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THEORETICAL ANALYSIS OF THE HYDRODYNAMIC FLOW
IN EXPLODING WIRE PHENOMENA

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Phenomena

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ABSTRACT

Theoretical calculations of exploding wire phenomena have been carried out with an IBM 704 Lagrangian code. The model for this analysis assumes instantaneous energy deposition in the wire. Calculations were made with different conditions. Among those presented are (1) different effective equations of state for the copper wire and (2) different equations of state for the surrounding air. The equations of state for the air are (a) a constant gamma-law gas and (b) the variable gamma-law gas calculated by F. R. Gilmore. The shock wave propagation in each is shown to require significantly different energies. The results are compared to calculations made without copper wire (energy assumed deposited in a volume of air equal to that of the wire) and with an analytic similarity solution for cylindrical shock waves. The propagation of the shock wave in air calculated with the Gilmore equation of state for air shows excellent agreement with that observed by F. D. Bennett. The second shock wave in the copper shows qualitative agreement with that observed by Bennett, but indicates that in the range of EWP, copper does not act like a constant gamma-law gas. As for similarity, it is shown that hydrodynamic flow during the explosion of wires is not similar.

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in Exploding Wire Phenomena

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I. INTRODUCTION

The complete analysis of exploding wire phenomena (EWP) from the instant current begins to flow represents an extremely difficult task for two major reasons, namely, (1) the present state of the art in analytical techniques and (2) the present level of knowledge of physical phenomena in the range of EWP. Nevertheless, present computational techniques with IBM 704 Digital Computers and more exact knowledge of the physical properties of air¹ make possible theoretical analysis of certain aspects of EWP with a high degree of accuracy within the framework of an idealized model.

The model used will be described as follows. At $t = 0$ it is assumed that the energy is deposited uniformly in the wire, which is surrounded by air at STP. End effects, radiation diffusion, thermoconductivity, real viscosity, mixing of metal vapor and air, and magnetic effects are neglected. Complete axial symmetry is assumed at all times. No dark pulse is considered. This paper will describe (1) the method of computation; (2) the results of several calculations; (3) an analytical similarity solution for cylindrical shocks; (4) a comparison of the numerical and analytical results; and, finally, (5) a comparison with experiment.

II. CALCULATIONS

The equations that were solved numerically with the IBM 704 Lagrangian

code used for this analysis are similar to those given by R. D. Richtmeyer² which are as follows

$$\frac{\partial u}{\partial t} = - \frac{1}{\rho_0} \left(\frac{\partial (p r)}{\partial r} \right) \quad (1)$$

$$\frac{\partial \epsilon}{\partial t} = \frac{p}{\rho_0} \frac{\partial \rho}{\partial t} \quad (2)$$

$$\frac{\partial R}{\partial t} = u \quad (3)$$

and with

$$\frac{1}{\rho} = \frac{1}{\rho_0} \left(\frac{\partial (R r)}{\partial r} \right) \frac{\partial R}{\partial r} \quad (4)$$

$$p = p(\xi, \rho) \quad (5)$$

where

$r = R$ = coordinates of a small fluid element at time $t = 0$

$R = R(r, t)$ is the R -coordinate of the same fluid element at time t

u = fluid velocity = $u(r, t)$

p = fluid pressure = $p(r, t)$

ρ = fluid density = $\rho(r, t)$

ϵ = internal energy per unit mass = $\epsilon(r, t)$

The boundary conditions used are

$$u(0, t) = 0$$

$$u(r, 0) = 0$$

$\epsilon(r, 0) = \epsilon_{cu}^0$ for $r \leq r_0$, the initial radius of the copper wire, and ϵ_{cu}^0 equals the energy density assumed in the wire.

$$\begin{aligned}
 &= \epsilon_{\text{air}}^0, \text{ for } r > r_0, \text{ the initial energy density in air} \\
 \rho(r,0) &= \rho_{\text{cu}}^0 \text{ for } r \leq r_0, \text{ where } \rho_{\text{cu}}^0 = \text{initial density of the wire} \\
 &= \rho_{\text{air}}^0 \text{ for } r > r_0, \text{ where } \rho_{\text{air}}^0 = \text{initial density of air.} \\
 p(r,0) &= f \left[\epsilon(r,0), \rho(r,0) \right]
 \end{aligned}$$

An explicit differencing scheme was used to solve Eqs. (1) through (4) similar to that shown by Richtmyer (Reference 2, p. 194) with the use of the code stabilizing artificial viscosity term added to the pressure^{3,4}

$$\eta = \begin{cases} -I \rho \frac{\partial u}{\partial r} \left| \frac{\partial u}{\partial r} \right|, & \text{if } \frac{\partial u}{\partial r} < 0 \\ 0, & \text{if } \frac{\partial u}{\partial r} \geq 0, \end{cases} \quad (5)$$

where I is an arbitrary constant.

