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PRELIMINARY REPORT
ON
MAGNETOGASDYNAMIC ASPECTS

by

D R Chenoweth
University of Illinois

August 1961

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D. R. Chenoweth
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SUMMARY

This report consists of three parts. The first part is a summary of the fundamentals of magnetofluid-mechanics which were used as a basis for this investigation. The second part is a brief review of the magnetofluidmechanics literature on flow modification that is relevant to this work. The last part presents an analytical investigation of some highly restricted magnetofluidmechanic flows.

The incompressible, electrically conducting flows are treated by continuum fluid mechanics using the assumption of small magnetic Reynolds number. Electrically conducting flow in the close vicinity of the axis of symmetry of a magnetic field similar to that of a single, circular current loop was studied under highly restrictive conditions. This example gave an indication of the effect which the magnetic field of a current loop has on the flow approaching it. Some magnetofluidmechanic vortex flows under the influence of nonuniform axial magnetic fields were also investigated. These flows displayed the dependence of the "effective" viscosity on electromagnetic parameters.

This material was prepared between June 1959 and June 1960.

INTRODUCTION

Although the subject of magnetofluidmechanics is still comparatively new to the aeronautical and thermodynamic fields, it has developed into an area of research with an extremely broad scope. It has brought into contact several fields which were not closely related before. To accurately describe the phenomenon which is present when an electrically conducting fluid is moving through electromagnetic fields represents an almost unprecedented mathematical challenge. The subject of magnetofluidmechanics has been described by many as an ideal playground for mathematicians. To an engineer or a physicist, it represents an obstacle both theoretically and experimentally. The stage was quickly reached where conventional methods were not sufficient to produce the desired results. But while new tools are being developed, the old methods must be crudely applied to gain at least a partial understanding of the problems at hand. Most of the results obtained in this field thus far have had a more limited range of applicability than has existed in fluid dynamics in the past. In fact, any result which is even remotely connected with reality can be regarded as useful, because to adopt an all-or-nothing attitude inevitably results in nothing. Many possible applications of magnetofluidmechanics have been proposed as a result of the research which has already taken place, and it is only natural to believe that a thorough knowledge of the subject would lead to many more practical uses of the complicated interactions which can exist.

The purpose of this work is to study the interactions which appear when some common magnetic field configurations are applied to a plasma column. The channeling effect which an axisymmetric magnetic field has on electrically conducting flow through the field is of particular interest in this investigation.

It is unfortunate that a text book which coherently summarizes the information and concepts which have been found useful in magnetofluidmechanic studies has not as yet appeared. Hence, one who has not been connected with the field since its rapid growth began has no choice but to survey the literature for bits of information which might be useful in the particular investigation being undertaken. Such a large body of literature has appeared on all phases of magnetofluidmechanics that a survey now amounts to considerable work. The first portions of this report are the result of the survey conducted in connection with this investigation.

The latter portions of this report are devoted to the study of some axisymmetric flows under the influence of axisymmetric magnetic fields. The electrically conducting fluid is assumed to be incompressible and to have small magnetic Reynolds number. In some cases, it is necessary to impose several additional restrictions in order to obtain explicit results. Since these flows are treated using the equations of continuum fluid mechanics, the results have a limited range of validity. Nevertheless, several interesting features of magnetofluidmechanic flow are displayed by the simple expressions obtained here. This work also illustrates how much one must sacrifice in the interest of obtaining analytical results.

NOMENCLATURE

a	"inflow-gradient" ⁷⁴
A	dimensionless parameter, $\delta \frac{R_e}{R_m} = \delta \frac{\eta}{\nu}$
$\bar{B}$	magnetic induction vector, $\mu_e \bar{H}$
C	dimensionless flow modification parameter, $C(H_o, \eta, U_o, R, C_1)$
C_j	integration constants ($j = 0, 1, 2, 3$)
$\bar{E}$	electric field intensity vector
$\bar{H}$	magnetic field intensity vector
H_o	characteristic magnetic field intensity
$\bar{J}$	current density vector
K	parameter with reciprocal length squared dimensions, $\frac{\beta}{R_o^2} \frac{R_e}{R_m}$
ℓ	some characteristic length
p	fluid static pressure
p_B	magnetic pressure, $\frac{\mu_e H_o^2}{8\pi}$
P	integration constant with dimensions of pressure
q	characteristic dynamic pressure, $\frac{1}{2} \rho_o U_o^2$
Q	dimensionless interaction parameter, $\frac{p_B}{q} B_m = \frac{\mu_e H_o^2 \ell}{\rho_o \eta U_o}$
R	radius of circular current loop
R_o	reference radius
R_e	fluid Reynolds number, $\frac{U_o \ell}{\nu}$
R_m	magnetic Reynolds number, $\frac{U_o \ell}{\eta}$
t	time
$\bar{U}$	fluid velocity vector

NOMENCLATURE (cont)

U_o	characteristic velocity
V_o	characteristic tangential velocity, $\frac{\Gamma}{R_o}$
α	dimensionless parameter, $A + 2$
β	dimensionless magnetic field distribution parameter
Γ	vortex strength parameter
δ	dimensionless magnetic field distribution parameter
η	magnetic diffusivity, $\frac{1}{\sigma\mu_e}$ (magnetic viscosity)
λ	dimensionless parameter, $\beta \frac{R_e}{R_m} = \beta \frac{\eta}{\nu}$
Λ	indicates a measure of ratio $\frac{p_e}{q}$, $C_1 \frac{H_o}{U_o}$
μ	absolute viscosity
ν	fluid kinematic viscosity, $\frac{\mu}{\rho_o}$
ρ_o	fluid density
σ	fluid electrical conductivity
Φ	integration constant labeling magnetic lines of force
Ψ	integration constant used as a "shape" function or used in two-dimensional case, to label streamlines
ω^2	parameter characterizing the size of the current loop

Cylindrical coordinate notation:

	Radial	Tangential	Axial
Coord. direction	r	ϕ	z
Velocity comp.	u	v	w
Mag. field comp.	H_r	H_ϕ	H_z

PRELIMINARY REPORT ON MAGNETOGASDYNAMIC ASPECTS

I. INTRODUCTION TO MAGNETOFLUIDMECHANICS INVESTIGATION

A. History

The early development of MFM (magnetofluidmechanics) was slow and sporadic because of the lack of practical applications to create an incentive for research in this area. However, in recent years, because of the many proposed engineering applications of MFM in aeronautical, thermonuclear, and astrophysics fields, considerable interest has developed; and MFM is now one of the most rapidly expanding fields of research.

The interaction between electrically conducting media and electromagnetic fields has been of interest for some time to astronomers working in the field of cosmical physics.^{1, 2} The major factor in the development of the formal basis for MFM was the realization by astrophysicists that conducting matter and magnetic fields are commonplace in the universe. The importance of the problem was evident because celestial magnetic fields would affect the motion of the charged particles in space and under certain conditions electromagnetic fields might even be much stronger than gravitational fields. It was thus determined, among other things, that electromagnetic fields play an important role in the features of the sun.

Astrophysicists had barely developed the cosmic electrodynamic theory of clouds of charged matter, when they found that the capability of launching bodies through this media existed. The body could, of course, contain a means of applying an electromagnetic field to the charged matter which increases the complexity of the problem. The importance of applications of MFM begins to appear at this point, since it is obvious that an environment consisting of charged particles, among other things, might be obtained in the flow behind the bow shock wave of a high-speed vehicle. Therefore, it is also conceivable that a similar environment might be generated in the internal flow of a test facility or a propulsion device of some type.

B. Applications

Although many investigations in MFM and plasma physics are fundamental in nature and are thus useful to all MFM engineering applications, some are quite specialized and restricted in scope to particular engineering applications. The aeronautical and thermonuclear applications of MFM are often subdivided into the categories of flow modification, containment, and propulsion.³⁻⁶ These are then further subdivided into other more specialized problems.

1. Flow Modification

For example, flow modification is usually separated into internal and external flow. The primary reason for making this subdivision is that problems in these two areas are usually given quite different analytical treatment, due to different mechanisms of plasma generation and the different boundary conditions required; the desired results are in many respects very nearly the same. For instance, the possibility of using electromagnetic fields to prevent a hot fluid from coming in contact with a wall (internal or external), which it would destroy through excessive heat transfer rates, is one of the most interesting applications of MFM.^{3, 6} The extent to which this "fluid suspension" process can be accomplished is, of course, primarily a function of the fluid electrical conductivity, the electromagnetic field strength, the

velocity of the fluid, and the density of the fluid. If the interaction between a magnetic field and a conducting fluid is very strong, the fluid will tend to follow the magnetic lines of force quite closely and in the limit the magnetic lines of force will act the same as invisible solid boundaries. Also skin friction (internal or external) may be reduced by judicious use of a magnetic field either to inhibit transition to turbulent flow or to alter the boundary-layer profile.^{3, 6}

In general, the net effect of such a magnetic field applied to either an external or an internal flow is an increase in vehicle drag or a decrease in thrust obtainable from a propulsion unit, respectively, because of the interference of the magnetic field with the flow. Since an increase in drag is easy to obtain, the possibility of using this effect to reduce the speed of a re-entry body by placing an electromagnet in the nose of the vehicle is interesting. However, since the drag due to the magnetic force is proportional to the relative flow velocity, it can be more or less expected that attempts to obtain large drag increases in the low-velocity region near the nose of a vehicle would not be too successful. The use of magnetic drag skirts at some distance from the nose looks more promising. Lift control through magnetic fields also offers an interesting possibility.⁶

Unfortunately, several additional problems arise when MFM effects are used. The system generating the magnetic field will probably need to be cooled, since its parts will dissipate power. Also, the fluid may be hot enough to radiate sufficient energy so that a sizable quantity of heat will be transferred even though the hot fluid is not in contact with the surface. In addition, the weight of the unit required to generate a magnetic field of sufficient strength may be too large to consider, or of such size that any advantages are nullified so far as application to missiles is concerned.

The first practical application of MFM was the design of a magnetic pump to place a conducting fluid in motion. Following this first application, there was considerable work done on electromagnetic flowmeters. Inspired by the degree of success which the early inventors enjoyed, investigators started new analytical and experimental studies of the interaction mechanisms which could be used to an advantage in proposed applications. Since then Poiseuille,^{7, 8} Couette,⁹ Stokes,^{10, 11} and some one-dimensional^{7, 8, 12-14} MFM flows have been theoretically investigated with considerable success. The flat plate and blunt body (stagnation point) MFM flows have also been investigated with an equal amount of success. However, these basic flows were all investigated for very special types of electromagnetic field configurations, and even then closed solutions were not obtained in many cases.

It has been found, in most cases, that one cannot get worth-while changes in drag or significant reduction of heat transfer rates without unrealistic magnetic field strengths, extremely high temperatures, or artificial increases in gas conductivity by seeding techniques.

Accurate information on the electrical conductivity of gases is indeed limited.¹⁵⁻²⁰ In addition, the air around a missile can become a conductor as a result of several mechanisms. For instance, the energy imparted to the air by the missile may be great enough to ionize it, the missile may be in the ionosphere where the air is ionized by radiation from the sun, or both. To further complicate the picture, ablation from the surface of the missile may add foreign material and charged particles to the boundary layer. Several special devices have also been suggested to increase the conductivity of the flow around a missile. There are, of course, other factors affecting ionization processes which are not fully understood; for example, the flow is sure to be in a non-equilibrium state, and methods for predicting various recombination rates which

are necessary to describe such a flow are inadequate. It is, therefore, difficult to estimate the flow conditions around a high-speed missile.

2. Containment

The problem of containment has perhaps been studied with more vigor than any other problem, mainly because of the hopes of releasing the energy of nuclear fusion and utilizing electromagnetic methods to contain extremely hot ionized gases.^{3,4} The question of how to contain a plasma at 10^8 °K is a challenging one, and some type of field may be the answer. For this purpose, the only two fields which seem reasonable are electric and magnetic fields. However, it appears that electric fields will not be very successful since a field which contains particles of one charge will not contain particles of the other type. A magnetic field seems to be the most promising possibility. Several methods of electromagnetically containing extremely hot ionized gases are under study.

The "magnetic mirror" concept, which is based on the principle that charged particles moving in a magnetic field tend to be reflected from regions of higher-than-average field strength (in other words, a particle moves in a magnetic field so as to keep its magnetic moment approximately constant), has received considerable attention.^{4,21} A magnetic mirror can be used to cap the ends of a solenoid, but a more subtle approach is to wrap the field lines around into a closed toroidal shape. This has objections because the magnetic field in a toroidal geometry is necessarily nonuniform. This trouble may be overcome by stretching and twisting the toroid toward a sort of figure-eight shape known as a "stellarator".²² The stellarator concept has been the subject of a great deal of analytical and experimental investigation. The two devices described so far have at least one element in common in that they attempt to confine a plasma by application of externally generated magnetic fields.

A third method for confinement of a plasma differs from those described above in that an internal magnetic field, produced by currents in the plasma, is used.^{4,23,24} The confinement mechanism is present because, owing to the magnetic attraction of parallel currents, there is the tendency for the discharge in a plasma to contract under the action of its self-generated magnetic fields. This phenomenon is commonly called the "pinch effect."

