ACCURATE ITERATIVE ANALYSIS OF THE K-V EQUATIONS
O. A. Anderson, Lawrence Berkeley National Laboratory, 1 Cyclotron Road, Berkeley, CA 94720 USA

Those working with alternating-gradient (A-G) systems look for simple, accurate ways to analyze A-G performance for matched beams. The useful K-V equations [1] are easily solved in the smooth approximation [2], [3], [4]. This approximate solution becomes quite inaccurate for applications with large focusing fields and phase advances. Results of efforts to improve the accuracy [5], [6] have tended to be indirect or complex. Our generalizations presented previously [7] gave better accuracy in a simple explicit format. However, the method used to derive our results (expansion in powers of a small parameter) was complex and hard to follow; also, reference [7] only gave low-order correction formulas.

The present paper uses a straightforward iteration method and obtains equations of higher order than shown in our previous paper.

The K-V equations for the envelopes \(a(z)\) and \(b(z)\) are

\[
\begin{align*}
\frac{\mathrm{d}a(z)}{\mathrm{d}z} &= -K(z) a + \frac{e^2}{n^2 A} + \frac{2Q}{A+b}, \\
\frac{\mathrm{d}b(z)}{\mathrm{d}z} &= +K(z) b + \frac{e^2}{n^2 A} + \frac{2Q}{A+b}
\end{align*}
\]

with input parameters: normalized beam current \(Q\); emittance \(\varepsilon\); and A-G focus function \(K(z)\). The zero of \(z\) is located at the midpoint of a quadrupole and \(K(z)\) is assumed here to be symmetric about \(z=0\), periodic over a cell length \(2L\), and antisymmetric about \(L/2\). Thus

\[
K(z-2L) = K(z), \quad K(-z) = K(z), \quad K(z-L) = -K(z). (3)
\]

We solve for the \(x\) and \(y\) beam envelopes \(a(z)\) and \(b(z)\), assumed to be matched to the lattice, i.e., periodic over \(2L\). To aid the solution of Eqs. (1) and (2), we define in Eqs. (4)–(19) the operators on even periodic functions \(\{\ldots\}, \{\ldots\}, J\), and \(J_l\); the even periodic functions \(h(z), g(z), \delta(z)\) and \(\rho(z)\); and the constants \(K, \alpha, \beta, \gamma, A, K_{\text{eff}}, \Phi, \text{and } p_m\). In Eq. (19), \(h_1\) is the first Fourier coefficient of \(h(z)\).

The operator \(\{\ldots\}\) performs an average over a cell length \(2L\), while the operator \(\{\ldots\}\) removes the average part of a periodic function: e.g., \(2\{\cos^2 x\} = 1 + \cos 2x = \cos 2x\). The operator \(\int f\) operates on periodic functions that have no average. It gives the repeated indefinite integral and removes the average part, if any, of the result.

DECOUPLING AND DECOMPOSING
With the quadrupole symmetries of Eq. (3), our matched beam assumption implies \(b(z) = \alpha_n(z+L)\), so that Eqs. (1) and (2) are decoupled. We have \(\alpha = A(1+p)\), \(b = A(1+p_0)\).

The Q terms in Eqs. (1) and (2) can be expanded as

\[
\frac{2Q}{A+b} = \frac{Q}{A}(1 - (p+p_0)/2 + \ldots) = \frac{Q}{A}(1 - k^2\delta(z) + \ldots),
\]

since [8]

\[
(p+p_0)/2 = k^2\delta(z) + \ldots.
\]

with \(\delta(z)\) [Eq. (11)] derived from the lattice waveform \(b(z)\).

This decouples Eqs. (1) and (2). After the decoupled version of Eq. (1) is solved for \(a(z)\), then \(b(z)\) is found by symmetry. Equation (2) is no longer needed.

Substituting \(a = A(1+p)\) in the first three terms of Eq. (1), expanding \(1/n^2\), dividing by \(A\), and using (21) and (15), the first K-V equation is equivalent to

\[
\rho(z) - kh(z) - kh(z)\rho + \frac{3}{2}(1 - \gamma \rho^2 - 6\rho^2 - 15\rho^4 + 15\rho^6 + 15\rho^8 + \ldots) + q(1 - k^2\delta(z)) = 0.
\]

To solve for the ripple \(\rho(z)\) and the mean radius \(A\) (which appears in the definitions of \(\alpha\) and \(q\)), we decompose Eq. (23) into a pair of equations. Averaging Eq. (23),

\[
0 = -k\rho_0 - \rho_{02} + 2\alpha_0k\rho_0 + -10\alpha_0k^2\rho_{00} + 5\alpha_0k^4\rho_{00} + q(1 - k^2\delta(z)) = 0.
\]

Subtracting Eq. (24) from (23),

\[
\rho'' = -kh(z) - k\rho_{00} - \alpha_0 + 2\alpha_0k^2\rho_{00} - 10\alpha_0k^4\rho_{00} + 10\alpha_0k^6\rho_{00} - qk^2\delta(z) = 0.
\]

with \(\{\ldots\}\) from Eq. (5). There are now two equations, each containing \(A_0\) and \(\rho_0\). Because of our periodicity constraint these have the essence of the K-V equations (1) and (2).