With an assumed energy deposition of 20 joules/cm in a 5 mil diameter copper wire, initial conditions are as follows:

$$r_0 = 0.00635 \text{ cm}$$

$$\epsilon_{\text{cu}}^0 = 2.210 \times 10^{12} \text{ ergs/cm}^3 = 2.210 \times 10^5 \text{ joules/cm}^3$$

$$\epsilon_{\text{air}}^0 = 2.50 \times 10^5 \text{ ergs/cm}^3 = 0.250 \text{ joules/cm}^3$$

$$\rho_{\text{cu}}^0 = 8.90 \text{ g/cm}^3$$

$$p_{\text{cu}}^0 = 1.105 \times 10^{12} \text{ dynes/cm}^2, \text{ for } \gamma_{\text{cu}} = 1.50$$

$$\rho_{\text{air}}^0 = 1.293 \times 10^{-3} \text{ g/cm}^3$$

$$p_{\text{air}}^0 = 10^6 \text{ dynes/cm}^2$$

$$\text{Initial volume} = 1.267 \times 10^{-4} \text{ cm}^3/\text{cm of wire}$$

$$\text{Initial mass of wire/cm} = 1.126 \times 10^{-3} \text{ g/cm of wire}$$

$$\begin{aligned}
 &\text{Radius of cylinder of air equal in mass to that of copper} \\
 &= 0.527 \text{ cm.}
 \end{aligned}$$

For the copper, the equation of state assumed is a simple gamma-law gas with effective values of $\gamma = 1.67, 1.5, 1.2, \text{ or } 1.1$. For air, calculations were made with a simple gamma-law gas equation of state with $\gamma = 1.4$ and the theoretical equation of state calculated by F. R. Gilmore⁴. His values of γ for various values of temperature and density ratios are given in Appendix A for convenience.

III. RESULTS OF EXPLODING WIRE CALCULATIONS

The first set of calculations presented are those made with an assumed gamma-law gas equation of state for the copper with $\gamma = 1.5$ and two different equations of state for air, namely, a constant gamma-law gas equation of state with $\gamma = 1.4$ and the theoretical equation of state calculated by F. R. Gilmore. Gilmore's equation of state assumes an ideal gas mixture in chemical equilibrium allowing for dissociation and ionization. Figure 1 shows the results in a plot of the position of the shock wave during the first $4 \mu\text{sec}$. Plotted is $(2R)^2$ vs t , where R is the position of shock front and t is time in μsec . Also shown are the experimental points of F. D. Bennett⁵ for the 5-mil copper wire. Behind the shock fronts are plotted the luminous "contact surface" from Bennett's experiment and the calculated metal vapor-to-air contact surface. (Hereafter, "contact surface" will refer to this metal vapor-to-air surface -- "luminous contact surface" will refer to Bennett's experimental surface). In a Lagrangian code, this contact surface is easily followed during the hydrodynamic flow. Physically, this surface would not be the ideal one represented in our calculations after 3 or $4 \mu\text{sec}$ but the calculated surface should be a good approximation up to that time and after 3 or so μsec , this calculated surface should approximate the points of mean density between air and copper. In both calculated shocks, the initial energy

deposition in the wire is 28 joules/cm. Figures 2 and 3 give pressure profiles and density ratio and velocity profiles, respectively, at 1.5 and 2.5 μ sec for the constant gamma-law air. Figures 4 and 5 give the corresponding profiles for the same times using the Gilmore equation of state for air (henceforth referred to as the "Gilmore air"). Figure 6 gives relative density profiles for the copper at 1.5 and 2.5 μ sec for the Gilmore air case only. Figure 7 presents values of gamma and the corresponding temperatures of the air at 1.5 μ sec.

In an attempt to reproduce the second shock wave observed by Bennett⁵ in the explosion of the 6.3-mil copper wire, the calculations were carried out to 15 μ sec using three different values of gamma for the copper, namely, 1.1, 1.2, and 1.67. The results are plotted in Figure 8 in R-vs-t curves. The calculated contact surfaces are also shown. Figure 9 shows pressure, velocity and relative density profiles in the copper gas, $\gamma = 1.2$, at $t = 5.03$, 8.89, 10.06 and 14.95 μ sec.

IV. ANALYTICAL SIMILARITY SOLUTIONS FOR A CYLINDRICAL SHOCK WAVE

In order to compare the above results to those expected from a similarity solution for a cylindrical shock wave, a generalized analytical solution, for plane, cylindrical or spherical shock waves was derived.* The method of integrating the equations was that used by R. Latter.⁶ Consequently, in Appendix B only the salient features of the analysis are given. The resulting pressure, density and particle velocity are given implicitly for

*After this analysis was completed, the author found the same work done by Akira Sakurai, J. of the Phys. Soc. of Japan, 10 827 (1955). This paper however, does not contain any calculations or curves.

cylindrical shock waves by the following expressions.

r = Eulerian coordinate

R = position of the shock front

$$x = \frac{r}{R}$$

$$p(r,t) = p_0 A^2 \frac{F(x)}{R^2}$$

$$\rho(r,t) = \rho_0 \psi(x)$$

$$u(r,t) = A \frac{\Phi(x)}{R}$$

$$\dot{R}R = A$$

where ρ_0 = density of the fluid in front of the shock wave = 1.293×10^{-3} g/cm³,
and A = constant = 2.337×10^5 cm²/sec for $\gamma = 1.4$ and $E = 28 \times 10^7$ ergs/cm
and (with $\gamma = 1.40$),

$$x = \frac{0.95401 \left| \frac{\Phi}{x} - \frac{1}{\gamma} \right|^{(\gamma-1)/2\gamma}}{\left| \frac{\Phi}{x} \right|^{1/2} \left| \frac{\Phi}{x} - \frac{2}{\gamma} \right|^{1/2}} \quad (7)$$

$$\frac{F(x)}{\psi(x)} = \frac{\gamma-1}{2\gamma} x^2 \frac{\left(\frac{\Phi}{x} \right)^2 \left(1 - \frac{\Phi}{x} \right)}{\left(\frac{\Phi}{x} - \frac{1}{\gamma} \right)} \quad (8)$$

$$\psi^{1-\gamma} = - \frac{0.067828}{x \left(\frac{\Phi}{x} - x \right) F(x)} \quad (9)$$

and

$$E = 2\pi \rho_0 A^2 \int_0^1 x \left(\frac{F}{\gamma-1} + \frac{1}{2} \psi \Phi^2 \right) dx \quad (10)$$

