility distribution for terrorist risk can be compared to the mean of the probability distribution for a random act. For example, risk from a specific terrorist act can be compared to risk from a specific random act (such as a tornado).

To rank the risk for many different terrorist scenarios, the scenarios can be ranked using the expected value interval. For example, the scenarios can be ranked by $E^*(x)$; ranking by the centroid is probably less appropriate because scenarios with similar centroids can have different uncertainties as indicated by the expected value interval.

Due to the large number of <attack, success, consequence> 3-tuples in Risk, calculations of both the necessity/possibility distribution and the expected value interval for Risk have been implemented in a computer code called PossibleRisk. Possible Risk was written in java 1.4.2 using the netbeans 3.5.1 integrated development environment.

Example 3

As an example of the evaluation of terrorist risk for one specific threat, assume the degrees of evidence in Figures 3.7, 3.8, and 3.9 are applicable to Attack, Success, and Consequence for that threat.

Figure 3.7. Degrees of Evidence for Attack for Example 3

Figure 3.8. Degrees of Evidence for Success for Example 3

Figure 3.9. Degrees of Evidence for Consequence for Example 3

Figure 3.10 shows the possibility distributions based on these degrees of evidence entered into the PossibleRisk code input data panel. Figure 3.11 shows the expected value intervals and the centroids for Attack, Success, Consequence, and Risk calculated by PossibleRisk. Figure 3.12 shows the possibility distribution for Risk produced by PossibleRisk.

ReportExample ✕

Scenario	Attack (per year)		Success (probability)		Consequence (\$ to repair)	
	Frequency (per y...	Possibility: [0,1]	Success (probab...	Possibility: [0,1]	Consequence (\$)	Possibility: [0,1]
Enter Description of Scenario Here!	0	1.0	0	0.0	1	0.0
	10 ⁵	0.2	10 ³	0.1	10	0.0
	10 ⁴	1.0	10 ²	0.8	10 ²	0.0
	10 ³	0.5	10 ¹	1.0	10 ³	0.0
	10 ²	0.5	1	1.0	10 ⁴	0.0
	10 ¹	0.0			10 ⁵	0.0
	1	0.2			10 ⁶	0.0
	10	0.0			10 ⁷	0.1
					10 ⁸	1.0
					10 ⁹	1.0
					10 ¹⁰	0.5
					10 ¹¹	0.0
					10 ¹²	0.0

Figure 3.10. Possibilities for Example 3

Figure 3.11. Expected Value Intervals and Centroids for Example 3

Figure 3.12. Possibility Distribution for Risk for Example 3

4 Fuzzy Sets and the Linguistic Technique

This section provides a brief summary of fuzzy sets and how possibility theory can be extended from crisp to fuzzy sets. Then, the linguistic technique is presented.

4.1 Fuzzy Sets

4.1.1 The Concept of a Fuzzy Set

For crisp sets, each element of X is either a member of a crisp subset of X or is a member of the crisp complement of that subset; all elements have degrees of membership of either 0 or 1 in any crisp subset. Elements of X can have partial membership, 0 to 1 inclusive, in a fuzzy subset of X ; it is possible for an element of X to have a nonzero degree of membership in both a fuzzy subset of X and in the complement fuzzy subset.

Fuzzy sets address the type of uncertainty called vagueness; that is, a fuzzy set represents an event whose outcomes cannot be precisely identified. Fuzzy sets are useful for evaluating linguistic terms. For example, consider the UOD to be “The 50 states in the United States”. Consider the subset associated with the linguistic “States in the South”. There is vagueness associated with “States in the South”. Is Texas in the South? Is Kentucky in the South? Using a fuzzy set to represent “States in the South”, the vagueness can be captured. Alabama can be assigned a degree of membership 1, Texas a degree of membership 0.7 (or some other value depending on the interpretation of Southern states), Kentucky 0.5, New York 0, Pennsylvania 0, etc. The fuzzy subset “States in the South” has each state in the United States with a degree of membership between 0 and 1: 1.0 for Alabama, 1 for Georgia, 1 for Mississippi, 0.7 for Texas, 0.5 for Kentucky, 0 for New York, etc.

4.1.2 Mathematics for Fuzzy Sets

Each element of X is assigned a *degree of membership* in a particular fuzzy subset of X by a membership function (mapping):

$$\mu_A : X \rightarrow [0,1] \text{ (Eq. 29)}$$

$\mu_A(x)$ is the degree of membership of an element x in the fuzzy subset A . [Klir and Folger] For a crisp set $\mu_A(x)$ is either 0 or 1, but for a fuzzy set it can be any value between 0 and 1 inclusive. The fuzzy set is characterized by the degree of membership of the elements of X in that fuzzy set. For a particular fuzzy set, the crisp set that contains all the elements of X with nonzero degree of membership in that fuzzy set is called the support for that fuzzy set.

