Air Core Cryogenic Magnet Coils for Fusion Research and High Energy Nuclear Physics Applications*

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In the fields of controlled fusion research and high energy nuclear physics it is becoming evident that means for the efficient generation of very high magnetic fields must be sought. Since ferromagnetic materials are of no help at the required fields (of order 10^5 gauss) one can only rely on the use of air core coils, so that the fields which can be reached depend only on the ampere-turns achievable and simple geometrical factors. In this case the limitations on attainable fields usually come down to a question of available electrical power, or, more fundamentally, to limitations imposed by heating of the coil conductors and problems of heat transfer within the coil. For laboratory-sized magnets, the limit is reached in magnets such as the Bitter magnet, where nearly the ultimate limit is reached in power density and heat transfer, in order to achieve steady fields of 10^5 gauss in volumes of order 1 liter.

The avenue to alleviating these problems has been open to us for a long time, but has been largely ignored, for a variety of reasons. It has been long known that at very low temperatures the electrical resistivity of many pure metals drops to a small fraction of its value at room temperature. This fact implies a correspondingly large drop in the volume resistivity losses of a coil, and has been exploited on a laboratory scale by several investigators.[1] The possible application of this same technique to the generation of high magnetic fields for fusion and accelerator applications has also been discussed

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1. For example, J. L. Olsen, Helvetica Physics, Acta 26, 798 (1953); Harold Furth and R. W. Wanick, Rev. Sci, Inst., 27, 195 (1956); H. Laquer and E. F. Hammel, Rev. Sci. Inst., 28, 875 (1957) and Proceedings of the 1956 Cryogenic Engineering Conference, September 5-7, 1956, pp. 117-119.

or suggested in a general way many times in the past.[2] What has apparently not been adequately noted is that dramatic effect which scaling to large magnet sizes would have on the practicality of applying cryogenic techniques to high field magnet designs. Since the strength of magnetic fields produced by any air core magnet can be shown to be proportional only to the current density and a linear dimension of the coil, uniformly scaling any air core coil in size produces a higher field strength for the same current density. This fact, coupled with the reduction of resistance attainable through cryogenics, means that very efficient generation of high magnetic fields should be possible with large cryogenic coils. Conversely, much of the potential gains of cryogenic techniques are lost if they are applied to small coils, for which designs such as the Bitter magnet, operating with conductors at room temperature, are probably superior.

Factors Which Influence The Efficiency

The possible quantitative advantages to be gained by lowering the temperature of air core coils will now be discussed. The degree of reduction of coil losses which can be obtained depends upon the temperature to which the coil is lowered, the purity of the metal, and the strength of magnetic field which is to be attained. Balanced against the reduction of coil losses achieved in this way is the fact that the lower the temperature at which operation is attempted the greater is the energy which will be lost irreversibly in running the refrigeration plant. Thus, it must be demonstrated that an overall gain can, in fact, be attained when this is taken into account. Whether a net reduction in energy losses is achievable depends on several factors.

The first factor to be considered is the energy necessarily required to be expended in the refrigeration plant in carrying the coil heat "uphill" from its operating temperatures to the temperature of the refrigerator heat sink (room temperature). This energy cost must clearly be added to the coil dissipation to obtain the total power loss to be ascribed to the production

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2. D. R. Wells, Project Matterhorn Technical Memo - NYO-6375 (September, 1956) and R. F. Post, UCRL-4231, (1954) and R. F. Post, Physical Rev., 69, 126 (1946).

of the confining magnetic field.

The "efficiency" of a refrigerator used to carry low temperature heat from the coil and deliver it at room temperature can be described in terms of the efficiency of an "ideal" (Carnot) refrigeration cycle, multiplied by a "mechanical" or "process" efficiency, η_R, which represents the effect of additional mechanical and thermal losses in the actual refrigeration system.

The Carnot efficiency relates the work, W, necessary to pump an amount of heat energy, Q, from a low temperature heat source at temperature T_0 to a higher temperature heat sink at temperature T_E, by use of an ideal heat engine. This is:

$$\left[\frac{Q}{W}\right]_{ideal} = \frac{T_0}{T_E - T_0} \qquad (1)$$

The actual amount of heat pumped between T_0 and T_e per unit energy is then given by:

$$\left[\frac{Q}{W}\right]_R = \eta_R \left[\frac{Q}{W}\right]_{ideal} = \eta_R \left[\frac{T_0}{T_E - T_0}\right] \qquad (2)$$

The total amount of energy lost to the heat sink is the sum of the heat pumped plus the work expended in pumping it. Thus:

$$W_{TOTAL} = Q + W_R$$

$$= Q\left[1 + \frac{1}{\eta_R}\,\frac{T_E - T_0}{T_0}\right] \qquad (3)$$

In the present case Q would be equal, in steady-state, to the actual joule heat released in the coil. This means that if the magnet losses are $P_m(T_0)$ watts at temperature T_0, then to find the total energy which must be expended in producing the magnetic field, one must multiply P_m by the "refrigeration factor"

$$G_R = \left[1 + \frac{1}{\eta_R}\,\frac{T_E - T_0}{T_0}\right] \qquad (4)$$

G_R is tabulated in Table I for various values of η_R, for a value of $T_E = 300^\circ K$ (27°C, i.e., "room temperature").

The overall gain (or loss) to be achieved by the use of refrigeration is proportional to the product of the mean resistivity of the conducting material used in the coil and the refrigeration factor, G_R. This defines an "effective resistivity" for the coil. In using this factor to determine the overall gain to be achieved by the use of refrigeration, we will be interested in two questions. The one is the choice of the particular metal from which to fabricate the coil. The other is the choice of optimum operating temperature for the coil. It will therefore be convenient to adopt a "standard" against which to compare. This standard will be taken to be pure copper at 300°K (27°C), which has a resistivity of 1.73×10^{-6} ohm-cm, designated by ρ_s. Thus, taking $T_E = 300^\circ K$, we may evaluate the ratio $\left[\rho(T_0)/\rho_s\right] G_R(T_0)$, as a function of temperature for any pure metal (including copper itself), to determine the reduction in power losses achievable by refrigeration.

It is clear that any metal for which the resistance does not drop to a value less than $(1/G_R)$ times the resistance of copper at 300°K cannot possibly offer a gain. Since the results to be obtained are obviously going to depend on the value of refrigerator mechanical efficiency assumed, a word needs to be added on this question. The value of η_R is limited primarily by the state of the refrigerator art at the temperature to be achieved, and the size of refrigerator units used. Today small units might typically have values of η_R of 0.25 or even lower, whereas it is possible to estimate that large, carefully designed, units might achieve mechanical efficiencies as high as 0.50 or possibly higher. This efficiency is in sharp contrast to that which would be achieved in early, small refrigerators. Low refrigerator efficiency has often led to discouraging results in earlier calculations of the possible gains from cryogenics in magnet coil design.

In determining $\rho(T_0)$ for use in this expression it is necessary to consider the factors which influence the resistivity of metals at low temperatures. The electrical resistivity of many nearly-pure metallic

TABLE I

C_R as a function of Temperature and α_R

η_R / T^oK	0.25	0.3	0.35	0.4	0.45	0.5
10	117	98	84	73	65	59
15	77	64	55	49	43	39
20	57	48	41	36	32	29
25	45	38	32	28	25	23
30	37	31	27	23	21	19
35	31	26	23	20	18	16

$$C_R = \left[1 + \frac{1}{\eta_R} \frac{T_E - T_o}{T_o} \right] \qquad T_E = 300^oK$$

elements in the annealed state is describable as the sum of three "components"[3], one associated with the pure element itself, which we shall call ρ_0, a second, arising from impurity atoms (or crystal lattice defects) which will be called ρ_i, and a third associated with the effect of a magnetic field, which we shall call ρ_B. For small impurity contents the three components add essentially independently, so that the total resistivity is simply:

$$\rho = \rho_o(T) + \rho_i + \rho_B \tag{5}$$

To a reasonable approximation the impurity resistivity term is independent of temperature, and thus appears merely as an additive constant (Matthiessen's Rule). The same is roughly true of the magneto-resistance term, ρ_B. The intrinsic resistivity, ρ_0, however, varies markedly with temperature, especially at low temperatures.

The variation of ρ_0 with temperature for certain nearly pure metals is remarkably accurately predicted over a wide range of temperature by a theoretical expression based on quantum mechanical calculations by Houston, Bloch and others. From this theory it is possible to derive a "universal resistivity curve" (the Bloch-Gruneisen function) which can be used to predict approximately the temperature dependence of the intrinsic resistivity of many pure metals (such as Cu, Al, Na, etc.) in terms of a characteristic "resistance temperature", Θ, which can be found for each metal. The B-G function will be of use in obtaining analytic expressions for the energy losses, and in predicting the optimum conductor material to be used in the coil. The values of Θ for pure metals are a few hundred degrees Kelvin, i.e., of the order of room temperature.

The universal resistivity curve is obtained by expressing temperature in dimensionless units, $t = T/\Theta$, and resistivity in the dimensionless unit $r = \rho(t)/\rho(\Theta)$, i.e., r is the ratio of resistivity at temperature t to that at temperature Θ. One significant feature of the theoretical curve is that at normal temperature ($t \simeq 1$) and down to temperatures of about 0.2 Θ, the

3. See for example, D. K. C. MacDonald, "Handbuch der Physik", Vol. 14

relative resistivity varies only linearly with temperature. In this range the B-G function is closely approximated by a simple linear function

$$r = 1.16t - 0.16 \qquad (6)$$

However, for temperatures below approximately $t = 0.15$, r varies as t^5, thus dropping rapidly to small values as the temperature approaches $0^\circ K$. It is in this range that the greatest gains from cooling are to be realized. The variation of r for $t < 0.10$ ($T < 0.1\ \Theta$) is given by the expression:

$$r = (525)\ t^5 \qquad (7)$$

In Table II values of the intrinsic resistivity of sodium relative to its resistivity at $273^\circ K$ ($0^\circ C$) are given, as calculated from the Bloch-Gruneisen theory. These are compared with various experimental data (after MacDonald[4]). The agreement is seen to be remarkable, even down to the lowest temperatures.