It has also been proposed that the plasma could be contained by the pressure of electromagnetic radiation.^{25,26}

All of the schemes discussed have been plagued by instabilities and high energy losses, which have severely limited efforts to attain sufficiently high temperatures to support continuous thermonuclear reactions. A charged particle moving in a magnetic field tends to circle about a magnetic line of force with the radius of the path of the particle decreasing with increasing magnetic field strength. The gas conditions necessary to produce a fusion reaction, if the magnetic field is strong enough to contain the gas, are such that the mean free path of the gas particles is much longer than the radius of the orbit for a charged particle in the magnetic field.⁴

When the pinch effect is considered from the idealized standpoint of steady-state conditions and constant current density, it appears at first to be the best method, but no steady-state pinch has yet been accomplished. One type of instability which arises with plasma columns is the so-called "kink instability," in which the resultant magnetic pressure is greater on the inside of the kink and a net force acts in such a direction as to increase the bend.⁴ Therefore, once a slight kink develops, it will grow in size until the discharge

breaks up and the plasma scatters. Another type of instability is "sausage instability," which is characterized by alternating large and small diameters of the plasma column.⁴ This instability tends to grow even faster than kink instability. Kink instabilities of long wave length can be eliminated by encasing the pinch within a conducting shell, and a longitudinal (axial) magnetic field helps to eliminate kink instabilities of short wave length.²⁷ Therefore, a region of complete stability can sometimes be obtained by using both of these methods.

The economics of the stabilized pinch are very favorable compared to those of the stellarator or magnetic mirror methods, chiefly because of the efficient way in which the magnetic field is produced. Not only is a plasma a better conductor than copper at thermonuclear temperatures but at these high temperatures the peak magnetic field occurs at the plasma surface, where it is needed, rather than in the coils. However, if the pinch cannot be stabilized, the situation is much less favorable.

Another proposed method, called the "picket fence," uses opposing windings.⁴ The major flaw here is that very large particle leaks occur at the junctions between opposing windings. To remedy this flaw, it has been proposed that one could use a "moving picket fence," i. e., interchange the direction of currents in the windings at high frequency. It is thought that this method has better stability than other systems.

The instability of plasma is not a problem which is encountered only in connection with containment, but it has more serious consequences there, perhaps, than it has in flow modification or propulsion problems.²⁸⁻³¹

3. Propulsion

There also appears the possibility of using MFM phenomena to produce or augment thrust. There are many proposed electromagnetic propulsion systems.³²⁻³⁵ Electrostatic acceleration of charged particles, say ions, is of considerable interest at present. The main problem here is the production of charged particles of one type and elimination of those of the other type (i. e., neutralization of the space charge). Another possibility which is receiving a great deal of attention is the acceleration of plasma. Two of the ways in which this can be accomplished is by changing or collapsing magnetic fields and by the interaction of electric current and a steady magnetic field. Since heat transfer rates are a problem, electromagnetic methods would probably be required to prevent excessive damage to the components of the propulsion unit, and these fields would interfere with the acceleration process and lower the thrust obtainable.

Most proposed methods of electromagnetic propulsion are characterized by the requirement of large amounts of electrical energy and low net thrust, but these disadvantages are offset somewhat by the very high exhaust velocities and the resulting high specific impulses obtainable.

It is also clear that a plasma accelerator, with electric potential reversed, becomes a generator of electric power; thus, there appears to be a possibility of using these effects to extract electric power directly from a flowing, conducting fluid.³⁶ A great deal of effort is being made to turn this proposal into a reality, and considerable progress is being made.

C. Fundamental Physics of MFM

1. Particle Physics

There are several approaches which can be taken to the theoretical treatment of ionized gases. The oldest and most fully developed of these approaches is that of particle physics. Considerable knowledge can

be gained by studying single-particle motion in an externally applied electromagnetic field, with no account being taken of collisions or plasma effects, i. e. , the influence of one particle on another through induced electric and magnetic fields. The study of individual particles frequently gives insight into the behavior of an ionized gas.²¹ However, such a study gives only qualitative results and does not give any quantitative information on specific gas properties because collisions and plasma effects must be taken into account.

In empty space, a charged particle under the influence of a magnetic field tends to move in a helical path about a magnetic line of force. It experiences a force directed toward the magnetic field line proportional to the vector product of its velocity vector and the local magnetic field vector.²¹ The radius of this helix is known as the radius of gyration, and the angular frequency at which this gyration takes place is known as the cyclotron or Larmor frequency. The electrons and ions of an ionized gas would perform this same circling motion if it were not for their collisions with other particles. But because of collisions, the particles begin to drift across the magnetic field. The nonisotropic nature of this motion caused by collisions gives rise to the body force, which is the vector product of the current density vector and the local magnetic field vector. In a rarefied gas, the circling motion becomes more important because of the greater time between collisions, and the drift of electrons in the direction of the electric field is thereby hindered. If the electrons can flow in the direction perpendicular to both fields, they do so; and they produce what is called the "Hall current." The electrical conductivity is greatly affected by the gas density and magnetic field strength. For strong fields the conductivity would effectively disappear at densities corresponding to altitudes of about 200,000 feet because of "ion slip," which will be discussed later. Fortunately, the conductivity is partially restored if no Hall current can flow; instead, a Hall potential is set up, and electrons again drift in the direction of the original field. Thus, there are possibilities of maintaining conductivity at high altitudes. This complicated phenomenon which causes serious loss of conductivity at low densities in a strong magnetic field poses a very complex problem in most practical cases since it depends on the detail of the geometry involved.^{6, 8, 30}

Fundamentally, the conductivity arises from the forces that act on charged particles in the electric field. Such electrostatic forces act in opposite directions on electrons and ions; the electrons, being more mobile, account for most of the conductivity. In a dense gas, the electrons experience collisions with atoms and ions, but an electron drift is set up which constitutes the current. Particle drifts in an ionized gas can also be caused by an unsteady magnetic field, an inhomogeneous magnetic field, or a gravitational field in addition to an electric field.

2. Flow Classification

When the various experimental research programs in MFM were started a few years ago, the aerodynamicist was, and still is to some extent, faced with the problem of producing and maintaining high electrical conductivity in gases. "Seeding" with something such as potassium ($\approx 0.1\%K$) is often suggested, but even then air does not conduct very well compared to metals at temperatures commonly encountered in aeronautical applications. However, arc heaters and many of the recently developed high-energy, chemically driven, and electromagnetic shock tubes promise to be excellent test facilities for simulating MFM flow in the high-speed flight regime.³⁷⁻⁴⁴ Generally, the fluids considered in aeronautical applications differ from those in thermonuclear applications in that the electrical conductivity of the fluid is of a different order of magnitude in the two problems. In many aeronautical applications, the currents flowing in the gas are very small compared to the current flowing in external wires. Therefore, in many cases the local magnetic field

in the flow field is essentially determined by the externally applied magnetic field. A rough indication of the magnitude of the portion of the local magnetic field due to gas currents is obtained by multiplying the magnetic Reynolds number by the applied magnetic field. Hence, one may at times neglect one or the other of these magnetic fields.

a. Assumptions usually made

The majority of the analyses in MFM have been carried out under the assumption that the electrical conductivity (or magnetic Reynolds number) of the fluid is either very small or very large. The effects in the intermediate range are generally not calculated even though they are actually the most important.

The assumption that the magnetic Reynolds number is very small is the same as assuming that the magnetic and electric fields induced by the fluid motion are negligible compared to the applied magnetic field which generally has its source external to the flow field. The fluid-dynamic and electromagnetic equations are then usually solved separately because, to this order, the magnetic field is assumed to be known throughout the flow field and satisfies the Laplace equation. The problem of finding a solution is therefore greatly simplified but by no means assured.²⁸

When the magnetic Reynolds number is taken to be infinite, the amount of simplification does not seem to be so great. The two sets of equations separate only in special cases. The magnetic field configuration is not known at the outset because the lines of force move with the fluid; the process is one in which the magnetic lines of force can be regarded as being frozen into the moving fluid.^{1, 28} The fluid can flow along but not across the magnetic lines of force in this limit. Quite often, then, the number of lines of force passing through a unit area is proportional to the density of the gas, since the magnetic field is compressed with it.

The magnetic field may be a function of time as well as space coordinates, and a steady configuration is achieved only when the streamlines and magnetic field lines become aligned as time increases indefinitely. If the time duration of the cycle (i. e., a pulse magnetic field) is so short and the velocity of the gas so high that the magnetic field cannot diffuse through the material, then the magnetic Reynolds number is effectively infinite.

The value of investigations at very small or very large magnetic Reynolds number is sometimes questioned; however, the characteristics of a given MFM problem can be bracketed by analyses made at these extremes, since solutions for arbitrary values of the fluid electrical conductivity are seldom found. It is often quite misleading to use the magnitude of the magnetic Reynolds number to imply the magnitude of the electrical conductivity of a fluid, because the actual MFM flow in question may be either low- or high-speed flow. Nevertheless, the two are sometimes used quite interchangeably in the literature, and thus, a certain amount of looseness may also be observed here. More correctly, the magnitude of the magnetic Reynolds number should be regarded as a measure of the potential available for interactions with applied fields and as a tool for estimating the magnitude of the induced fields in the fluid due to its motion.

b. Useful parameters

It has already been mentioned that the magnetic field strength has considerable effect on the electrical conductivity of a gas. For weak magnetic fields, the current is parallel to the electric field and is proportional to it; the proportionality constant is the ordinary electrical conductivity. The effect of this parameter

is taken into account in terms of the magnetic Reynolds number. At higher magnetic field intensities, the electrons may make several cyclotron orbits between collisions. In this case, the electrons tend to drift in a direction perpendicular to both the electric and magnetic field. Hence, as it has already been pointed out, a current perpendicular to the electric field (Hall current) is produced. It is thus obvious that another parameter is needed to take account of this phenomenon. The parameter is called the Hall coefficient, and is the product of the cyclotron frequency of an electron and the mean free time between collisions; thus, the Hall coefficient is equal to the tangent of the angle between the current and the electric field.⁶ At very high magnetic field intensities, the electrons and ions in the gas can be held so strongly by the magnetic field that there is a relative drift between the ionized and un-ionized components of the gas. This so-called "ion slip" tends to reduce the effective conductivity of the gas, as the emf in the gas depends on the velocity of the ionized components.⁶

It is sometimes convenient and physically significant to use a parameter which describes more closely the strength of the interaction between the magnetic field and the fluid motion, thus giving a more accurate measure of the flow modification. Such a parameter is usually constructed by taking the ratio of the magnetic pressure to the dynamic pressure times the magnetic Reynolds number. Hence, for small values of this interaction parameter, the flow is only barely disturbed by the magnetic field, while for large values of the parameter, the flow will tend to follow the magnetic lines of force more closely. In other words, a typical ratio of the electric body force to the inertial force is given by this interaction parameter. A typical ratio of the electric body force to the viscous force, the square root of which is commonly called the Hartmann number, is given by the product of the interaction parameter with the fluid Reynolds number. Therefore, for large values of these ratios, one can justify neglecting the inertia or the viscous force terms in some analytical investigations.

Another parameter which is useful in some investigations is the ratio of the flow speed to the speed of propagation of hydromagnetic waves, called Alfven waves. This parameter is called the Alfven number, or sometimes it is referred to as the magnetic Mach number. The speed of propagation is usually termed the Alfven velocity, and it is the velocity of propagation of perturbations in the magnetic field strength along the magnetic field lines of force in an inviscid fluid with infinite electrical conductivity.^{5, 28} In addition to hydromagnetic waves, there are two other types of waves, electrostatic and electromagnetic, which are important in the study of oscillatory motions in a plasma.²¹

A quantity which is sometimes used in discussions of ionized gases is the "Debye shielding distance"; it is simply the relaxation distance which an electron can be expected to diffuse into a plasma. In fact, it is sometimes used to define a plasma, an ionized gas being called a plasma only if the Debye shielding distance is small compared to other characteristic lengths of interest.²¹ Quoting Spitzer:²¹

In plasmas produced within the laboratory, the Debye shielding distance is important in that it gives roughly the thickness of the sheath which develops whenever the plasma is in contact with a solid surface. Without such a sheath a plasma, in the absence of a magnetic field, would lose electrons much more rapidly than positive ions, owing to the greater electron velocity.

D. Basis of Theory

Because some of the outer shell electrons are detached in a gas at high temperature and the gas is thus partially ionized, the gas becomes an electric conductor, and the possibility of interactions with electromagnetic fields is admitted. This possibility has had the effect of uniting two branches of physics, gas dynamics and electromagnetic theory. Before the development of MFM these subjects had few points of contact other than the similarity of some of their mathematical formalism. However, the simplicity of electromagnetic theory, which depends upon its linearity, disappears when the theories are related, because of the essential nonlinearity of gas dynamics; and even more nonlinear terms are encountered.

1. Methods of Treatment

If a continuum point of view is emphasized, the theory is applicable only to liquids and to dense ionized gases. In addition, continuum theory is valid only when electrical neutrality is preserved in the gas and the mass stress-tensor is isotropic. Whenever these criteria break down, i. e., the gas is at low pressure or non-equilibrium conditions exist, the continuum theory of gases is usually replaced by the kinetic theory of gases. The main objection to treating plasma motion with KTG is that the long-range characteristics of the Coulomb interaction forces between particles are neglected. Unfortunately, long-range forces play a very important role in the dynamics of a plasma. There have been efforts to alleviate this problem using a quantum statistical mechanics approach, or what might here be called quantum MFM. However, the quantum MFM theory has not as yet reached the stage where it is useful and dependable.

