UCRL-5447

UNIVERSITY OF CALIFORNIA
Lawrence Radiation Laboratory
Livermore, California

Contract No. W-7405-eng-48

HYDROMAGNETIC STABILITY OF A DIFFUSE LINEAR
PINCH

William A. Newcomb

August 1959

Printed for the U. S. Atomic Energy Commission

Proposed running head:

Hydromagnetic Stability

Send proofs to:

William A. Newcomb
Lawrence Radiation Laboratory
University of California
P. O. Box 808
Livermore, California

HYDROMAGNETIC STABILITY OF A DIFFUSE LINEAR PINCH*

William A. Newcomb

Lawrence Radiation Laboratory, University of California
Livermore, California

ABSTRACT

The hydromagnetic energy principle is applied to the derivation of necessary and sufficient conditions for the hydromagnetic stability of a linear pinch with distributed plasma current (a diffuse linear pinch). The results are quite general in that the axial and azimuthal components of the magnetic field, which determine the structure of the pinch completely, are treated as arbitrary functions of distance from the axis. For purposes of illustration, the general results are applied to the limiting case of a pinch with the plasma current confined to an infinitely thin layer (a sharp pinch).

* This work was performed under auspices of the U. S. Atomic Energy Commission.

HYDROMAGNETIC STABILITY OF A DIFFUSE LINEAR PINCH*

William A. Newcomb

Lawrence Radiation Laboratory, University of California
Livermore, California

1. INTRODUCTION

We shall investigate the hydromagnetic stability of a diffuse linear pinch (diffuse in the sense of a spread-out plasma current distribution; the sharp pinch, in which the plasma current is confined to an infinitely thin layer, is included as a limiting case.). For this purpose we take as an idealized model any cylindrically symmetrical and infinitely long configuration of plasma and magnetic field with the following properties: (1) The plasma has infinite conductivity. (2) The magnetic field has axial and azimuthal components, but no radial component. (3) The plasma stress tensor is isotropic;¹ we may therefore speak of a scalar plasma pressure P . (4) The plasma pressure gradient is balanced by the magnetic force $\underline{J} \times \underline{B}$, where $\underline{J} = \nabla \times \underline{B}$ is the plasma current density. (5) The system is bounded on the outside by a perfectly conducting wall of radius b . (6) It may also be bounded on the inside by a perfectly conducting wire of radius a . If so, it is called a tubular pinch; if not, a columnar pinch. In either case, if r = distance from the axis, the pinch occupies the space $a < r < b$, where $a = 0$ for a columnar pinch. (7) There are no vacuum regions.² There may, however, be regions

¹ The isotropy holds only for the equilibrium state; it is not preserved by small oscillations.

² This restriction will be removed later on. (See Theorem 14 below.)

in which the plasma pressure is negligibly small even though the conductivity is still infinite.

Let us introduce cylindrical coordinates r, θ, z about the axis of symmetry. The field component B_r vanishes, and the components B_θ and B_z depend only on r . (The field lines, consequently, form a system of coaxial helices with pitch $2\pi r B_z / B_\theta$.) The current density in the plasma is determined by Maxwell's equation $\underline{J} = \nabla \times \underline{B}$; the radial component J_r therefore vanishes, and the other components are given by

$$J_\theta = -\frac{dB_z}{dr}; \quad J_z = \frac{1}{r} \frac{d}{dr} (r B_\theta). \quad (1)$$

The radial distribution of the plasma pressure P is governed by the magnetohydrostatic equilibrium condition $\nabla P = \underline{J} \times \underline{B}$, which reduces to

$$\frac{dP}{dr} + B_z \frac{dB_z}{dr} + \frac{B_\theta}{r} \frac{d}{dr} (r B_\theta) = 0. \quad (2)$$

If B_θ and B_z are given as functions of r , then J_θ , J_z and P , the other quantities characterizing the equilibrium state, are determined by Eqs. (1) and (2), except for an additive constant in P . Since we are not assuming any particular boundary condition to be satisfied by P at the walls, the additive constant will remain undetermined, but it will be clear later on that the stability of a pinch does not depend on this constant. For our purposes, therefore, the structure of a pinch can be specified completely by giving the functions $B_\theta(r)$ and $B_z(r)$ between the limits a and b . We shall not assume any special form for these functions. Instead, we shall allow them to be completely arbitrary in order to obtain results general enough to be applied to any diffuse linear pinch.³

³As is now evident, we are using the term "diffuse pinch" in such a way as to include a variety of configurations that would not ordinarily be described as pinches, e.g., stellarator-type configurations in which the plasma is contained by a purely axial field.

So far we have considered only the equilibrium state of a pinch, but in order to derive stability criteria we must also examine the behavior of small displacements from this equilibrium state. According to hydromagnetic theory, any such displacement can be described by a single vector function of position, the displacement $\underline{\xi}$ of the plasma itself (1). The displacement in the magnetic field, for example, is determined by the fact that the field lines are constrained to move with the plasma (2, 3).

Our treatment will be based on the hydromagnetic energy principle, which was first derived in its most general form by Bernstein, Frieman, Kruskal, and Kulsrud (1).⁴ According to this principle, a system is stable if a certain energy integral $W(\underline{\xi})$ is positive for every displacement $\underline{\xi}$ satisfying the boundary conditions, unstable if there exists a $\underline{\xi}$ for which $W(\underline{\xi})$ is negative. The energy integral is given by

$$W(\underline{\xi}) = \frac{1}{2} \int d^3x \left\{ Q^2 + \underline{J} \cdot \underline{\xi} \times \underline{Q} + (\underline{\nabla} \cdot \underline{\xi}) \underline{\xi} \cdot \underline{\nabla} P + \gamma P (\underline{\nabla} \cdot \underline{\xi})^2 \right\}, \quad (3)$$

where $\underline{Q} = \underline{\nabla} \times (\underline{\xi} \times \underline{B})$ and γ is the ratio of specific heats. Ordinarily, the most straightforward way of applying the energy principle is to impose a conveniently chosen normalization condition on $\underline{\xi}$ and then to minimize $W(\underline{\xi})$ with respect to $\underline{\xi}$; the system is then stable if the minimum value is positive and unstable if it is negative.

The energy principle says nothing about the marginal case where a non-trivial $\underline{\xi}$ exists for which $W(\underline{\xi}) = 0$, but none for which $W(\underline{\xi}) < 0$. When this happens the system can be either stable, unstable, or neutrally stable, as we can see from the simple example of a particle moving in the one-dimensional potential $V(x) = \alpha x^2 + \beta x^4$. The term αx^2 corresponds to $W(\underline{\xi})$, and the

⁴See also Lundquist (4) for an earlier and less general form of the energy principle.

equilibrium point at $x = 0$ is stable or unstable according as α is positive or negative. If $\alpha = 0$, the equilibrium is stable, unstable, or neutrally stable according as β is positive, negative, or zero. Fortunately, we shall soon see that the question of what happens in the marginal case is of no physical interest for a linear pinch.

We note, by way of justification, that the hydromagnetic energy principle is, in the case of a diffuse linear pinch with an isotropic plasma stress-tensor,⁵ identical with the energy principle derived from the Boltzmann equation in the limit of small m/e (6, 7, 8, 9).⁶ Nevertheless, it does not take into account the effects of runaway electrons (10), negative Landau damping (11), or nonadiabatic particle motions (12), all of which may give rise to instabilities that are not predicted by our analysis.

It follows from symmetry considerations that the small-amplitude motions of a linear pinch can be analyzed into normal modes for which ξ_r , $i\xi_\theta$ and $i\xi_z$ are real functions of r multiplied by $\exp(im\theta + ikz)$. We can therefore limit ourselves to displacements of this form without loss of generality. Furthermore, the original derivation of the energy principle (1), which refers to the class of all possible displacements, remains valid even if the class of displacements is restricted by fixing m and k . A separate stability criterion is thus obtained for each set of values of m and k .

⁵The diffuse pinch with a nonisotropic stress tensor will be treated by Baños and Schwartz (5).

⁶Since the Boltzmann equation accounts for the deviations from isotropy associated with small oscillations, the agreement between the two energy principles implies that the stability criterion is unaffected by these deviations. This is true because the deviations vanish for the marginally stable normal modes, which form the boundary between stable and unstable regions. (An exceptional case to which these statements do not apply will be discussed in the next section.)

It is very easy to minimize the energy integral $W(\xi)$ with respect to the components ξ_θ and ξ_z . The result is a reduction of W to the one-dimensional form (13)

$$W(\xi) = \frac{\pi}{2} \int_a^b r dr \Lambda\left(\xi, \frac{d\xi}{dr}\right), \quad (4)$$

where ξ is an abbreviation for the radial component ξ_r and Λ is a certain quadratic form in ξ and $d\xi/dr$ with m and k as parameters. The pinch is then stable for given values of m and k if $W(\xi)$ is positive for every ξ and unstable if there exists a ξ for which $W(\xi)$ is negative. For a certain discrete set of values of m and k there will exist a nontrivial ξ for which $W(\xi)$ vanishes, but none for which $W(\xi)$ is negative. We do not know whether the pinch is stable for these marginal values of m and k , but it does not really matter because the pinch will definitely be stable for neighboring values of k on one side of the marginal value and unstable for neighboring values on the other side. For all practical purposes, therefore, the stability condition resulting from Eq. (4) may be regarded as necessary and sufficient.