It is obvious from the nature of eq. (7) that if the behavior predicted by the Bloch-Gruneisen theory were the whole story, the intrinsic resistivity could be reduced to an arbitrarily low figure merely by dropping the temperature, so that the energy losses in a high field magnet coil could be made arbitrarily low. However, at very low temperatures the impurity and magneto-resistivity terms may become important, so that it is necessary carefully to consider these addiational loss factors in order to assess the eventual gains which actually could be achieved.

In choosing the best metal or metals for use in large magnet coils, some preliminary criteria can be applied immediately. These are that the metal should be inexpensive and readily obtainable in pure form. Beyond this, it is clear that the most important requirements are that the metal should possess a low intrinsic resistance and the lowest possible magneto-resistance coefficient. In scanning the list of elements, it appears that there are

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4. D. K. C. MacDonald, Proc. Roy. Soc., A, 202, 103 (1950)

TABLE II

T^oK	r_{calc}	$r_{observed}$ (1)	(2)	(3)
273.2	1.0000	1.0000		
170.9	0.5672		0.5672	
108.7	0.3135		0.3168	
90.0	0.2600			
87.8	0.2279	0.2279		
77.6	0.1860	0.1849		
56.8	0.1022		0.1055	
20.4	0.00327	0.0034		0.00326
15.95	0.00100			0.00098
14.1	0.00055			0.00051
13.1	0.000384			0.000365
11.05	0.000154			0.000173
9.65	0.00009_3			0.00010_2
8.1	0.00004			0.00005
4.2	0.00000+			0.00000

1. Meissner and Voigt (1930)
2. Waltier and Kamerlugh Onnes (1924)
3. MacDonald and Mendelssohn (1950)

three prime candidates. These are, copper, aluminum and sodium. Of the three, copper is most attractive in terms of ease of use, and its ready availability in high purity form. Insufficient data exists to properly evaluate aluminum, but it appears to be somewhat superior to copper in the overall gains which could be expected. But if sodium can be used, it hold promise of far greater gains than can ever be achieved with copper.

The approximate resistance temperatures for copper, aluminum and sodium are respectively, 300°, 390° and 200° Kelvin. Thus, referring to the Bloch-Gruneisen curve, it can be seen that for $t < 0.1$ (temperatures lower than 30°K, 39°K or 20°K, respectively) the intrinsic resistance of each of these metals will fall rapidly to a small fraction of its value at room temperature. It is at these temperatures, and lower, where we can expect to achieve the greatest gain from cooling. The resistivity of copper at $t = 1$ (300°K) is 1.73×10^{-6} ohm-cm, that of aluminum at its characteristic temperature (390°K) is 3.87×10^{-6} ohm-cm and that of sodium at 200°K ($t = 1$) is 3.08×10^{-6}.

Thus at 20°K, ($t = 0.067$) from (25) the intrinsic resistance of copper is 7.1×10^{-4} of its value at room temperature, i.e., only 1.2×10^{-9} ohm-cm. Similarily, aluminum at 25°K ($t = 0.064$) has a predicted intrinsic resistivity of 2.2×10^{-9} ohm-cm. At 10°K ($t = 0.05$) sodium has an intrinsic resistivity of 0.5×10^{-9} ohms, or about 5×10^{-4} of the resistivity of copper at room temperature.

Consider now the magneto-resistance effect. This effect arises from additional small scattering losses imposed on the conduction electrons of a metal when they move in the presence of a strong magnetic field and the resulting Hall electric fields. In general, the effect, only important at high fields and is most pronounced when the direction of current flow is perpendicular to the field direction, as it is in magnet coils of conventional design. For current flow parallel to the direction of the field, (such as might be encountered in so-called "force-free" magnet coils[5]) the effect is substantially lower, no Hall potentials being generated in establishing the flow.

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5. H. Furth and R. W. Waniek, op. cit.

Except in respect to gross details, the theory of the magneto-resistive effect does not adequately predict the actual values observed. The qualitative behavior to be expected is an increase in resistivity which is quadratic with field at low fields, becomes linear at higher fields and finally saturates to a constant value at very high fields. Since the effect is very small, it is only observable at very low temperatures, so that experimental work is scanty in this field, and fraught with difficulty. Fortunately, however, for the metals which will be discussed in this study, relatively good data exists. Recent data on the magneto-resistance coefficient of copper and sodium exists for transverse magnetic fields in the general range of interest for this study[(6)]. In the range of the most accurate measurements, a nearly linear variation of ρ_B with B is found. The fields necessary to produce the predicted saturation effects have only barely been reached in these experiments. However, for the purpose of this study, a <u>linear</u> variation of ρ_B for all B values above those for which direct measurements exist will be assumed. This may result in somewhat overestimating the effect of magneto-resistivity at the highest fields used. But note that, in applying the results of laboratory measurements of magneto-resistance to predict the magneto-resistance effect on a large coil, it is necessary to assume that an equally favorable situation with regard to the setting up of Hall potentials can exist in the large coil as that which existed in the measurements. This will, no doubt, require experimentation and care in the design of the coil, since little information exists at present on this question.

In Table III some experimental values of ρ_B for Cu and Na are given as a function of B. It is clear from the data that sodium exhibits a far lower transverse magneto-resistance effect than copper.

It is also clear that, although it is negligible at low fields, at magnetic fields of 10^5 gauss or more, ρ_B represents a substantial, if not dominant, part of the resistivity at low temperature. However, to calculate quantitatively

6. See for example, D. K. C. MacDonald, "Handbuch der Physik", Vol. 14

TABLE III

	Cu		Na	
B	ρ_B	ρ_B/B	ρ_B	ρ_B/B
gauss	ohm-cm	ohm cm/gauss	ohm-cm	ohm-cm/gauss
10^4	4.8×10^{-9}(a)	4.8×10^{-13}	1.7×10^{-10}(c)	1.7×10^{-14}
2×10^4	9.6×10^{-9}(a)	4.8×10^{-13}		
3×10^4			5.2×10^{-10}(c)	1.7×10^{-14}
1.5×10^5	6.6×10^{-8}(b)	4.4×10^{-13}		

$\overline{\rho_{B/B}} = 4.7 \times 10^{-13}$ ohm-cm/gauss $\qquad \overline{\rho_{B/B}} = 1.7 \times 10^{-14}$ ohm-cm/gauss

(a) R. G. Chambers, Proc. Roy. Soc. A, 238, 344 (1956)
Cu @ 4^oK, $\rho(4)/\rho(300) = (1/825)$

(b) S. C. Olsen and L. Rinderez, Nature, 173, 682 (1954)
Cu @ 4^oK, $\rho(4)/\rho(300) = (1/125)$

(c) D.K.C. MacDonald, Handbuch der Physik, Vol. 14, p. 182
Na @ 4.2^oK, $\rho(4)/\rho(275) = 2 \times 10^{-4}$

the effect which magneto-resistance has on the resistance losses in a magnet coil, we need to consider the actual problem in somewhat more detail. In particular, in calculating magneto-resistance effects in a coil, we need to take into account the variation of magnetic field inside the windings themselves, since the actual increase in mean resistivity is thereby reduced. The amount of this reduction depends on the distribution of magnetic field within the windings, which in turn depends on the distribution of current.

<u>Magnet Losses</u>

Magnet coil losses represent the summation of resistive losses in each volume element of the coil[7]. Each volume element dissipates electrical energy at a rate determined by the product of the volume resistivity of the conducting material and the square of current density, i.e., as ρj^2. We will assume that either the current varies sufficiently slowly in time or that the conductors are sufficiently well subdivided that skin effects may be ignored. (This assumption is, however, not without its economic consequences.) The total energy dissipated by the coil is then simply the integral of ρj^2 over the volume of the magnet.

$$P_{MAG} = \int_v \rho j^2 \, dv \qquad (8)$$

If ρ is measured in ohm-cm and i in amperes/cm^2 then the power density of resistive losses has the units of watts/cm^3. Values of ρj^2 of the order of 10 watts/cm^3 are encountered in typical industrial magnet designs. Generally speaking, ρ is independent of current density, but may vary with position in the coil for reasons which will be discussed. Therefore, in the typical case where the current density has the same value throughout the magnet,

$$P_M = i^2 \cdot \int_v \rho dv = \bar{\rho} j^2 V \qquad (9)$$

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7. For the pioneer treatment of the general problem of high field magnets and magnet losses see F. Bitter, Rev. Sci. Inst., <u>7</u>, 379 (1936), <u>8</u>, 318 (1937) and <u>10</u>, 373 (1939). The treatment given here differs mainly in the type of problem treated, rather than in any fundamental way.

where $\overline{\rho} = 1/v \int \rho dv$, is the mean resistivity of the coil.

On the other hand, the magnetic field produced by the coil is the vector sum of the field produced by each current carrying volume element. Considering the case of long cylindrical coils, the field produced by the coil will be purely in the axial (z) direction and will simply be the summation of the contributions from successive cylindrical shells of the coil, each of which contributes a field increment

$$dB_z = \frac{4\pi}{10} \, j dr \tag{10}$$

if j is expressed in amperes per square centimeter.

Therefore,

$$B_z = \frac{4\pi}{10} \int_{r_1}^{r_2} j(r) dr \tag{11}$$

r_2 and r_1 are the outside and inside radial dimensions of the coil, respectively.

In optimizing the coil design, we will be interested in maximizing the magnetic energy density produced by the coil per unit total power consummed.