The resulting curves for $F(x)$, $\psi(x)$, and $\frac{1}{2}(x)$ are shown in Figure 10. Also shown are the results of a numerical solution of the similarity equations of S. C. Lin.⁷ For values of $\frac{1}{2}/x$ near $1/\gamma$, Eq. (7) reduces to (for x near zero)

$$\frac{1}{2} \approx \frac{1}{\gamma} x + 0.1319 x^2 \quad (11)$$

This expression for $\frac{1}{2}$ substituted into Eqs. (8) and (9) will give the limiting expressions for F and ψ near the origin. Integrating Eq. (10) using the curves in Figure 10 and Eq. (11), yields

$$E = 3.965 \rho_0 A^2 \quad (12)$$

and from $Rt = A$,

$$R = (2At)^{1/2} = 1.0035 \left(\frac{E}{\rho_0} \right)^{1/4} t^{1/2} \quad (13)$$

From Eq. (13), a $(2R)^2$ vs- t graph with R in cm and t in μ sec should give a slope $m = 1.878$, if $E = 28$ joules/cm and $\rho_0 = 1.291 \times 10^{-3}$ g/cm³.

To check this approach with the numerical solution where the energy is deposited in a finite volume, a problem was run as described in Section II except that copper was replaced by air. Figures 11 through 15 show the results. Both a constant gamma-law air and the Gilmore air were used.

V. DISCUSSION OF THEORETICAL RESULTS

The theoretical results shown in Figs. 1 and 11 indicate quite conclusively that the similarity solution is, as expected, inadequate to explain the hydrodynamic flow in exploding wire phenomena. Furthermore, the assumption of a constant gamma-law equation of state for the air is also inadequate. The details of the deviations are brought out by comparison of various combinations of results. For Example, Figs. 2, 10, and

12 point up the differences in the magnitudes of the shock-front pressure as well as the variation of pressure behind the shock front. Figures 3, 10, 13 and 14 show the differences in the densities and velocities. And so on.

The next point of interest is just how accurate are the numerical calculations. The results that will give an answer are the computations with the energy deposition in air alone and using the constant gamma-law equation of state for air with $\gamma = 1.4$. Using Eq. 12, with $E = 28 \times 10^7$ ergs/cm and $\rho_0 = 1.293 \times 10^{-3}$ g/cm³, yields $A = 2.337 \times 10^5$ cm²/sec. This leads to shock-front pressures and velocities of

$$p(R) = \frac{5.864 \times 10^7}{R^2} \quad (14)$$

$$u(R) = \frac{1.947 \times 10^6}{R} \quad (15)$$

where R is the shock-front position in cm. Using the results of Lin's numerical solution to the above similarity equations, the equation corresponding to Eq. 12 is

$$E = 3.85 \rho_0 A^2$$

from which

$$A_L^2 = 5.62 \times 10^{10}$$

and

$$A_L = 2.37 \times 10^5,$$

where the subscript L is used to indicate Lin's result. The equations corresponding to Eqs. 14 and 15 are, respectively,

$$p_L(R) = \frac{6.05 \times 10^7}{R^2} \quad (16)$$

$$u_L(R) = \frac{1.97 \times 10^6}{R} \quad (17)$$

Table I gives the values of $p_{\max}$, $u_{\max}$, and $\psi_{\max}$ for two values of R . The theoretical $\psi_{\max}$ is the boundary condition on ψ at the shock front and equals $\gamma + 1/\gamma - 1 = 6.00$ for $\gamma = 1.4$. These similarity results are compared with the present calculations in the code. During the first $4 \mu\text{sec}$, the values of $\psi_{\max}$ on the print outs varied from about 5.4 to 5.922. Clearly, the differencing mesh used is quite adequate.

TABLE I

Comparison of Similarity Results with Code

	$t = 1.5 \mu\text{sec}$ $R = 0.854 \text{ cm}$			$t = 2.5 \mu\text{sec}$ $R = 1.097 \text{ cm}$		
	Code	Lin	Rouse	Code	Lin	Rouse
$p_{\max}/10^8$	0.823	0.830	0.807	0.490	0.504	0.490
$u_{\max}/10^6$	0.235	0.231	0.235	0.184	0.180	0.185
$\psi_{\max}$	5.71	6.00	6.00	5.59	6.00	6.00

Referring again to Fig. 11, the slope of the curve using the constant γ -law air is, from the present analysis, 1.878 and from Lin's analysis, 1.90. To the accuracy of Lin's work, this agrees very well. However, evidence that 1.878 is nearer the limiting value for the hydrodynamic flow is this. It was noticed that as the number of zones used in the initially cold air was increased from 20 to the present 100 out to a radius of 1.5 cm, the slope of the curve $(2R)^2$ vs t in Fig. 11 for $\gamma = 1.4$ decreased from 2.19 to 1.90 and the intercept with the t -axis increased from about -0.2 to $-0.02 \mu\text{sec}$. In Fig. 17 a plot of the reciprocal of the number of zones and the corresponding slopes for the runs with constant γ -law air and Gilmore air indicate that the limit of an infinite number of zones would result in a slope, m , between 1.85 and 1.90. Therefore, with respect to the present analytical results the calculated shock-front plotted as $(2R)^2$ vs t will

lead to an error in the slope of - 1 percent. This certainly is quite sufficient for this investigation.

