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Newton Descent Observer for Nonlinear Discrete-Time Systems

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Abstract—This paper presents a global Newton Descent single step observer for nonlinear discrete time systems. The stability conditions for the observer require that a bounded step Armijo condition be met at each step to ensure stability. The stability and performance of the observer are demonstrated for linear and nonlinear systems.

I. INTRODUCTION

The Newton-Descent single step observer is a full state observer for nonlinear discrete time systems. It is based upon a numerical approach that assures at each time step, i.e., iteration, the improvement of the state estimate for the nonlinear system based upon an output error objective function. The algorithm is presented in its general nonlinear form. Assuming that the nonlinear system is fully observable over the past $N$ time steps and that the linearized system and observability matrices are bounded, the convergence of the observer estimate to the actual state trajectory can be assured with proper selection of the Newton step length parameter. The step size factor must meet both the the Newton descent criteria based upon application of Armijo rule and a stability criteria that mandates it remain sufficiently close to unity to ensure that the scaled system is stable at each iteration. The Newton Descent observer is implemented through transformation of the nonlinear system into a linear time varying one with a corresponding expression of the observer algorithm.

II. BACKGROUND

Many of the methods for designing and implementation of control laws for linear and nonlinear systems using state space methods rely upon full accessibility of the controller to all the state variables of the system. While conceivably all the states of a system could be obtained through measurement, more often than not, physical constraints or cost make it unreasonable to do so. Instead, the states of the system are most often estimated by an observer using the available outputs from the system.

A. Kalman Observers

Kalman[1] and Luenberger[2] developed the classic linear state estimation techniques for stochastic and deterministic systems. Both approaches use an accurate model of the system and output injection to drive the estimation error to zero. Extension of these methods has become one of the prime starting points for nonlinear observers, generally relying on linearization of nonlinear system along a steady state trajectory. Examples of nonlinear extensions in model predictive control efforts include the Luenberger extensions of Soroush[3],and the extended Kalman extensions of Jazwinski [4] and of Bequette[5].

The second major route toward nonlinear system estimation arises out of nonlinear controls, typified by the works of Isidori [7], Khalil[8] and others. These methods generally take a nonlinear state space model and transform it into a linear structure to which a linear observer strategy may be applied.

A third major form of state estimation for nonlinear systems has evolved from optimization methods. While primarily associated with model predictive control development, there have been separate important efforts by others. The model predictive control related efforts have, in many cases, complemented the researcher’s work in predictive control development. Examples of these include the works of Rawlings[9], Mayne [12], and Rao[11]. In [13] Moraale and Grizzle present a general use, nonlinear Newton observer. It is these optimization based approaches that form the principle foundation for the gradient based state estimation method proposed herein.

B. Nonlinear Discrete-Time Systems

Consider the discrete nonlinear system

$$x_{k+1} = f(x_k, u_k), \quad y_k = h(x_k, u_k),$$

(1)

where $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$, and $y_k \in \mathbb{R}^p$ are the state, control, and output at the current time step $k$ with the initial state condition $x_0$. The system measurement trajectories over past $N$ time steps are given by the input and output vectors $U_k \in \mathbb{R}^{N \times m}$ and $Y_k \in \mathbb{R}^{N \times p}$

$$U_k = \begin{bmatrix} u_{k-N+1} \\ u_{k-N+2} \\ \vdots \\ u_k \end{bmatrix},$$

(2)

$$Y_k = \begin{bmatrix} y_{k-N+1} \\ y_{k-N+2} \\ \vdots \\ y_k \end{bmatrix}.$$  

(3)

The predictive $N$ step output sequence $\phi_k$ equal to $\phi(x_{k-N+1}, U_k)$ evolves from the system state at time step $x_k$.
of the initial state estimate and the greater the number of iterations demanded for observer convergence. For an initial state estimate at time step \( k \) meeting the criteria, the Newton refinement will quadratically converge to a a refined estimate of the past system state. As the system evolves, this refined solution, propagated forward in time, becomes the starting estimate for the next time step.