Now the magnetic energy density of the field is equal to $B_z^2/8\pi$ dynes/cm^2. Thus, from (2)

$$\frac{B_z^2}{8\pi} = S_m = \frac{2\pi}{100} \left[\int_{r_1}^{r_2} j(r) dr \right]^2 \quad \text{dynes/cm}^2 \tag{12}$$

To produce this magnetic pressure requires the expenditure of resistive losses as predicted by (8). Since we are interested in maximizing the magnetic energy density pressure per unit of energy dissipation, i.e., maximizing (12) relative to (8) , we write (8) in terms of resistive losses per unit length of the coil, $\rho_M = \dot{W}_M/L$, where L is the magnet length. The ratio of (12) to (8) becomes

$$F_m = \left[\frac{B^2/8\pi}{P_m}\right] = \left[\frac{\int_{r_1}^{r_2} j\,dr}{100\int_{r_1}^{r_2} \rho j^2\, r\,dr}\right] \quad \text{dynes cm}^{-2}/\text{watt cm}^{-1} \qquad (13)$$

F_m measures the efficiency of production of the magnetic field pressure by the coil. F_m can be written in the form which is independent of the actual coil dimensions and the magnitude of the current density. Let $u = r/r_1$, (i.e., measure all dimensions in units of the inner radius of the coil) and similarly express the variation of the current density in units of the mean current density $j(r) = \bar{j}x(r)$. F_m then takes the form:

$$F_m(\alpha, x) = \frac{\left[\int_1^{\alpha} x\,d\,u\right]^2}{100\int \rho\, x^2\, u\,d\,u} = \frac{\left[\int_1^{\alpha} x\,d\,u\right]^2}{100\,\bar{\bar{\rho}}\,(x)} \qquad (14)$$

where $\alpha = r_2/r_1$, the ratio of inner and outer coil radii, and $\bar{\bar{\rho}}(x)$ is the weighted mean resistivity of the coil.

We see immediately that F_m does not depend on the actual coil dimensions at all, but depends only on the ratio of the inner and outer coil radius and the relative spatial variations of resistivity and current density. Therefore, F_m may be used as a figure of merit for coil design which is independent of coil dimensions. Stating the result another way, at constant magnetic field the coil losses per unit length are independent of magnet size, provided the magnet is scaled proportionately. This latter fact points up the advantages which are inherent in the use of large magnets to produce high magnetic fields[8]

8. The calculation given here applies to a long solenoid. In the paper which follows this one, (UCRL-5631-T, "The Design of Large Cryogenic Magnet Coils", C. E. Taylor, Proceedings of the 1959 Cryogenic Engineering Conference, September, 1959), a more detailed calculation, pertaining to a coil of finite length is given.

If we take the current density to be proportional to some power of the relative radius, i.e., $x = u^n$, then F_m takes the form:

$$F_m(\alpha, u^n) = \frac{1}{50} \left(\frac{\alpha^{n+1} - 1}{\alpha^{n+1} + 1} \right) \frac{1}{\bar{\rho}\,(u^n)} \qquad (15)$$

For $x = 1$ (current density independent of position within the coil) this reduces to

$$F_m(\alpha, 1) = \frac{1}{50} \left(\frac{\alpha - 1}{\alpha + 1} \right) \frac{1}{\bar{\rho}}$$

$$\bar{\rho} = \left(\frac{2}{\alpha^2 - 1} \right) \int_1^{\alpha} \rho u du \qquad (16)$$

Constant current density does not necessarily correspond to the optimum coil design, as will be later discussed. Nevertheless, taking this as an assumption, we see that the maximum value of F_m, occuring when $\alpha \to \infty$ (infinite outer coil radius), is $F_m(\infty, 1) = 1/50\bar{\rho}$. At $\alpha = 3$, $F_m = 1/100\bar{\rho}$, within a factor of two of its limiting value.

The volume of the coil per unit length can also be readily expressed in terms of α. This is

$$v = \pi(r_2^2 - r_1^2) = \pi r_1^2 (\alpha^2 - 1) \qquad (17)$$

Since the cost of the magnet will be approximately proportional to its volume, it can be seen that it may not often be worthwhile to increase the value of α much beyond 3.

From the definition of F_m, the power per unit length required to produce a given value of magnetic pressure is given by:

$$P_m = \frac{B_o^2}{8\pi} \cdot \frac{1}{F_m} = \mathcal{S}_m \cdot \frac{1}{F_m} \quad \text{watts/cm} \qquad (18)$$

Alternatively, the magnetic pressure produced by the expenditure of P_m watts/cm is

$$S_m = F_m P_m \qquad \text{dynes/cm}^2 \qquad (19)$$

Also, from (8), (17) and (18), the power dissipated per unit volume of magnet material for a given plasma pressure is given by: (uniform current density case)

$$\left(\frac{P_m}{V}\right) = \frac{50}{\pi} \quad \frac{\rho}{r_1^2} \cdot \frac{S_m}{(\alpha-1)^2} \qquad \text{watts/cm}^3 \qquad (20)$$

The marked effect of scaling up the dimension r_1, or of increasing α, in reducing the dissipation per unit volume of conductor is apparent. This will be of importance in later considerations of the question of heat transport within the magnet.

We can now apply the method used in obtaining equation (14) for the coil figure of merit, F_m, to calculate the magneto-resistance correction for a long solenoid. We have, in general, that

$$F_m(\alpha,x) = \frac{\left[\int_1^\alpha x \, du\right]^2}{100 \, \bar{\rho}(x)} \qquad (21)$$

We wish to evaluate $\rho(x)$ in the presence of magneto-resistance. If we take $\rho = \rho_o + \rho_B$ with $\rho_B = \gamma B$, i.e., proportional to B. Then:

$$\rho(x) = \int_1^\alpha \left[\rho_o + \alpha \, B(u)\right] x^2 \, udu \qquad (22)$$

where B(u) is given, as can be seen from (11) by

$$B(u) = 4\pi \int_{r_1}^{r_2} j(r)\, dr = B_o \frac{\int_u^{\alpha} x\, d\, u'}{\int_1^{\alpha} x\, d\, u'} \qquad (23)$$

In the case of uniform current distribution, x = 1, $B(u) = B_o \left[\alpha - u/\alpha - 1\right]$ and

$$\bar{\rho} = \rho_o + \frac{\alpha B_o}{3} \cdot \left[\frac{\alpha + 2}{\alpha + 1}\right] = \rho_o + \bar{\rho}_B \qquad (24)$$

Since $\gamma B_o = \rho_B$ $(B = B_o)$ we see that the mean magneto-resistivity is related to the magneto-resistance at B_o through:

$$\bar{\rho}_B = \rho_B \cdot \frac{1}{3} \left[\frac{\alpha + 2}{\alpha + 1}\right] \qquad (25)$$

i.e., the mean magneto-resistance is about 1/3 of the peak magneto-resistance value.

We are now in a position to evaluate theoretically the influence of magneto-resistance on the energy dissipation in a long solenoid. Using the values of ρ_B from Table II and the values of $\rho_o(T)$ from the B-G curve we may plot the mean coil resistivity as a function of temperature and magnetic field. This is done in Fig. (1) (a) and (b) for copper and sodium respectively. The data are plotted as ratios to the standard resistivity, ρ_s, of copper at 300°K. A value of α (ratio of inner to outer coil radius) of 2 has been assumed in calculating these curves. It can be seen, however, from (25) that these results are not particularly sensitive to the value of α chosen. In these curves the role of impurity resistivity has been ignored. The conditions under which this assumption is valid are discussed later. It should also be recognized that some uncertainty, perhaps as large as a factor of two, still exists in the values of ρ_B which should apply in large conductors, so that the magneto-resistance contribution to the calculated losses may be in error by this amount.

Taking a value of B_o of 10^5 gauss, for example, it is clear that theoretical reductions in the mean resistivity of about 100 in the case of copper, and more than 1000 in the case of sodium, seem possible. Since these factors appear directly in the rates of heat dissipation per unit volume in the body of the coil, they will have a profoundly favorable influence on the problems of cooling and heat transfer within the coil. This is one of the most important features of a large cryogenic coil.

To calculate the overall reduction in power loss associated with the production of the magnetic field at any given coil temperature, we need only multiply the results of Fig. (1) by the values of G_R at that temperature, Table I. This has been done in Figs. (2), (3) and (4) for copper and sodium for different values of B_o and for different assumed refrigerator mechanical efficiencies η_R.

It can be seen from the curves that unless a sufficiently low temperature is attained, the use of refrigeration will only result in an increase in the net power loss (relative to copper at room temperature) since the energy required to run the refrigerator is not compensated sufficiently by the reduction in the coil losses. It will also be noted that all the curves exhibit minima, thus defining optimum temperatures at which to operate. For example, from the figure, it can be seen that at 10^5 gauss, the optimum operating temperature for copper coils is about $30^\circ K$, at which point an overall reduction in net power loss per unit volume of magnet conductor of about 2 is achievable.

For sodium, the potential gains are much more striking. If we take $\eta_R = 0.5$, the overall power losses predicted are only 4% of the standard value, a factor of reduction of 25 to 1. This occurs at $10^\circ K$, however, which will accentuate the refrigeration problem. Even at $\eta_R = 0.25$, however, the overall losses are only 8% of the standard.

In addition to the reductions shown in these figures, some additional "concealed" gains exist. Because of the greatly reduced internal heat transfer problems which result from the low volume dissipation rates associated with operation at cryogenic temperatures, it should be possible substantially to decrease the fraction of magnet volume devoted to coolant passages. This would increase the effective current density in the magnet, producing a higher magnetic field per unit magnet volume. The additional gain over ordinary high field coils achievable in this way can be estimated to be between 40% and a factor 2.