2. Restriction to Continuum Flow

One must learn to handle highly simplified cases before one can hope to treat more realistic problems. It is in this frame of mind that the following work has been done. The discussions and investigations which follow will be confined to continuum flow theory and therefore, will have a limited range of application. Probably the weakest assumption of continuum theory, with respect to describing plasma motion, is that of equilibrium flow, because this condition seldom exists. Although in some cases this assumption is not a bad approximation, it should be pointed out that in general a basic property of plasma is its tendency toward electrical neutrality. For example, if the number of electrons and the number of ions per unit volume differ considerably, electrostatic forces yield a potential energy per particle greater than the thermal energy. Unless some special mechanism is present to support it, the charged particles will move in such a way as to reduce the potential energy and restore electrical neutrality.

a. Mathematical difficulty

The MFM equations relating the electric and magnetic fields in a fluid conductor to the local fluid velocity, pressure, and temperature present a problem of considerable mathematical complexity. This complexity arises mainly from the non-linearity of the equations. There are three basic types of non-linear terms. The most familiar of these non-linear terms, $(\bar{\mathbf{U}} \cdot \nabla) \bar{\mathbf{U}}$, is the inertia force term present as a carryover from the non-magnetic momentum equation. An additional non-linear term, $\bar{\mathbf{J}} \times \bar{\mathbf{H}}$, or for the steady case, $\bar{\mathbf{H}} \times \text{curl } \bar{\mathbf{H}}$, is added to the momentum equation for MFM to take account of the electromagnetic forces. The electric body force, $\bar{\mathbf{J}} \times \bar{\mathbf{H}}$, is the force which makes an electric motor operate, and thus one has considerable control over it. The third non-linear term, $\text{curl} (\bar{\mathbf{U}} \times \bar{\mathbf{H}})$, is present in the magnetic diffusivity equation and arises from the electromagnetic induction. The induced electromotive force, $\bar{\mathbf{U}} \times \bar{\mathbf{H}}$, is the force which makes a generator function. Hence, one might say that the mathematical difficulties are threefold compared to the already complex non-magnetic problem.⁸

Moreover, an additional contribution, J^2/σ , the Joule heating term, must be added to the energy equation to take account of the heating due to the electrical resistance of the fluid.⁴⁶ The main difficulty which arises with this term is that we must know the electric field, the magnetic field, and the fluid velocity in order to evaluate the current density. The difficulty in determining the electrical conductivity of a plasma in motion, a major problem in itself, has already been discussed.

Another problem which causes considerable trouble in accurately describing a plasma in motion stems from the large variations in fluid properties which are due to the extremely large temperature gradients and the resulting density gradients usually present in a plasma. The fluid properties (i. e., viscosity, heat conductivity, and electrical conductivity) could be reasonably treated as constants in low temperature flows. The complications which are added by considering these properties as variables are unmanageable without considering the effects of MFM. Therefore, the models which are considered theoretically in MFM are usually much more idealized than those considered in fluid dynamics in the past.

b. Methods of obtaining results

When one is faced with the task of solving a given MFM problem analytically, the first question which must be answered is what assumptions should be made in order to find with a reasonable amount of effort a solution which will be useful in obtaining a better understanding of the original problem.

The geometric configurations of the flow field and the magnetic field are both subject to simplifying restrictions. It has already been pointed out that the flow properties are necessarily subject to highly restrictive assumptions. In other words, approximations may be made in either the fluid dynamic or the electromagnetic equations or both, or in their boundary conditions. In general, the geometry of the flow field and of the applied electromagnetic field should be kept simple enough so that the influence of the additional assumptions is not masked by the complexity of the field configurations. Obviously, the key to finding complete solutions lies in choosing simple field configurations with boundaries designed so that almost all of the interesting quantities are known a priori.

Usually after one has applied all of the restrictions which it is felt can reasonably be made for a particular MFM problem, there still remains a set of non-linear partial differential equations which do not lend themselves to quick and easy extraction of useful results. It is often a major undertaking to obtain any useful results at all.

II. BRIEF REVIEW OF RELEVANT LITERATURE

This investigation is primarily concerned with flow modification problems and the useful results which can be derived from MFM flow modification. As has already been pointed out, it is convenient to separate flow modification into external or internal flow. These can be further subdivided according to the particular flow field, electromagnetic field, boundary conditions, and other restrictive assumptions which are made.

A. External Flow

1. Flat-Plate

The flow of an electrically conducting fluid over a flat-plate has received a great deal of attention and study with regard to the effects which an applied magnetic field has on skin friction, total drag, heat-transfer, and transition to turbulent flow. These studies have been very important because the flat-plate configuration

is one of the few configurations which can be treated with some degree of rigor.⁴⁸ In achieving to some extent a rigorous treatment of the flat-plate problem, it was necessary to consider the incompressible boundary-layer equations and a uniform parallel magnetic field transverse to the plate. In general, skin friction and heat-transfer can be reduced by this method at the expense of an increase in total drag. In some cases, the transverse magnetic field may inhibit transition to turbulent flow; however, situations may arise in which it destabilizes the boundary-layer flow. The special case where the conducting fluid flow was perpendicular to an infinite flat-plate gave stagnation-point results which were useful in preliminary studies of MFM flow near the stagnation-point on the blunt nose of a re-entry body.^{49, 50} The promising results obtained for this problem lead to more elaborate investigations of MFM flow about the forward regions of a re-entry body.

The flat-plate MFM boundary-layer investigation has been extended to a study of compressible flow with a non-uniform parallel magnetic field transverse to both the plate and the flow direction. This study uses numerical methods and assumes the magnetic Reynolds number to be small.⁵¹

2. Hemispherical Blunt Body

The next, seemingly natural, step which was taken in the study of MFM stagnation-point flow was the investigation of the flow about a hemispherical blunt nose with a uniform "radial" magnetic field.⁵²⁻⁵⁵ The flow was considered to be hypersonic, incompressible, and inviscid with small magnetic Reynolds number. Study of the behavior of the bow shock wave as a function of magnetic field strength was especially interesting. The shock front was also assumed to be hemispherical; it was found that, at least theoretically, this "idealized" magnetic field configuration affected the bow shock wave detachment distance, commonly called the standoff distance. However, it was also concluded that relatively large values of the interaction parameter would be required to obtain the desired results.

In an effort to provide a more realistic magnetic field configuration, the problem just discussed was investigated using a magnetic dipole located at the center of curvature of the blunt hypersonic body and its shock front.^{56, 57} The result was increases in stand-off distance and pressure relief on the body with increasing interaction parameter. However, it was noted that both of these effects were relatively independent of the magnetic Reynolds number. Likewise, the heat-transfer rate was found to be almost independent of the magnetic Reynolds number, although it depended strongly on the interaction parameter.

Another paper discloses that a "frozen region" around a magnetic dipole in hypersonic flow prevents the conducting fluid from penetrating into the immediate neighborhood of the dipole.⁵⁸ The flow was considered to be two-dimensional and to have very large magnetic Reynolds number. The existence of such a frozen region is significant since it might serve to protect a re-entry body from the high-temperature flow behind the bow shock wave. The problem is whether or not the hydrodynamic pressure outside the frozen region can be balanced by the hydrodynamic plus the magnetic pressure inside the region. It was concluded that the "frozen region" could exist even in hypersonic flow, but that it might be necessary to use the so-called "seeding technique" to produce high enough electrical conductivity to give the desired results. Some work has also been done on magnetic boundary layers and magnetic detachment of MFM flow.⁵⁹⁻⁶¹

It has been shown that a magnetic dipole can be used as a rough approximation of the external magnetic field variation of certain solenoids. This fact admits the possibility of attempting experimental verification of the theoretical results obtained using a magnetic dipole; at least one such attempt has been made using an electromagnetic shock tube as a means of producing hypersonic ionized air flow.⁶² Reasonable agreement

between theory and experiment was obtained, considering the vast differences which existed between the theoretical and laboratory models. It was found that for an interaction parameter of 69, the bow shock wave stand-off distance could be increased as much as 7.5 times; this was the maximum value of the interaction parameter obtained in the tests.

B. Internal Flow

1. Compression

Many plasma generating devices inherently operate at relatively low pressures. Since it is sometimes desirable to work with a plasma at considerably higher pressures, it is interesting to investigate some methods of compressing plasma by means of magnetic fields. It is obvious from what has already been said that a plasma will be compressed if it is being forced (i. e., accelerated with a pulsed magnetic field configuration) to flow into a region of higher magnetic field strength. This is true because the lines of force will be converging, and plasma motion across these lines of force is resisted; thus a retarding force in the direction of the plasma motion is present in addition to the compressing force.

Although one must, in general, take account of collisions and plasma effects, it is still instructive to look at this problem from the viewpoint of particle physics. Evidence indicates that the behavior of plasmas in which the energy density is low compared to that of the magnetic fields can be predicted from single-particle adiabatic theory.⁶³

When a charged particle is orbiting about a magnetic line of force, the constancy of the magnetic moment indicates that the radius of gyration is inversely proportional to the square-root of the magnetic field intensity. However, even for single-particle motion, the constancy of the magnetic moment is approximately true only if the magnetic field intensity changes slowly in space and time. This means that the magnetic field must be approximately constant over distances of the order of the radius of gyration and for time intervals of the order of the Larmor period.²¹ Actually, even for a plasma in a magnetic field which is constant over space and time, the magnetic moment is changed as a result of collisions and plasma effects.

Regardless of these implications, some elaborate experiments have been conducted on low energy density plasmas using pulsed, high-intensity magnetic fields, and the results agreed reasonably well with theory, assuming the magnetic moment to be constant.

In one experiment, several solenoids of decreasing diameter were joined end to end about a common axis.⁶³ The magnetic field strength was increased for each solenoid as its diameter decreased. The magnetic field of these solenoids served as a guide for the plasma as it was moved through the vacuum chamber in which the experiment was performed. The plasma was injected at the end of the largest solenoid and then forced along the guiding magnetic field to the small end by a series of pulsed magnetic fields.

From single-particle theory using the magnetic moment as a constant, one would conclude that the radius of the plasma column should be reduced by approximately the square-root of the ratio of the final-to-initial magnetic field intensity of the solenoids; if no particles were lost, the density of the plasma should be increased by the ratio of the final-to-initial magnetic intensity. Within the limitations of the apparatus used for this experiment,⁶³ the results did in fact indicate a plasma behavior much like that predicted from single-particle theory.

It should be pointed out, on the other hand, that in nearly all practical applications of MFM the requirement of low-energy density plasma with respect to the energy density of the magnetic fields involved will not be satisfied. In addition, even larger departures from the other restrictions which were necessary for this example will be encountered, so that, in general, particle physics still must be used only to obtain qualitative results.

2. Channel Flows

The type of MFM flow which has been investigated most and from which the most useful information has been obtained is channel flow. The MFM channel flows have received so much attention because a relatively rigorous treatment has been possible for most of the channel flow variations which were subject to simple electromagnetic field configurations.

The one-dimensional MFM channel flow investigations have contributed a great deal of the fundamental information which is necessary to the basic understanding of MFM effects. In most of the investigations, viscosity and heat transfer were neglected; some of the investigations were carried out under the assumption that the total electric field was also identically zero. The latter assumption reduces the complexity of the equations a great deal, and the results are not very difficult to interpret because the problem has been reduced to an isoenergetic flow problem. However, this assumption is actually quite restrictive since this electric field contains both the applied and induced fields. Nevertheless, the results are quite informative and should be stated here.¹³ It was found that the effect of a steady magnetic field on one-dimensional isoenergetic gas flow is similar to friction due to viscosity and that sufficiently large field strengths will produce choking. In other words, the friction force is replaced by the magnetic body force. Thus, the maximum effect of a magnetic field on isoenergetic, steady, compressible gas flow is to cause the Mach number at the exit of the field to be equal to unity.

A more general, one-dimensional treatment in which the electric field is not neglected but in which the magnetic and electric fields are applied externally at right angles to one another has been presented.^{7,8,12} The primary objective of these investigations was to study how effectively the electrically conducting gas flow through a channel can be accelerated by use of such a field configuration. The approximation of quasi-one-dimensional gasdynamics was used, so that the mean values of the fluid properties at each station along the channel were considered, and the channel's cross-sectional area changed only slowly along its length. The equations for this case could not, in general, be integrated for the interesting quantities, but it was possible to obtain explicit expressions for the gradients of the velocity and Mach number along the channel. From these expressions, three critical speeds were obtained and the character of the possible types of flows were then determined in terms of these critical speeds.

More information was obtained from the equations for this problem by assuming a channel of constant area. In fact, results were obtained which directly contradicted those obtained for isoenergetic flow: it was found that in special cases the Mach number may diverge from unity. In addition, two separate special cases indicated that smooth passage through the sonic speed was possible in both directions.

Considerable information was also obtained from these equations for a case where the area of the channel was assumed to vary exactly at the rate which was necessary to produce constant-density-channel flow. The solution obtained for that case was only one of the large number of solutions for incompressible flow.