The discrete time nonlinear observer based upon the iterative Newton method is constrained at each time step to begin the refinement with a state estimate sufficiently proximate to the actual state, such that a full Newton step is possible. This limitation may be overcome by considering an alternate algorithm based upon the Newton descent numerical method. In this approach, a reduced step size in the Newton direction is permitted, to ensure reduction of the residual output error. The refinement step of the Newton observer (9) is modified by the insertion of the step size parameter \( \alpha_i \in [0, 1] \) to become

\[
\xi^{i+1} = \xi^i - \alpha_i \nabla \phi(\xi^i)^{-1}(\phi(\xi^i, U_k) - Y_k). 
\]

The step size is chosen at each iteration, based upon the Armijo criteria to guarantee that the iteration results in a significantly smaller output residual. The evaluation of \( \alpha_i \) at each iteration is most often accomplished via a line search. The drawback associated with this approach is that the guaranteed quadratic convergence rate of the original iterative Newton observer is not sustained, and only asymptotic convergence of the iterative Newton descent approach can be assured[14].

At each time step both iterative forms for the iterative Newton nonlinear observer seek to converge to a state estimate that matches the actual system state with a predefined precision. In general, a minimum number of iterations at each time step. This number [2] will vary according to the degree of nonlinearity of the estimated system.

### III. Newton-Descent Single Step Observer

As the system state and observer estimates evolve, these sequences sweep out a pair of trajectories. One may consider an observer strategy that seeks convergence of the observer trajectory from an arbitrary initial condition to the state trajectory over a finite time interval. At each time step, a singular adjustment in the state estimate is made, based upon the past \( N \) input and output measures, driving the observer estimate closer to the actual system state. The Newton descent single step observer proposed herein considers this approach.

#### A. Nonlinear Systems Observer Algorithm

The Newton descent single step observer is distinguished from the iterative forms by diminution of the latter’s iterative refinement step (12) to a single algebraic relationship.

\[
\hat{x}_{k-N+1|k-N+1} = \hat{x}_{k-N+1|k-N} - \alpha_k L_k (\hat{\phi}_{k|k} - Y_k),
\]

where \( L_k \) is the gradient operator, the step length scaling factor based upon the Newton descent observer step size

\[
\phi_k = \begin{bmatrix}
h(x_{k-N+1}, u_{k-N+1}) \\
h(f(x_{k-N+1}, u_{k-N+1}), u_{k-N+2}) \\
\vdots \\
h(f(...(f(x_{k-N+1}, u_{k-N+1}),...)), u_k)
\end{bmatrix}
\]

Assuming that the system is perfectly modeled, the output measurement trajectory exactly equals the corresponding output prediction

\[
Y_k = \phi(x_{k-N+1}, U_k).
\]
estimated time varying system, such that full column rank, the derivative operator
server at the current time step and equation set:

\[ \hat{\phi}_k = \begin{bmatrix} h(x_{k-N+1}|k-N, u_{k-N+1}) \\ h(f(x_{k-N+1}|k-N, u_{k-N+1}), u_{k-N+2}) \\ \vdots \\ h(f(...(f(x_{k-N+1}|k-N, u_{k-N+1})...), u_k) \end{bmatrix} \] (14)

B. Linear Time Varying Approximation of Nonlinear System

The nonlinear system (1) may be linearized along the trajectories of state and control \((x_k, \hat{x}_k, u_k)\) forming the linear time varying (LTV) system

\[ x_{k+1} = A_k x_k + B_k u_k, \quad y_k = C_k x_k \] (15)

where \(A_k\) and \(B_k\) are the state and control Jacobians of the system equation and \(C_k\) is the state Jacobian of the output equation

\[ A_k = \frac{\partial f}{\partial x}(x_k, u_k), \quad \hat{A}_k = \frac{\partial f}{\partial \hat{x}}(\hat{x}_k, u_k) \]
\[ B_k = \frac{\partial f}{\partial u}(x_k, u_k), \quad \hat{B}_k = \frac{\partial f}{\partial \hat{u}}(\hat{x}_k, u_k) \]
\[ C_k = \frac{\partial h}{\partial x}(x_k, u_k), \quad \hat{C}_k = \frac{\partial h}{\partial \hat{x}}(\hat{x}_k, u_k). \] (16)