Some additional gains could conceivably be achieved by redistribution of the current density in the coil so as to diminish the effect of the magneto-resistance. The gains achievable in this way are slight, but may be of some interest at the highest magnetic fields. The effects of changing the distribution of current in a solenoid are most easily calculated in terms of the effect on the coil figure of merit F_m. From (22) and (23) it can be shown that when a linear magneto-resistance effect is present, if the current density is assumed to vary as u_n, then

$$B(u) = B_0 \left[\frac{\alpha^{n+1} - u^{n+1}}{\alpha^{n+1} - 1} \right] \tag{26}$$

and therefore:

$$F_m(\alpha, u^n) = \frac{1}{50\,\rho_0} \left[\frac{\alpha^{n+1} - 1}{\alpha^{n+1} + 1} \right] \frac{1/(n+1)}{\left[1 + \frac{b}{3} \left(\frac{\alpha^{n+1} + 2}{\alpha^{n+1} + 1} \right) \right]} \tag{27}$$

when $b = (\rho_B/\rho_0)$ is large, small improvements in F_m can be obtained by using positive n values[9]. These values of n correspond to the physical situation of distributing more of the current in the outer portions of the coil, where the field tends to be weaker. The gains which can be achieved in this way can be seen from Fig. (5) which presents the ratio of F_m for $n > 0$ to F_m for $n = 0$, for various values of α. Although the small reduction in coil

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9. This is in contrast to the Bitter magnet. In such a magnet design, because of the unimportant role of magneto-resistance at ordinary temperatures the most efficient coil is one in which $n = -1$, i.e., $j \sim 1/r$. See D. R. Wells, op. cit.

losses achieved in this way would seldom justify the added difficulty of tailoring the current density, in some cases the advantage of the somewhat more favorable distribution of mechanical stresses which results from re-distributing the current in this way may provide an additional incentive.

Another way, briefly mentioned earlier, in which some gains might be achieved could be through the use of "force-free" coil configurations[10], in which the direction of flow of the current is assumed to be parallel to the direction of the field inside the coil. Since the longitudinal magneto-resistance coefficient is always smaller than the transverse m- r coefficient, it would appear that an improvement might be effected. However, force-free coils are less efficient in producing a given vacuum magnetic field than a simple solenoid. Therefore, until more accurate values for the longitudinal m- r coefficient, are obtained and a detailed calculation is done, it is not possible to demonstrate that a net gain could in fact be achieved.

Calculation of Required Purity of Conductor Metal

To effect substantial gains from the cooling of magnet coils high purity metals are required. Fortunately, the metals which have been discussed, namely copper, aluminum and sodium, possess some advantages in this direction. For example, copper, being a critical material for the electrical and electronic industries, is already available commercially in high purity form. Sodium also seems to possess some advantages from the purity standpoint. To quote MacDonald[11] in an article on the electrical properties of sodium, "----- on all counts, sodium appears of all metals to be the most nearly "ideal" in (electrical) behavior -------. Sodium also occupies a unique place among the metals in accepting no other metal, (even of the alkali group) into stable solid solution; this is presumably why even readily available commercial sodium is of rather high purity." MacDonald later qualifies his remarks (with reference to the solubility of potassium in sodium), but the fact remains that the technological problem of obtaining high purity sodium is not onerous.

10. Furth and Waniek, op. cit.
11. D. K. C. MacDonald, Proc. Roy. Soc., 221, 534 (1954).

The presence of magneto-resistance has the effect of somewhat diminishing purity requirements, especially in the case of copper. That is, whenever, the impurity resistance term is small compared to the magneto-resistance term, we may disregard it. The condition for this to be true can be estimated from the magneto-resistance coefficient and the known effect of particular impurities on the resistance[(12)]. Data exists for the effect of adding known quantities of various impurities to copper. In Fig. (6) curves are plotted from these data (assuming Matthiessen's Rule) which give the allowable impurity content vs. magnetic field, for various impurities, in concentrations such that the impurity resistivity contribution is 10% of the peak magneto-resistance. At this impurity level and below, the impurity, resistance should be small enough to be ignored compared to magneto-resistance. For fields of 5×10^4 gauss or greater, the required purities are of the same order as those attained in high conductivity commercial copper.

Comparable data does not seem to exist for sodium, but it is predictable from conductivity theory that the addition of impurities will result in increases in resistivity which are of the same order as those in copper. However, since the magneto-resistance effect in sodium is about a factor of 20 less than that in copper, it can be expected that the purity requirements for sodium will be more severe by about the same factor. This still does not appear to be a requirement outside the capabilities of modern ultra-pure metal technology. In measurements reported by MacDonald[(13)] on the resistivity of alkali metals, low impurity resistance contributions were found, for specimens with spectroscopically determined purities of 99.95%. But Horsley[(14)] reports practical methods of refining sodium to purities of 99.9995% or 100 times higher purity than those reported by MacDonald in his experiments.

In the calculations to be given below it is assumed that the conductor material of the coil is of high purity and is used in the wholly annealed state. This condition can probably be closely approached, especially with

12. D. K. C. MacDonald, Handbuch der Physik, op. cit., pg. 189, and A. N. Gennetsen, Handbuch der Physik, op. cit., pg. 210.
13. D. K. C. MacDonald and K. Mendelssohn, "The Resistivity of Na at Low Temperatures", Proc. Roy. Soc., A, 202, 103 (1950)
14. G. W. Horsley, "The Purification of Sodium by Vacuum Distillation", A.E.R.E., Report M/R, 1152 (1953)

sodium, which is self annealing at low temperature. However, the fact that there still exist uncertainties of perhaps a factor of two in the basic magneto-resistance data on sodium, and the fact that really large bulk samples have not yet been tested, means that one must expect the present calculations to be uncertain by about the same amount, i.e., by about a factor of two, up or down, at high field values. It is not inconceivable that even larger discrepancies then this might arise, but it does not appear possible for them to be as large as an order of magnitude.

<u>Numerical Evaluation of Power Losses to Maintain Confining Field</u>

We may now evaluate the total power required to maintain the confining magnetic field, using the results which have been derived in preceding sections. It is convenient again to present these with reference to a "standard" coil, the dissipation of which can then be multiplied by appropriate factors to determine actual coil dissipation and net energy expenditures. The standard we shall use is a long cylindrical solenoid consisting of solid copper (coil packing fraction = 1.0) carrying a uniform current density at 300°K. The dissipation per centimeter of this coil then may be found from the expression (16) for F_m. In Fig. (7) the "standard" solenoid dissipation per centimeter is plotted <u>vs</u>. magnetic field for various values of α. It can be seen that at 300°K, coil dissipation factors are very large, amounting to some 103 kilowatts/cm at 100 kilogauss for $\alpha = 2$.

To find the dissipation of a refrigerated coil, it is only necessary to multiply the power loss of the standard coil by: (1) the reciprocal of the packing fraction and (2) the relative resistivity as given by the curves of Fig. (1). Consider, for example, a long solenoid with sodium conductors, for which $\alpha = 2$ and the packing fraction is 0.9. If operated at 10°K, at a field of 10^5 gauss the coil dissipation turns out to be only 86 watts/cm, or less than 1/1000 of the dissipation of the "standard" coil.

Similarly a copper solenoid with packing fraction of 0.9, $\alpha = 2$, operated at 30°K and producing a field of 10^5 gauss, dissipates 1800 watts/cm or about 1.8% of the power dissipated by the standard coil.

To find the overall energy cost of maintaining the magnetic field it is only necessary to multiply the coil dissipations by the value of G_R appropriate to the operating temperature. Alternatively, the dissipation rate of the "standard" coil can be multiplied by the reduction factors of Fig. (2,3,4). Thus for the sodium solenoid example above, if $\eta_R = 0.5$, the overall power loss including refrigeration turns out to be 4.6 kilowatts/cm or only 4.5% of the power loss of the standard solenoid. These calculations are of course theoretical and therefore as yet unproved experimentally, for the reasons discussed earlier.

It should be noted that in all cases of interest the factor G_R will be substantially larger than one. For example, in the Na example above, $G_R = 59$. Thus the actual power required to produce the field is only 1/58th or 1.7% of the power required by the refrigeration plant. It follows that any inefficiencies which may occur in the magnet power supply are of negligible importance in the overall power balance, as long as they do not appear as heat in the cryogenic circuit.

It should be recognized that the volume heat dissipation rates which will be encountered in the design of these refrigerated coils will normally be very small compared to usual practice. For example, in the sodium coil above, if the inner diameter of the coil were 100 cm, the dissipation rate per unit volume of conductor would be only about 0.004 watts/cm^3 or less than <u>one millionth</u> of that which is successfully used in high field magnets (such as the Bitter magnet) in the laboratory. This means that relatively simple gas cooling systems, using helium as the coolant, should be more than adequate to cope with the heat transfer problem. Aiding the heat transfer problem is the fact that the thermal conductivity of the conductors will become substantially higher at low temperatures, relative to values at room temperature. For example, at 10°K, the thermal conductivity of sodium is 20 watts/cm degree[15], as compared with a value of only 1.3 watts/cm degree at room temperature.

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15. D. K. C. MacDonald, G. K. White and S. B. Woods, Proc. Roy. Soc. A, <u>235</u>, 358 (1956)

In the discussion of coil losses, mention should be made of the problem of mechanical stresses arising from the large magnetic forces associated with the use of high magnetic fields. In small coils used in the laboratory this can become a serious problem at fields above a few hundred thousand gauss. However, some of these problems would appear to be substantially less severe for cryogenically cooled coils. First, the yield strength and ultimate tensile strength of all metals increases substantially at low temperatures, compared to values at room temperature. For ordinary metals, the increase amounts to about a factor of two, so that copper, for example, becomes comparable in yield and tensile strength to mild steel.[16] Although the strength of sodium also increases by about the same factor, it is still small, so that additional reinforcement would be required.