Now, using a more realistic equation of state for air, the Gilmore results, the propagation of the shock wave is reduced considerably. Assuming the energies are proportional to m^2 , it would take about 46.2 joules/cm using the Gilmore air - a 65% increase - in order to match the slope of the constant gamma-law air. Surprisingly, though, Figs. 15 and 16 show that the flow behind the shock front is close to similarity, except for the magnitudes at the shock front.

Turning now to the exploding wire model, the addition of the copper had the opposite effect on the shock wave propagation that the Gilmore air had. The results, shown in Fig. 1, shows the increase in the shock wave propagation. With constant gamma-law air, it would take a 39% increase in the energy deposition in the similarity solution for air alone to match the shock wave propagation with copper. With Gilmore air alone, it would take an 85% increase in the energy to match the corresponding results with copper! In fact, assuming that the relative shock wave energies are proportional to m^2 and starting with $m = 1.90$ corresponding to an energy of 28 joules/cm, decreasing 28 joules/cm by 60.6% to 17 joules/cm to allow for the use of the Gilmore air and then adding 82% of this or 13.9 joules/cm, to allow for the copper, results in 30.9 joules/cm. Going directly from constant gamma air alone to the copper and Gilmore air by increasing the 28 joules/cm by 11.1% gives 31 joules/cm. Consequently, the error in interpreting the propagation of shock waves of the strength discussed here produced by exploding wires in terms of similarity results amounts to an error of about 10% in spite of the gross misrepresentation of the physics!

The next interesting aspect of this exploding wire analysis is the calculation of the second shock front observed by Bennett in his photograph

of the exploded 6.3-mil copper wire. The results are shown in Fig. 8 in a plot of r vs t for the contact surface and the second shock front where the theoretical curves are shown for calculations using three different values of effective γ for the copper, namely, 1.67, 1.2, and 1.1. Figure 9 shows profiles of the pressure, velocity, and relative density out to the contact surface at time $t = 5.03, 8.89, 10.06, \text{ and } 14.95 \mu \text{ sec}$ for the case, $\gamma_{\text{cu}} = 1.2$. It appears that the observed wedge luminosity is due to the reflection of the weaker inward moving shock at the axis at about $2.5 \mu \text{ sec}$ with a receding shock front moving outward from the axis. This verifies Bennett's interpretation.

Of greatest interest about the calculated second shock is the dependence of its propagation on the equation of state assumed for the copper. Clearly, the real γ of the equation of state for copper would vary, but with little information on this quantity, calculations with constant γ s and good fits with experimental data should give a hint of the values of γ in the range of EWP. The present calculations by themselves however, are insufficient to determine the temperature associated with this second shock. For equating $(\gamma - 1) \epsilon_{\text{cu}}$ to nkT , one is still left with two unknowns, n and T . An additional bit of information that may help to pin down an associated temperature is a measurement of the radiant energy emitted by the exploding wire as a function of time. Matching this with calculations using temperature-dependent opacities for the copper and the air in a consistent calculation of the hydrodynamic flow perhaps should lead to a more exact understanding of the equation of state of the metal in the temperature range of exploding wire phenomena. Naturally, other materials besides copper can be subjected to the same experimental and theoretical analysis.

VI. DISCUSSION OF EXPERIMENTAL RESULTS

In Fig. 1, the experimental points of Bennett's case (No. 83) are plotted. Shown are his points for the shock front and the luminous contact surface. Through the shock-front points is drawn a line parallel to the theoretical line. It was located by giving more weight to the points beyond $2.4 \mu\text{sec}$. Dr. Bennett's data gives a slope of 2.00 ± 0.05 . Translating the theoretical line to the right $0.14 \mu\text{sec}$ would cause the theoretical and experimental lines to coincide. Hence, if the wire was completely gasified $0.14 \mu\text{sec}$ after the close of the switch, the shock front in air would propagate exactly as calculated in the present work. The difference is probably due partly to the time dependence of the energy deposition as is evident from the voltage oscilloscope traces shown in the references.

Another factor affecting the coincidence of the calculated and experimental shock fronts is the equation of state of the copper. The calculations above using an effective gamma of copper equal to 1.2 resulted in the displacement of the linear portion of the theoretical curves in Fig. 1 to the right. When experimentally measured power-vs-time data is obtained simultaneously with shock-front measurements, then calculations with this time-dependent energy dump will help separate the effect of the equation of state of the copper on the propagation of the shock front in air. When such data is available, more realistic calculations can be made with a time-dependent energy input.

To normalize the luminous contact surface to the calculated contact surface, the calculated points should be translated to the right $0.14 \mu\text{sec}$ as indicated by the short arrows. This shows that after about $1.5 \mu\text{sec}$, the calculated surface falls considerably behind the luminous surface. At $1.64 \mu\text{sec}$, the luminous surface is at about $r = 0.678 \text{ cm}$. In Fig. 7, the temperature profiles from the calculations at $1.5 \mu\text{sec}$ - equivalent to the measured surface at about $1.64 \mu\text{sec}$ - indicates that out to about 0.71 cm

the air is hot enough to emit. However, the energy flux needed to affect the film with grain densities different from background would not extend that far and could be near the experimental luminous surface. This problem needs further theoretical and experimental study.

The second shock wave of interest is the one evident in reference 5 as a luminous cone receding from the axis beginning at about $8.5 \mu\text{sec}$. The spread of the measured edges of this shock front is indicated by the experimental points in Fig. 8. Note the agreement with the calculated second shocks in three different areas: (1) with $\gamma = 1.67$ between 4.5 to $7.75 \mu\text{sec}$; (2) with $\gamma = 1.1$ near $8.5 \mu\text{sec}$ and, (3) with $\gamma = 1.2$ between 9.5 and $15 \mu\text{sec}$. Considering the model used in these calculations and the manner in which the equation of state of the copper is handled, this agreement is interesting. Clearly, a constant gamma-law gas equation of state for the copper is not sufficient to explain the behavior of this second shock. This problem will be given more study.