In this framework, the Newton descent single step observer at the current time step \(k\) can be represented by the equation set:

\[ \hat{x}_{k-N+1}|k-N = \hat{A}_{k-N} \hat{x}_{k-N}|k-N + \hat{B}_{k-N} u_{k-N}, \] (17)
\[ \hat{x}_{k-N+1}|k-N+1 = \hat{x}_{k-N+1}|k-N - \alpha_k L_k(\hat{\phi}_k - Y_k), \] (18)
\[ \hat{x}_k = \Delta_k \cdot \hat{x}_{k-N+1}|k-N+1 + \Psi_k \cdot U_k. \] (19)

Assuming the gradient of the estimated output vector has full column rank, the derivative operator \(L_k\) may be set equal to its pseudo-inverse.

\[ L_k = \nabla \hat{\phi}_k \] (20)

Define \(\hat{O}_k \in \mathbb{R}^{Np \times N}\) as the extended observability matrix and \(\hat{\Gamma}_k \in \mathbb{R}^{Np \times N_m}\) as the input sequence matrix for the estimated time varying system, such that

\[ \hat{O}_k = \begin{bmatrix} \hat{C}_{k-N+1} \\ \hat{C}_{k-N+2} \hat{A}_{k-N+1} \\ \vdots \\ \hat{C}_{k} \hat{A}_{k-1} \ldots \hat{A}_{k-N+1} \end{bmatrix}, \quad N > n \] (21)

and

\[ \hat{\Gamma}_k = \begin{bmatrix} 0 & \ldots & 0 & 0 \\ \hat{C}_{k-N+2} \hat{B}_{k-N+1} & \ldots & 0 & 0 \\ \hat{C}_{k-N+3} \hat{A}_{k-N+2} \hat{B}_{k-N+1} & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \hat{C}_{k} \hat{A}_{k-1} \ldots \hat{A}_{k-N+2} \hat{B}_{k-N+1} & \ldots & \hat{C}_{k} \hat{B}_{k-1} & 0 \end{bmatrix} \] (22)

Then the estimated system output trajectory for the discrete time system is given by

\[ \hat{\phi}_k = \hat{O}_k \hat{x}_{k-N+1}|k-N + \Gamma_k U_k. \] (23)

The system and input coefficient matrices \(\Delta_k\) and \(\Psi_k\) defining the propagation step of the observer to the current state estimate are given by

\[ \Delta_k = \hat{A}_{k-1} \hat{A}_{k-2} \ldots \hat{A}_{k-N+1} \] (24)
and

\[ \Psi_k^T = \begin{bmatrix} B_k^T \hat{A}_{k-N+1} \hat{A}_{k-N+2} \ldots \hat{A}_{k-1} \\ B_k^T \hat{A}_{k-N+2} \hat{A}_{k-N+3} \ldots \hat{A}_{k-1} \\ \vdots \\ B_k^T \hat{A}_{k-2} \hat{A}_{k-1} \\ B_k^T \hat{A}_{k-1} \\ 0 \end{bmatrix}. \] (25)

C. Linear Time Invariant Systems

The Newton-Descent observer reduces to its simplest form for discrete linear time invariant (LTI) system.

\[ x_{k+1} = A x_k + B u_k, \quad y_k = C x_k, \] (26)

where \(x_k \in \mathbb{R}^n, u_k \in \mathbb{R}^m\) and \(y_k \in \mathbb{R}^p\). The most salient difference brought by the LTI system is the constancy of the system matrices. The constant system and output matrices \((A, B, C)\) represent both the system and the estimated system dynamics.