Several years ago Rosenbluth (14) investigated the hydromagnetic stability of a sharp columnar pinch⁷ with respect to displacements that are continuous across the current layer. He later found, however, that the restriction to continuous displacements is not correct, and by dropping this restriction he was able to derive a new and more stringent stability criterion (21). From the standpoint of the energy principle the difficulty amounts to this (22): some of the displacements for which the energy integral is negative in a diffuse pinch become discontinuous in the limit of an infinitely thin current layer. This is why the new stability criterion is more stringent than the old one - the class of allowed displacements is wider, since it now contains discontinuous

⁷Other treatments of the sharp columnar pinch are reported in References (15) to (20).

as well as continuous displacements. The most striking feature of the new stability criterion is that it depends on the detailed structure of the current layer; one cannot simply treat J_θ and J_z as delta functions. For this reason it can only be derived by treating the sharp pinch as a limiting case of the diffuse pinch. This circumstance alone lends great interest to the theory of diffuse-pinch stability.

Suydam (13) has applied the hydromagnetic energy principle to a generalized diffuse pinch.⁸ After reducing the energy integral to the one-dimensional form (4), he obtained the following necessary condition for stability:

$$\frac{r}{8} B_z^2 \left(\frac{d \log \mu}{dr} \right)^2 + \frac{dP}{dr} > 0, \quad (5)$$

where $\mu(r) = B_\theta / r B_z$. A pinch is stable for all values of m and k only if this inequality is satisfied for every value of r between a and b . (The quantity μ is simply 2π divided by the pitch of a field line.)

Suydam's necessary condition for stability is obviously not sufficient. Consider, for example, a sharp pinch that is unstable according to Rosenbluth's criterion. If the current layer is of infinitesimal thickness δ , then the first term in (5), which is necessarily positive, is of order δ^{-2} while the second term is only of order δ^{-1} . The pinch therefore satisfies Suydam's condition in spite of its instability.⁹

⁸For further information on the stability of a generalized diffuse pinch, see Refs. (17) and (18). Stability criteria have also been derived for diffuse pinches with special current distributions, e.g., uniform J_z and vanishing J_θ (17, 18, 23).

⁹Exceptional cases arising when $d\mu/dr$ vanishes at some point in the current layer are considered briefly in Section VI. (Points where B_z vanishes give no trouble because $d \log \mu / dr$ becomes infinite.)

What is needed is a necessary and sufficient condition for stability in a form suitable for numerical calculation, and our main purpose will be to derive one by minimizing $W(\xi)$. We shall start in Section II by minimizing with respect to the components ξ_θ and ξ_z , thus reducing W to the one-dimensional form expressed by Eq. (4). The following comparison theorem will be an immediate consequence: A linear pinch is stable for all values of m and k if and only if it is stable for $m = 1$, $-\infty < k < \infty$ and for $m = 0$ in the limit as $k \rightarrow 0$. Another immediate consequence will be that the stability criterion is independent of γ , the ratio of specific heats. The next step, minimization with respect to ξ_r , will involve two difficulties. First, the Euler-Lagrange equation for this minimization problem is singular at certain values of r between a and b , and second, there are cases of interest in which the relevant Euler-Lagrange solution gives not the minimum but only a stationary value for W . The first difficulty will be surmounted by proving that W can be minimized separately in each of the intervals between singular points, and the second by proving that the pinch is unstable whenever the Euler-Lagrange solutions fail to give a minimum. The minimization will be completed in Section IV after a discussion in Section III of some elementary properties of the Euler-Lagrange solutions. Then, in Section V, the necessary and sufficient condition for stability¹⁰ will be given in several forms, all of them depending on the behavior of the Euler-Lagrange solutions. Since the Euler-Lagrange equation can always be solved numerically, we will have found a definite procedure for determining the stability of any pinch configuration that has been specified by giving B_θ and B_z as functions of r . Finally, for illustrative purposes, we shall treat the sharp pinch in Section VI as a limiting

¹⁰ This condition will be a slight generalization of the one given by Rosenbluth (21) on the basis of a heuristic argument involving marginal stability.

case, obtaining stability criteria in closed form. The results of that section will agree with those of Rosenbluth for the sharp columnar pinch (21) and with those of Kaufman and Newcomb for the sharp tubular pinch (22).

II. REDUCTION OF THE ENERGY INTEGRAL TO A ONE-DIMENSIONAL FORM

As we have seen in the introduction, it is permissible to restrict our attention to displacements for which ξ_r , $i\xi_\theta$ and $i\xi_z$ are real functions of r multiplied by $\exp(im\theta + ikz)$, thus obtaining a separate stability criterion for each set of values of m and k . Assuming for the time being that m and k do not both vanish, and dropping the exponential factor, let us transform to the real variables ξ, η, ζ :

$$\xi = \xi_r \quad (6a)$$

$$\eta = \nabla \cdot \underline{\xi} - \frac{1}{r} \frac{d}{dr} (r\xi_r) = \frac{im}{r} \xi_\theta + ik\xi_z \quad (6b)$$

$$\zeta = i(\underline{\xi} \times \underline{B})_r = i\xi_\theta B_z - i\xi_z B_\theta \quad (6c)$$

(The special case $m = k = 0$ will be treated later on.) The cylindrical components ξ_θ and ξ_z are given in terms of η and ζ by

$$\xi_\theta = -i \frac{kr\zeta + rB_\theta \eta}{krB_z + mB_\theta}; \quad \xi_z = i \frac{m\zeta - rB_z \eta}{krB_z + mB_\theta} \quad (7)$$

We substitute these expressions for ξ_θ and ξ_z in the energy integral as given by Eq. (3), using only the real part of the complex $\underline{\xi}$ because $W(\underline{\xi})$ is a quadratic rather than a linear form, and eliminating $\underline{J}$ and ∇P with the help of Eqs. (1) and (2). The result, after some tedious but straightforward algebra, is

$$W(\xi, \eta, \zeta) = \frac{\pi}{2} \int_a^b r dr \left\{ \Lambda \left(\xi, \frac{d\xi}{dr} \right) + \gamma P \left[\eta + \frac{1}{r} \frac{d}{dr} (r\xi) \right]^2 + \frac{k^2 r^2 + m^2}{r^2} \left[\zeta - \zeta_0 \left(\xi, \frac{d\xi}{dr} \right) \right]^2 \right\}, \quad (8)$$

where

$$\Lambda(\xi, \frac{d\xi}{dr}) = \frac{1}{k^2 r^2 + m^2} \left[(krB_z + mB_\theta) \frac{d\xi}{dr} + (krB_z - mB_\theta) \frac{\xi}{r} \right]^2 + \left[(krB_z + mB_\theta)^2 - 2B_\theta \frac{d}{dr}(rB_\theta) \right] \frac{\xi^2}{r^2}, \quad (9)$$

$$\zeta_0 \left(\xi, \frac{d\xi}{dr} \right) = \frac{r}{k^2 r^2 + m^2} \left[(krB_\theta - mB_z) \frac{d\xi}{dr} - (krB_\theta + mB_z) \frac{\xi}{r} \right]. \quad (10)$$

As Rosenbluth has pointed out (24), one can now obtain a very simple sufficient condition for the stability of a tubular pinch. Since every term in the energy integral is positive-definite except possibly the term in Λ containing $B_\theta d(rB_\theta)/dr$, a tubular pinch is stable for all m and k if $|rB_\theta|$ is an everywhere-decreasing function of r . (This condition, of course, could never be satisfied in a columnar pinch.)¹¹

A pinch is stable for specified values of m and k if and only if $W(\xi, \eta, \zeta)$ is positive for every set of trial functions (ξ, η, ζ) with ξ satisfying the boundary conditions¹²

$$\xi(a) = \xi(b) = 0. \quad (11)$$

Let us define

$$W(\xi) = \text{Min}_{\eta, \zeta} W(\xi, \eta, \zeta). \quad (12)$$

Then $W(\xi, \eta, \zeta)$ is positive for every (ξ, η, ζ) if and only if $W(\xi)$ is positive for every ξ .

¹¹ Since Suydam's condition is necessary for stability, it should obviously be a consequence of Rosenbluth's condition. That this is so can easily be verified by transforming the inequality (5) into

$$rB_z^2 [d \log(\mu r^4)/dr]^2 > 8B_\theta^2 d \log(rB_\theta)/dr.$$

¹² The inner boundary condition must be modified slightly in the case of a columnar pinch (See Section V).