The second factor which alleviates the mechanical stress problem is that in large coils the mechanical stresses, which are distributed over the entire coil volume, are generally reduced when the size and relative radial thickness of the coil is increased. The net body force per unit volume exerted by the magnetic field is proportional to the gradient of the magnetic energy density within the coil windings. In a long solenoid, the main forces are therefore radial and are given by[17]:

$$dF = \frac{d}{dr} \frac{B^2(r)}{8\pi} \quad (\text{dynes/cm}^2) \text{ per cm} \qquad (28)$$

If the current density varies within the coil so that $x = u^n$ then

$$dF = \frac{1}{r_1} \frac{B_o^2}{4\pi} \left(\frac{(n+1)}{(\alpha^{n+1} - 1)}\right)^2 u^n \left(\alpha^{n+1} - u^{n+1}\right) \qquad (29)$$

This has its maximum value at $u = \alpha\ (n/2n+1)\ (n/n+1)$, where it is equal

16. R. J. Corruccini, Chemical Engineering Progress, 53, pp. 262, 342 and 397 (1957)
17. D. R. Wells, op. cit., has given a detailed analysis of the mechanical stresses in Bitter magnets, a problem which is related to the one treated here.

to

$$dF_{max} = \frac{1}{r_1}\left(\frac{B_o^2}{4\pi}\right) \cdot \frac{(n+1)^2\, n^{(n/n+1)}}{(2n+1)\ (2n+1/n+1)} \qquad \frac{\alpha^n}{\left(\alpha^{n+1} - 1\right)} \tag{30}$$

If n = 0 (uniform current distribution) dF_{MAX} occurs at the inner boundary and is equal to

$$dF_o = \frac{1}{r_1}\left(\frac{B_o^2}{4\pi}\right)\left(\frac{1}{\alpha - 1}\right) \tag{31}$$

If in equation (30), n = 1, corresponding to a redistribution of the current toward the outer radius, the maximum body force will occur within the coil, where for $\alpha = 2$, it will reach a peak value only 52% of the peak value for n = 0. This may be a worthwhile gain to exploit if operation at very large fields is contemplated.

In summary, it has been shown that cryogenic techniques offer the possibility of substantially improving the efficiency and practicality of generating high magnetic fields in air-core coils of large size. Overall reductions in power requirements of as high as 25, by comparison with conventional coils, are predicted, provided high purity conductors and efficient refrigeration cycles are used. Potential uncertainties in the basic conductivity data used in these configurations, however, introduces an uncertainty of about a factor of 2 in the expected gains.

The achievement of a factor of 10 or more reduction in the power requirements of large, high field air core magnets should have a substantial effect on their cost and practicality for high energy accelerators and related activities. Looking to the future, such gains can be shown also to be of potentially great practical importance if fusion power reactors using externally generated magnetic fields to confine the reacting plasma can be perfected.

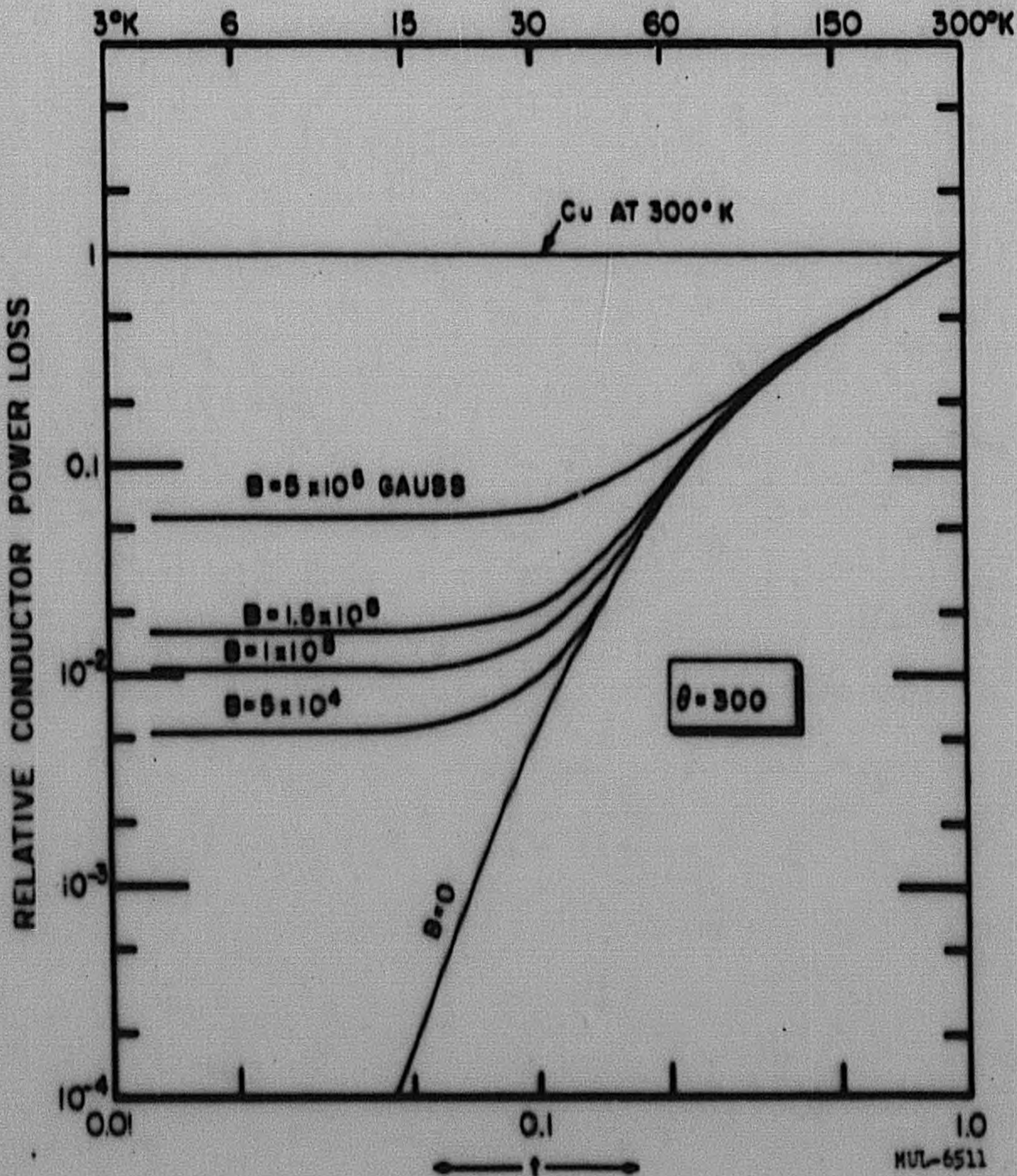

3°K
6
15
30
60
150
300°K
Cu AT 300° K
1
0.1
B=5x10^5 GAUSS
B=1.6x10^5
B=1x10^5
10^-2
B=5x10^4
θ=300
10^-3
B=0
10^-4
0.01
0.1
1.0
t
RELATIVE CONDUCTOR POWER LOSS
MUL-6511

Fig. 1a

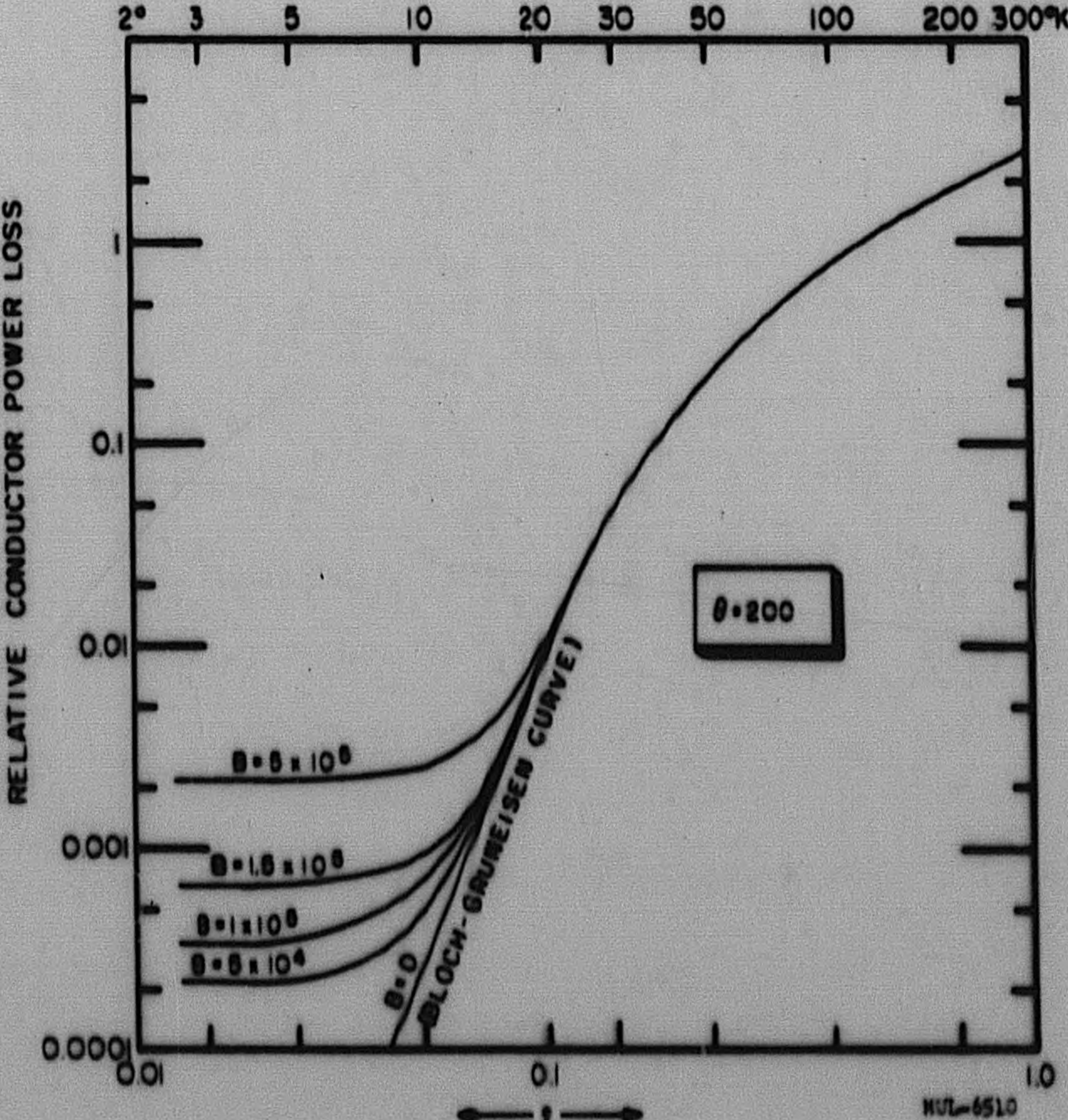

2° 3 5 10 20 30 50 100 200 300°K
RELATIVE CONDUCTOR POWER LOSS
1
0.1
0.01
0.001
0.0001
0.01
0.1
1.0
θ=200
B=0 (BLOCH-GRUNEISEN CURVE)
HUL-6510