This characteristic extends to the coefficient matrices associated with the output sequence \(\phi\). Exemplifying this, the extended observability matrix for both the system and the observer based estimated system is given by

\[ O = \hat{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{N-1} \end{bmatrix}, \quad N \geq n. \] (27)

Incorporating these factors, the Newton descent single step observer algorithm for an LTI system becomes

\[ \hat{x}_{k-N+1}|k-N = A \hat{x}_{k-N+1}|k-N + B u_{k-N}, \] (28)
\[ \hat{x}_{k-N+1}|k-N+1 = \hat{x}_{k-N+1}|k-N - \alpha L(\hat{\phi}_k - Y_k), \] (29)
\[ \hat{x}_k = A^{N-1} \hat{x}_{k-N+1}|k-N+1 + \Psi \cdot U_k. \] (30)

IV. NEWTON DESCENT SINGLE STEP OBSERVER STABILITY

The principle motivation for the Newton Descent single step nonlinear observer lies in its potential to provide an observer algorithm whose estimates of system state will reliably converge to the actual system state trajectory over a finite sequence of time steps, and remain stable as the system evolves over time. This behavior requires that the actual system always remain bounded, regardless of stability, and that the observer remain stable for all measured inputs and
outputs of the actual system. In this section the stability of the Newton Descent single step observer is considered.

The development of the stability criteria for the Newton Descent single step observer begins with the consideration of the LTI system. This is followed by a similar development for nonlinear systems, allowing the added complexities of the nonlinear application to be clearly identified.

A. Observer Stability for LTI System

Consider the discrete linear time invariant system without input.

\[ x_{k+1} = Ax_k, \quad y_k = Cx_k \]  

(31)

where \( x_k \in \mathbb{R}^n \) and \( y_k \in \mathbb{R}^p \).

For the Newton Descent single step observer, the convergence of the estimated state trajectory to that of the LTI system can be assured through the selection of a step size \( \alpha \) for the Newton refinement that meets the following stability criteria.

**Theorem 4.1:** Given the discrete time linear time invariant system \( (A, B, C) \) with the spectral radius \( \rho(A) \), an initial state \( x_0 \), and measurable and observable past input and output trajectories of length greater than or equal to \( N \). The Newton Descent single step observer of horizon length \( N \), with an arbitrary estimated initial system state \( \hat{x}_{0|0} \) will stably converge to the system state trajectory if the refinement step size parameter \( \alpha \in (0, 1) \) is selected, such that

\[ (1 - \alpha) \rho(A) < 1 \]  

(32)

is satisfied.

**Proof:** For a linear stable system (\( \rho(A) < 1 \)), the selection of any value \( \alpha \in (0, 1) \) will result in exponential convergence. A full Newton step is always convergent and the estimated trajectory will converge to the actual trajectory in a single time step (iteration). The smaller the value of \( \alpha \), the slower the convergence rate. For an unstable LTI systems (\( \rho(A) \geq 1 \)), the step length \( \alpha \) must exceed a minimal value

\[ \alpha > (\rho(A) - 1)/\rho(A). \]  

(33)

A smaller value for \( \alpha \) will result in divergence of the estimated state trajectory.

Define the state estimation error

\[ \delta x_k = \hat{x}_{k|k} - x_k. \]  

(34)

Then propagation of error between the past estimated and actual system states is given by:

\[ \delta x_{k-N+1} = (I - \alpha LO)A\delta x_{k-N}. \]  

(35)

This system is stable if and only if \( (I - \alpha LO)A \) is a stable matrix, i.e., all eigenvalues lie within the unit circle. If the system is observable, then \( O \) is full column rank, and \( L \) may be chosen as the left inverse of \( O \): \( L = O^T \). In this case, the observer system \( M \) can always be stabilized

\[ M = (I - \alpha LO)A = (1 - \alpha)A \]  

(36)

if \( \alpha \) is chosen sufficiently close to \( 0 \), such that

\[ (1 - \alpha \lambda) \rho(A) < 1 \]  

(37)

where \( \rho(A) \) is the spectral radius of \( A \). This relation forms the stability criteria for all LTI systems, and can be satisfied regardless of the original system stability.