ITERATIVE SOLUTION OF K-V EQUATIONS
On the right of Eq. (25), the \(k\rho(z)\) term dominates the terms involving the unknown function \(\rho(z)\). They are omitted for the initial integrations, which give \(\rho_0\). Then we insert \(\rho_0\) into (25) and integrate again to get \(\rho_{00}\). The process is repeated for \(\rho_{00}\). The resulting terms of greatest significance are:

\[
\rho_0 = -k\rho_0,
\]

\[
\rho_{00} = \rho_0 + \alpha\rho_{00} + \frac{10}{3}\alpha k^4\rho_{00},
\]

\[
\rho_{00} = \rho_0 - \alpha^2 k^2\rho_0 - 2\alpha k^3\rho_{00}.
\]

To complete the approximate solution of the K-V equations, \(\rho(z)\) from Eq. (26) is put in the matching equation (24). From

<table>
<thead>
<tr>
<th>Table 1: Definitions to be used in this paper</th>
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<tbody>
<tr>
<td>(f) (\equiv) ((1/2L)\int_{-L}^{L} f(z),dz)</td>
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<tr>
<td>(\delta(z) \equiv \int f(z),dz)</td>
</tr>
<tr>
<td>(\rho(z) \equiv (a(z) - A)/A)</td>
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<tr>
<td>(h(z) \equiv K(z)/A)</td>
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<tr>
<td>(g(z) \equiv \int h(z),dz)</td>
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<tr>
<td>(K_{\text{eff}} \equiv k^2\beta^2)</td>
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<td>(K_{\text{eff}} \equiv k^2\beta^2)</td>
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<tr>
<td>(\Phi \equiv 3k^2\beta^2)</td>
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<tr>
<td>(h_1 \equiv k^2/\pi^2)</td>
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</table>
Eq. (26) we discarded items, such as $2\alpha k \eta p^2$, that would give terms in (24) higher than third power in the parameters $k, \alpha$, and $q$. A miniscule term, $qk^2 / \delta(z)$, is also omitted.

The order of a term in the matching equation is reckoned by counting the number of factors $k, \alpha$, and $q$. These would become small parameters in a non-dimensional formalism [8]. Here, we prefer to retain physical units for length $z$, etc.

Inserting Eq. (26) into Eq. (24) yields seven terms through third order. Some terms combine, with result

$\beta_2 = 2.7 \times 10^{-3}$

(4)

First, second, and third-order results for $A_1$ from (33), (32), and (30), are plotted in Fig. 1a. The smooth approximation is relatively inaccurate except near the point where its error curve crosses the 0% line.

**MAXIMUM RADIUS**

Knowing the matched mean radius $A_1$, one can complete the solution for the envelope $a(z) = A_1[1 + p(z)]$ using $p(z)$ from Eq. (26); $b(z)$ can be found by changing the sign of the terms that contain odd powers of $k$.

Some terms of Eq. (26) can be written in exact form [8] for models such as FODO, but Fourier expansion is more useful in general:

$$h(z) = h_1 \left[ \frac{\cos \pi \frac{3z}{L} + \frac{1}{3} c_3 \cos \frac{3\pi z}{L} + \frac{1}{3} c_5 \cos \frac{5\pi z}{L}}{\frac{1}{3}} \right].$$ (34)

Values (usually of order unity) of $h_1$ and $c_n$ for both FODO and smooth profiles are given in Ref. [8]. With the definition

$$\beta_1 = 3 \frac{1}{2} \frac{\varepsilon_2^2}{\pi^2 A_1^4}$$ (35)

we have

$$s_{11}^\text{max} = A_1 \left[ 1 + p_m(1 + 1/27 c_3 + 1/125 c_5) + 1/8 p_m^2 (1 + 25/54 c_5) \right]$$ (36)

using results from Ref [8]. The accuracy of Eq. (36) is shown in Fig. 1b, along with that of the truncations.
This equation is used to calculate $\sigma_\Omega$ as a function of the strength of the quadrupole field gradient. Figure 2b shows its accuracy and also illustrates the second-order case

$$\sigma_\Omega = 2L(\kappa_\Omega^{(0)})^{1/2}[1 + \frac{1}{2}\Phi + \frac{5}{8}\Phi^2].$$

(43)

This equation is used to calculate $\sigma_\Omega$ as a function of the strength of the quadrupole field gradient. Figure 2b shows its accuracy and also illustrates the second-order case

$$\sigma_\Omega = 2L(\kappa_\Omega^{(0)})^{1/2}[1 + \frac{1}{2}\Phi].$$

(44)

and the smooth approximation,

$$\sigma_s = 2L(\kappa_s^{(0)})^{1/2}.$$  

(45)

ACKNOWLEDGEMENTS


REFERENCES

[8] For derivation of equations, etc., see the appendices to: O.A. Anderson, LBNL report LBNL-57388, 2005 (to be submitted to Accel. and Beams).