For union, intersection, and complement of fuzzy sets the standard definitions are:

$$\begin{aligned}\mu_{A \cup B}(x) &= \max[\mu_A(x), \mu_B(x)] \\ \mu_{A \cap B} &= \min[\mu_A(x), \mu_B(x)] \\ \mu_A^-(x) &= 1 - \mu_A(x)\end{aligned} \quad \text{(Eq. 30)}$$

where A and B are fuzzy subsets of X . Other definitions can be used as discussed in the references. The other definitions used for union and intersection are such that the

standard max. union is a lower bound and the standard min. intersection is an upper bound. [Klir and Yuan, Chapter 3] That is, other definitions for union and intersection are constrained by:

$$\begin{aligned}\mu_{A \cup B}(x) &\geq \max[\mu_A(x), \mu_B(x)] \\ \mu_{A \cap B} &\leq \min[\mu_A(x), \mu_B(x)] .\end{aligned}\quad (\text{Eq. 31})$$

The height of a fuzzy set is the maximum degree of membership of any element in the set. If the height is 1, the fuzzy set is said to be normal; if the height is less than 1, the fuzzy set is said to be subnormal.

A fuzzy set A on X can be represented as a family of crisp sets. An α -cut of a fuzzy set is a crisp set defined as:

$$A_\alpha = \{x \in X \mid \mu_A(x) \geq \alpha\} \quad (\text{Eq. 32})$$

The family of all α -cuts, $C(A)$, is a representation of A using crisp sets: [Dubois and Prade, Section 1.4]

$$C(A) = \{A_\alpha \mid \alpha \in]0,1]\} . \quad (\text{Eq. 33})$$

Defuzzification is a process for providing a single measure for a fuzzy result. [Klir and Yuan] Typically, defuzzification is applied to a fuzzy number, and numerous different techniques for defuzzification have been developed. One widely used defuzzification measure is the centroid method, so named because it is formulated equivalent to a center of mass calculation in classical mechanics. Consider a fuzzy number N. The centroid of N is the real number:

$$\bar{z} = \frac{\sum_{\text{all real numbers}} \mu_N(z) \cdot z}{\sum_{\text{all real numbers}} \mu_N(z)} \quad (\text{Eq. 34})$$

4.2 Fuzzified Possibility Theory

In the linguistic technique, we wish to apply possibility measures to fuzzy sets; such a process is sometimes called “fuzzified evidence theory”. [Klir and Yuan, Section 9.5]

The possibility the fuzzy event A (fuzzy subset A) in X will occur is:

$$\Pi(A) = \max_{x \in X} \{\min[\pi(x), \mu_A(x)]\} \quad (\text{Eq. 35})$$

as developed in Dubois and Prade. [Dubois and Prade, Section 1.7] Also, Klir uses this approach in an example application. [Klir and Folger, Section 6.2]

Zadeh developed Equation 35 by considering the overlap between the random fuzzy set (event) A with degree of membership $\mu_A(x)$, and the specific fuzzy set- call it F- defined by the possibility measure Π on X. (For any possibility measure Π there exists an equivalent fuzzy set F such that: $\mu_F(x) = \pi(x)$). [Dubois and Prade, Section 1.7])

Equation 35 can be physically understood as an extension of Equation 17 as follows. Equation 17 is for A being a crisp subset of X, so for any x an element of X that element is either in A (degree of membership in A is 1) or is not in A (degree of membership in A is 0). If A is fuzzy, then any element x in X can be in A with degree of membership $\mu_A(x)$ in $[0, 1]$, so the degree of degree of membership must be considered and all elements in X must be considered with their respective degree of membership.

Similarly, the necessity that the fuzzy event A will occur is:

$$N(A) = \min_{x \in X} \{ \max [(1 - \pi(x), \mu_A(x))] \} \quad (\text{Eq. 36})$$

4.3 Approximate Reasoning

In the numerical technique, Equation 1 was evaluated by numerically convoluting possibility distributions for fuzzy quantities. In the linguistic technique, numerical convolution cannot be used as linguistic terms (represented by fuzzy sets) are used instead of fuzzy quantities.

Therefore, we need a technique for “convoluting” linguistic terms and the technique is approximate reasoning.