1) Estimate Error Dynamics - LTI System: The dynamics of the current state estimate follows immediately from that refinement stability requirement.

\[ \hat{x}_k - x_k = A^{N-1}\delta x_{k-N+1} \]

\[ = A^{N-1}M A \delta x_{k-N+1} \]

\[ = A^{N-1}M (A^{N-1})^\dagger (\hat{x}_{k-1} - x_{k-1}) \]  

(38)

Since \( A^{N-1}M (A^{N-1})^\dagger \) has the same eigenvalues as \( M \), the stability condition remains the same.

2) The Newton Descent Property - LTI System: In an unconstrained minimization problem

\[ \min_{x \in \mathbb{R}^n} e : \mathbb{R}^n \to \mathbb{R}, \]  

(39)

descent algorithm must execute a global strategy that ensures the chosen iterate decreases the objective function value at each step

\[ e(x_{k+1}) \leq e(x_k). \]  

(40)

In the Newton descent single step observer, this strategy is implemented by reducing the norm of the output residuals.

\[ e(\hat{x}_{k-N|k-N}) = \frac{1}{2} \| \hat{x}_{k-N|k-N} - Y_k \| \]  

(41)

The Newton descent observer algorithm must guarantee at each time step \( k \) that the output measurement residual error decreases.

\[ \| \hat{e}(\hat{x}_{k-N+1|k-N+1}) - Y_k \| \leq \lambda \| \hat{e}(\hat{x}_{k-N+1|k-N}) - Y_k \|. \]  

(42)

To meet this requirement, the value of parameter \( \lambda \) must be demonstrated to be sufficiently small, and its relationship to the Newton refinement step length variable \( \alpha \) determined.

For the LTI system, suppose \( \alpha \) is chosen so that the output error is reduced. Then the objective reduction requirement is given by

\[ \| O \hat{x}_{k-N+1|k-N+1} - Y_k \| \leq \lambda \| O \hat{x}_{k-N|k-N} - Y_k \|. \]  

(43)

Substituting the system output sequence for \( Y_k \) (5) and applying the mean value theorem to each side of the relation, the output residual error may be expressed in terms of the state estimation error.

\[ | \delta x_{k-N+1} | \leq \lambda \| A \| | \delta x_{k-N} | \]  

(44)

If \( \lambda \) is sufficiently small, such that

\[ \lambda \| A \| < 1, \]  

(45)

then the estimation error converges asymptotically to zero.

The Newton refinement step defines the relation of the state estimation error to the step length parameter \( \alpha \)

\[ O \hat{x}_{k-N+1|k-N+1} - Y_k = O(I - \alpha LO)A\delta x_{k-N}. \]  

(46)
Forming the norm of each side, this relationship reduces to
\[ |\delta x_{k-N+1}| = (1 - \alpha)||A|||\delta x_{k-N}| \]  
(47)

Comparing the result with previous (44), the relation between the two parameters is found.
\[ \lambda = 1 - \alpha \]  
(48)

In the numerical solution of nonlinear equations involving the Newton descent algorithm, the assurance of the error contraction is normally achieved through satisfaction of the Armijo condition alone. This mandates the selection of a coefficient \( \alpha_k \in [0, 1] \) such that the error norm is decreased at each iteration. In the Newton descent observer there is the additional constraint that the selected parameter value must be sufficiently close or equal to unity to meet the observer stability relationship (31). For an LTI system, one is always assured of being able to find such an \( \alpha \) regardless of the original system stability. Moreover, the fixed nature of the system matrix \( A \) allows a single value of the step parameter to be selected, i.e., \( \alpha_k = \alpha \) equal to a constant, for any given system and output trajectory.