More recently, some simple, closed-form solutions have been obtained for constant-temperature MFM channel flow with crossed electric and magnetic fields.^{64, 65} However, it was necessary to assume a particular velocity variation along the channel length; explicit expressions for the current density, pressure, channel area, and magnetic field were then obtained in terms of distance along the channel. The applied electric field was also assumed to be constant. However, it was later pointed out that the solutions can be extended to include special variations of the electric field and electrical conductivity. Another paper has dealt with some cases involving cylindrical symmetry where zero temperature gradient was assumed.⁶⁶

The two-dimensional flow of a viscous, incompressible, electrically conducting fluid through an infinitely long channel (Poiseuille flow) was first treated by Hartmann several years ago and is often called Hartmann Channel Flow. The magnetic field was uniform and was imposed perpendicular to the parallel insulating walls. Because of the simplicity of the differential equations and the boundary conditions, closed form solutions were obtained. Since the velocity profile was flattened with increasing magnetic field strength, it departed greatly from the familiar parabolic profile of Poiseuille flow. It was found that the velocity profile shape did not depend on an electric field applied perpendicular to both the flow direction and the magnetic field, but that the electric field had only a scaling effect on the velocity profile. The expression obtained for the pressure contained a separate term resulting from the magnetic field; therefore this term was called the "magnetic pressure." In addition, the distortion of the magnetic field by the fluid motion was illustrated very well by the solutions for this example.^{7, 28}

Hartmann's solution has been repeated in slightly modified forms many times in the literature during the last few years. However, there are several channel flow variations which have added useful information. One of these flows is a Poiseuille flow, like the Hartmann Channel Flow, except that the uniform magnetic field was rotated 90° but still kept transverse to the flow direction.⁷ This configuration illustrated how an electric field is induced by the fluid motion; it also showed how much more complicated a simple MFM flow can become when only a small change is made.

Another channel flow variation is obtained when the length of the channel over which the magnetic field is applied is limited to a finite length.⁴⁷ Some solutions to this problem have been obtained using series expansions and iteration techniques. Solution by this method is very long and complex, but considerable simplification is obtained if the channel is assumed to have infinite height.

A channel flow which is rather interesting is one in which the magnetic field is applied impulsively to the flow.⁴⁷ Most investigations of this type have been carried out under the assumption of large magnetic Reynolds number. If the fluid is a perfect conductor, the magnetic lines of force are effectively frozen in and convected downstream with the flow. If the magnetic Reynolds number is large, but not infinite, the stretching of the magnetic field lines cannot go on indefinitely, but at some time after application of the magnetic field a relative motion or slip will develop between the fluid and the lines of force; thus, equilibrium will eventually be established when the lines of force slip at the fluid velocity and become stationary relative to the walls. A solution can be found for this problem if the fluid is assumed to have zero viscosity.

A set of exact solutions has been obtained for laminar, steady-state, axial MFM flow in an annular channel for the case of a radial applied magnetic field.⁶⁷ Expressions giving the velocity profile and the distortion of the magnetic field were obtained. When the inside radius of the annulus is allowed to become very large, these expressions simplify into a special case to the Hartmann Channel Flow equations. Solutions

such as this are very interesting but have the disadvantage of being very difficult to study experimentally because of the type of magnetic field configuration assumed.

An axisymmetric MFM channel flow has been investigated for a particular set of highly restrictive assumptions.⁶⁸ The channeling effect of the magnetic field produced by a circular current loop on steady, compressible MFM flow at low magnetic Reynolds number in a circular channel of constant radius, co-axial with the magnetic field, was analyzed. The gas was assumed to be inviscid and non-heat conducting; unfortunately the electric field was also assumed to be zero, thus restricting the investigation to an iso-energetic flow. The equations were treated, in the supersonic flow case, by the method of characteristics, and results were obtained numerically. Both the subsonic and supersonic flow cases were linearized for small values of the interaction parameter (i. e., the flow properties were expanded in series in terms of the interaction parameter), and some formal solutions were obtained. An experimental investigation with a shock-tube showed qualitative agreement with the trends indicated by the theory.

The channel flow model has also been utilized both theoretically and experimentally for investigation of the effects which a magnetic field has on transition to turbulence. In general, when the magnetic field is transverse to the flow direction, the change brought about in the velocity distribution may make the flow more stable or unstable to transition to turbulence. In the case of a two-dimensional channel, it seems that a stabilizing effect is obtained; however, this does not seem to be the case for all flow problems. In many cases, if it is possible to apply a coplanar magnetic field, a stabilization is achieved as the motion across the magnetic lines of force is resisted. In other words, if the flow direction is parallel to the magnetic lines of force, no interaction takes place unless some deviation or instability arises. If there is a deviation in flow direction, a resisting force will be induced. The resisting force will be proportional and opposite to the velocity component transverse to the lines of force and will tend to damp out the initial instability. In general, these stability studies are much more complicated than the other channel flow investigations.

MFM channel flow has been investigated experimentally to some extent using the shock tube as a tool for producing the ionized gas.³⁰ The magnetic field was created first and then the shock was fired through it. A strongly ionizing shock produced outside a nearly axial magnetic field will channel along the axis using the magnetic field as a "wall." More specifically, there is a region along the axis where the axial field has been removed by the plasma; there is also a region of mixed plasma and field, called the "sheath." A third region near the tube wall is observed where there is increased axial magnetic field and no plasma. The interface between the plasma and the magnetic field was observed to be an unstable region. It was stated that the channel radius depended on the axial magnetic field intensity, while the sheath thickness did not vary noticeably with the field strength.

III. INVESTIGATION OF SOME MFM FLOWS

In the literature which has been reviewed for this work, numerical methods were usually resorted to when the magnetic fields were more complicated than uniform, parallel, magnetic fields. In this work, some attempts are made to obtain analytical results by making further restrictions and by removing boundary conditions. It is not expected to obtain useful numerical results from such an approach, but indications of the behavior of MFM flow can perhaps be obtained for some more complicated magnetic fields.

A. Basic MFM Equations

The MFM equations which are necessary for this investigation are given here, while those, such as the energy equation, which have not been used specifically are omitted. The limitations that are imposed by the equations, and which must be observed throughout the investigation, are also presented. Standard notation is used, and if symbols are employed which are not defined, their definition can be found in the list of symbols.

The fluid is assumed to be a homogeneous and isotropic medium; it is assumed that the collision frequencies in the ionized gas are large compared to the gyro-frequencies of the charged particles and that the plasma is electrically neutral in the macroscopic sense. Thus, the transport coefficients are scalar quantities and the macroscopic equations of continuum fluid mechanics are valid. Such electromagnetic effects as coulomb interactions, displacement currents, convection currents, Hall currents, and the relative slip between charged and neutral particles are not taken into account. The fluid viscosity, μ , and density, ρ_o , are assumed to be constant; thus, the kinematic viscosity, ν , is constant. The fluid magnetic permeability, μ_e , and electrical conductivity, σ , are also assumed to be constant; thus, the magnetic diffusivity, η , is also a constant. In addition to being incompressible, the flow is assumed to be steady. These assumptions are expressed by the following equations:

$$\rho_o = \text{const.}, \quad \nu = \frac{\mu}{\rho_o} = \text{const.}, \quad \mu_e = \frac{B}{H} = \text{const.}, \quad \eta = \frac{1}{\sigma\mu_e} = \text{const.}, \quad \frac{\partial}{\partial t} \equiv 0.$$

When ion slip, Hall currents, and convection currents are neglected, Ohm's law can be written as

$$\bar{J} = \sigma [\bar{E} + (\bar{U} \times \bar{B})], \quad (1)$$

which is good for conductors that are electrically neutral; it is therefore a good approximation for most ionized gases since the electrons and ions are usually not separated very far and one can think of the charges as forming doublets, i. e., the fluid is said to be "polarized."

Three of Maxwell's steady state electromagnetic equations which are needed are

$$\nabla \times \bar{H} = \bar{J}, \quad (2)$$

$$\nabla \times \bar{E} = 0, \text{ and} \quad (3)$$

$$\nabla \cdot \bar{H} = 0. \quad (4)$$

The first and second of these three equations are sometimes called Ampere's law and Faraday's law, respectively.

The continuity and momentum equations for a steady, viscous, incompressible, electrically conducting fluid are written

$$\nabla \cdot \bar{U} = 0 \quad (5)$$

and

$$\rho_o (\bar{U} \cdot \nabla) \bar{U} + \nabla p = \mu \nabla^2 \bar{U} + \bar{J} \times \bar{B}, \quad (6)$$

where the last term is the electromagnetic body force.

To put these equations into the form in which they will be used, the vector identity

$$(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} \equiv -\mathbf{C} \times (\mathbf{A} \times \mathbf{B}) \equiv (\mathbf{A} \cdot \mathbf{C}) \mathbf{B} - (\mathbf{B} \cdot \mathbf{C}) \mathbf{A} \quad (7)$$

is needed. $\bar{E}$ and $\bar{J}$ are eliminated from Equations 1 to 3 and Equation 7 is used to expand the cross-products. In the light of Equations 4 and 5 the equation which relates the local magnetic field to the motion of the fluid is

$$\eta \nabla^2 \bar{H} = (\bar{U} \cdot \nabla) \bar{H} - (\bar{H} \cdot \nabla) \bar{U}. \quad (8)$$

This is called the diffusivity equation or sometimes the equation of motion of the magnetic field because of its similarity to the equation of motion of the fluid.

The last term in Equation 6 can also be expanded by using Equation 2 to eliminate $\bar{J}$ and then using Equation 7. Thus, Equation 6 becomes

$$\rho_o (\bar{U} \cdot \nabla) \bar{U} + \nabla p = \mu \nabla^2 \bar{U} + \mu_e \left[(\bar{H} \cdot \nabla) \bar{H} - \frac{1}{2} \nabla H^2 \right] \quad (9)$$

where H^2 is the sum of the squares of each of the components of the magnetic field intensity vector.

It should be pointed out that Equations 8 and 9 have been derived and published in more general forms with generalized coordinates using tensor notation.^{69, 70} These equations will not be repeated here, but Equations 8 and 9 can be obtained immediately from these references in any coordinate system which seems to be best suited to a particular MFM configuration.

1. Specific MFM Field Equations

For this work it was decided, because of the common occurrence of axisymmetric flows in practical applications, that the investigations should be conducted in a cylindrical coordinate system. Another reason for using this coordinate system is that much of the work published has been conducted for parallel, uniform magnetic fields applied transverse to the flow direction using rectangular coordinates. Cylindrical coordinates are more suitable for the treatment of flows with axisymmetric magnetic fields and also for investigation of some vortex flows, as very little has been done on this type of MFM flow.

The coordinates r , ϕ , and z are used to denote the radial, tangential, and axial directions, respectively; correspondingly u , v , w and H_r , H_ϕ , H_z are the components of the velocity and magnetic field intensity, respectively, in the directions indicated above. This work will be concerned only with axisymmetric configurations which are steady.

$$\frac{\partial}{\partial t} \equiv 0 \quad \frac{\partial}{\partial \phi} \equiv 0. \quad (10)$$

When these restrictions are applied to the equations given in the references already mentioned,^{69, 70} Equations 4, 5, 8, and 9 may be written expanded in component form as

$$\frac{\partial}{\partial r} (rH_r) + \frac{\partial}{\partial z} (rH_z) = 0 \quad (11)$$

$$\frac{\partial}{\partial r} (ru) + \frac{\partial}{\partial z} (rw) = 0 \quad (12)$$

$$r: \frac{DH_r}{DT} = \frac{\mathcal{D}u}{\mathcal{D}T} + \eta \left(\nabla_c^2 H_r - \frac{H_r}{r^2} \right) \quad (13)$$

$$\phi: \frac{DH_\phi}{DT} + \frac{vH_r}{r} = \frac{\mathcal{D}V}{\mathcal{D}T} + \frac{H_\phi u}{r} + \eta \left(\nabla_c^2 H_\phi - \frac{H_\phi}{r^2} \right) \quad (14)$$

$$z: \frac{DH_z}{DT} = \frac{\mathcal{D}w}{\mathcal{D}T} + \eta \nabla_c^2 H_z \quad (15)$$

$$r: \rho_o \left(\frac{Du}{DT} - \frac{v^2}{r} \right) + \frac{\partial p}{\partial r} = \rho_o \nu \left(\nabla_c^2 u - \frac{u}{r^2} \right) + \frac{\mu_e}{4\pi} \left(\frac{\mathcal{D}H_r}{\mathcal{D}T} - \frac{H_\phi}{r} - \frac{1}{2} \frac{\partial H^2}{\partial r} \right) \quad (16)$$

$$\phi: \rho_o \left(\frac{Dv}{DT} + \frac{uv}{r} \right) = \rho_o \nu \left(\nabla_c^2 v - \frac{v}{r^2} \right) + \frac{\mu_e}{4\pi} \left(\frac{\mathcal{D}H_\phi}{\mathcal{D}T} + \frac{H_r H_\phi}{r} \right) \quad (17)$$

$$z: \rho_o \frac{Dw}{DT} + \frac{\partial p}{\partial z} = \rho_o \nu \nabla_c^2 w + \frac{\mu_e}{4\pi} \left(\frac{\mathcal{D}H_z}{\mathcal{D}T} - \frac{1}{2} \frac{\partial H^2}{\partial z} \right) \quad (18)$$

where the notation

$$\begin{aligned} \frac{D}{DT} &\equiv u \frac{\partial}{\partial r} + w \frac{\partial}{\partial z} \\ \frac{\mathcal{D}}{\mathcal{D}T} &\equiv H_r \frac{\partial}{\partial r} + H_z \frac{\partial}{\partial z} \end{aligned} \quad (19)$$