The indicated minimization with respect to η and ζ is trivial. Since the terms involving η and ζ are positive-definite and can be made to vanish by setting

$$\eta = -\frac{1}{r} \frac{d}{dr} (r\xi), \quad (13a)$$

$$\zeta = \zeta_0 \left(\xi, \frac{d\xi}{dr} \right), \quad (13b)$$

we obtain

$$W(\xi) = \frac{\pi}{2} \int_a^b r dr \Lambda \left(\xi, \frac{d\xi}{dr} \right). \quad (14)$$

Equation (13a) is equivalent to $\nabla \cdot \underline{\xi} = 0$; hence W is minimized by incompressible displacements. Furthermore, Λ does not contain γ , so that the stability criterion is independent of γ . (Other features of the motion do, however, depend on γ .) In particular, by choosing $\gamma = \infty$ we see that the stability criterion is unchanged by replacing the plasma with an incompressible fluid.¹³

A more convenient form of the energy principle is obtained by integrating Eq. (14) by parts to eliminate the term in $\xi d\xi/dr$:

$$W(\xi) = \frac{\pi}{2} \int_a^b dr \left[f \left(\frac{d\xi}{dr} \right)^2 + g \xi^2 \right], \quad (15)$$

where f and g , functions of r with m and k as parameters, are given by

$$f = \frac{r(krB_z + mB_\theta)^2}{k^2 r^2 + m^2}, \quad (16)$$

¹³ This result has also been obtained on the basis of a marginal stability argument by Shafranov for the special case of a diffuse pinch with a uniform current distribution (23) and by Dungey and Loughhead for a sharp pinch (25). Tayler has raised certain objections to the marginal stability argument (26); although his objections are justified, they do not apply to our derivation, which is based on the energy principle.

$$g = \frac{1}{r} \frac{(krB_z - mB_\theta)^2}{k^2 r^2 + m^2} + \frac{1}{r} (krB_z + mB_\theta)^2 - \frac{2B_\theta}{r} \frac{d}{dr} (rB_\theta) - \frac{d}{dr} \left(\frac{k^2 r^2 B_z^2 - m^2 B_\theta^2}{k^2 r^2 + m^2} \right). \quad (17)$$

The coefficient f is never negative, but g may have either sign. Another useful expression for g is obtained with the help of Eq. (2):

$$g = \frac{2k^2 r^2}{k^2 r^2 + m^2} \frac{dP}{dr} + \frac{1}{r} (krB_z + mB_\theta)^2 \frac{k^2 r^2 + m^2 - 1}{k^2 r^2 + m^2} + \frac{2k^2 r}{(k^2 r^2 + m^2)^2} (k^2 r^2 B_z^2 - m^2 B_\theta^2). \quad (18)$$

Excluding $m = 0$, let us compare displacements with different values of m but with the same value of $q = k/m$. If we make the substitution $k = mq$ in Eqs. (16) and (17), we find that the second term in Eq. (17) is the only one that depends on m . Since this term is positive-definite and proportional to m^2 , the least stable displacements must be those with $m = \pm 1$. Consequently, if a linear pinch is stable for $m = 1$, $-\infty < k < \infty$, it is also stable for all higher values of m . (We are using the fact that $W(\xi)$ is unchanged when m and k both change sign.)

If $m = 0$, Eq. (15) reduces to

$$W(\xi) = W_0(\xi) + \frac{\pi k^2}{2} \int_a^b r dr B_z^2 \xi^2, \quad (19)$$

where

$$W_0(\xi) = \frac{\pi}{2} \int_a^b dr \left[r B_z^2 \left(\frac{d\xi}{dr} \right)^2 + \left(\frac{B_z^2}{r} + 2 \frac{dP}{dr} \right) \xi^2 \right]. \quad (20)$$

The term containing k^2 is positive-definite; hence the pinch is stable for all k if it is stable in the limit as $k \rightarrow 0$.

Finally, we must consider displacements for which m and k both vanish.¹⁴ These displacements can be divided into two classes, those for which ξ_r vanishes identically and those for which ξ_θ and ξ_z vanish identically. We have $W(\xi) = 0$ for displacements of the first class, which are therefore marginally stable. But since it is physically obvious that these displacements are not only marginally but neutrally stable, we can forget about them. Displacements of the second class involve radial expansions and compressions of the plasma, so that $\nabla \cdot \xi$ cannot vanish. Equation (15) is therefore not applicable, and we may expect the stability criterion for this case to differ from the criterion for $m = 0, k \rightarrow 0$. By direct substitution in Eq. (3) we obtain, after an integration by parts,

$$W = \frac{\pi}{2} \int_a^b dr \left[r(B_z^2 + B_\theta^2 + \gamma P) \left(\frac{d\xi}{dr} \right)^2 + 2(B_z^2 + \gamma P) \xi \frac{d\xi}{dr} + (B_z^2 - B_\theta^2 + \gamma P) \frac{\xi^2}{r} \right], \quad (21)$$

where ξ is again an abbreviation for ξ_r . (Since ξ_θ and ξ_z do not enter, there is nothing analogous to the minimization with respect to η and ζ .) Let us subtract Eq. (20) from Eq. (21). Then, using Eq. (2) and integrating by parts, we obtain

$$W - W_0 = \frac{\pi}{2} \int_a^b r dr \left[B_\theta^2 \left(\frac{d\xi}{dr} - \frac{\xi}{r} \right)^2 + \gamma P \left(\frac{d\xi}{dr} + \frac{\xi}{r} \right)^2 \right], \quad (22)$$

¹⁴ This is the exceptional case mentioned in footnote 6; for these displacements the hydromagnetic energy principle does not agree with the energy principle derived from the Boltzmann equation.

which is necessarily positive. Consequently, if the pinch is stable for $k \rightarrow 0$, it is also stable for $k = 0$.¹⁵

The results of the last three paragraphs are summarized in the following theorem.¹⁶

Theorem 1. A linear pinch is stable for all values of m and k if and only if it is stable for $m = 1$, $-\infty < k < \infty$ and for $m = 0$, $k \rightarrow 0$.

This theorem obviously allows a considerable simplification in the testing of a pinch for stability. If we wish to find out whether a specified pinch is stable for all values of m and k , we need only examine the special cases mentioned in the theorem. It will sometimes be of interest, however, if a pinch is known to be unstable, to find out exactly which values of m and k give rise to the instability. For this reason we shall continue to state our results in terms of a general m without restricting ourselves to $m = 0$ and $m = 1$.

III. PROPERTIES OF THE EULER-LAGRANGE SOLUTIONS

We have now reduced the energy integral $W(\xi)$ to the simple form given by Eq. (15). If we try to minimize this form with respect to ξ , we are led to the Euler-Lagrange equation

$$\frac{d}{dr} \left(f \frac{d\xi}{dr} \right) - g\xi = 0, \quad (23)$$

¹⁵ This conclusion is not affected by the discrepancy between the two energy principles, since Kruskal and Oberman have shown (7) that hydromagnetic stability is always a sufficient condition for stability according to the Boltzmann equation.

¹⁶ This theorem, in a somewhat weaker form, was first proved by Rosenbluth for the special case of a sharp columnar pinch (14); this is the paper in which attention was restricted to displacements that are continuous across the current layer.

the solutions of which give stationary, although perhaps not minimal, values for $W(\xi)$. This equation has a singular point wherever f vanishes, and we can see from Eq. (16) that this happens wherever

$$krB_z + mB_\theta = 0, \quad (24)$$

or in terms of the quantity μ entering into Suydam's condition, wherever $k + m\mu = 0$. (The singular points can also be characterized as those values of r for which $m\theta + kz$, the phase of the displacement, is constant along a line of force.¹⁷) For given values of m and k , let $r_s^{(1)}, r_s^{(2)}, \dots, r_s^{(q)}$ be the singular points. (The number and location of the singular points are, of course, functions of m and k .) These points divide the interval $a < r < b$ into $q + 1$ subintervals $a < r < r_s^{(1)}, r_s^{(1)} < r < r_s^{(2)}, \dots, r_s^{(q)} < r < b$, which we shall call the independent subintervals.

In general it is not possible to continue an Euler-Lagrange solution past a singular point.¹⁸ For this reason we cannot speak of an Euler-Lagrange solution in the entire interval $a < r < b$; each solution is defined in only one of the independent subintervals. We shall see in the next section, however, that the energy integral can be minimized in each of these subintervals separately, so that there will never be any need to continue a solution past a singular point.

As is now evident, we shall have occasion to evaluate the energy integral not only over the full interval $a < r < b$, but also over subintervals

¹⁷ In a columnar pinch the point $r = 0$ is also singular. We shall consider the effect of this singular point in Section V, restricting ourselves in the meantime to singular points that satisfy Eq. (24).

¹⁸ This is true only because ξ is a real variable. If it were complex we could go around the singular point by analytic continuation, but the resulting solutions would generally be multivalued.