B. Observer Stability for Nonlinear System

Consider the discrete time nonlinear system (1) at time step \( k \) without input
\[ x_{k+1} = f(x_k), \quad y_k = h(x_k), \quad x_{k-N} = x_0. \]  
(49)

For a nonlinear system, if the system and observer dynamics and observability grammians are bounded, it is possible to define the stability conditions for the Newton refinement such that convergence is assured.

1) The Newton Descent Property - Nonlinear System: Consider the discrete time nonlinear system (49) with measurable and observable past input and output trajectories and the corresponding Newton Descent single step observer of horizon length \( N \), with an arbitrary estimate of the initial system state \( \hat{x}_{0|0} \).

\textbf{Theorem 4.2:} Assume that the system dynamics based upon both the actual and estimated state trajectories are bounded, such that the Lipschitz condition
\[ ||\nabla f(\chi_k)|| \leq \mu_k, \quad \chi \in [x_k, \hat{x}_k] \]  
(50)
is satisfied. Similarly, assume that the actual and estimated systems are observable, such that \( \nabla \phi \) is full rank, and that the extended observability grammian \( \nabla \phi^T \nabla \phi \) is bounded and positive definite.

\[ \nu_{1k} I < \nabla \phi(\chi_k)^T \nabla \phi(\chi_k) < \nu_{2k} I, \quad \chi \in [x_k, \hat{x}_k] \]  
(51)

If a Newton refinement step size \( \alpha_k \) can be found, such that the output residual error factor \( \lambda_k \) satisfies the condition
\[ \frac{\lambda_k^2 \nu_{2k} \mu_k^2}{\nu_{1k}} \leq 1, \]  
(52)
then the estimated state trajectory is convergent to the actual system trajectory.

\textbf{Proof:} Given the initial estimated state \( x_{0|0} \) and the initial propagation of that estimate
\[ x_{1|0} = f(x_{0|0}), \]  
(53)
it is desired that the estimated state be refined such that the state estimation error is reduced
\[ |\hat{x}_{1|1} - x_1| \leq \xi |\hat{x}_{0|0} - x_0| \]  
(54)
where the \( \xi \leq 1 \). If the gradient of the past estimated output horizon \( \nabla \phi(\hat{x}_{1|0}) \) is full column rank, then the state estimate refinement is obtained from the Newton descent observer
\[ \hat{x}_{1|1} = \hat{x}_{1|0} - \alpha_1 \nabla \phi(\hat{x}_{1|0})^T (\phi(\hat{x}_{1|0}) - \phi(x_1)) \]  
(55)
which must satisfy the output measurement error constraint
\[ ||\phi(\hat{x}_{1|1}) - \phi(x_1)||^2 \leq \lambda_1^2 ||\phi(\hat{x}_{1|0}) - \phi(x_1)||^2 \]  
(56)
for \( \lambda_1^2 \leq 1 \).

By assumption the actual and estimate system dynamics and observability grammians are bounded
\[ ||\nabla f|| \leq \mu_1, \quad \mu > 0 \]  
(57)and
\[ \nu_{11} I \leq \nabla \phi^T \nabla \phi \leq \nu_{21} I \]  
(58)
where \( 0 < \nu_{11} < \nu_{21} \). From the mean value theorem, one can write
\[ \delta x_{1|1} \nu_{11} \delta x_1 \leq \delta x_{1|1} \nabla \phi^T \nabla \phi \delta x_1 \]  
and
\[ \delta x_{0|0} \nabla f^T \nabla \phi \delta x_0 \leq \delta x_{0|0} \mu_{11} \nu_{21} \delta x_0. \]

As a result
\[ \delta x_{1|1} \nu_{11} \delta x_1 \leq \lambda_1^2 \delta x_{0|0} \mu_{11} \nu_{21} \delta x_0 \]  
\[ \frac{\lambda_1^2 \mu_k^2 \nu_{21}}{\nu_{1k}} \leq 1 \Rightarrow \lambda_1 \leq \frac{1}{\mu_1} \sqrt{\frac{\nu_{11}}{\nu_{21}}} \]  
(60)