$$\nabla_c^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$$

$$H^2 \equiv H_r^2 + H_\phi^2 + H_z^2$$

is identical to that in the reference.⁶⁹

2. Approach to Solution of MFM Field Equations

It is of primary interest to determine the effect which a given applied magnetic field has on a corresponding flow field configuration. When a fluid having large electrical conductivity is flowing at high velocity, an applied magnetic field distribution will be grossly distorted. Although the two effects can never be entirely separated, the problem of primary concern here is the modification of the flow by the magnetic field rather than the distortion of the magnetic field by the flow. Specifying that the applied magnetic field is approximately equal to the magnetic field distribution throughout the flow field is equivalent to assuming that the magnetic Reynolds number of the flow is much less than unity. This is the approach which will be used in this work, and Equations 11 through 19 will be utilized to determine the flow fields corresponding to given magnetic field configurations. If one wants to be unrealistic, one may remove the restriction on the magnetic Reynolds number and say that somehow it is known that the magnetic field distribution throughout the flow is approximately that which is used to determine the flow field; however, at large R_m one has no way of knowing how to apply a magnetic field which, when distorted, will give the local distribution used in the equations. Therefore, the assumption of a small magnetic Reynolds number will be retained so that magnetic fields which closely approximate some realistic field configurations can be specified. It should be pointed out that the terms containing R_m are not going to be neglected as is usually the case when an approach such as this is used. It is felt that there will be circumstances under which these terms may be significant, and that they certainly will always give an indication of the effect which the magnetic field has on the flow. Such an approach can be expected to indicate only first order effects of moderately weak MFM interactions. It should also be noted that the electrical conductivity of the fluid is not restricted by the assumption of small R_m , as much as it might at first be thought, because the assumption of constant density has already confined the investigation to the low velocity regime. Thus, the electrical conductivity, σ , may in general take on significant values without violating $R_m \ll 1$, since high-speed flows are excluded. No restrictions will be explicitly placed on the magnitude of the interaction parameter, Q , but physically possible magnitudes of the magnetic field intensity combined with the constant density assumption and $R_m \ll 1$ actually place an upper limit on Q of the order of unity.

A further means of simplifying the equations before any specific examples are investigated is the reduction of the applied magnetic fields to two-component fields by assuming that no H_ϕ component exists. This, of course, would not be possible if large currents were passed through the flow parallel to the z -axis, because a significant H_ϕ field component would exist and the pinch effect would have to be taken into account. The axisymmetric magnetic fields which are of interest for applying to the flow fields in this investigation do not contain an H_ϕ component.

B. Circular Current Loop Case

It is interesting to investigate the effects which the magnetic field due to a single circular loop of current has on electrically conducting flow approaching the current loop symmetrically about the axis of the loop. When one investigates the expressions describing the magnetic field of such a current loop, one finds that they do not lend themselves to use in an analytical investigation; it is found that they contain complete elliptic integrals of the first and second kinds.^{71, 72} It would at first seem that numerical methods are the only way to treat such a problem. However, there seems to be some hope when Equation 15

$$u \frac{\partial H_Z}{\partial r} + \frac{\partial H_Z}{\partial z} = H_r \frac{\partial w}{\partial r} + H_z \frac{\partial w}{\partial z} + \eta \left(\frac{\partial^2 H_Z}{\partial r^2} + \frac{1}{r} \frac{\partial H_Z}{\partial r} + \frac{\partial^2 H_Z}{\partial z^2} \right) \quad (20)$$

is examined in detail.

Consider a current loop of radius R whose axis of symmetry coincides with the z -axis. When one restricts the region of interest to $r \ll R$, one would expect $H_r \ll H_z$ and $\frac{\partial H_Z}{\partial r} \ll \frac{\partial H_Z}{\partial z}$ except in the vicinity of the origin; correspondingly, one might expect a flow field in the region $r \ll R$ with $u \ll w$ and $\frac{\partial w}{\partial r} \ll \frac{\partial w}{\partial z}$, where the last of these inequalities is probably the weakest assumption. It is therefore observed that under these conditions $u \frac{\partial H_Z}{\partial r}$ is certainly negligible compared to $\frac{\partial H_Z}{\partial z}$, and likewise, $H_r \frac{\partial w}{\partial r}$ compared to $H_z \frac{\partial w}{\partial z}$. It is, of course, obvious that the smaller the radial distance r is compared to the current loop radius R , the better these assumptions become; on the other hand, they do not become extremely bad as far as the magnetic field is concerned until $\frac{r}{R}$ exceeds about $1/3$.^{68, 71, 72}

In view of the argument just presented, it seems reasonable to investigate the region where $r \ll R$ under the assumptions that $H_z = H_z(Z)$ and $w = w(z)$ in that region of the flow field. These assumptions reduce Equation 20 to

$$\frac{dw}{dZ} - \frac{w}{H_Z} \frac{dH_Z}{dZ} = -\frac{\eta}{H_Z} \frac{d^2 H_Z}{dZ^2} \quad (21)$$

the solution of which can immediately be written as

$$w = H_Z \left\{ \eta \int \frac{-1}{H_Z} \frac{d^2 H_Z}{dZ^2} dZ + C_1 \right\} \quad (22)$$

where C_1 is the constant of integration. If Equation 22 is inserted into Equation 12 one obtains

$$u = \frac{r\eta}{2} \left\{ \frac{1}{H_Z} \frac{d^2 H_Z}{dZ^2} - C_1 \frac{dH_Z}{dZ} + \frac{dH_Z}{dZ} \int \frac{1}{H_Z^2} \frac{d^2 H_Z}{dZ^2} dZ \right\} + \frac{C_2}{r} F(z) \quad (23)$$

where C_2 is an integration constant. However, it should be noted that once a magnetic field is specified, Equations 22 and 23 do not constitute a solution unless they also satisfy Equation 13, from which $F(z)$ can possibly be determined.

It has been assumed that if any physical walls exist, they are removed from the region around the axis of symmetry a distance of the order of R , such that their effects on the region of interest can be ignored.

In choosing an $H_z(Z)$ distribution to correspond roughly to that of a current loop, it is desired that the accuracy of the expression be the greatest in the region very close to the axis of symmetry. Thus, $H_z(Z)$

is chosen to be exactly the variation along the axis of symmetry; it will therefore be approximately correct a short distance off the z-axis. When one inserts

$$H_Z = H_O R^3 (R^2 + Z^2)^{-3/2} \quad (24)$$

into Equation 11, the other component

$$H_r = 3/2 H_O R^3 r z (R^2 + Z^2)^{-5/2} + C_O / r f(z) \quad (25)$$

is obtained. However, the boundary condition $H_r = 0$ when $r = 0$ is a necessary condition. The integration constant C_O is therefore zero.

$$C_O = 0 \quad H_r = 3/2 H_O R^3 r z (R^2 + Z^2)^{-5/2} \quad (26)$$

Equations 24 and 26 are the approximate magnetic field components for a current loop of radius R located at $Z = 0$ and whose axis of symmetry coincides with the z-axis; they are assumed to be valid in the region $r \ll R$ around the axis of symmetry.

The explicit equations for W and u can now be obtained by inserting Equations 24 and 26 into Equations 22 and 23. The function $F(z)$ is found when the expression for u is substituted into Equation 13.

$$w = \left\{ \eta \left[\frac{9R^2 \ln(Z + \sqrt{R^2 + Z^2})}{(R^2 + Z^2)^{3/2}} - \frac{6Z}{R^2 + Z^2} \right] + \frac{C_1 H_O R^3}{(R^2 + Z^2)^{3/2}} \right\} \quad (27)$$

$$u = \left\{ 3/2r \left[\frac{9\eta R^2 Z \ln(Z + \sqrt{R^2 + Z^2})}{(R^2 + Z^2)^{5/2}} - \eta \frac{(R^2 + 2Z^2)}{(R^2 + Z^2)^2} + \frac{Z C_1 H_O R^3}{(R^2 + Z^2)^{5/2}} \right] + \frac{C_2}{r} (R^2 + Z^2)^{3/2} \right\}. \quad (28)$$

If one prefers that the z-axis not act as a sink or source for flow, the boundary condition $u = 0$ at $r = 0$ is used and the integration constant C_2 is found to be zero. Equations 27 and 28 can be written in a shorter, dimensionless form using the identity

$$\ln\left(\frac{Z}{R} + \sqrt{1 + \frac{Z^2}{R^2}}\right) = \sinh^{-1}\left(\frac{Z}{R}\right) \quad (29)$$

and by dividing through by a reference velocity U_o ,

$$\frac{w}{U_o} = \left\{ \frac{\eta}{U_o} \left[\frac{9R^2 \sinh^{-1}\left(\frac{Z}{R}\right)}{(R^2 + Z^2)^{3/2}} - \frac{6Z}{R^2 + Z^2} \right] + \frac{R^3 C}{(R^2 + Z^2)^{3/2}} \right\} \quad (30)$$

$$\frac{u}{U_o} = \left\{ \frac{3}{2} \frac{r}{U_o} \left[\frac{9\eta R^2 Z \sinh^{-1}\left(\frac{Z}{R}\right)}{(R^2 + Z^2)^{5/2}} - \frac{\eta(R^2 + 2Z^2)}{(R^2 + Z^2)^2} + \frac{ZR^3 C U_o}{(R^2 + Z^2)^{5/2}} \right] \right\} \quad (31)$$

where the dimensionless parameter C involves several of the constants characterizing the flow field.

When the behavior of Equations 30 and 31 is studied, it is found that C certainly plays the role of a flow modification parameter of some type. The parameter C could perhaps be correlated with the constants on which it depends by experimental investigations. However, it should be pointed out that C involves most of the same constants which are contained in the interaction parameter Q ; in addition, C contains the integration constant C_1 .

It is found that C can be most conveniently written as the sum of two parameters such that

$$C = C(H_o, \eta, U_o, R, C_1) = \left[\frac{\omega^2}{R_m} + \Lambda \left(C_1, \frac{H_o}{U_o} \right) \right] \quad (32)$$

where ω^2 is a positive parameter characterizing the size of the current loop, and Λ is a parameter which obviously indicates a measure of the ratio of the magnetic pressure to the dynamic pressure. In fact, if

$$C_1 = \frac{\mu_e}{4\pi\rho_o} \frac{H_o}{U_o},$$

then Λ is identically $\frac{p_B}{q}$ such that the product $CR_m = \omega^2 + Q$, where Q is the interaction parameter. It is this form of CR_m that will be arbitrarily used in most of the discussion of Equations 30 and 31.

If the dimensionless parameter R_m , the magnetic Reynolds number, is substituted into Equations 30 and 31 using the radius R as the characteristic length,

$$\frac{w}{U_o} = \frac{H_Z}{H_o} \left\{ C - \frac{3}{R_m} \left[\frac{2Z}{R^2} \sqrt{R^2 + Z^2} - 3 \sinh^{-1}\left(\frac{Z}{R}\right) \right] \right\} \quad (33)$$

$$\frac{u}{U_o} = \frac{H_r}{H_o} \left\{ C - \frac{1}{R_m} \frac{R^2 + 2Z^2}{R^2 Z} \sqrt{R^2 + Z^2} - 9 \sinh^{-1}\left(\frac{Z}{R}\right) \right\} \quad (34)$$

where

$$\frac{H_z}{H_o} \text{ and } \frac{H_r}{H_o}$$

are given by Equations 24 and 26, respectively. In some cases it is convenient to evaluate C by letting $w = W_o$ at $Z = 0$ and then

$$C = \frac{W_o}{U_o}.$$

This would give

$$W_o = W_o(H_o, \eta, R, C_1) = W_o(R_m, \omega^2, \Lambda), \quad (35)$$

which is certainly a reasonable indication. The equations could be investigated using W_o for the parameter indicating the strength of interaction, since W_o would perhaps be easier to determine experimentally.

Because H_z and w have been assumed functions of Z only, an expression for the pressure can be obtained from Equation 18:

$$p = P + f(r) - \frac{\rho_o w^2}{2} - \frac{\mu_e}{8\pi} H_r^2 - 2\rho_o v \frac{u}{r}. \quad (36)$$

The constant P and the function $f(r)$ are not known, but a good indication of the pressure variation in the z -direction is given by Equation 36 when the expressions for u , w , and H_r are inserted.

In order to analyze this flow field thoroughly, it is necessary to look at the shape of the flow given by

$$\frac{dr}{dz} = \frac{u}{w}, \quad (37)$$

which does not give the streamlines because they are undoubtedly three-dimensional, a tangential velocity component in general being present. When the ratio $\frac{u}{w}$ is formed, it at first appears very unlikely that Equation 37 can be integrated; however, in this particular case, a very simple relation,

$$\frac{dw}{dz} = -\frac{2u}{r}, \quad (38)$$

exists between u and w . The combination of Equation 37 and 38 allows integration so that

$$\Psi = \left(\frac{r}{R}\right)^2 \left(\frac{w}{U_o}\right) \quad (39)$$

is obtained, where constant values of Ψ give the shape of the flow field, and Ψ might be called a "shape" function. Similarly, the magnetic lines of force,

$$\Phi = \left(\frac{r}{R}\right)^2 \left(\frac{H_Z}{H_0}\right), \quad (40)$$

are given by constant values of Φ .