Consider two (different) universes of discourse X and Y.

The Cartesian product of X and Y is a set defined as:

$$X \times Y = \{ \langle x, y \rangle \mid x \in X \text{ and } y \in Y \} \quad (\text{Eq. 37})$$

where $\langle x, y \rangle$ denotes a 2-tuple.

A binary relation, R, is defined as a subset of the Cartesian product $X \times Y$; the elements of a binary relation are specific 2-tuples from $X \times Y$ selected based on the properties of the relation. (If for every x there is a unique y, a relation is called a function.) A crisp relation is a special case for which the degree of membership of the tuples in the relation is restricted to $\{0, 1\}$; a fuzzy relation is a relation for which the degree of membership of the tuples in the relation is $[0, 1]$.

For a fuzzy relation in general, there is no specific formula for the degree of membership of its tuples; the degrees of membership are as required to satisfy the relation. Consider the following example from Klir and Folger: $X = \{\text{New York City, Paris}\}$ and $Y = \{\text{Beijing, New York City, London}\}$ where the relation $R(X, Y)$ is “very far”. This fuzzy relation can be expressed as: [Klir and Folger, Section 3.1]

$$R(X, Y) = 1/\text{NYC, Beijing} + 0/\text{NYC, NYC} + 0.6/\text{NYC, London} + 0.9/\text{Paris, Beijing} + 0.7/\text{Paris, NYC} + 0.3/\text{Paris, London}.$$

Of interest to us are fuzzy relations formed from fuzzy sets. Let A and B be fuzzy sets of X and Y, respectively.

Let $R = A \times B$ be the fuzzy relation of interest and interpret ‘x’ as ‘min’: [Dubois and Prade, Section 1.8] [Ross Fuzzy Logic]

$$\mu_{A \times B}(x, y) = \min[\mu_A(x), \mu_B(y)] . \quad (\text{Eq. 38})$$

The Cartesian coproduct $A + B$ is defined as:

$$A + B = \{ \langle x, y \rangle \mid x \in A \text{ } y \in Y \cup \langle x, y \rangle \mid x \in X \text{ } y \in B \} . \quad (\text{Eq. 39})$$

Interpreting ‘+’ as ‘max’: [Dubois and Prade, Section 1.8]

$$\mu_{A+B}(x, y) = \max[\mu_A(x), \mu_B(y)] . \quad (\text{Eq. 40})$$

It is always true that:

$$\begin{aligned} N(A \times B) &= \min[N(A), N(B)] \\ \Pi(A + B) &= \max[\Pi(A), \Pi(B)] \end{aligned} \quad (\text{Eq. 41})$$

X and Y are said to be *noninteractive in a possibilistic sense* under a relation $R = X \times Y$ if and only if: [Klir and Yuan, Section 7.3]

$$\pi_R(x, y) = \min[\pi_X(x), \pi_Y(y)] \text{ for all } x \in X, y \in Y \quad (\text{Eq. 42})$$

where π_R is the possibility distribution for R and the marginal possibility distributions are defined as:

$$\begin{aligned} \pi_X(x) &\equiv \max_{y \in Y} [\pi_R(x, y)] \text{ and} \\ \pi_Y(y) &\equiv \max_{x \in X} [\pi_R(x, y)] . \end{aligned} \quad (\text{Eq. 43})$$

For any A and B subsets of X and Y , respectively, $A \times B$ is a subset of $X \times Y$; if X and Y are noninteractive, then A and B are said to be noninteractive.

If A and B are noninteractive:

$$\begin{aligned} N(A + B) &= \max[N(A), N(B)] \text{ and} \\ \Pi(A \times B) &= \min[\Pi(A), \Pi(B)] . \end{aligned} \quad (\text{Eq. 44})$$

Approximate reasoning is a way to combine fuzzy events from different UODs.

Consider a rule of the form “ $C = A \times B$ ” where A , B , and C are fuzzy sets. Assuming that A and B are noninteractive, Equation 44 can be used to evaluate the rule for Π .⁸ (Similarly, Equation 41 can be used to evaluate the rule for N , but we will not be using necessity in the linguistic evaluation.)

Now consider a set of I disjunctive rules for C : “ $C = A_1 \times B_1$ ” or “ $C = A_2 \times B_2$ ”, ..., or “ $C = A_i \times B_i$ ”; the rules for C can be represented as: $C = \bigcup_{\text{all } i} A_i \times B_i$.