VII. SUMMARY AND CONCLUSIONS

1. Dr. Bennett's technique of rendering visible the shock trajectory in air appears to be quite accurate.
2. Hydrodynamic flow during the explosion of wires is not similar. The relatively small error obtained by the use of the similarity solution to determine the energy of the shock wave in air is quite accidental: the large effects of the exact equation of state of the air and the presence of the wire material tend to cancel each other.
3. The theoretical model for exploding wire phenomena presented in Section I of this paper appears to be quite satisfactory for accurate calculations of two aspects of the problem, namely, the propagation of the shock wave in air and the propagation of a second shock wave that occurs in the metal vapor. The theoretical equation of state for air calculated by

F. Gilmore removes the uncertainty inherent in assuming that the air acts as a constant gamma-law gas with a gamma of 1.4 in the range of exploding wire phenomena. An important remaining theoretical uncertainty is the equation of state of the wire material. However, more observations of the second shock wave and calculations such as those in this paper should give information in various temperature and pressure ranges. In addition, accurate measurements of the metal vapor to air contact surface by some clever technique during the early stages of the explosion coupled with similar calculations should yield results in a different temperature and pressure range. Possibly a radiation flux measurement could also serve as a boundary condition on temperature calculations. And of course, all of the calculations would have to be consistent with an accurate temperature determination as, for example, with some spectroscopic technique. Theoretical and experimental work in this area should be continued.

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APPENDIX A. The Gilmore Equation of State for Air

For convenience, some of Gilmore's results in reference 1 are reproduced here. Shown are his Fig. 6 and Tables 11, 14, and 15.

Fig. 6. Effective specific heat ratio for shocks.

TABLE 11. PRESSURE OF DRY AIR IN CHEMICAL EQUILIBRIUM

TEMPERATURE °K	DENSITY RATIO ρ/ρ_0									
	10	10 ⁰	10 ^{-1/2}	10 ⁻¹	10 ^{-3/2}	10 ⁻²	10 ^{-5/2}	10 ⁻³	10 ⁻⁴	10 ⁻⁵
PRESSURE, ATMOSPHERES										
1000	3.663 ⁺¹	3.663 ⁰		3.663 ⁻¹		3.663 ⁻²		3.663 ⁻³	3.663 ⁻⁴	3.663 ⁻⁵
2000	7.322 ⁺¹	7.330 ⁰		7.330 ⁻¹		7.330 ⁻²		7.337 ⁻³	7.366 ⁻⁴	7.454 ⁻⁵
3000	1.101 ⁺²	1.107 ⁺¹		1.123 ⁰		1.166 ⁻¹		1.246 ⁻²	1.309 ⁻³	1.327 ⁻⁴
4000	1.499 ⁺²	1.557 ⁺¹		1.561 ⁰		1.743 ⁻¹		1.774 ⁻²	1.800 ⁻³	1.863 ⁻⁴
5000	1.965 ⁺²	2.097 ⁺¹		2.196 ⁰		2.266 ⁻¹		2.396 ⁻²	2.717 ⁻³	3.251 ⁻⁴
6000	2.478 ⁺²	2.634 ⁺¹	8.547 ⁰	2.784 ⁰	9.172 ⁻¹	3.077 ⁻¹	1.054 ⁻¹	3.698 ⁻²	4.207 ⁻³	4.366 ⁻⁴
7000	3.016 ⁺²	3.256 ⁺¹	1.084 ⁺¹	3.671 ⁰	1.267 ⁰	4.401 ⁻¹	1.506 ⁻¹	4.970 ⁻²	5.120 ⁻³	5.202 ⁻⁴
8000	3.631 ⁺²	4.103 ⁺¹	1.414 ⁺¹	4.921 ⁰	1.692 ⁰	5.644 ⁻¹	1.831 ⁻¹	5.864 ⁻²	6.004 ⁻³	6.360 ⁻⁴
12,000	7.506 ⁺²	8.602 ⁺¹		9.062 ⁰		9.776 ⁻¹		1.159 ⁻¹	1.479 ⁻²	1.699 ⁻³
18,000	1.360 ⁺³	1.506 ⁺²		1.831 ⁺¹		2.320 ⁰		2.578 ⁻¹	2.627 ⁻²	2.689 ⁻³
24,000	2.041 ⁺³	2.519 ⁺²		3.166 ⁺¹		3.455 ⁰		3.642 ⁻¹	4.201 ⁻²	4.943 ⁻³
PRESSURE, DYNES PER SQUARE CENTIMETER										
1000	3.711 ⁺⁷	3.711 ⁺⁶		3.711 ⁺⁵		3.711 ⁺⁴		3.711 ⁺³	3.711 ⁺²	3.711 ⁺¹
2000	7.417 ⁺⁷	7.425 ⁺⁶		7.425 ⁺⁵		7.425 ⁺⁴		7.432 ⁺³	7.462 ⁺²	7.551 ⁺¹
3000	1.115 ⁺⁸	1.121 ⁺⁷		1.138 ⁺⁶		1.181 ⁺⁵		1.262 ⁺⁴	1.326 ⁺³	1.344 ⁺²
4000	1.518 ⁺⁸	1.577 ⁺⁷		1.683 ⁺⁶		1.766 ⁺⁵		1.797 ⁺⁴	1.823 ⁺³	1.887 ⁺²
5000	1.991 ⁺⁸	2.124 ⁺⁷		2.225 ⁺⁶		2.295 ⁺⁵		2.427 ⁺⁴	2.752 ⁺³	3.293 ⁺²
6000	2.510 ⁺⁸	2.668 ⁺⁷	8.658 ⁺⁶	2.820 ⁺⁶	9.291 ⁺⁵	3.117 ⁺⁵	1.068 ⁺⁵	3.706 ⁺⁴	4.852 ⁺³	4.423 ⁺²
7000	3.055 ⁺⁸	3.298 ⁺⁷	1.098 ⁺⁷	3.719 ⁺⁶	1.233 ⁺⁶	4.458 ⁺⁵	1.526 ⁺⁵	5.035 ⁺⁴	5.187 ⁺³	5.270 ⁺²
8000	3.678 ⁺⁸	4.156 ⁺⁷	1.432 ⁺⁷	4.985 ⁺⁶	1.714 ⁺⁶	5.717 ⁺⁵	1.855 ⁺⁵	5.940 ⁺⁴	6.082 ⁺³	6.443 ⁺²
12,000	7.604 ⁺⁸	8.714 ⁺⁷		9.180 ⁺⁶		9.903 ⁺⁵		1.174 ⁺⁵	1.468 ⁺⁴	1.721 ⁺³
18,000	1.378 ⁺⁹	1.526 ⁺⁸		1.855 ⁺⁷		2.350 ⁺⁶		2.612 ⁺⁵	2.661 ⁺⁴	2.724 ⁺³
24,000	2.068 ⁺⁹	2.552 ⁺⁸		3.207 ⁺⁷		3.500 ⁺⁶		3.689 ⁺⁵	4.256 ⁺⁴	5.007 ⁺³