The determination of the tangential velocity component, v , is not so simple as that of the other velocity components. The expressions for H_Z , H_r , w , and u given by Equations 24, 26, 33, and 34, respectively, must be used with Equations 14 and 17; the first partial derivatives of v can be explicitly obtained when $v = 0$ and can be written as

$$\frac{\partial v}{\partial Z} = v f_1(z) \quad (41)$$

$$\frac{\partial v}{\partial r} = \frac{v}{r} f_2(z) \quad (42)$$

where

$$f_1(z) = \frac{1}{r} \left\{ \frac{2uH_r}{H_Z u - H_r w} \right\} \quad (43)$$

$$f_2(z) = \left\{ \frac{wH_r + uH_Z}{wH_r - uH_Z} \right\} \quad (44)$$

from which it was not possible to obtain a closed form solution for v .

Since it was felt that some knowledge of the behavior of v would be necessary for accurate interpretation of the flow, Equations 41 and 42 were investigated in the vicinity of the origin. In other words, $\frac{Z}{R}$ was assumed small enough that $\left(\frac{Z}{R}\right)^2$ could be neglected compared to unity. The linearized equations for H_Z , H_r , w , u , and $f_1(z)$ become

$$H_Z \cong H_0 \quad (45)$$

$$H_r \cong H_0 \left(3/2 \frac{Zr}{R^2} \right) \quad (46)$$

$$\frac{w}{U_0} \cong \left[C + \frac{3}{R_m} \frac{Z}{R} \right] \quad (47)$$

$$\frac{u}{U_o} \approx \frac{3}{2} \frac{r}{R} \left[C \frac{z}{R} - \frac{1}{R_m} \right] \quad (48)$$

$$f_1(z) \approx 3 \frac{z}{R^2} \left[1 - \frac{z}{R} C R_m \right] \quad (49)$$

which gives

$$V = \Gamma F(r) \exp \left[\left(\frac{z}{R} \right)^2 \left(\frac{3}{2} - \frac{z}{R} C R_m \right) \right] \quad (50)$$

using Equation 41. The last term in the exponent of Equation 50 could also be neglected in the special case when C is of the same order as $\frac{z}{R}$. It should be pointed out that Equation 50 indicates a local minimum when $z = 0$, as it is expected to do. When one looks at Equations 45 to 50, one must keep in mind that they are good only near the origin, or false conclusions will certainly be drawn from them. Equation 50 also indicates a local maximum at $\frac{z}{R} = \frac{3}{2} \frac{1}{C R_m}$, but this is not a valid result unless $C R_m$ is large.

1. Discussion of Solution

It has already been pointed out that the restrictions made on the magnetic field configuration do not become especially bad until $\frac{r}{R}$ exceeds about 1/3, and that the approximation is better the closer one gets to the axis of symmetry. An idea of what the assumed magnetic field configuration looks like can be obtained by looking at Figure 1. When the magnetic field intensity given by Equations 24 and 26 is compared to that which is numerically computed from the exact equations,⁶⁸ one concludes that for the purposes of this work the restrictions on the magnetic field model should not hamper the investigation to any large extent. The assumption that $w = w(z)$ can be expected to cause peculiar behavior when w becomes zero since when this happens, a boundary surface will appear perpendicular to the z -axis across which there will be no flow. However, as long as one keeps in mind that the region of interest is restricted to the vicinity of the axis of symmetry, some useful information may be obtained.

It must be understood that the solution is not valid when R_m is identically zero. This is merely a characteristic of the approach used here. All initial conditions, including a prescribed non-magnetic flow solution, were discarded in the interest of obtaining an explicit solution. When R_m is zero the equations indicate that any non-magnetic flow may be present, because the magnetic field does not affect the flow. This solution may be regarded as a first order indication of the effects of the specified magnetic field on the moving, electrically conducting fluid. The solution must be used with small but non-zero values for R_m . The rigorous approach to a solution when R_m is assumed small was attempted and abandoned because no explicit information could be obtained from the complex equations. This method consisted of expansion of the dependent variables in a power series in terms of R_m in which the leading term represented the non-magnetic contribution and the following term represented the first-order perturbation due to electromagnetic effects, etc.

The transcendental nature of Equations 33 and 34 due to the presence of the inverse hyperbolic sine function complicates the study of the behavior of these equations. A detailed investigation indicates that the

product CR_m plays a very important role with respect to control of the flow. One can become thoroughly confused if one starts plotting flow fields for arbitrary values of CR_m . In order to illustrate this point, it is useful to write Equations 33 and 34 in the form

$$\frac{w}{U_o} = \frac{1}{R_m} \frac{H_z}{H_o} \left\{ CR_m + F_1\left(\frac{z}{R}\right) \right\} \quad (51)$$

$$\frac{u}{U_o} = \frac{1}{R_m} \frac{H_r}{H_o} \frac{R}{z} \left\{ CR_m \frac{z}{R} + F_2\left(\frac{z}{R}\right) \right\} \quad (52)$$

where

$$F_1\left(\frac{z}{R}\right) = \left\{ 9 \sinh^{-1}\left(\frac{z}{R}\right) - 6 \frac{z}{R} \sqrt{1 + \left(\frac{z}{R}\right)^2} \right\} \quad (53)$$

$$F_2\left(\frac{z}{R}\right) = \left\{ 9 \frac{z}{R} \sinh^{-1}\left(\frac{z}{R}\right) - \left[1 + 2 \left(\frac{z}{R}\right)^2 \right] \sqrt{1 + \left(\frac{z}{R}\right)^2} \right\} \quad (54)$$

and where the functions $\frac{H_z}{H_o}$ and $\frac{H_r}{H_o}$ are given by Equations 24 and 26, respectively.

The functions $F_1\left(\frac{z}{R}\right)$ and $F_2\left(\frac{z}{R}\right)$ are plotted versus $\frac{z}{R}$ in Figure 2. It is immediately obvious that when CR_m is less than unity, there are three values of $\frac{z}{R}$ which reduce Equation 51 to zero. Since $\frac{u}{U_o}$ will not necessarily be zero, the flow will be turned parallel to boundary surfaces which are perpendicular to the z -axis at these points. It is also obvious that the location of the boundary surfaces depends entirely on CR_m . Figure 2 and Equation 51 lead to the conclusions that there are four flow fields when CR_m is less than unity, three when CR_m equals unity, two when CR_m is greater than plus or minus unity, and one when CR_m is infinite. The plus or minus sign of CR_m values determines the direction of the flow at the origin.

The fact that mathematically several flow fields can be present in the vicinity of the current loop would at first seem to preclude any practical application of this example. However, CR_m is not necessarily less than unity when H_o is zero. In fact, the parameter ω^2 may be quite large depending on the size of the current loop; hence, ω^2 can be used to characterize the geometry of the flow before interaction with the magnetic field. The procedure which will be used in this investigation will be to pick a value for ω^2 which gives a suitable flow configuration and then this configuration will be observed for various values of R_m and Q . It will be supposed that ΛR_m is identical to Q even though the equations indicated that they may only be proportional to each other. The fact that one boundary surface will always be present for finite values of CR_m does limit somewhat the study of the flow through the current loop because the flow leaving the loop is turned 90 degrees by the boundary surface. However, some information can be obtained from studying the effect that the magnetic field of a current loop which is located behind the surface has on the flow approaching that surface. A flow somewhat similar to this has already been discussed,⁴⁹ but a more complicated behavior is presented here. The flow field corresponding to this type of flow is the one which never contains the origin for any positive value of CR_m . If ΛR_m is regarded as the interaction parameter, the position of the surface when Q

is zero corresponds to stagnation-point flow against a solid surface with no magnetic field. In this application the parameter ω^2 therefore determines the distance between the current loop and the solid surface. As CR_m increases, the boundary surface moves against the flow when the flow is opposite the direction of the magnetic field and one might consider this behavior to be analogous to MFM flow detachment or a frozen region.^{58,61} The flow shape is plotted in Figure 3 for several values of CR_m , and it is noted that large changes in CR_m are necessary to produce significant changes in position of the boundary surface. The exact location of the boundary surfaces cannot be explicitly determined because Equation 51 is transcendental. Since the addition of a positive value of CR_m to $F_1\left(\frac{z}{R}\right)$ amounts to an upward translation of the curve for $F_1\left(\frac{z}{R}\right)$, an idea of the motion of the boundary surface is obtained by observing the curve for $F_1\left(\frac{z}{R}\right)$ in Figure 2. The slope of $F_1\left(\frac{z}{R}\right)$ increases as $\frac{z}{R}$ goes to infinity; therefore it is obvious that its intercept with the $\frac{z}{R}$ axis, which uniquely determines the boundary surface location, will move more rapidly for small values of CR_m than it will as CR_m gets larger. In the limit, nevertheless, the boundary surface will be moved to infinity, which corresponds to infinite interaction with the magnetic field. However, it must be kept in mind that these equations have been obtained using the small R_m assumption; this, in fact, limits the strength of interaction which is physically possible to fairly small values even with large magnetic fields. The limiting case obtained with an infinite interaction corresponds to having an infinite magnetic pressure or an infinite magnetic field which is, of course, impossible. If one associates ΛR_m identically with the interaction parameter Q , one observes that practical interactions, i. e., $Q \approx 1$, displace the boundary surface only about 5/100 of the current loop radius R . This would seem to be a reasonable order of magnitude for the flow to be detached at the stagnation-point of a blunt body. This argument is, of course, based on hypothetical circumstances and the flow on the other side of the boundary surface is being ignored. In addition, there is no physical proof that such a phenomena actually can exist in flow with small R_m .

It would be particularly interesting if some information could be obtained about electrically conducting flow through the current loop. Just as $F_1\left(\frac{Z}{R}\right)$ characterized the behavior of w , $F_2\left(\frac{Z}{R}\right)$ characterizes the behavior of u . However, there is a function which is added to $F_2\left(\frac{Z}{R}\right)$ and not a constant; the function is a straight line through the origin with slope CR_m . Since this straight line always passes through the origin, it is impossible to translate the hump of $F_2\left(\frac{Z}{R}\right)$, which is below the z -axis, to a position above the axis. Therefore, it seems that for any finite radius from the axis of symmetry at the origin there must always be a negative radial component of velocity. This states that the throat of the channel is never at Z equal to zero except for infinite CR_m .

The flow coming from plus or minus infinity does not go through the loop or pass the origin unless CR_m is greater than minus or plus unity, respectively. In addition, for values of CR_m slightly greater than unity, the flow turns outward before turning inward to go through the loop. It is obvious that one must select a value of ω^2 such that this undersirable phenomenon does not appear when $Q = 0$ if one would like the flow shape to be parallel to or converging on the axis of symmetry before the interaction takes place. It is interesting to note that for CR_m greater than about 4, there is no outward flow as the fluid approaches the current loop. This corresponds to having the straight line $CR_m \frac{Z}{R}$ through the origin with a large enough slope so that when it is added to $F_2\left(\frac{Z}{R}\right)$, the large hump above the z -axis on one side of the origin is completely smoothed out or cancelled. Thus, if ω^2 is greater than or equal to 4 this problem is eliminated. When this procedure is used, it is obvious from Figure 4 that the configuration before interaction corresponds to a converging-diverging channel flow.

In the limit, when the only remaining boundary surface has been shifted to infinity, the flow behaves as expected by flowing parallel to the magnetic lines of force forming a nozzle shape symmetric about the origin. This case is of mathematical interest only, as it is not physically possible. It is felt that some useful information can be obtained from the flow through the loop even though it does misbehave as it flows out of the loop. In fact, the flow shape plotted in Figure 4 does collaborate some statements which have been made about subsonic linearized flow through a current loop--the statements, for instance, that the flow will not be symmetric about the current loop and that most of the channeling effect for weak interactions will occur in the region following the current loop.⁶⁸ Only in the limit does the flow field become symmetric and follow the shape of the magnetic lines of force.

The plot of $F_1\left(\frac{Z}{R}\right)$ would at first suggest that W becomes infinite at large distances from the origin; however, it must be noted that $\frac{H_Z}{H_0}$ is multiplied times $F_1\left(\frac{Z}{R}\right)$, and since $\frac{H_Z}{H_0}$ goes to zero at large distances from the origin, this reduces W to a slowly varying function a few current loop diameters up and down stream. Figure 4 also illustrates that most of the effects of the magnetic field are confined to within plus or minus one current loop diameter of the origin, and that free stream conditions can be assumed to exist a few current loop diameters upstream; thus, it is not particularly important what the flow does at infinity.

Figure 4 illustrates the effects of the interaction on the flow shape in the vicinity of the axis of symmetry. It shows that the downstream region is first affected by the magnetic field; then, for larger interactions, the upstream region is affected more and more. Although some evidence of a contraction of the flow at the current loop is observed, the streamlines mostly appear to have been shifted toward the z -axis; this is due to their closeness to the axis of symmetry. Actually, of course, one would expect the streamlines to be altered more by the magnetic field when r is of the order of R , because the magnetic field intensity increases as one proceeds away from the origin toward the loop of current, and larger H_r components are encountered at greater distances from the axis of symmetry. This investigation gives only a mathematical indication of the effects of the magnetic field on the region around the axis of symmetry.