Using Equation 44 with Equation 14:

⁸ The rules of interest in this application are of the form $C = A \times B$. These rules are simpler than rules involving implication. [Darby Methodology]

$$\Pi(C) = \max_{all\ i} \{ \min[\Pi(A_i), \Pi(B_i)] \} . \quad (Eq. 45)$$

In general, a complete set of rules has more than one C, each C formed as just discussed. That is, the complete set of rules for ‘n’ different fuzzy subsets is of the form: $\{C_j \mid j = 1, 2, \dots, n\}$. Equation 45 can be used to calculate the possibility for each of the C_j .

4.4 The Linguistic Technique

Fuzzy sets are useful for the evaluation of terrorist risk in that Risk can be evaluated using linguistic (fuzzy) descriptions of Attack, Success, and Consequence. Figures 4.1, 4.2, and 4.3 provide fuzzy sets for these terms.

Figure 4.1. Degrees of Membership (μ) in Fuzzy Sets for Attack

Figure 4.2. Degrees of Membership (μ) in Fuzzy Sets for Success

Figure 4.3. Degrees of Membership (μ) in Fuzzy Sets for Consequence

For the linguistic evaluation of Equation 1, approximate reasoning is used to combine Attack x Success into Loss, and to combine Loss x Consequence into Risk. It is assumed that Success|Attack is noninteractive with Attack, and that Loss is noninteractive with Consequence|Success.

Loss is evaluated as Attack x Success using the rules given in Table 4.1.

Table 4.1. Rule Set for “Loss”

		Success		
		Low	Average	High
Attack	Unlikely	Small	Small	Moderate
	Credible	Moderate	Moderate	Large
	Likely	Moderate	Large	Large

Using Equation 45, the possibility (Π) for each of the fuzzy sets of Loss (Small, Moderate, and Large) is evaluated as follows:

$$\Pi(Loss_j) = \max_{all\ rules} \{ \min[\Pi(Attack), \Pi(Success)] \}$$

where j ranges over the three fuzzy subsets of Loss: Small, Moderate, and Large.

Risk is evaluated as Loss x Consequence using the rules given in Table 4.2.

Table 4.2. Rule Set for “Risk”

		Consequence			
		Minor	Medium	Major	Catastrophic
Loss	Small	Low	Low	Medium	High
	Moderate	Medium	Medium	High	High
	Large	Medium	High	High	Very High

Using Equation 45, the possibility (Π) for each of the fuzzy sets of Risk (Low, Medium, High, and Very High) is evaluated as follows:

$$\Pi(Risk_j) = \max_{all\ rules} \{ \min[\Pi(Loss), \Pi(Consequence)] \}$$

where j ranges over the four fuzzy subsets of Risk: Low, Medium, High, and Very High.

The application of the linguistic technique for evaluating Equation 1 for terrorist risk consists of three steps:

1. Assign degrees of evidence to the discrete sets representing Attack, Success, and Consequence.
2. Calculate possibilities for all the linguistic terms (fuzzy subsets) for Attack, Success, and Consequence.
3. Calculate possibilities for the linguistic terms (fuzzy subsets) for Risk.

The first step is performed in the same manner as in the numerical technique. The second step is performed using Equation 35. The third step is performed by applying Equation 45 to the rules in Tables 4.1 and 4.2.

To provide a point estimate centroid measure for Risk, the fuzzy sets on Risk are assigned numerical weights given in Table 4.3.

Table 4.3 Weights for Fuzzy Sets of Risk

Fuzzy Set for Risk	Weight
Low	2
Medium	4
High	16
Very High	64

Using the centroid approach discussed in Section 3.2.1.2, the centroid of Risk is taken to be:

$$C_{Risk} = \frac{\sum_{i=Low, Medium, High, Very High} \Pi_i * Weight_i}{\sum_{i=Low, Medium, High, Very High} \Pi_i} . \quad (Eq. 46)$$

PossibleRisk has implemented a linguistic evaluation of terrorist risk using the fuzzy sets and rules just presented.

Example 4

Assume that the degrees of evidence for the treat of concern are the same as used in Example 3, Figures 3.7 through 3.9, thereby resulting in the possibilities for the elements in Attack, Success, and Consequence as shown in Figure 3.10. This completes the first step of the linguistic technique for Example 4.

The second step of the linguistic technique is to calculate possibilities for all the fuzzy subsets for Attack, Success, and Consequence, where the fuzzy sets are defined in Figures 4.1 through 4.3. This is accomplished by using Equation 35. For example, the possibility for the fuzzy set Unlikely for Attack is calculated as:

$$\Pi_{Attack}(Unlikely) = \max_{a \in \{0, 10^{-5}, 10^{-4}, 10^{-3}, 10^{-2}, 10^{-1}, 1, 10\}} [\min\{\pi(a), \mu_{Unlikely}(a)\}]$$

where the π 's are from Figure 3.10 and the μ 's are from Figure 4.1.