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Note: For each value, the power of ten is indicated by a superscript.

Table 11. Pressure of dry air in chemical equilibrium.

TABLE 14. NUMBER OF MOLES PER MOLE OF COLD AIR

$$Z = 1.9910 \sum_1 n_1 = PV/RT$$

TEMPERATURE °K	DENSITY RATIO ρ/ρ_0									
	10	10 ⁰	10 ^{-1/2}	10 ⁻¹	10 ^{-3/2}	10 ⁻²	10 ^{-5/2}	10 ⁻³	10 ⁻⁴	10 ⁻⁵
1000	1.0000	1.0000		1.0000		1.0000		1.0000	1.0000	1.0000
2000	1.0000	1.0000		1.0001		1.0007		1.0017	1.0055	1.0170
3000	1.0023	1.0072		1.0018		1.0010		1.1341	1.1912	1.2069
4000	1.0226	1.0620		1.1341		1.1896		1.2107	1.2262	1.2715
5000	1.0728	1.1448		1.1990		1.2370		1.3081	1.4835	1.7752
6000	1.1275	1.1984	1.2298	1.2669	1.3196	1.4001	1.5163	1.6543	1.9143	1.9864
7000	1.1761	1.2699	1.3364	1.4315	1.5623	1.7162	1.8570	1.9334	1.9970	2.0088
8000	1.2390	1.4001	1.5259	1.6794	1.8261	1.9259	1.9763	2.0010	2.0487	2.1700
12,000	1.708	1.957		2.062		2.224		2.636	3.365	3.865
18,000	2.063	2.284		2.775		3.518		3.910	3.984	4.078
24,000	2.322	2.865		3.602		3.930		4.143	4.778	5.623

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Note: For each value, the power of ten is indicated by a superscript.

Table 14. Number of moles per mole of cold air.

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TABLE 15. EFFECTIVE SPECIFIC HEAT RATIO FOR SHOCK WAVES

$$\gamma' = 1 + PV/E = 1 + P/\rho E$$

TEMPERATURE °K	DENSITY RATIO ρ/ρ_0									
	10	10^0	$10^{-1/2}$	10^{-1}	$10^{-3/2}$	10^{-2}	$10^{-5/2}$	10^{-3}	10^{-4}	10^{-5}
1000	1.3802									
2000	1.3377	1.3378		1.3375		1.3361		1.3326	1.3221	1.2941
3000	1.3036	1.2967		1.2780		1.2398		1.1950	1.1731	1.1650
4000	1.2694	1.2447		1.2135		1.1967		1.1900	1.1807	1.1591
5000	1.2449	1.2229		1.2087		1.1938		1.1650	1.1251	1.09635
6000	1.2350	1.2178	1.2077	1.1944	1.1770	1.1567	1.1364	1.1198	1.1029	1.09939
7000	1.2286	1.2028	1.1845	1.1639	1.1442	1.1289	1.1195	1.1144	1.1112	1.1084
8000	1.2176	1.1808	1.1607	1.1439	1.1326	1.1268	1.1237	1.1221	1.1178	1.1081
12,000	1.1764	1.1608		1.1530		1.1398		1.1191	1.1016	1.09495
18,000	1.1898	1.1751		1.1531		1.1362		1.1308	1.1294	1.1251
24,000	1.1950	1.1781		1.1650		1.1594		1.1513	1.1273	1.1096

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Note: For each value, the power of ten is indicated by a superscript.

Table 15. Effective specific heat ratio for shock waves.

APPENDIX B. Generalized Analytical Similarity Solutions for Strong Shock Waves.

A generalized similarity solution of the hydrodynamic equations in Eulerian form can be derived in the manner that R. Latter used for spherical waves.⁶ Hence, only the salient features of the analysis will be given with the resulting calculations and curves for plane, cylindrical, and spherical waves.