One thing which is particularly disturbing about the equations obtained here is the problem of keeping all of the assumptions and restrictions satisfied simultaneously. For instance, to obtain velocities which do not violate the constant density assumption, while using very small values for R_m and staying close to the axis of symmetry, becomes somewhat of a problem. It was noticed, however, that if a constant parameter is placed in the exponents of Equations 24 and 26, this parameter appears in the equations in such a way that if it is very small, the problem of making the initial assumptions compatible is eliminated. Looking at Equations 24 and 26, one can see that this amounts to restricting the magnetic field to a slow variation with distance. This will not be done here because the magnetic field configuration would then depart too greatly from the current loop case. However, it will be found that this is characteristic of the approach used here.

C. Study of Some Vortex Flows

In the current loop example it was particularly unfortunate that very little information was obtained about the tangential component of velocity. In addition, it would be interesting to investigate the effect of allowing H_Z to be a function of the radial distance r , since in the current loop case $H_Z = H_Z(Z)$. It is for these reasons that some flows which are more simple than the first example are now investigated.

There are several reasons why MFM vortex flow is important. An electrically conducting fluid which is flowing around or which is being channeled by an axisymmetric magnetic field will, in general, experience induced rotational motion. Another reason is that missiles are usually intentionally spinning, hence if a magnetic field is attached to the missile it is being applied to a vortex flow. In addition, an efficient method for suppressing heat transfer is by injection of cool fluid tangentially into a channel around a plasma column causing it to rotate so that the cooler, more dense plasma is retained at the periphery of the plasma column. Therefore, it is likely that MFM vortex flow will necessarily exist in applications to re-entry bodies and in cooling of nuclear rockets.⁷³ Moreover, the working fluid of most laboratory plasma generators is usually injected tangentially about the plasma column so that the internal flow of a plasma generator is often highly rotational and considerable vorticity is still present in the plasma stream coming from the generator. It might be attempted to stabilize the plasma column by use of an axial magnetic field in addition to the attempts to channel the exhaust flow by a solenoidal magnetic field.

1. Power Law Radial Variation of H_Z

Consider the magnetic field

$$H_r = H_\phi = 0 \quad H_Z = H_Z(r) = H_o \left(\frac{r}{R_o} \right)^\delta \quad (55)$$

where R_o is a reference radius at which the magnetic field intensity is H_o (see Figure 5a). For such a magnetic field, Equation 15 becomes

$$u \frac{\partial H_Z}{\partial r} = H_Z \frac{\partial w}{\partial z} + \eta \left(\frac{\partial^2 H_Z}{\partial r^2} + \frac{1}{r} \frac{\partial H_Z}{\partial r} \right) \quad (56)$$

where $\frac{\partial w}{\partial z}$ can be removed using Equation 12:

$$\frac{\partial w}{\partial z} = - \left[\frac{\partial u}{\partial r} + \frac{u}{r} \right] \quad (57)$$

$$\frac{\partial u}{\partial r} + u \left[\frac{1}{r} + \frac{1}{H_Z} \frac{\partial H_Z}{\partial r} \right] - \frac{\eta}{H_Z} \left(\frac{\partial^2 H_Z}{\partial r^2} + \frac{1}{r} \frac{\partial H_Z}{\partial r} \right) = 0 \quad (58)$$

When Equation 55 is inserted into Equation 58, the equation

$$\frac{\partial u}{\partial r} + \frac{u}{r} (1 + \delta) - \frac{\eta \delta^2}{r^2} = 0 \quad (59)$$

is obtained, the solution of which is

$$u = \frac{\delta \eta}{r} + F(z) \left(\frac{r}{R_o} \right)^{-(\delta+1)} = U_o \left\{ \frac{R_o}{r} \frac{\delta}{R_m} + \frac{F(z)}{U_o} \left(\frac{r}{R_o} \right)^{-(\delta+1)} \right\} \quad (60)$$

where $F(Z)$ is an unknown function of Z . Insertion of Equation 60 into Equation 12 gives

$$w = F(r) + \delta \left(\frac{r}{R_o} \right)^{-(\delta+2)} \int F(z) dz \quad (61)$$

where $F(r)$ is an unknown function of r . From Equation 14 it is obvious that

$$\frac{\partial v}{\partial z} = 0; \quad (62)$$

therefore Equation 17 becomes

$$u \left(\frac{\partial v}{\partial r} + \frac{v}{r} \right) = v \left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} - \frac{v}{r^2} \right), \quad (63)$$

but since $V = V(r)$, $F(Z)$ is chosen to be a constant.

$$F(z) = C_o \frac{u}{U_o} = \frac{\delta}{R_m} \frac{R_o}{r} + \frac{C_o}{U_o} \left(\frac{r}{R_o} \right)^{-(\delta+1)} \quad (64)$$

$$w = F(r) + C_o \delta z \left(\frac{r}{R_o} \right)^{-(\delta+2)} \quad (65)$$

If the boundary condition $w = 0$ at $z = 0$ is applied, one obtains $F(r) = 0$; and if the boundary condition $u = 0$ when $r = \infty$ is used, C_o must be zero, except in special cases which will not be considered here.

$$w = 0 \quad \frac{u}{U_o} = \frac{\delta}{R_m} \frac{R_o}{r} \quad (66)$$

Inserting Equation 66 into Equation 69 gives

$$\frac{d^2 v}{dr^2} + \frac{dv}{dr} \frac{1}{r} \left(1 - \frac{\delta R_o U_o}{v R_m} \right) - v \frac{1}{r^2} \left(1 + \frac{\delta R_o U_o}{v R_m} \right) = 0. \quad (67)$$

But $R_e = \frac{R_o U_o}{v}$; therefore Equation 67 becomes

$$v'' + v' \frac{1}{r} (1 - A) - v \frac{1}{r^2} (1 + A) = 0 \quad (68)$$

where $A = \delta \frac{R_e}{R_m}$. Equation 68 can be simplified considerably by making a transformation on the dependent variable v .

$$v = Y r^{1/2 (A-1)} \quad (69)$$

transforms Equation 68 to

$$Y'' = \frac{1}{4r^2} (A+1)(A+3)Y \quad (70)$$

whose solution can be written

$$Y = C_1 r^{1/2 (A+3)} + C_2 r^{-1/2 (A+1)} \quad (71)$$

where C_1 and C_2 are integration constants. Inserting Equation 71 into Equation 69 gives

$$v = C_1 r^{A+1} + C_2 r^{-1} = \frac{\Gamma}{r} \left[1 + C_3 \left(\frac{r}{R_o} \right)^\alpha \right] \quad (72)$$

where

$$\alpha = A + 2 = 2 + \delta \frac{R_e}{R_m} = 2 + \delta \eta/\nu \quad (73)$$

and two new constants, C_3 a dimensionless constant and Γ the vortex strength, are defined. If the boundary condition $v = 0$ at $r = R_o$ is used, then $C_3 = -1$.

$$v = \frac{\Gamma}{r} \left[1 - \left(\frac{r}{R_o} \right)^\alpha \right]. \quad (74)$$

Using the relation $\frac{1}{r} \frac{dr}{d\phi} = \frac{u}{v}$, the streamlines

$$\Psi = \frac{R_m \Gamma}{U_o \delta R_o} \left[\ln \left(\frac{r}{R_o} \right) + \frac{C_3}{\alpha} \left(\frac{r}{R_o} \right)^\alpha \right] - \phi \quad (75)$$

are obtained for this flow pattern, where constant values of Ψ label the different streamlines.

For any case where

$$u = u(r) \quad w = w(z) \quad v = v(r) \quad H_Z = H_Z(r) \quad H_r = H_\phi = 0 \quad (76)$$

an expression for the pressure can be obtained easily from Equations 16 and 18.

$$p = P + \rho_o v \left[\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} \right] - \frac{\rho_o}{2} [u^2 + w^2] + \rho_o \int \frac{v^2}{r} dr - \frac{\mu_e}{8\pi} H_Z^2 \quad (77)$$

where P is an integration constant. In view of Equation 12 the second term on the right side of Equation 77 is zero.

Inserting Equations 66 and 72 into Equation 77 gives

$$p = P - \frac{\rho_o U_o^2}{2} \left(\frac{\delta}{R_m} \frac{R_o}{r} \right)^2 - \frac{\mu_e}{8\pi} H_o^2 \left(\frac{r}{R_o} \right)^{2\delta} - \frac{\rho_o}{2} \left(\frac{\Gamma}{r} \right)^2 \left\{ 1 - \frac{4C_3}{\alpha-2} \left(\frac{r}{R_o} \right)^\alpha - \frac{C_3^2}{\alpha-1} \left(\frac{r}{R_o} \right)^{2\alpha} \right\}, \quad (78)$$

which can be put into a dimensionless form by dividing through by the magnetic pressure,

$$p_B = \frac{\mu_e}{8\pi} H_o^2, \quad (79)$$

giving

$$\frac{p}{p_B} = \frac{P}{p_B} - \frac{\delta}{Q R_m} \left(\frac{R_o}{r} \right)^2 - \left(\frac{r}{R_o} \right)^{2\delta} - \frac{R_m}{Q} \left(\frac{R_o}{r} \right)^2 \left\{ 1 - \frac{4C_3}{\alpha-2} \left(\frac{r}{R_o} \right)^\alpha - \frac{C_3^2}{\alpha-1} \left(\frac{r}{R_o} \right)^{2\alpha} \right\} \quad (80)$$

where the equations

$$Q = \frac{p_B}{q} R_m \quad q = \frac{1}{2} \rho_o U_o^2 \quad V_o = \frac{\Gamma}{R_o} = U_o \quad (81)$$

have been used.

It is obvious from Equation 75 and Figure 6 that streamlines for this example are spirals with limiting cases of radial flow and pure circulation. The usefulness of this example is impaired by the fact that it is two-dimensional and that the origin is a singular point (see Figure 5). Nevertheless, it appears that the flow for $r > R_o$ might contribute some information. However, for any physical application, it would be necessary to consider only negative values of δ , i. e., inward flow, as the tangential velocity becomes infinite for outward flow when r goes to infinity. In fact, the cases where $\alpha < 1$ are the most reasonable flows because the expression

$$\left(\frac{r}{R} \right)_{V \max.} = \left(\frac{1}{1-\alpha} \right)^{1/\alpha} \quad (82)$$

indicates that a local maximum for V exists only when $\alpha < 1$. When α is examined, one concludes it is most probable that $\alpha < 0$ in any physical application because $\frac{R_e}{R_m}$ will, in general, be a very large number and therefore extremely small values for δ would be required to make $\alpha > 0$. On the other hand, it will at the same time be necessary to restrict δ to be of the same order of magnitude as R_m such that velocities will not be obtained which violate the constant density assumption. If δ is restricted to small values, it is obvious that the magnetic field intensity can vary only slowly with distance, for $r > R_0$.

The best interpretation of the portion of the flow field $r > R_0$, when $\delta < 0$, seems to be that fluid is being drawn radially from a MFM vortex flow outside $r = R_0$. The circle at $r = R_0$ can be regarded as a porous wall with a low pressure region inside, because the tangential velocity is zero on the surface. The amount of fluid crossing $r = R_0$ is of course directly related to the magnetic field distribution. It is interesting to note that a case exists, at least mathematically, in which the combination of the fluid electrical conductivity and viscosity coefficient along with the right magnetic field distribution makes $\alpha = 0$ and all circulation is completely stopped and only the radial flow is present. The fluid is stagnant at infinity and is being drawn inside the circle at $r = R_0$ with a finite radial velocity determined by δ and R_m . If the magnitude of $\frac{\delta}{R_m}$ is increased, i. e., the sink velocity is increased, circulation is induced in the fluid directly outside $r = R_0$ with the fluid at infinity remaining stagnant. In the limit when $\alpha = -\infty$, potential flow results. This limit can mathematically be reached in several ways, but since it involves infinite velocity, it can not be physically attained. On the other hand, if the sink velocity is decreased, a circulation is induced which is in the opposite direction to the vortex just discussed. The vortex created in this case behaves much differently from the one which was already discussed. In the limit when $\delta = 0$ a pure circulation is present since there is no radial velocity. This again is not a physically interesting limit because infinite circulation is present at infinity.

It was already pointed out that the most interesting flow is obtained for $\alpha < 0$. The behavior of the flow in this case illustrates several differences between a non-magnetic spiral flow and one which is controlled electromagnetically. For instance, in a non-magnetic viscous spiral flow, the radius where maximum velocity occurs would be primarily a function of the viscosity coefficient. Here, for MFM spiral flow, it is a function of the magnetic field distribution, the fluid electrical conductivity, and the viscosity coefficient. In other words, the "effective" viscosity of the fluid is a function of the parameters ν , η , and δ . Mathematically, at least, MFM spiral flow can exhibit strange behavior under certain conditions. Part of this behavior can be attributed to the approach used here, as no initial conditions for non-magnetic flow were used; thus when $R_m = 0$, the solution is not valid and predicts that any non-magnetic solution is possible. Nevertheless, for finite values of the parameters involved, this solution certainly gives an indication of some of the phenomenon which can be expected in MFM flow. In Figures 5 and 6 unrealistic values for some of the parameters were used in order to best illustrate the nature of the solution.