$r_1 < r < r_2$, where $a < r_1 < r_2 < b$. We therefore introduce the notation

$$W(r_1, r_2; \xi) = \frac{\pi}{2} \int_{r_1}^{r_2} dr \left[f \left(\frac{d\xi}{dr} \right)^2 + g \xi^2 \right]. \quad (25)$$

The displacement ξ in Eq. (25) may be an Euler-Lagrange solution if there are no singular points between r_1 and r_2 , in which case W is especially easy to evaluate:

$$\begin{aligned} W(r_1, r_2; \xi) &= \frac{\pi}{2} \int_{r_1}^{r_2} dr \left[f \left(\frac{d\xi}{dr} \right)^2 + \frac{d}{dr} \left(f \frac{d\xi}{dr} \right) \xi \right] \\ &= \frac{\pi}{2} \int_{r_1}^{r_2} dr \frac{d}{dr} \left(f \xi \frac{d\xi}{dr} \right) = \frac{\pi}{2} f \xi \frac{d\xi}{dr} \Big|_{r_1}^{r_2}. \end{aligned} \quad (26)$$

In particular, if r_1 and r_2 are vanishing points of $f\xi$, the energy integral is equal to zero.

Let us now examine the behavior of the Euler-Lagrange solutions in the neighborhood of a singular point r_s . (We are, of course, interested only in one-sided neighborhoods of r_s , for we have seen that an Euler-Lagrange solution is defined in only one of the independent subintervals.) If $x = |r - r_s|$, the coefficients f and g are approximated in the neighborhood of r_s by

$$f = \alpha x^2, \text{ with } \alpha > 0, \quad (27)$$

and

$$g = \beta, \quad (28)$$

where α and β are independent of x , and the Euler-Lagrange equation reduces to

$$\alpha \frac{d}{dx} \left(x^2 \frac{d\xi}{dx} \right) - \beta \xi = 0. \quad (29)$$

The solutions are x^{-n_1} and x^{-n_2} , where n_1 and n_2 are the roots of the indicial equation

$$n^2 - n - \frac{\beta}{\alpha} = 0. \quad (30)$$

(We are neglecting the marginal case where $n_1 = n_2$.)

The condition for n_1 and n_2 to be real and unequal is that $\alpha + 4\beta$ should be positive. If $\alpha + 4\beta$ is negative, the roots of the indicial equation are complex conjugates, and the real solutions of the Euler-Lagrange equation are

$$\xi = x^{-n} + x^{-n*} \quad \text{and} \quad \xi = i(x^{-n} - x^{-n*}). \quad (31)$$

These solutions are oscillatory in the neighborhood of r_s .

Using the fact that $k + m\mu = 0$ at a singular point, we can obtain the expressions for α and β from Eqs. (16) and (18):

$$\begin{aligned} \alpha &= \frac{r_s}{k^2 r_s^2 + m^2} \left(kr \frac{dB_z}{dr} + kB_z + m \frac{dB_\theta}{dr} \right)_{r=r_s}^2 \\ &= \frac{r_s B_\theta^2 B_z^2}{B^2} \left(\frac{d \log \mu}{dr} \right)^2, \end{aligned} \quad (32a)$$

$$\beta = \frac{2B_\theta^2}{B^2} \frac{dP}{dr}, \quad (32b)$$

where $B^2 = B_\theta^2 + B_z^2$. The condition for nonoscillatory solutions, $\alpha + 4\beta > 0$, then reduces to Suydam's condition:

$$\frac{r}{B} B_z^2 \left(\frac{d \log \mu}{dr} \right)^2 + \frac{dP}{dr} > 0, \quad (33)$$

where all quantities are evaluated at r_s . We shall prove later on that a pinch with oscillatory solutions is unstable, thus verifying that Suydam's condition is necessary for stability.

Let us suppose that Suydam's condition is fulfilled at r_s , so that n_1 and n_2 are real. It follows from Eq. (30), the indicial equation, that $n_1 + n_2 = 1$, and we shall take $n_2 > n_1$. The solution x^{-n_2} is necessarily infinite at r_s , but x^{-n_1} can be either infinite or zero, depending on whether

n_1 is positive or negative. Furthermore, $W(\xi)$ converges at r_s for the solution x^{-n_1} but diverges for x^{-n_2} . We shall say that an Euler-Lagrange solution is small at r_s if it behaves like a constant multiple of x^{-n_1} in the neighborhood of r_s . Now let $\xi_s(r)$ be an Euler-Lagrange solution that is small at r_s , and let $\xi(r; r_1)$ be one that vanishes at the nonsingular point r_1 . These solutions are uniquely determined except for normalization factors, and it is easily seen that with a proper choice of the normalization factors we have

$$\xi_s(r) = \lim_{r_1 \rightarrow r_s} \xi(r; r_1). \quad (34)$$

Thus the smallness of a solution at a singular point is analogous to the vanishing of a solution at a nonsingular point.

We shall find that the stability criteria depend critically on the existence or nonexistence of points at which certain Euler-Lagrange solutions vanish. The following theorem, a special case of Sturm's separation theorem (27), will then be of interest.

Theorem 2. If $\xi_a(r)$ and $\xi_b(r)$ are any two linearly independent solutions of the Euler-Lagrange equation in the same independent subinterval I , and if $\xi_a(r)$ vanishes at r_1 and r_2 , two distinct interior points of I , then $\xi_b(r)$ vanishes at some point between r_1 and r_2 . (We shall assume that r_1 and r_2 are consecutive vanishing points of ξ_a ; it is clear that this involves no loss in generality.)

Proof: Since $\xi_a(r)$ and $\xi_b(r)$ are solutions of the Euler-Lagrange equation, we have

$$(f\xi_a')' - g\xi_a = 0, \quad (35)$$

$$(f\xi_b')' - g\xi_b = 0. \quad (36)$$

Multiplying Eqs. (35) and (36) by ξ_b and ξ_a respectively, and then subtracting, we obtain

$$fw' + f'w = 0, \quad (37)$$

where $w = \xi_b \xi_a' - \xi_a \xi_b'$, the Wronskian of the two solutions. It follows that wf is constant, so that w must have the same sign everywhere. We have $w(r_1) = \xi_b(r_1) \xi_a'(r_1)$ and $w(r_2) = \xi_b(r_2) \xi_a'(r_2)$. Clearly, $\xi_a'(r_1)$ and $\xi_a'(r_2)$ have opposite signs; hence $\xi_b(r_1)$ and $\xi_b(r_2)$ also have opposite signs, and $\xi_b(r)$ must vanish somewhere between r_1 and r_2 .

Corollary 2-1. Let I be an independent subinterval with a singular endpoint r_s , suppose that Suydam's condition is fulfilled at r_s , and let $\xi_s(r)$ be an Euler-Lagrange solution that is small at r_s . Then, if $\xi_s(r_2) = 0$, where r_2 is an interior point of I , every other solution vanishes at some point between r_s and r_2 .

Proof: If r_1 is an interior point of I that is close to r_s , it follows from Eq. (34) that $\xi(r; r_1)$ vanishes not only at r_1 but also at some point close to r_2 . The corollary is then obtained as a limiting case of the theorem if $\xi(r; r_1)$ is used in the place of $\xi_a(r)$.

IV. THE MINIMIZATION OF $W(\xi)$

We must find out whether $W(\xi)$ is positive for every physically admissible trial function $\xi(r)$. These functions, in addition to satisfying the boundary conditions (11), must all be continuous, since they represent displacements of a real fluid, the plasma. We can, however, extend the class of admissible trial functions by allowing finite jumps at the singular points. To see this, consider a $\xi(r)$ that varies rapidly in the neighborhood of a singular point r_s . More precisely, suppose that ξ changes by a finite amount in the interval from $r_s - \eta$ to $r_s + \eta$, where η is very small. In this

interval f and $d\xi/dr$ are of order η^2 and η^{-1} respectively, so that $W(r_s - \eta, r_s + \eta; \xi)$ is of order η . If we take the limit as $\eta \rightarrow 0$, then ξ becomes discontinuous and $W(a, b; \xi)$ approaches a finite limit. This means that we can allow displacements with finite jumps at the singular points, since displacements of this type can be approximated to any desired degree of accuracy by continuous displacements. (We cannot, of course, allow finite jumps at nonsingular points; they would make infinite contributions to W .)

We shall say that a pinch is stable in an independent subinterval I if the energy integral is positive over I for every ξ . We then have

Theorem 3. A linear pinch is stable for specified values of m and k if and only if it is stable in each of the independent subintervals.

Proof: Let r_s be a singular point, and write $W = W_a + W_b$, where W_a and W_b are the contributions from the intervals $a < r < r_s$ and $r_s < r < b$ respectively. (It is clearly sufficient to consider only one of the singular points.) We shall prove that W is positive for every ξ if and only if W_a and W_b are individually positive for every ξ . The "if" part of this statement is obvious, and to prove the "only if" part, suppose that a ξ exists for which W_a is negative. Then define ξ_1 as follows

$$\xi_1(r) = \begin{cases} \xi(r), & a < r < r_s \\ 0, & r_s < r < b. \end{cases} \quad (38)$$

(The displacement ξ_1 will in general have a finite jump, but we have seen that this is permissible at a singular point.) We now have

$$W(a, b; \xi_1) = W(a, r_s; \xi_1) = W(a, r_s; \xi) < 0, \quad (39)$$

which is what we set out to prove.