Setting
\[ \xi^2 = \frac{\lambda_1^2 \mu_k^2 \nu_{21}}{\nu_{1k}} \]  
(61)
and taking the square root of both sides, the desired result is obtained. As the Newton descent observer enforces the reduction in the output error at each time step \( k \), by induction, the convergence of the estimated trajectory to the actual state sequence is assured. \( \diamond \)
2) Newton Refinement Step Size Determination: The Newton descent observer requires the determination of the step size scalar \( \alpha_k \) for the refinement step. Consider the following development.

\[
\delta x_1 = \dot{x}_{1|0} - \alpha_1 \nabla \phi(\dot{x}_{1|0})^T (\dot{\phi}(\dot{x}_{1|0}) - \phi(x_1)) - x_1 \quad (62)
\]

As \( \dot{x}_{1|0} = f(\dot{x}_{0|0}) \) and \( x_1 = f(x_0) \), the following relations may be developed using the mean value theorem.

\[
\|f(\dot{x}_{0|0}) - f(x_0)\| = \|\nabla f(x_k)\|\|\dot{x}_{0|0} - x_0\| \quad (63)
\]

\[
\|\dot{\phi}(\dot{x}_{0|0}) - \phi(f(x_0))\| = \|\nabla \phi(x_j)\|\|\nabla f(x_k)\|\|\delta x_0\| \quad (64)
\]

Applying the bounds of (57) and (58) and these relations to the norm of (62), the evolution of the state estimate error is obtained.

\[
|\delta x_1| = \mu_1 (1 - \alpha_1 \frac{\nu_{21}}{\nu_{11}}) |\delta x_0| \quad (65)
\]

From relation (60)

\[
|\delta x_1| \leq \mu_1 \lambda_1 \sqrt{\frac{\nu_{21}}{\nu_{11}}} |\delta x_0| \quad (66)
\]

Comparing (65) and (66), the relation between the step size \( \alpha_1 \) and \( \lambda_1 \) is obtained

\[
\alpha_1 \geq (1 - \frac{1}{\mu_1}) \sqrt{\frac{\nu_{11}}{\nu_{21}}} \quad (67)
\]

For stable discrete time nonlinear systems \( (\mu < 1) \), \( \alpha \) is given by

\[
0 < \alpha_1 \leq 1 \quad (68)
\]

For unstable discrete time systems \( (\mu \geq 1) \), \( \alpha \) must satisfy

\[
\sqrt{\frac{\nu_{11}}{\nu_{21}}} (1 - \frac{1}{\mu_1}) < \alpha_1 < 1 \quad (69)
\]

V. NEWTON DESCENT SINGLE STEP OBSERVER SIMULATIONS

The Newton Descent single step observer is distinguished by its global stability allowing it to provide convergent estimates of system state for stable and unstable, linear and nonlinear systems based upon a finite number of past input and output measures. This performance is evaluated in the following simulations.

A. LTI System Simulations

The performance and stability of the Newton Descent single step algorithm is applied to an unstable discrete LTI system. The stability of the Newton descent observer is evaluated for particular choices of the step size parameter \( \alpha \).

Consider the unstable LTI system without input

\[
A = \begin{bmatrix} 1.5 & 0.3 \\ 0.2 & 0.8 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}. \quad (70)
\]

The eigenvalues of \( A \) are \((1.5772, 0.7228)\) and its spectral radius equals 1.5812. The initial condition for the actual system is \( x_0 = [-20, 5] \); the estimated initial condition for the same system is \([10, -1]\). The stability criteria (33) requires the value of \( \alpha \) to exceed 0.3676.