Fig. 1b

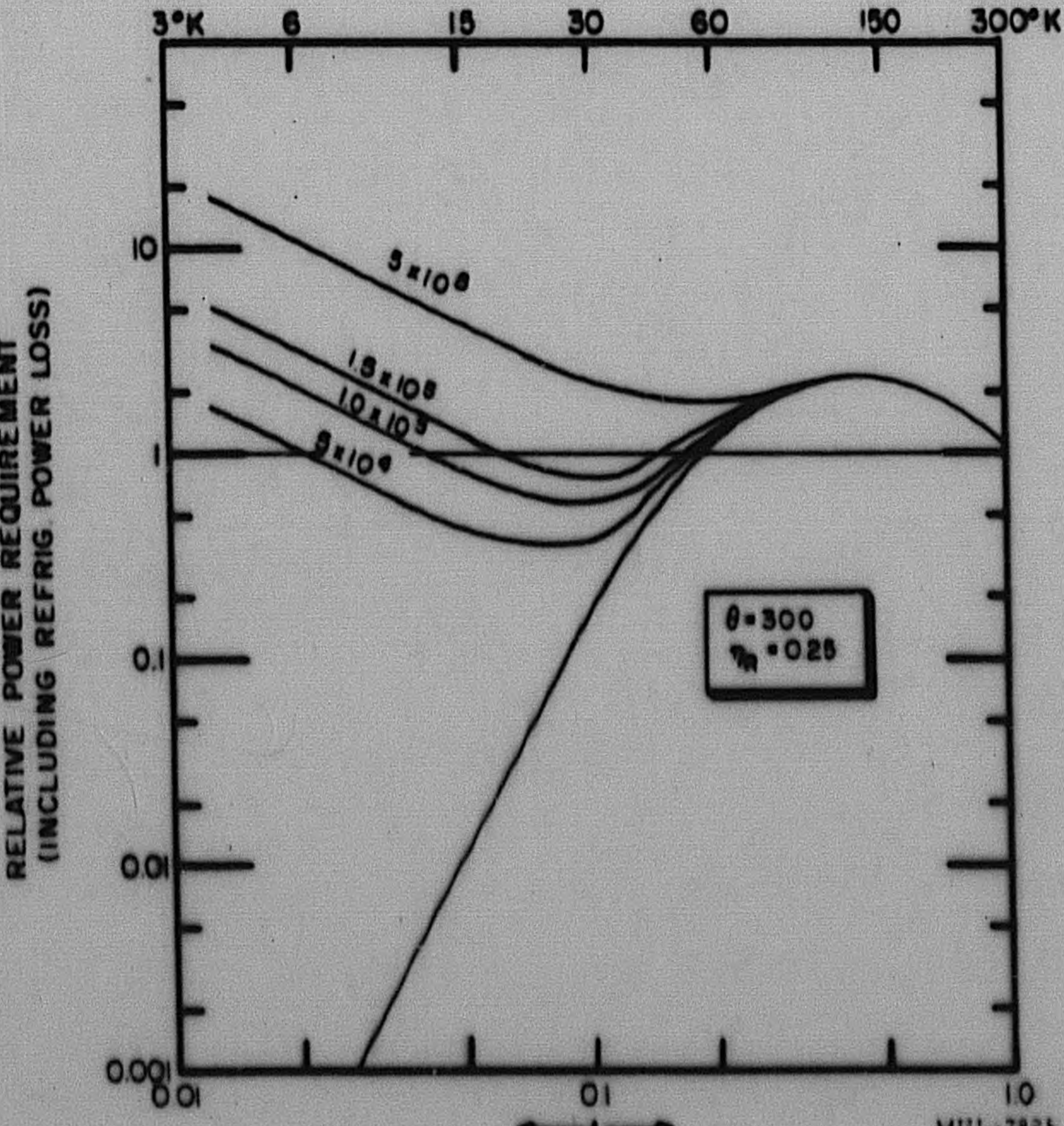

3°K
6
15
30
60
150
300°K
RELATIVE POWER REQUIREMENT
(INCLUDING REFRIG POWER LOSS)
10
1
0.1
0.01
0.001
5×10⁶
1.5×10⁵
1.0×10⁵
5×10⁴
θ=300
η_R=0.25
0.01
0.1
1.0
t
MUL-7825

Fig. 2

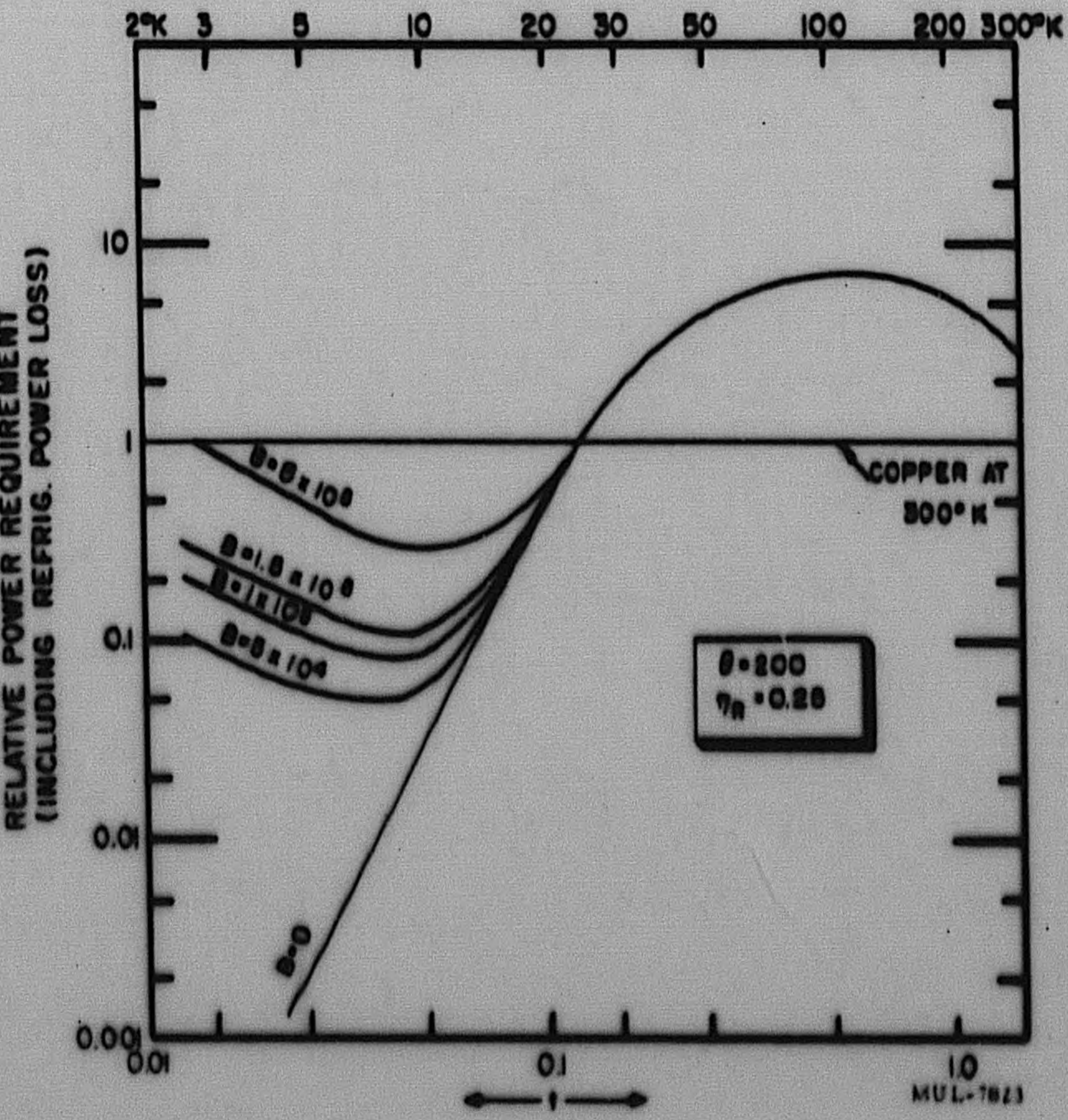

2°K
3
5
10
20
30
50
100
200
300°K
RELATIVE POWER REQUIREMENT
(INCLUDING REFRIG. POWER LOSS)
10
1
0.1
0.01
0.001
0.01
0.1
1.0
COPPER AT
300°K
t