This theorem enables us to minimize $W(\xi)$ for the complete interval $a < r < b$ by minimizing it in each of the independent subintervals separately. We are not yet prepared, however, to minimize $W(\xi)$ for an interval with singular endpoints. Let us therefore consider the following preliminary minimization problem: If r_1 and r_2 are nonsingular points of the same independent subinterval, find the displacement $\xi(r)$ that minimizes $W(r_1, r_2; \xi)$ while satisfying the boundary conditions

$$\begin{aligned}\xi(r_1) &= c_1, \\ \xi(r_2) &= c_2.\end{aligned}\tag{40}$$

Since the minimizing ξ , if there is one, will make W stationary, we know that it must be an Euler-Lagrange solution. Let $\xi_1(r)$ and $\xi_2(r)$ be the Euler-Lagrange solutions satisfying

$$\xi_1(r_1) = 0, \quad \xi_1'(r_1) = 1, \tag{41}$$

$$\xi_2(r_1) = 1, \quad \xi_2'(r_1) = 0. \tag{42}$$

Now if $\xi_1(r_2) \neq 0$, there will exist a unique Euler-Lagrange solution $\xi_0(r)$ satisfying the boundary conditions (40); it is given by

$$\xi_0(r) = \frac{c_2 - c_1 \xi_2(r_2)}{\xi_1(r_2)} \xi_1(r) + c_1 \xi_2(r). \tag{43}$$

We shall prove that $W(r_1, r_2; \xi)$ is minimized by $\xi_0(r)$ if $\xi_1(r)$ never vanishes in the interval $r_1 < r < r_2$.¹⁹ As usual, it does not matter what happens in the marginal case where $\xi_1(r)$ vanishes at r_2 , and we shall prove that W is not minimized by $\xi_0(r)$ when $\xi_1(r)$ vanishes anywhere between r_1 and r_2 .

Suppose first that $\xi_1(r)$ never vanishes in the interval $r_1 < r \leq r_2$.

To prove that $\xi_0(r)$ minimizes the energy integral we shall define an auxiliary

¹⁹ This condition for minimization is a special case of Jacobi's condition (28).

integral $W^\dagger(r_1, r_2; \xi)$, called the Hilbert invariant integral,²⁰ with the following properties: $W^\dagger(\xi)$ has the same value for all ξ satisfying (40), $W^\dagger(\xi) < W(\xi)$ for every such ξ other than ξ_0 , and $W^\dagger(\xi_0) = W(\xi_0)$. It will then follow that the inequality

$$W(\xi) \geq W^\dagger(\xi) = W^\dagger(\xi_0) = W(\xi_0) \quad (44)$$

holds for every ξ that satisfies (40), and this will prove that W is indeed minimized by ξ_0 .

Before defining the Hilbert integral, let us consider the singly infinite family of functions

$$\xi_A(r) = A\xi_1(r) + c_1\xi_2(r), \quad (45)$$

where A is an arbitrary constant. These functions are obviously solutions of the Euler-Lagrange equation, and ξ_0 is one of them. Their graphs, which we shall call characteristic curves, cover the strip $r_1 < r \leq r_2$ in the r, ξ plane (See Fig. 1). This is true only because $\xi_1(r)$ never vanishes in the interval $r_1 < r \leq r_2$; if it did vanish at some point r_0 of that interval, then the characteristic curves would all intersect at the point $(r_0, c_1\xi_2(r_0))$, and none of them would pass through any other point on the line $r = r_0$ parallel to the ξ axis (See Fig. 2).

We now define $p(r, \xi)$, a function of position in the r, ξ plane, as the slope of the characteristic curve passing through the point (r, ξ) . This function is defined everywhere in the strip $r_1 < r \leq r_2$. On the line $r = r_1$, as is evident from Fig. 1, we have $p \rightarrow \pm\infty$, except at the point (r_1, c_1) , where p takes on finite values depending on the direction of approach.

²⁰ The use of a Hilbert integral is a standard technique in the calculus of variations. An elementary exposition of this technique is given in Ref. (28).

Fig. 1, illustrating Theorem 4. The curve C_0 is the graph of $\xi_0(r)$, and C_1 is the graph of any other function $\xi(r)$ with the same values at the endpoints.

Fig. 2. Characteristic curves for an interval failing to satisfy the condition for minimization. The curve C_0 is the graph of $\xi_0(r)$.

Every characteristic curve satisfies the differential equations

$$d\xi = pdr, \quad (46)$$

$$d(fp) - g\xi dr = 0, \quad (47)$$

of which the first expresses the definition of p and the second the fact that the characteristic curves represent Euler-Lagrange solutions. Also, since p is a function of r and ξ , we have

$$dp = \frac{\partial p}{\partial r} dr + \frac{\partial p}{\partial \xi} d\xi. \quad (48)$$

Eliminating the differentials from Eqs. (46), (47) and (48), we obtain the following useful relation between the partial derivatives of p :

$$\frac{\partial}{\partial r}(fp) + fp \frac{\partial p}{\partial \xi} - g\xi = 0. \quad (49)$$

The Hilbert integral can now be defined by

$$W^\dagger(C) = \frac{\pi}{2} \int_C [(fp)^2 + g\xi^2] dr + 2fp(d\xi - pdr), \quad (50)$$

where C is a curve lying in the part of the r, ξ plane bounded by the lines $r = r_1$ and $r = r_2$. This definition, of course, can be applied only to curves for which the integral converges. For this reason the left endpoint of C , if it is on the line $r = r_1$, must be the point (r_1, c_1) . But this condition, since it is satisfied by the graph of any function $\xi(r)$ satisfying the boundary conditions (40), will not cause any trouble. It is also satisfied by the characteristic curves, for which $W^\dagger$ is obviously equal to W .

Let us write the Hilbert integral in the form

$$W^\dagger(C) = \frac{\pi}{2} \int_C (F_r dr + F_\xi d\xi), \quad (51)$$

where

$$F_r = g\xi^2 - fp^2, \quad F_\xi = 2fp. \quad (52)$$

From Eqs. (49) and (52) we obtain

$$\frac{\partial F}{\partial \xi} \frac{r}{\xi} - \frac{\partial F}{\partial r} \frac{\xi}{r} = 0. \quad (53)$$

The Hilbert integral is therefore path-independent; its value depends only on the endpoints of C .

Let C_0 be the graph of $\xi_0(r)$, and let C_1 be the graph of any other function $\xi(r)$ satisfying the boundary conditions (40). We have $W^\dagger(C_1) = W^\dagger(C_0)$ because $W^\dagger$ is path-independent, and $W^\dagger(C_0) = W(C_0)$ because C_0 is a characteristic curve. Furthermore,

$$\begin{aligned} W(C_1) - W^\dagger(C_1) &= \frac{\pi}{2} \int_{C_1} \left\{ \left[f \left(\frac{d\xi}{dr} \right)^2 + g\xi^2 \right] dr \right. \\ &\quad \left. - (fp^2 + g\xi^2) dr - 2fp(d\xi - pdr) \right\} \\ &= \frac{\pi}{2} \int_{C_1} f \left(\frac{d\xi}{dr} - p \right)^2 dr, \end{aligned} \quad (54)$$

which is always positive because f is always positive. As we have already seen, it follows immediately that $W(C_1) > W(C_0)$, which proves that ξ_0 is the minimizing displacement. We state our result as a theorem:

Theorem 4. If r_1 and r_2 are nonsingular points of the same independent subinterval, and if the nontrivial Euler-Lagrange solutions that vanish at r_1 never vanish in the interval $r_1 < r \leq r_2$,²¹ then the energy integral $W(r_1, r_2; \xi)$ is smaller for an Euler-Lagrange solution $\xi_0(r)$ than it is for any other $\xi(r)$ with the same boundary values.

Corollary 4-1. Under the conditions of Theorem 4, $W(r_1, r_2; \xi)$ is positive for every nontrivial $\xi(r)$ that satisfies the boundary conditions $\xi(r_1) = \xi(r_2) = 0$.

²¹ Since these solutions are constant multiples of each other, this condition will hold either for all of them or for none.

Proof: By Theorem 4, we have $W(\xi) > W(\xi_0)$, and $W(\xi_0) = 0$ because ξ_0 , the Euler-Lagrange solution that vanishes at both endpoints, must vanish identically.

Theorem 4 states that W is locally minimized by the Euler-Lagrange solutions; i. e., it is minimized whenever r_1 and r_2 are sufficiently close together. Specifically, they are sufficiently close together when r_2 is to the left of the next vanishing point, if one exists, of the nontrivial Euler-Lagrange solutions that vanish at r_1 . We shall see below that this condition on the closeness of r_1 and r_2 is, if the marginal case is neglected, not only sufficient for minimization but also necessary.

The endpoints r_1 and r_2 do not enter symmetrically into the statement of Theorem 4 and its corollary. Nevertheless, their roles can be interchanged because of Theorem 2. We can also obtain from Theorem 2 a symmetrical form of the criterion of closeness for r_1 and r_2 : r_1 and r_2 are sufficiently close for W to be minimized by ξ_0 if there is no nontrivial solution of the Euler-Lagrange equation that vanishes at two distinct points between r_1 and r_2 . We shall, however, continue to state our results in a form that is unsymmetrical with respect to the two endpoints; the roles of the endpoints can always be interchanged by an application of Theorem 2 or its corollary.