The simulation of the unstable linear system allows two of the principle characteristics of the Newton descent observer to be clearly seen. The first is the capability of the observer estimate to map the actual state trajectory. For a step length \( \alpha \) selection of 0.5, Fig.(1) displays the asymptotic convergence of state estimate to the actual system trajectory. The monotonic decrease in the output residual error objective evaluation as proscribed by the Newton descent criteria is shown in Fig.(2).

The LTI example also verifies the stability rule (33). Selecting a step length of 0.1 that violates the criteria, the resulting observer state estimates diverge as seen in the trajectory of the state variable Fig.(3).

B. Nonlinear System Simulation

The stability and performance of the Newton descent single step observer is demonstrated for a selected initial condition outside the convergence region of the full step Newton observer.
The Newton single step full state observers are applied to an inverted pendulum-cart system. This is a fourth order system given by the following set of dynamic equations

\[
\begin{align*}
    x_{k+1} &= x_k + hv_k \\
    v_{k+1} &= v_k + h \frac{100\omega_k^2 \sin(\theta_k) - 25g \sin(2\theta_k) + 5}{(2 + 50(1 - \cos^2(\theta_k)))} \\
    \theta_{k+1} &= \theta_k + h\omega_k \\
    \omega_{k+1} &= \omega_k - h \frac{50\omega_k^2 \sin(2\theta_k) - 52g \sin(\theta_k) + 5\cos(\theta_k)}{(4 + 100(1 - \cos^2(\theta_k)))}
\end{align*}
\]

(71)

where \(x_k\) and \(v_k\) are the horizontal position and velocity of the cart, \(\theta_k\) and \(\omega_k\) are the angular position and velocity of the attached pendulum, \(k\) is the time step, and \(h\) is the sample time interval. The inverted pendulum for the simulations described herein, is characterized by an unstable system matrix whose norm exceeds the value of one. A sinusoidal force is applied to the cart to obtain system motion. The Newton single step observers measure the sinusoidal input and the output position and angle for their estimation.

1) Newton Descent Observer: Consider the inverted pendulum system above. The actual system initial state is \(x = [0 \ 1 \ \pi \ 1]^T\). The observer has a horizon of length 4 and it is initialized to the estimated system state \(x = [50 \ 50 \ 50 \ 50]^T\). The step size \(\alpha\) is set equal to 0.77. The sampling rate equals 1 kHz. The run interval is 0.1 second.

The spectral radius of the estimated system quickly converges from a value of 14 to that of the actual system near the value 1.8 as shown in Fig.(4) within the initial ten time steps of observer operation. This behavior characterizes the trajectories of all the estimated state variables.

The observability matrices of both the actual and estimated systems remain full rank and the grammian bounded throughout the time interval, satisfying the observability requirement for the observer.

The maximum value for Newton descent scaling factor \(\lambda_k\) at each time step (63), and the minimum step length requirement (73) for \(\alpha\) in the Newton refinements of the observer are compared against their actual values in Fig.(5) and (6). The actual value of the descent scaling factor is the ratio of the objective residual errors before and after each step. The requirements for \(\lambda\) and \(\alpha\) are met throughout the interval.

2) Full Step Newton Observer: For the full step Newton observer, the step length \(\alpha\) is set equal to unity. All other conditions of the system and estimator are identical to those of the previous simulation.

The difference in observer performance is seen in the failure of the estimated system spectral radius to converge to the actual system trajectory Fig.(7). The characterizes the behavior of the state variable estimates and associated system and observability metrics. While the step length factor \(\alpha\) in Fig.(9) satisfies the necessary minimum condition of (69), the necessary step reduction requirement of (60) is not.
VI. CONCLUSIONS AND FUTURE WORKS

In this paper a new observer is presented for nonlinear discrete time systems. The observer uses a finite horizon, single step, Newton descent algorithm, to estimate the state trajectories of both stable and unstable, linear and nonlinear systems. The stability conditions for this observer have been defined. Its accuracy and performance have been compared with a full step Newton observer for several discrete time systems.

REFERENCES