Fig. 3

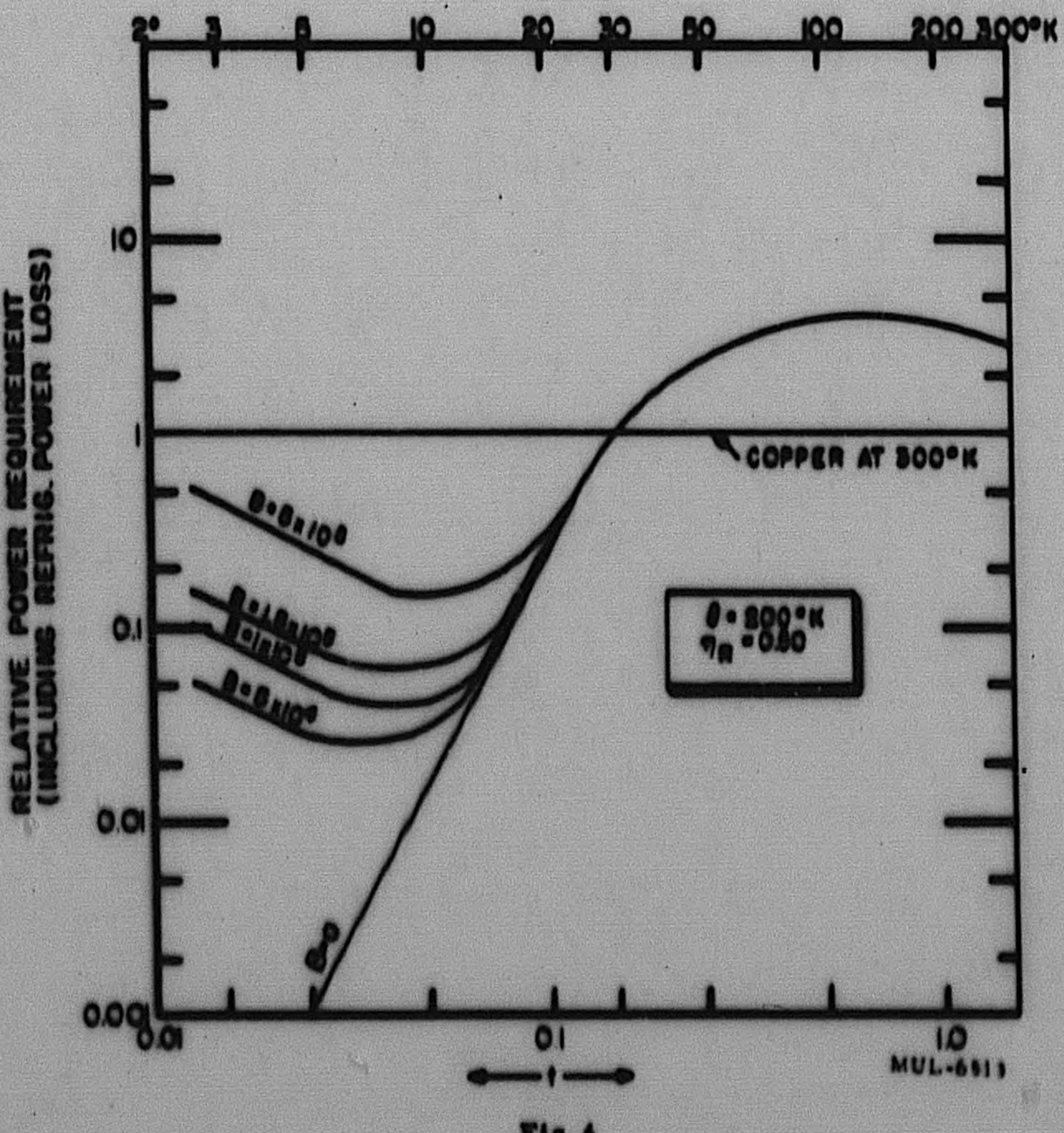
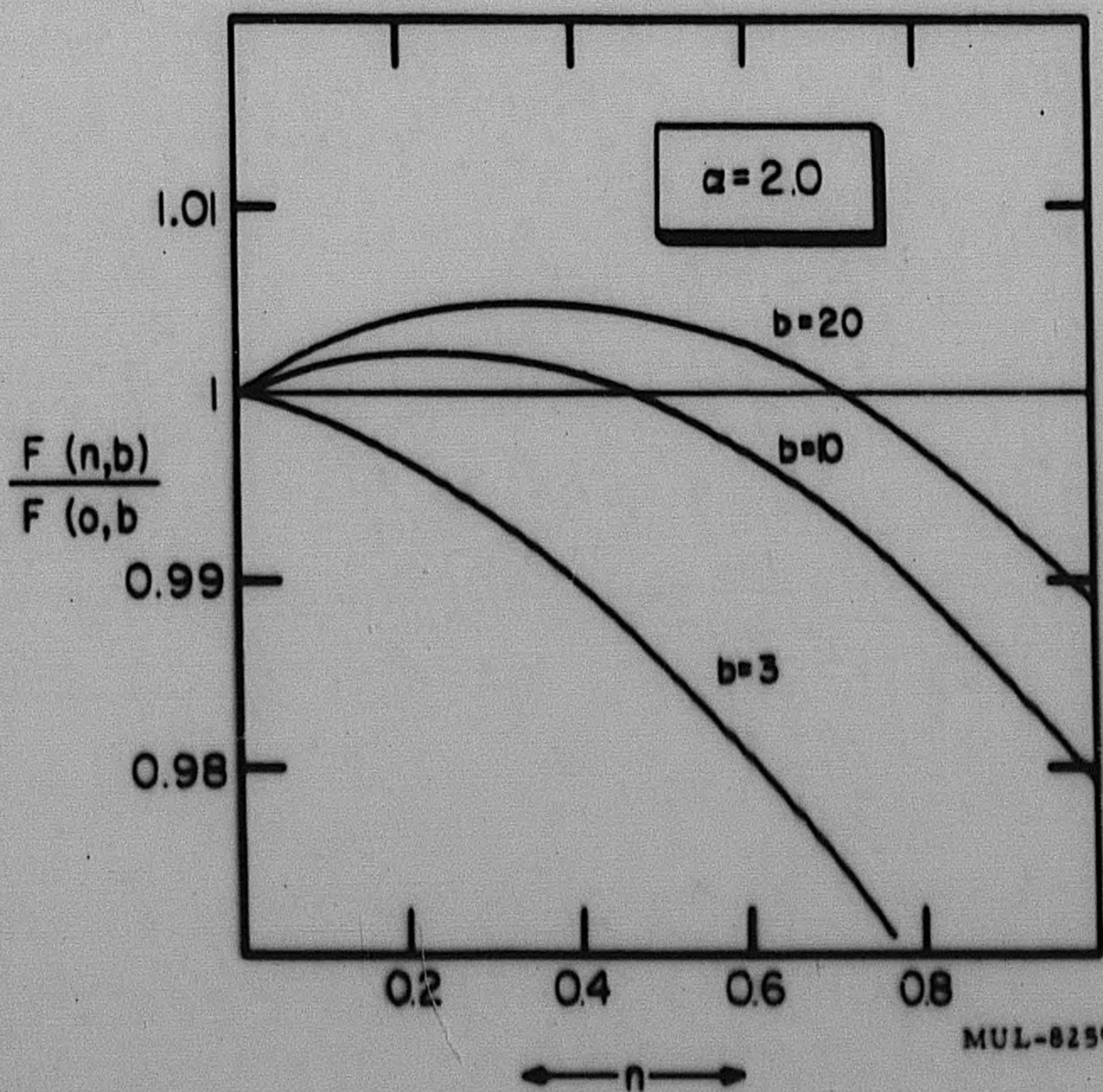
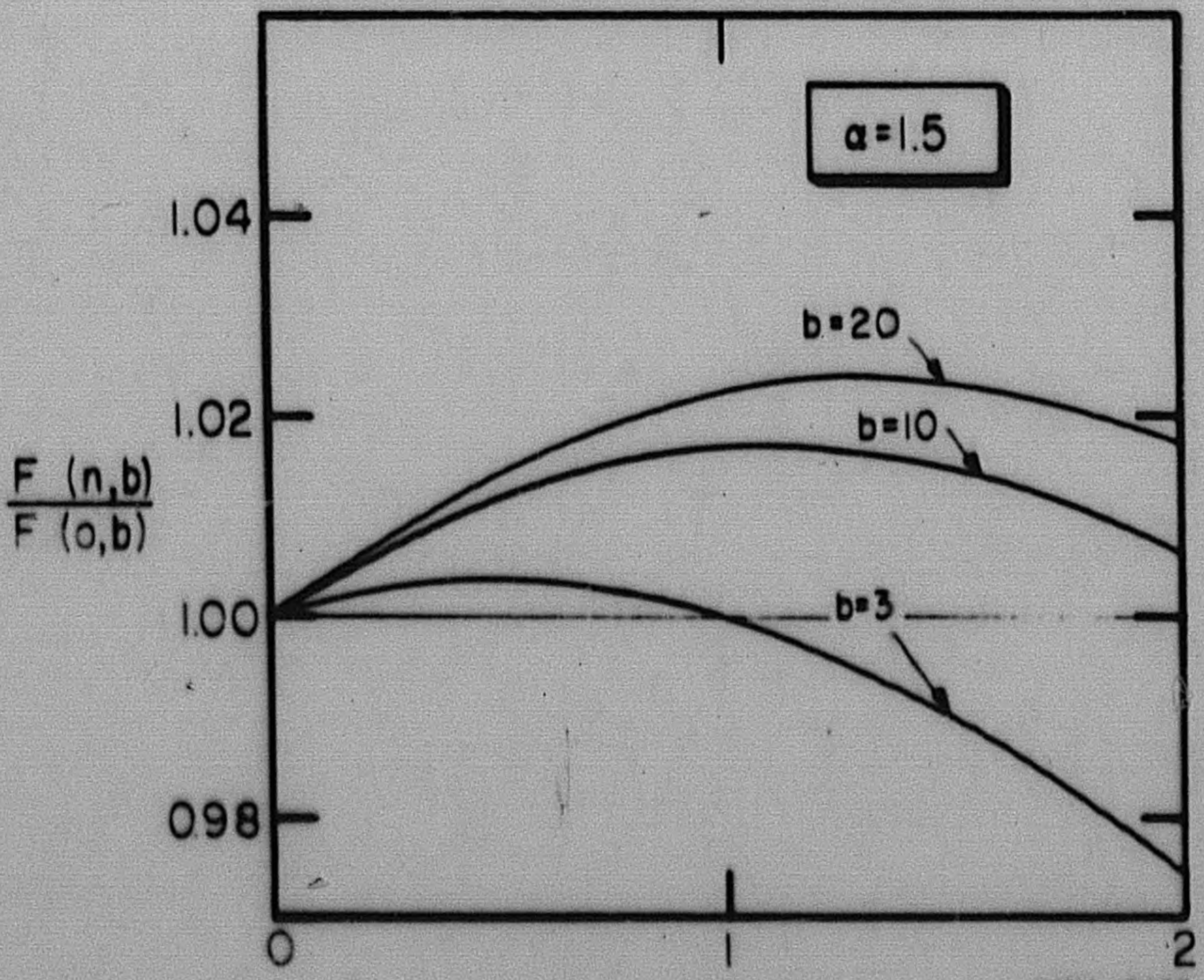