Now suppose that $\xi_1(r)$, the Euler-Lagrange solution satisfying Eqs. (41), vanishes at some point r_0 between r_1 and r_2 . The characteristic curves for this case are shown in Fig. 2. It is clear, first of all, that p is infinite on the line $r = r_0$, so that the proof of Theorem 4 does not apply here. In the place of Theorem 4 we have

Theorem 5. If r_1 and r_2 are nonsingular points of the same independent subinterval, and if the nontrivial Euler-Lagrange solutions that vanish at

r_1 also vanish at some point r_0 between r_1 and r_2 , then for any Euler-Lagrange solution $\xi_0(r)$ there exist functions $\xi(r)$ with the same boundary values and with $W(r_1, r_2; \xi) < W(r_1, r_2; \xi_0)$.

Proof: Let the Euler-Lagrange solution $\xi_0(r)$ be represented by the curve (1 2 3 4 5) in Fig. 3, let the arc (1 6 7 3) be another characteristic curve between r_1 and r_0 , and let $\xi_1(r)$ be the function represented by the broken curve (1 6 7 3 4 5). The Hilbert integral has the same properties as before if it is restricted to curves lying entirely within the strip $r_1 \leq r \leq r_0$. Applying it to the characteristic curves (1 2 3) and (1 6 7 3), we obtain $W(1 6 7 3) = W^\dagger(1 6 7 3) = W^\dagger(1 2 3) = W(1 2 3)$, so that $W(r_1, r_2; \xi_1) = W(r_1, r_2; \xi_0)$. Now replace the segment (7 3 4) with the arc (7 8 4) representing an Euler-Lagrange solution. Because of the local minimization property of Euler-Lagrange solutions, we can take the points 7 and 4 so close together that $W(7 8 4) < W(7 3 4)$. Let $\xi_2(r)$ be represented by the curve (1 6 7 8 4 5). Then $W(r_1, r_2; \xi_2) < W(r_1, r_2; \xi_1) = W(r_1, r_2; \xi_0)$, which means that W is not minimized by $\xi_0(r)$.

The following corollary shows that the pinch is unstable whenever the Euler-Lagrange solutions fail to minimize the energy integral.

Corollary 5-1. Under the conditions of Theorem 5, there exists a $\xi(r)$ that makes $W(r_1, r_2; \xi)$ negative and satisfies the boundary conditions $\xi(r_1) = \xi(r_2) = 0$.

Proof: As in Corollary 4-1, the Euler-Lagrange solution $\xi_0(r)$ that vanishes at r_1 and r_2 vanishes identically. Hence $W(r_1, r_2; \xi_0) = 0$, and according to Theorem 5 there exists a $\xi(r)$ vanishing at r_1 and r_2 for which W takes on a smaller value. (A displacement of this type is shown in Fig. 4.)

Corollary 5-2. Under the conditions of Theorem 5, the minimum value of $W(r_1, r_2; \xi)$ with respect to the class of displacements satisfying the boundary conditions (40) is minus infinity.

Fig. 3. Trial functions used in the proof of Theorem 5.

Fig. 4, illustrating Corollary 5-1. The segments (1234) and (356) represent Euler-Lagrange solutions, and $\xi(r) = (12356)$ is a displacement for which $W(r_1, r_2; \xi)$ is negative.

Proof: By hypothesis, the Euler-Lagrange solutions that vanish at r_1 also vanish at some point r_0 between r_1 and r_2 . Let us pick a point r_3 between r_0 and r_2 (See Fig. 5). Then, according to Theorem 2, the Euler-Lagrange solutions that vanish at r_3 also vanish at some point r_4 between r_1 and r_0 . Theorem 5 can then be applied to any interval $r_4 < r < r_5$, where r_5 is between r_3 and r_2 , and Corollary 5-1 guarantees the existence of a displacement $\xi_1(r)$ that vanishes outside this interval and makes the energy integral negative. Now let $\xi_2(r)$ be any displacement that satisfies the boundary conditions (40) and vanishes between r_4 and r_5 , and let $\xi_A(r) = A\xi_1(r) + \xi_2(r)$. Since the functions $\xi_1(r)$ and $\xi_2(r)$ do not overlap, we have

$$W(r_1, r_2; \xi_A) = A^2 W(r_1, r_2; \xi_1) + W(r_1, r_2; \xi_2). \quad (55)$$

The displacement $\xi_A(r)$ satisfies the boundary conditions (40), and it follows from Eq. (55) that the energy integral is negatively infinite in the limit as $A \rightarrow \infty$.

Let us now define a regular singular point as a singular point at which Suydam's condition is fulfilled, so that small solutions exist. The next two theorems are the analogues of Theorem 4 for intervals with singular end-points.

Theorem 6. Let r_1 be a regular singular point and r_2 a nonsingular point of one of the two independent subintervals adjacent to r_1 , say the one extending to the right. If the nontrivial Euler-Lagrange solutions that are small at r_1 never vanish in the interval $r_1 < r \leq r_2$ (cf. footnote 21), and if $\xi_0(r)$ is one of these solutions, then $W(r_1, r_2; \xi)$ has a smaller value for $\xi_0(r)$ than it does for any function $\xi(r)$ satisfying the conditions

$$\xi(r) \text{ bounded, } r_1 \leq r \leq r_2, \quad (56a)$$

$$\xi(r_2) = \xi_0(r_2), \quad (56b)$$

Fig. 5, illustrating Corollary 5-2. The dotted curves represent Euler-Lagrange solutions, and the solid curves represent the displacements $\xi_1(r)$ and $\xi_2(r)$ mentioned in the text.

with the exception of ξ_0 itself if $n_1 < 0$, since ξ_0 satisfies the conditions (56) in that case. (Notice that there is no restriction on the value of ξ at the singular point r_1 , except that it should be finite.)

Proof: Let us make use of the Euler-Lagrange solutions that are small at r_1 to define a family of characteristic curves as in the proof of Theorem 4 (See Fig. 6). These solutions are given in the neighborhood of r_1 by

$$\xi = \xi_A(r) \mp A(r-r_1)^{-n_1}, \quad (57)$$

where A is some constant; hence

$$\begin{aligned} p(r, \xi) &= \xi'_A(r) \mp -An_1(r-r_1)^{-n_1-1} \\ &\mp -n_1\xi/(r-r_1). \end{aligned} \quad (58)$$

Thus p becomes infinite as $r \rightarrow r_1$, but the integrand in the Hilbert integral $W^\dagger$ remains finite because f is proportional to $(r-r_1)^2$. Furthermore, there is no other value of r for which p becomes infinite, since the relevant Euler-Lagrange solutions never vanish between r_1 and r_2 . It follows that $W^\dagger(C)$ is well-defined for any curve C in the strip $r_1 \leq r \leq r_2$ of the r, ξ plane. As before, we can prove that $W^\dagger(C)$ depends only on the endpoints of C , and that $W(C) \geq W^\dagger(C)$, the equality sign holding only when C is a characteristic curve. Now let $\xi_0(r_2) = c$, let C_1 be the graph of any function $\xi(r)$ satisfying the conditions (56), and let C_0 be the graph of $\xi_0(r)$ plus a vertical segment joining the left endpoint of C_1 to the point (r_1, M) , where

$$M = \begin{cases} 0 & \text{if } n_1 < 0, \\ +\infty & \text{if } n_1 > 0 \text{ and } c > 0, \\ -\infty & \text{if } n_1 > 0 \text{ and } c < 0. \end{cases} \quad (59)$$

It is easily verified that $W^\dagger$ vanishes for the vertical segment. Then

$W(r_1, r_2; \xi) = W(C_1) > W^\dagger(C_1) = W^\dagger(C_0) = W(r_1, r_2; \xi_0)$, as we set out to prove.

MUL-7629

Fig. 6, illustrating Theorem 6. The curve C_0 is the graph of $\xi_0(r)$ plus a segment of the line $r = r_1$, and C_1 is the graph of some function $\xi(r)$ satisfying the conditions (56).

Corollary 6-1. Under the conditions of Theorem 6, $W(r_1, r_2; \xi)$ is positive for every nontrivial $\xi(r)$ that vanishes at r_2 and remains bounded between r_1 and r_2 .

Theorem 7. If r_1 and r_2 are successive singular points, both of them regular, and if the nontrivial Euler-Lagrange solutions that are small at r_1 never vanish in the interval $r_1 < r \leq r_2$, then $W(r_1, r_2; \xi)$ is positive for every nontrivial $\xi(r)$ that remains bounded in that interval.