The results are as follows:

$$\Pi_{Attack}(Unlikely) = 1$$

$$\Pi_{Attack}(Credible) = 0.5$$

$$\Pi_{Attack}(Likely) = 0.5$$

$$\Pi_{Success}(Low) = 0.1$$

$$\Pi_{Success}(Average) = 0.8$$

$$\Pi_{Success}(High) = 1$$

$$\Pi_{Consequence}(Minor) = 0$$

$$\Pi_{Consequence}(Medium) = 0.1$$

$$\Pi_{Consequence}(Major) = 1$$

$$\Pi_{Consequence}(Catastrophic) = 0.5$$

The third step of the linguistic technique is to evaluate the rules for Loss and Risk given in Tables 4.1 and 4.2, respectively, using Equation 45. For example, the possibility for the fuzzy set Moderate for Loss is:

$$\begin{aligned} \Pi_{Loss}(Moderate) &= \max[\min\{\Pi_{Attack}(Unlikely), \Pi_{Success}(High)\}, \\ &\min\{\Pi_{Attack}(Credible), \Pi_{Success}(Low)\}, \min\{\Pi_{Attack}(Credible), \Pi_{Success}(Average)\}, \\ &\min\{\Pi_{Attack}(Likely), \Pi_{Success}(Low)\}] \\ &= 1 \end{aligned}$$

The results are as follows:

$$\Pi_{Loss}(Small) = 0.8$$

$$\Pi_{Loss}(Moderate) = 1$$

$$\Pi_{Loss}(Large) = 0.5$$

$$\Pi_{Risk}(Low) = 0.1$$

$$\Pi_{Risk}(Medium) = 0.8$$

$$\Pi_{Risk}(High) = 1$$

$$\Pi_{Risk}(Very High) = 0.5$$

Using Table 4.3 and Equation 46 the centroid point estimate for risk is 21.417.

The results for risk for Example 4 calculated using PossibleRisk are shown in Figures 4.4 and 4.5. Figure 4.5 shows the result as a possibility distribution over a real number representing the weights assigned to the fuzzy sets of risk.

Figure 4.4. Linguistic Results for Example 4

Figure 4.5 Linguistic Risk Possibility Distribution for Example 4

4.5 Use of LEDTools

As discussed in Section 1, the LEDTools code can be used to perform the same linguistic evaluation as performed in PossibleRisk. [LEDTools] LEDTools is a general purpose linguistic evaluation tool and allows user defined universes of discourse and user defined approximate reasoning rules, whereas PossibleRisk uses predefined universes of discourse (risk, attack, success, loss, and consequence) and predefined rules.

This section summarizes the use of LEDTools for Example 4 of Section 4.4.

Figure 4.6 is the linguistic risk model created using LEDTools. Figure 4.7 is an example of the creation of the UOD for Consequence. Figure 4.8 is an example of the creation of the rule base for Risk. Finally, Figure 4.9 is the result of the LEDTools evaluation for risk.

Figure 4.6. Risk Model Created with LEDTools

Consequence

Linguistics and Degrees of Membership

Linguistics	Degree
Minor	0.00
Medium	0.10
Major	1.00
Catastrophic	0.50

Insert

Delete

Move cursor with arrow keys & Enter

Normalize

OK Cancel

Figure 4.7. UOD for Consequence Created with LEDTools

Risk

Loss		Consequence			
		Minor	Medium	Major	Catastrophic
Small	Low	Low	Medium	High	
Moderate	Medium	Medium	High	Very High	
Large	Medium	High	High	Very High	

OK Cancel

Figure 4.8. Rule Base for Risk Created with LEDTools

Figure 4.9. Result for Risk from LEDTools

5 Summary

This report summarizes two techniques that have been developed and used to evaluate risk of various terrorist scenarios to support NISAC analyses for 2004. These techniques are based on the LANL LED methodology.

Both techniques use possibility theory. The numerical technique uses fuzzy quantities and the linguistic technique uses fuzzy sets to represent linguistic terms. Approximate reasoning is used in the linguistic technique.

Both techniques have been implemented in PossibleRisk, a java code. The linguistic technique can also be implemented in the more general purpose LEDTools code.

Examples using both PossibleRisk and LEDTools are provided in this report.

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