2° 3 5 10 20 30 50 100 200 300°K
RELATIVE POWER REQUIREMENT
(INCLUDING REFRIG. POWER LOSS)
10
1
0.1
0.01
0.001
0.01
0.1
1.0
COPPER AT 300°K
θ = 0
t
MUL-6513

Fig. 4



MUL-8259

Fig. 5a



MUL-8258

Fig. 5b

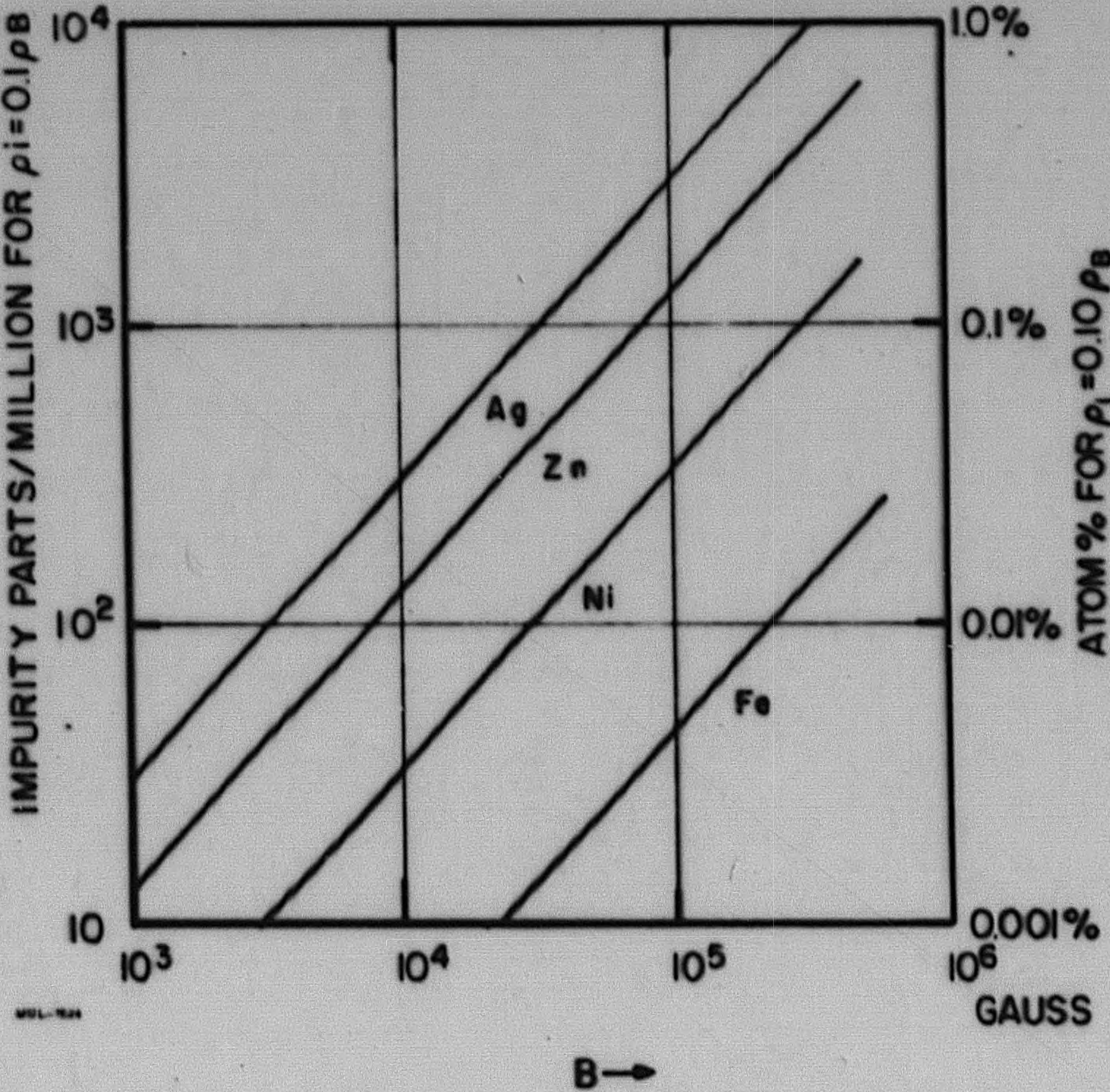

IMPURITY PARTS/MILLION FOR ρi=0.1ρB
ATOM % FOR ρi=0.10ρB
10⁴
10³
10²
10
1.0%
0.1%
0.01%
0.001%
10³
10⁴
10⁵
10⁶
GAUSS
B→
Ag
Zn
Ni
Fe

Fig. 6

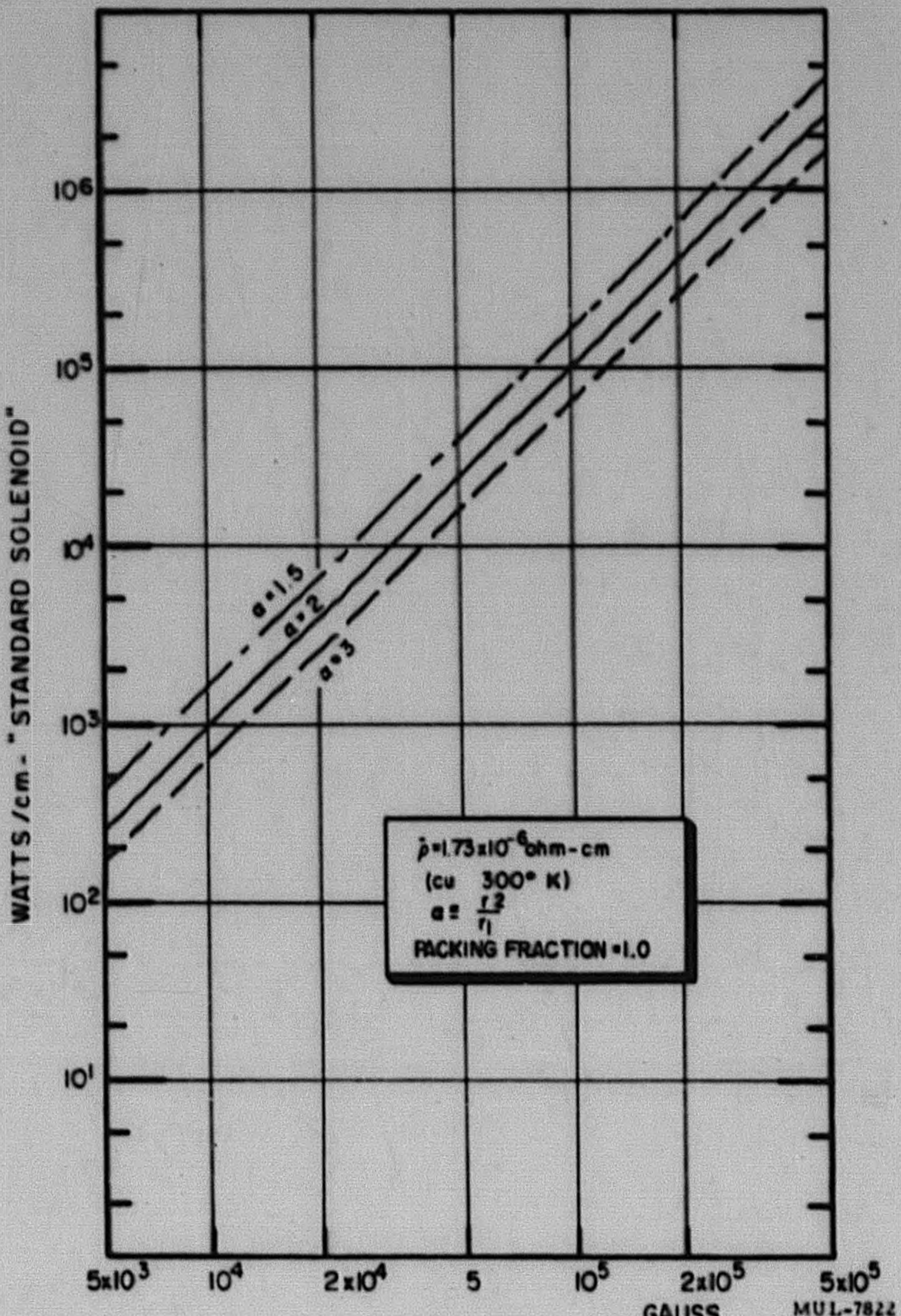

WATTS/cm- "STANDARD SOLENOID"
10^6
10^5
10^4
10^3
10^2
10^1
a=1.5
a=2
a=3
ρ=1.73x10^-6 ohm-cm
(cu 300° K)
a= r2/r1
PACKING FRACTION =1.0
5x10^3
10^4
2x10^4
5
10^5
2x10^5
5x10^5
GAUSS
MUL-7822

Fig. 7