Proof: The characteristic curves are again defined by the Euler-Lagrange solutions that are small at r_1 . In the neighborhood of r_2 these solutions are linear combinations of x^{-n_1} and x^{-n_2} , but x^{-n_2} dominates. Therefore, by analogy with Eq. (58), $p(r, \xi)$ is approximately equal to $n_2 \xi / (r_2 - r)$, and the Hilbert integral converges at r_2 as well as at r_1 . The rest of the proof is the same as it was for Theorems 4 and 6, except that C_0 is now defined as follows: If $\xi(r_1) = c_1$ and $\xi(r_2) = c_2$, then C_0 is composed of the segments from (r_1, c_1) to $(r_1, 0)$, from $(r_1, 0)$ to $(r_2, 0)$, and from $(r_2, 0)$ to (r_2, c_2) . (The curve C_0 represents a degenerate Euler-Lagrange solution; an Euler-Lagrange solution that is finite at both of the singular endpoints must vanish identically between the endpoints.)

Theorem 7 and Corollaries 4-1 and 6-1 give conditions under which the minimum value of W is zero. This does not mean that the stability is marginal, however, because the minimum value is attained only when ξ vanishes identically; we have proved that W is positive for every nontrivial ξ .

The reader may wonder why it was necessary to prove each theorem separately; why not prove Theorem 7, for example, by a straightforward limiting process applied to Corollary 4-1? The reason is that such a limiting process would show, not that $W(\xi)$ is positive for every nontrivial ξ , but

only that is is non-negative.²² The limiting process does, however, lead directly to the analogue of Theorem 5 for an interval with singular endpoints:

Theorem 8. Suppose that r_1 is a singular point, and that r_2 is a point, either singular or nonsingular, of the independent subinterval bounded on the left by r_1 . Then the minimum value of $W(r_1, r_2; \xi)$ for the class of displacements satisfying the boundary conditions (40) is minus infinity in each of the following cases: (1) r_1 is irregular, and (2) r_1 is regular, and the nontrivial Euler-Lagrange solutions that are small at r_1 vanish at some point between r_1 and r_2 .

Proof: In each of the cases mentioned there exist Euler-Lagrange solutions that vanish at least twice between r_1 and r_2 . This follows in the first case from the oscillatory nature of the solutions in the neighborhood of r_1 and in the second from Corollary 2-1. We can therefore find a subinterval with nonsingular endpoints for which Theorem 5 is applicable, and the present theorem follows immediately.

To summarize the results of this section, let $W_{\min}(r_1, r_2; c_1, c_2)$ be the minimum value of $W(r_1, r_2; \xi)$ for the class of displacements satisfying the boundary conditions (40), let $\xi_0(r)$ be the Euler-Lagrange solution that satisfies the boundary conditions

$$\xi(r_i) = c_i \quad \text{if } r_i \text{ is a nonsingular point (i = 1 or 2),} \quad (60a)$$

$$\xi(r) \text{ is small at } r_i \quad \text{if } r_i \text{ is a regular singular point,} \quad (60b)$$

²² For the same reason, separate minimization theorems should be stated and proved for intervals with the singular endpoint $r = 0$ in a columnar pinch (See below, Section V).

and let $\xi_1(r)$ be any nontrivial Euler-Lagrange solution that satisfies the boundary condition

$$\xi(r_1) = 0 \quad \text{if } r_1 \text{ is a nonsingular point,} \quad (61a)$$

$$\xi(r) \text{ is small at } r_1 \quad \text{if } r_1 \text{ is a regular singular point.} \quad (61b)$$

(We note that the solutions $\xi_0(r)$ and $\xi_1(r)$ are undefined when r_1 is an irregular singular point, and that $\xi_0(r)$ vanishes identically when r_1 and r_2 are both singular.) It has been shown that

$$W_{\min}(r_1, r_2; c_1, c_2) = W(r_1, r_2; \xi_0) \quad (62)$$

if $\xi_1(r)$ never vanishes in the interval $r_1 < r \leq r_2$. This minimum value (which is independent of c_1 when r_1 is singular) is attained only when $\xi(r) = \xi_0(r)$ if r_1 and r_2 are both nonsingular, and not at all if at least one of them is singular, since $\xi_0(r)$ then fails to satisfy the boundary conditions (40). Finally, we have shown that $W_{\min} = -\infty$ in every non-marginal case not covered by Eq. (62), i. e., whenever r_1 is an irregular singular point, and whenever $\xi_1(r)$ vanishes somewhere between r_1 and r_2 . Because of this last result, the derivation of stability criteria is not hindered by failure of the Euler-Lagrange solutions to minimize the energy integral - the pinch is obviously unstable whenever this happens.

V. STABILITY CRITERIA

The following theorem, an immediate consequence of Corollary 5-1, gives a necessary condition for stability.

Theorem 9. For specified values of m and k , a linear pinch is unstable in an independent subinterval I if there exists a nontrivial Euler-Lagrange solution that vanishes at two distinct interior points of I .

Corollary 9-1. A linear pinch is stable (1) for specified values of m and k , or (2) for all values, only if Suydam's condition is fulfilled (1) at every singular point, or (2) everywhere.²³

Parts (1) and (2) of this corollary follow respectively from the oscillatory character of the Euler-Lagrange solutions in the neighborhood of an irregular singular point and from the fact that every point is singular for some values of m and k . The next corollary is the limiting form of Theorem 9 obtained by an application of Corollary 2-1.

Corollary 9-2. For specified values of m and k , let I be an independent subinterval with a regular singular endpoint r_1 . Then the pinch is unstable in I if the nontrivial Euler-Lagrange solutions that are small at r_1 vanish at any interior point of I .

We now have necessary conditions for stability, and also, for a tubular pinch, a simple sufficient condition (See Section II). Assuming that there is no more than one singular point, Rosenbluth (21) has given the following necessary and sufficient condition: If there is no singular point (Case 1), then the nontrivial Euler-Lagrange solutions that vanish at $r = a$ do not vanish anywhere in the interval $a < r < b$; and if there is a singular point, say at r_s (Case 2), then the nontrivial solutions that vanish at $r = a$, and the ones that vanish at $r = b$, do not vanish anywhere in the intervals $a < r < r_s$ and $r_s < r < b$ respectively. He derives this condition by means of a marginal-stability argument, which he states only for Case 1. Since it is not entirely clear how this argument is to be extended to Case 2, it seems desirable to

²³ Although Suydam's condition, as expressed by the inequality (33), refers to a point that is singular only for certain specified values of m and k , it does not involve those values explicitly. It is this circumstance that makes possible such a simple result concerning stability for all values of m and k .

give a more rigorous proof, which we shall now do. Also, we shall make use of the small solutions to take account of situations where there is more than one singular point.

The following theorem and its corollaries give the necessary and sufficient conditions for stability of a tubular pinch, the columnar pinch requiring separate treatment because of the singular point at $r = 0$.²⁴ To simplify the statement of these conditions, the definition of smallness is extended to nonsingular points as follows: An Euler-Lagrange solution $\xi(r)$ is small at a nonsingular point r_1 if $\xi(r_1) = 0$.

Theorem 10. For specified values of m and k , a tubular pinch is stable in an independent subinterval I if and only if (1) Suydam's condition is fulfilled at the left endpoint²⁵ if that point is singular (i.e., not equal to a), and (2) the Euler-Lagrange solutions that are small at the left endpoint never vanish in the interior of I . (We are, as usual, neglecting the marginal case where these solutions are small at the right endpoint of I .)

Proof: The "only if" part follows immediately from Theorem 9 and its corollaries. The "if" part, since every displacement must satisfy the boundary conditions (11), follows from Corollaries 2-1 and 6-1 if I is bounded on the left by a and on the right by a singular point, from Corollary 6-1 if it is bounded on the left by a singular point and on the right by b , from Theorem 7 if it is bounded by two singular points, and from Corollary 4-1 if there are

²⁴ The analysis of singular points given in Section III is not applicable at $r = 0$ because Eqs. (27) and (28) are not valid there.

²⁵ Although Suydam's condition is needed at the left endpoint to guarantee the existence of small solutions, it need not be stated explicitly for the right endpoint because it follows from (2). For the same reason, it will not be mentioned at all in Corollary 10-2.

no singular points, so that I is bounded by a and b . (Our use of Theorems 6 and 7 implies that we are restricting ourselves to trial functions that are bounded in I . This restriction is legitimate even though the Euler-Lagrange solutions are unbounded, for $\xi(r)$ represents a physical displacement of the plasma and must therefore be finite even at the singular points.)

Combining this result with Theorem 3, we obtain

Corollary 10-1. A tubular pinch is stable if and only if (1) Suydam's condition is fulfilled at every point, and (2) for all values of m and k , and for every independent subinterval I , the Euler-Lagrange solutions that are small at the left endpoint of I never vanish in the interior of I .

According to Theorem 2 and its corollary, the left endpoint of I can be replaced in Theorem 10 and in Corollary 10-1 by the right endpoint. Also, the stability criterion can be given a symmetrical form:

Corollary 10-2. A tubular pinch is stable for specified values of m and k if and only if there is no nontrivial Euler-Lagrange solution that vanishes twice in the same independent subinterval. Thus the stability condition given by Theorem 9 is not only necessary but also sufficient.