The equations to be integrated are

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} = - \frac{1}{\rho} \frac{\partial p}{\partial r} \quad (B1)$$

$$\frac{\partial p}{\partial t} + u \frac{\partial p}{\partial r} = - \rho \left(\frac{\partial u}{\partial r} + \delta \frac{u}{r} \right) \quad (B2)$$

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial r} = \frac{\gamma p}{\rho} \left(\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial r} \right) \quad (B3)$$

where the terms are as defined in Section II with $\epsilon = \frac{p}{(\gamma - 1)\rho}$ and where δ equals 1 for a plane wave, 2 for a cylindrical wave, and 3 for a spherical wave. The following transformations are made:

$$x = \frac{r}{R}$$

$$\frac{\partial}{\partial t} \rightarrow \frac{\partial}{\partial t} - \frac{x \dot{R}}{R} \frac{\partial}{\partial x}$$

$$\frac{\partial}{\partial t} \rightarrow \frac{1}{R} \frac{\partial}{\partial x}$$

$$p(r, t) = p_0 \frac{f(x)}{R^{2-\gamma}}$$

$$\rho(r, t) = \rho_0 \psi(x)$$

$$u(r, t) = \frac{\phi(x)}{R^\alpha}$$

With the assumption that

$$R^2 \dot{R} = A, \text{ a constant,}$$

the above equations reduce to

$$-A_{\delta} (\alpha \phi + x \dot{\phi}) + \dot{\phi} \phi' + \frac{f'}{\psi} = 0 \quad (B4)$$

$$-A_{\delta} x \psi' + \phi \psi' + \psi \phi' + \delta \frac{\psi \phi}{x} = 0 \quad (B5)$$

$$-A_{\delta} 2 \alpha f' + (\phi - xA) f' + \frac{\gamma f}{\psi} (xA - \phi) \psi' = 0 \quad (B6)$$

where the prime indicates differentiation with respect to x and the dot differentiation with respect to t .

The corresponding energy integrals for the three waves becomes

$$\begin{aligned} E_{\delta=0} &= \frac{R \rho_0}{R^2 \alpha} \int_0^1 \left(\frac{1}{2} \phi^2 + \frac{f}{\gamma-1} \right) dx \\ E_{\delta=1} &= 2\pi \rho_0 \frac{R^2}{R^2 \alpha} \int_0^1 x \left(\frac{1}{2} \psi \phi^2 + \frac{f}{\gamma-1} \right) dx \\ E_{\delta=2} &= 4\pi \rho_0 \frac{R^3}{R^2 \alpha} \int_0^1 x^2 \left(\frac{1}{2} \psi \phi^2 + \frac{f}{\gamma-1} \right) dx. \end{aligned} \quad (B7)$$

Hence, to have the total energy independent of time, when

$$\delta = 0, \quad \alpha = 1/2$$

$$\delta = 1, \quad \alpha = 1$$

$$\delta = 2, \quad \alpha = 3/2.$$

Now let

$$F = \frac{f}{A_{\delta}^2}, \quad \Phi = \frac{\phi}{A_{\delta}}, \quad \text{and } \Psi = \psi,$$

then Eqs. (B4), (B5) and (B6) become

$$- (\alpha \Phi + x \dot{\Phi}) + \dot{\Phi} \Phi' + \frac{F'}{\Psi} = 0 \quad (B8)$$

$$(\frac{1}{2} - x) \psi' + \psi \frac{\delta}{x} = 0 \quad (B9)$$

$$-2\alpha F + (\frac{1}{2} - x) F' + \gamma F(x - \frac{1}{2}) \frac{\gamma'}{\gamma} = 0 \quad (B10)$$

and Eq. (B7) becomes

$$E = \left(2\pi\delta + 0 \right) \int_0^1 dx \int_0^1 \frac{1}{x} \left(\frac{1}{2} \psi^2 + \frac{F}{\gamma-1} \right) \quad (B11)$$

where

$$\begin{aligned} \delta_0 &= 1 \text{ when } \delta = 0 \\ &= 0, \text{ when } \delta \neq 0. \end{aligned}$$

Using latter's technique (and with considerable algebra) the three integrals of Eqs. (B8), (B9) and (B10) are

$$\frac{x^\delta (\frac{1}{2} - x) F}{\gamma\delta - 1} = a, \quad (B12)$$

$$F \frac{1}{2} - (x - \frac{1}{2}) \left(\frac{F}{\gamma-1} + \frac{1}{2} \psi^2 \right) = c, \quad (B13)$$

$$x = \frac{K_\delta \left| \frac{1}{x} - \frac{1}{\gamma} \right|^{n_\delta}}{\left| \frac{1}{x} \right|^{\frac{2}{3+\delta}} \left| \frac{1}{x} - \frac{3+\delta}{\gamma+1+\delta(\gamma-1)} \right|^{m_\delta}}, \quad (B14)$$

where, for $\delta = 0$

$$n_0 = \frac{\gamma-1}{2\gamma-1}, \quad m_0 = \frac{5\gamma-4}{3(2\gamma-1)} \quad (B15)$$

for $\delta = 1$

$$n_1 = \frac{\gamma-1}{2\gamma}, \quad m_1 = 1/2 \quad (B16)$$

for $\delta = 2$

$$n_2 = \frac{\gamma-1}{2\gamma+1}, \quad m_2 = \frac{13\gamma^2 - 7\gamma + 12}{5(2\gamma+1)(3\gamma-1)} \quad (B17)$$