2. Exponential Radial Variation of H_Z

Consider the magnetic field

$$H_r = H_\phi = 0 \quad H_Z = H_0 \exp \left[-\frac{\beta}{2} \frac{r^2}{R_0^2} \right] \quad (83)$$

where R_o is a reference radius, and the magnetic field is H_o at $R = O$ (see Figure 7a). Equation 58 holds for this case also. Thus

$$\frac{\partial u}{\partial r} + u \left[\frac{1}{r} - \frac{\beta r}{R_o^2} \right] - \eta \left[\frac{\beta^2 r^2}{R_o^4} - \frac{2\beta}{R_o^2} \right] = 0. \quad (84)$$

The solution is

$$u = -\frac{\beta \eta r}{R_o^2} + \frac{F(Z)}{r} \exp \left[\frac{\beta r^2}{2R_o^2} \right] \quad (85)$$

where $F(Z)$ is an unknown function of Z . The axial velocity component is obtained by using Equation 12:

$$w = \frac{2Z U_o \beta}{R_o R_m} - \frac{\beta}{R_o^2} \exp \left[\frac{\beta r^2}{2R_o^2} \right] \int F(z) dz + F(r) \quad (86)$$

where $F(r)$ is an unknown function of r . Again it is obvious from Equation 14 that $\frac{\partial v}{\partial z} = 0$; thus, from Equation 17 one observes that $u = u(r)$, and this requires that $F(Z)$ be a constant.

$$F(Z) = C_o \quad w = \frac{2Z}{R_o} \frac{U_o \beta}{R_m} - \frac{\beta}{R_o^2} \exp \left[\frac{\beta r^2}{2R_o^2} \right] C_o Z + F(r). \quad (87)$$

The boundary condition $u = 0$ at $r = 0$ requires $C_o = 0$, and if $w = 0$ at $Z = 0$ it is found that $F(r) = 0$.

$$\frac{u}{U_o} = -\frac{\beta}{R_m} \frac{r}{R_o} \quad (88)$$

$$\frac{w}{U_o} = \frac{2\beta}{R_m} \frac{Z}{R_o} \quad (89)$$

Inserting the expression for u into Equation 63 gives

$$v'' + v' \left(\frac{1}{r} - rK \right) - \frac{v}{r} \left(\frac{1}{r} + rK \right) = 0 \quad (90)$$

where $K = \frac{\beta}{R_o^2} \frac{R_e}{R_m}$. A transformation on the dependent variable

$$v = X \frac{1}{\sqrt{r}} \exp \left[-\frac{Kr^2}{4} \right] \quad (91)$$

transforms Equation 90 into

$$X'' = X \frac{1}{4} \left\{ \frac{3}{r^2} + K^2 r^2 \right\} \quad (92)$$

whose solution is

$$X = \frac{1}{\sqrt{r}} \left\{ C_1 \exp \left[\frac{K}{4} r^2 \right] + C_2 \exp \left[-\frac{K}{4} r^2 \right] \right\} \quad (93)$$

where C_1 and C_2 are integration constants. Insertion of Equation 93 into Equation 91 gives

$$v = \frac{1}{r} \left\{ C_1 + C_2 \exp \left[-\frac{K}{2} r^2 \right] \right\} \quad (94)$$

or

$$v = \frac{\Gamma}{r} \left\{ 1 + C_3 \exp \left[-\frac{K}{2} r^2 \right] \right\} \quad (95)$$

where two new constants, a dimensionless parameter C_3 and the vortex strength Γ , have been defined. This solution can be reduced to a form which makes it more interesting than the preceding example by letting $C_3 = -1$ so that v is zero when r is zero.

$$v = \frac{\Gamma}{r} \left\{ 1 - \exp \left[-\frac{K}{2} r^2 \right] \right\} \quad (96)$$

Equation 77 also holds for this solution, so that, in principle, an explicit expression for the pressure throughout the flow can be written; however, the integral

$$\int \frac{v^2}{r} dr \quad (97)$$

cannot be evaluated in closed form for Equation 96. The shape of the flow

$$\Psi = Zr^2 \quad (98)$$

is given by constant values of Ψ , the shape function.

It is obvious from Equation 98 that u and W are components of an axisymmetric MFM stagnation-point flow. It should be pointed out that this particular MFM flow has resulted in a solution which is quite similar to one obtained in a paper on a non-magnetic vortex flow investigation.⁷⁴ The velocity components obtained here are related identically to those in the reference,⁷⁴ when the so-called "inflow-gradient", a ,

is evaluated such that

$$a = \frac{\eta \beta}{R_o^2} = \frac{\beta}{R_m} \left(\frac{U_o}{R_o} \right) \quad (99)$$

expressed in terms of the quantities used here. Examination of Figure 7 leads one to conclude that this solution to exist physically only if $\beta > 0$, which corresponds to $a > 0$. This restricts the investigation to axisymmetric stagnation-point flow which can flow in the inward direction only. With this restriction in mind, Rott⁷⁴ defines a quantity, r^* , which he calls the "viscous radius":

$$r^* = \sqrt{\frac{2\nu}{a}} = R_o \sqrt{\frac{2\nu}{\beta \eta}} = R_o \sqrt{\frac{2}{\lambda}}, \quad (100)$$

and he points out that if $r \gg r^*$, potential flow dominates; if r is of the same order as r^* , then viscous effects are felt; if $r \ll r^*$ then nearly rigid rotation is found. Therefore, the magnitude of r^* indicates the size of the viscous core of the vortex. This leads one to the conclusion that the magnetic field distribution and the magnitude of the fluid electrical conductivity in addition to the viscosity coefficient, control the size of the viscous core in this example. Figure 7d illustrates that the viscous core gets smaller with increasing λ because the location of the maximum tangential velocity is shifted toward the origin as λ increases. Equation 98 and Figure 7 indicate that for $\beta > 0$ the fluid is coming from infinity and is exhausted along the axis of symmetry primarily inside the viscous core. This action is similar to the ordinary bathtub vortex. Rott discussed the viscous core of a tornado in some detail using the non-magnetic solution corresponding to these equations.⁷⁴ The difference here is that the fluid is electrically conducting and the flow is controlled entirely by the magnetic field distribution parameter. When β is zero, the fluid is stagnant; when β is finite, the fluid is in motion; and for increasing β , larger velocity components are possible. In fact, it will be necessary to restrict β to small values in this solution if it is desired that the solution remain valid at reasonable distances from the origin, because $R_m \ll 1$ and velocities compatible with the constant density assumption are required. Even then, if R_e takes on normal values, the viscous radius will be very small. In Figure 7 unrealistic values for the parameters were used in order to illustrate the nature of the solution.

The best interpretation of this flow seems to be that the flow is injected with infinite radial velocity at infinity into the magnetic field, and the flow is then turned parallel to the magnetic lines of force along the axis of symmetry. The behavior at infinity need not be considered however, since in any application of this example the fluid would be injected at a finite radius with finite velocity components. For instance, β , giving the magnetic field distribution would be known, and once a fluid with a given electrical conductivity is selected, the velocity components u and w are known; but V is not completely known until the fluid viscosity coefficient is also specified.

Small values for β restrict the magnetic field intensity gradient to small values; thus, it seems to be characteristic of the approach being used in these examples that in order for the assumptions which were made at the outset to be compatible, the magnetic field must vary slowly with distance.

One immediately thinks of applying this example to the flow in a plasma generator chamber or to the problem of cooling nuclear rockets since flows similar to this are encountered. This application is hindered by the fact that the fluid injected in these cases is initially not electrically conducting and that it reaches its maximum temperature on the axis of symmetry. In addition, the large temperature gradient certainly invalidates the constant density assumption. The most serious objection is that an H_ϕ components of magnetic field must be considered in such an application. Under the assumptions made here, this solution would be applied best to the flow of something like mercury. Some experiments have been performed by Lehnert in which mercury, between two coaxial cylinders in a magnetic field parallel to their axis, was observed when the cylinders were rotating relative to each other. A pronounced "magnetic rigidity" was observed to be induced by the action of the magnetic field.²⁸ An effect similar to this can be seen to exist here, because the viscous radius increases as β decreased, i. e. , as the hump in the magnetic field distribution is broadened. This corresponds to the fact that for large β the magnetic field is essentially being applied to a small region around the axis of symmetry, and therefore a strong MFM interaction can occur only in this region. For small β , the magnetic field is applied over a much larger region, thus depending on the so-called "magnetic viscosity", η , as well as ν . The viscous core is much larger and a strong MFM interaction has occurred over a much larger region. This illustrates quite well the so-called "magnetic rigidity" introduced by the action of the magnetic field. In fact, this solution indicates that a parallel magnetic field applied coaxial to a plasma column which is rotating will do much to destroy this rotational motion and the cooling action which goes with this motion. Such a magnetic field will also do much to hinder the exhaust flow of the plasma within the viscous core of the vortex in a plasma generator. Hence, if these two beneficial effects are used together, one must seek an optimum between them.

A few words are needed about this solution in the limiting cases which can occur mathematically and sometimes physically. The effect of β has already been indicated to be, in the limit of $\beta \gg 0$, essentially a reduction of the flow to the inviscid case since no magnetic field is being applied to the flow except at the origin, where an infinite gradient is present. Therefore the effect of MFM interaction is hardly felt elsewhere. This is a limit which is not physically possible because of the infinite gradient in the magnetic field at the origin. Thus the fact that the equations for the velocity components predict very large velocities should be of no concern. When $\beta = 0$ it has been indicated that the magnetic field is applied uniformly over the entire flow field and that its interference with the flow has completely eliminated all motion, or in other words, the viscous core of the vortex has been extended to infinity.

Of course, the only effect which ν has when it goes to zero is a reduction of the solution to a potential flow solution, which is not of any importance.

The limit when $R_m \gg 1$ has been excluded by the assumption $R_m \ll 1$, but it should be pointed out that the behavior in this limit is as expected because, since the magnetic field cannot be distorted here, one would expect all motion to be eliminated.

The limit when $R_m = 0$ is at first the most disturbing case, since the equations for u and W are unbounded. However, V in Equation 96 reduces to the inviscid case. This is just additional evidence that η , which is often called the "magnetic viscosity" because it plays a role in the "equation of motion" of the magnetic field similar to the role played by ν in the equation of motion of the fluid, has an effect on fluid motion in a magnetic field which is inverse to the effect of ν on the fluid motion. The behavior when $R_m = 0$

might be best interpreted in terms of η , which can be considered as a measure of how much fluid motion is possible without distorting the magnetic field beyond a certain amount; thus, when $\eta = \infty$, an infinite amount of motion is possible since the magnetic field cannot be distorted by the motion of the fluid and the fluid experiences no interference or resistance from the magnetic field. Therefore, the equations indicate an inviscid behavior at this limit which is the same as that at $\beta = \infty$. This may also be explained by the fact that no initial boundary conditions for non-magnetic flow existed for this solution; and hence when $R_m = 0$, the solution indicates that any non-magnetic flow can exist since it is not affected by the presence of the magnetic field. Thus, R_m must be small but non-zero, $0 \neq R_m \ll 1$, for this solution to be physically meaningful. Strange behavior may also be obtained when two or more of the limits are present simultaneously as indeterminate expressions may be encountered.

D. Conclusions

The highly restricted solutions presented here have indicated some interesting features of several types of MFM flow. The solutions were obtained through the diffusivity equation without any initial boundary conditions for non-magnetic flow. Since a rigorous approach was not used, the solutions do not behave correctly in all cases. The solutions obtained seemed to indicate that in order for the assumptions which were made to be compatible, the magnetic field intensity must vary slowly with distance.

One possible interpretation of the stagnation-point flow associated with the current loop example permitted a discussion of the phenomenon of MFM "flow detachment". The flow through the current loop illustrated that the downstream region is affected the most for weak interactions and that the upstream region is affected more and more for larger interactions; the flow is channeled into a symmetric configuration only in the limit of an infinite interaction.

The current loop solution was not hindered in any way by the approximations made in the magnetic field except for the restriction on the region of application of the approximation. However, the restriction $W = W(z)$ did seriously affect the solution's behavior, and much better results would be obtained if this restriction alone were removed by means of numerical methods. It should be pointed out, on the other hand, that if a numerical method were to be used, one would probably be wise to use exact values for the magnetic field components so that the results would be valid over a larger region. The constant density assumption should also be removed if possible so that higher velocities could be tolerated. It would probably be unwise to attempt to remove the small magnetic Reynolds number assumption, at least until the validity of the results at low magnetic Reynolds number are proved.

The vortex flow solutions displayed the dependence of the "effective" viscosity on the fluid electrical conductivity, the gradient in the magnetic field intensity, and the viscosity coefficient. In fact, the ratio η/ν times a parameter describing the gradient in the magnetic field intensity seems to be a characteristic combination in some types of MFM flow. The solutions indicated that the "inflow gradient" and the size of the viscous core of a MFM vortex can be completely controlled electromagnetically. These solutions were simple enough so that one could probably obtain some corresponding solutions using a more rigorous approach with reasonable initial non-magnetic boundary conditions.

It is concluded that interactions between electrically conducting flow and a given magnetic field configuration can be studied to some degree by the method used here; however, if more informative results are desired, one must use a more rigorous approach and resort to numerical methods to complete the investigation.

Figure 1 -- Current Loop Magnetic Field Approximation

Figure 2 -- Functions Characterizing Velocity Components u and w in Current Loop Example

Figure 3 -- Stagnation-Point Flow Shape Associated With Current Loop Solution

Figure 4 -- Shape of Flow Through Current Loop ($r \ll R$)

Figure 5 -- Two-Dimensional MFM Vortex Solution Plotted Versus Radial Distance

Figure 6 -- Two-Dimensional Vortex Flow Field

Figure 7 -- Three-Dimensional MFM Vortex Solution Plotted Versus Radial and Axial Distance

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