The constant of integration in Eq. (B13) is zero from the boundary condition

$$\bar{\Phi}(0) = 0.$$

The constants a and K_s are determined from the boundary conditions

$$\bar{\Phi}(1) = \frac{2}{\gamma + 1}$$

$$F(1) = \frac{2}{\gamma + 1}$$

$$\psi(1) = \frac{\gamma + 1}{\gamma - 1}$$

With the assumed value of $\gamma = 1.40$ and taking $\bar{\Phi}/x$ as the independent variable with $1/\gamma \leq \frac{\bar{\Phi}}{x} \leq 1$, the results are shown in Table B1 and Figs. B1 and B2. For these computations,

$$a = -0.067828,$$

$$K_0 = 0.87373,$$

$$K_1 = 0.95401$$

and

$$K_2 = 1.00734.$$

Table BI Similarity Solution for $\gamma = 1.40000$

$\delta = 0$					$\delta = 1$				$\delta = 2$			
Plane Wave					Cylindrical Wave				Spherical Wave			
t/x	x	ϕ	ψ	F	x	ϕ	ψ	F	x	ϕ	ψ	F
0.71600	0.37568	0.26899	0.19176	0.32838	0.53771	0.38500	0.08978	0.31497	0.62841	0.44994	0.068890	
0.71667	0.40416	0.28965	0.23117	0.32970	0.56355	0.40388	0.11411	0.31642	0.65768	0.47134	0.08236	
0.71750	0.43208	0.31002	0.27460	0.33137	0.58824	0.42206	0.14217	0.31798	0.67906	0.48722	0.10406	
0.71857	0.46066	0.33102	0.32448	0.33353	0.61292	0.44043	0.17594	0.32017	0.69963	0.50273	0.13208	0.31316
0.73000	0.61581	0.44954	0.72187	0.35808	0.73810	0.53881	0.48411	0.34497	0.80438	0.58720	0.39221	0.33194
0.73658	0.66632	0.49079	0.91880	0.37357	0.77610	0.57166			0.83220	0.61298	0.55772	0.35371
0.71000	0.78557	0.59703	1.64030	0.43851	0.86129	0.65458	1.32791	0.42674	0.89805	0.68252	1.20226	0.42016
0.76111	0.79000	0.60128	1.67763	0.44204	0.86434	0.65786	1.36436	0.43034	0.90038	0.68529	1.23768	0.42362
0.77174	0.82919	0.63992	2.05784	0.47828	0.89094	0.68758	1.74191	0.46740	0.92046	0.71036	1.6093	0.46091
0.78404	0.86914	0.68144	2.56451	0.52668	0.91704	0.71900	2.26565	0.51801	0.94022	0.73717	2.1267	0.51113
0.79792	0.90954	0.72573	3.25381	0.59158	0.94348	0.75282	2.98505	0.58398	0.95946	0.76557		
0.81320	0.95019	0.77269	4.21943	0.67969	0.96905	0.78803	4.03009	0.67523	0.97816	0.79544	3.9284	0.67062
0.81531	0.95558	0.77909	4.37473	0.69354	0.97239	0.79280	4.20084	0.68961	0.98055	0.79944	4.10559	0.68533
0.82646	0.98337	0.81271	5.30895	0.77499	0.98941	0.81770	5.24118	0.77451	0.99282	0.82053	5.18386	0.77135
0.83333	1.0000	0.83333	6.00000	0.83333	1.00000	0.83333	6.0000	0.83333	1.00000	0.83333	6.0000	0.83333

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Fig. 1. $(2R)^2$ vs t for three shock waves. The two theoretical calculations were carried out with $\gamma_{\text{air}} = 1.4$ and with the Gilmore¹ air. The experimental shock is plotted from data furnished through courtesy of Dr. F. D. Bennett⁵ of the Ballistic Research Laboratories.

Fig. 2. Pressure profiles at two times $\gamma_{\text{air}} = 1.4$.

Fig. 3. Relative density and velocity profiles at two times $\gamma_{air} = 1.6$.

Fig. 4. Pressure profiles at two times - Gilmore air.

Fig. 5. Relative density and velocity profiles at two times - Gilmore air.

Fig. 6. Relative density profiles in copper at two times - Gilmore air.

Fig. 7. Gamma and temperature profiles in air at $t = 1.5 \mu\text{sec}$.

Fig. 8. R vs t for the second shocks and contact surfaces for three calculations with $\gamma_{cu} = 1.1, 1.2$, and 1.67 . The experimental data is furnished through the courtesy of Dr. F. D. Bennett² of the Ballistic Research Laboratories.

Fig. 9. Pressure, velocity, and relative density profiles in the copper at four times.

Fig. 10. Analytical similarity solution. Also shown are points calculated by S. C. Lin⁷ with a numerical solution.

Fig. 11. $(2R)^2$ vs t for two shock waves in air. The same energy is assumed deposited in air. Calculations done with a constant gamma-law equation of state for air and with the Gilmore equation of state for air.

Fig. 12. Pressure profiles at two times with air alone $\gamma_{AIR} = 1.4$.

Fig. 13. Relative density profiles at three times with air alone $\gamma_{\text{air}} = 1.4$.

Fig. 14. Velocity profiles at two times with air alone $\gamma_{\text{air}} = 1.4$.

Fig. 15. Pressure profiles at three times with air alone + Gilmore air.

Fig. 16. Relative density profiles at two times with air alone.

Fig. 17. Slope, m , vs $1/N$ where N equals the number of zones used in the initially cold air. The zones are within a radius of 1.5 cm.

Fig. B1. Analytical similarity solution for Φ and ψ vs x for plane, cylindrical, and spherical shock waves.

Fig. B2. Analytical similarity solutions for F vs x for plane, cylindrical, and spherical shock waves.

END