As we have already stated, a columnar pinch differs from a tubular pinch in certain respects. First, the left-hand boundary condition is not $\xi(0) = 0$ but

$$\xi(0) = 0 \quad \text{if } m \neq \pm 1, \quad (63a)$$

$$\xi'(0) = 0 \quad \text{if } m = \pm 1. \quad (63b)$$

Second, Suydam's condition cannot be fulfilled at $r = 0$, since the left-hand side of the inequality (33) necessarily vanishes there; and third, the minimization theorems of the last section are not applicable to an interval bounded on the left by $r = 0$. But these differences are not essential. As one can easily verify, the Euler-Lagrange solutions never oscillate at $r = 0$, so that

Suydam's condition does not need to be fulfilled; and special minimization theorems can be proved for intervals $0 < r < r_2$ by the same methods that were used for intervals $r_1 < r < r_2$.²⁶ Without going into details, we merely give the result.

Theorem 11. Necessary and sufficient conditions for the stability of a columnar pinch are given by Theorem 10 and its corollaries with the following modifications: (1) An Euler-Lagrange solution is said to be small at $r = 0$ if it satisfies the boundary conditions (63). (2) Although Suydam's condition must still be fulfilled for every $r > 0$, it need not be fulfilled at $r = 0$.

Corollary 11-1. A columnar pinch has the same stability criterion as a tubular pinch in the limit as $a \rightarrow 0$, $B_\theta(a) = 0$. Thus the stability of a columnar pinch is not changed if an infinitely thin wire with no current is placed along the axis.

Proof: Although it is obvious that the wire can have no effect on stability for $m \neq \pm 1$, we might expect it to stabilize the $m = \pm 1$ displacements because of the added constraint that ξ must vanish at a conducting surface. That such a stabilizing effect does not in fact occur will be shown by examining the Euler-Lagrange solutions in the limit as $a \rightarrow 0$. For $m = \pm 1$ these solutions behave in the neighborhood of $r = 0$ like $1 + Ar^2$ and $1/r^2$, where A is some constant. For small but finite a , therefore, the solutions that vanish at $r = a$ are constant multiples of

$$\xi(r) = 1 + Ar^2 - \frac{a^2}{r^2} (1 + Aa^2). \quad (64)$$

²⁶ The cases $m = 0$, $m = \pm 1$, and $|m| > 1$ must be treated separately, since f and g depend on different powers of r in these three cases.

In the limit as $a \rightarrow 0$ this solution approaches $1 + Ar^2$, which satisfies Eq. (63b), so that the stability criterion of Theorem 10 approaches that of Theorem 11. (See Addendum, page 57.)

Theorems 10 and 11 solve the problem that was posed in the introduction. That is to say, they furnish a definite procedure for determining stability, since the question whether an Euler-Lagrange solution vanishes anywhere in the interior of I can always be answered by solving the Euler-Lagrange equation numerically. Furthermore, Theorem 1 enables us, if we are not interested in the stability of individual values of m and k but only in whether the pinch is stable for all values, to restrict our attention to the special cases $m = 0$, $k \rightarrow 0$ and $m = 1$, $-\infty < k < \infty$.

Another necessary and sufficient condition for stability is given by

Theorem 12. For specified values of m and k , a linear pinch is stable in an independent subinterval I if and only if: (1) Suydam's condition is fulfilled at the endpoints r_1 and r_2 if they are singular and if $r_1 \neq 0$. (2) If $\xi_1(r)$ and $\xi_2(r)$ are the Euler-Lagrange solutions satisfying

$$\xi_1(r) \text{ small at } r_1, \quad \xi_1(r_0) = 1, \quad (65a)$$

$$\xi_2(r) \text{ small at } r_2, \quad \xi_2(r_0) = 1, \quad (65b)$$

where r_0 is some interior point of I , and if $\xi_0(r)$ is defined by

$$\xi_0(r) = \begin{cases} \xi_1(r), & r_1 < r < r_0, \\ \xi_2(r), & r_0 < r < r_2, \end{cases} \quad (66)$$

then $\xi_0(r)$ does not vanish anywhere in the interior of I . (3) The energy integral satisfies the inequality $W(r_1, r_2; \xi_0) > 0$, which is equivalent because of Eq. (26) to

$$\xi_1'(r_0) > \xi_2'(r_0). \quad (67)$$

Proof: It follows immediately from Theorem 9 and its corollaries that parts (1) and (2) are necessary, and the necessity of part (3) is obvious. To prove sufficiency, we note first of all that $W(r_1, r_2; \xi)$ is positive whenever $\xi(r_0) = 0$ because of Corollaries 4-1 and 6-1, so that our attention can be restricted to displacements for which $\xi(r_0) \neq 0$, and in particular to the ones that are normalized to unity at r_0 . We shall therefore minimize $W(r_1, r_2; \xi)$ with respect to the class of displacements satisfying the normalization condition $\xi(r_0) = 1$, the boundary condition

$$\xi(r_1) = 0 \quad \text{if } r_1 = a, \quad (68a)$$

$$\xi(r_1) \text{ finite} \quad \text{if } r_1 \text{ is a singular point,} \quad (68b)$$

and a similar boundary condition at r_2 . Since $\xi_0(r)$ does not vanish in the interior of I , it follows from Theorem 4 if $r_1 = a$ and from Theorem 6 if r_1 is singular that $W(r_1, r_0; \xi)$ is minimized by ξ_0 . By applying Theorem 2 and its corollary we see that $W(r_0, r_2; \xi)$ is also minimized by ξ_0 . The trial function ξ_0 therefore minimizes W over the entire interval I , and the pinch is stable if and only if $W(r_1, r_2; \xi_0) > 0$.

VI. THE SHARP PINCH

Let us now suppose that the plasma current is restricted to a layer of thickness δ and apply Theorem 12 in the limit as $\delta \rightarrow 0$ to obtain the necessary and sufficient condition for stability of a sharp pinch. We assume that the current layer extends from r_{0i} to r_{0e} , where $r_{0e} - r_{0i} = \delta$, and that the functions B_θ , B_z and P are continuous at r_{0i} and r_{0e} . These functions are of order unity throughout the layer, but their derivatives are of order δ^{-1} . The regions $a < r < r_{0i}$ and $r_{0i} < r < b$ contain vacuum fields with components $B_{\theta i} r_{0i}/r$, B_{zi} and $B_{\theta e} r_{0e}/r$, B_{ze} respectively. Furthermore, the

plasma pressure P is uniform in these regions. Equation (2) can be integrated across the layer in the limit as $\delta \rightarrow 0$, and we obtain the pressure balance equation

$$P_i + \frac{1}{2} B_i^2 = P_e + \frac{1}{2} B_e^2, \quad (69)$$

where $B^2 = B_\theta^2 + B_z^2$ and the subscripts i and e refer to r_{0i} and r_{0e} respectively.

Suydam's condition is given by the inequality (5), which is obviously satisfied throughout the constant pressure regions. It can also be written in the form

$$\frac{rB^2}{8\sin^2\phi} \left(\frac{d\phi}{dr} \right)^2 + \left[\frac{dP}{dr} - \frac{B^2}{4} \frac{\cos\phi}{\sin\phi} \frac{d\phi}{dr} \right] + \frac{B^2}{8r} \cos^2\phi > 0, \quad (70)$$

where $\phi = \tan^{-1}(B_\theta/B_z)$ is the angle between the magnetic field and the z axis. In the current layer the three terms of (70) are of order δ^{-2} , δ^{-1} and 1 respectively, and the first term is necessarily positive; hence Suydam's condition is fulfilled everywhere unless $d\phi/dr$ vanishes at some point where P is a decreasing function of r .²⁷ Let us therefore assume that $d\phi/dr$ never vanishes in a region of decreasing pressure. (The quantity $d\phi/dr$ measures the rate of rotation of the magnetic field as we pass through the layer, and the vanishing of $d\phi/dr$ means that the field reverses its sense of rotation.)

The next step in applying Theorem 12 is to find the Euler-Lagrange solutions $\xi_1(r)$ and $\xi_2(r)$ in each independent subinterval. Suppose first that there is at least one singular point in the layer but none in the constant-pressure interval $a < r < r_{0i}$, let r_s be the first singular point, and consider

²⁷ We shall not worry about what happens when B^2 vanishes at some point r_1 . Since B^2 is positive on each side of r_1 , any pinch with $B^2(r_1) = 0$ can be regarded as the limiting form of a pinch in which B^2 never vanishes. For the same reason, we can assume that $d\phi/dr$ changes sign whenever it vanishes.

the independent subinterval $a < r < r_s$, taking $r_0 = r_{0i}$. In the constant-pressure part of this interval we can transform the Euler-Lagrange equation into Bessel's equation by a change of variable, obtaining the general solution

$$\xi(r) = \frac{kr}{krB_{zi} + \frac{mr_{0i}}{r} B_{\theta i}} \left[c_1 I'_m(kr) + c_2 K'_m(|kr|) \right], \quad (71)$$

where c_1 and c_2 are constants of integration. Our notation for Bessel functions is the same as Dwight's (29). Using the boundary condition $\xi_1(a) = 0$, we find that

$$\xi_1(r) = \frac{ckr}{krB_{zi} + \frac{mr_{0i}}{r} B_{\theta i}} \left[I'_m(kr) K'_m(|ka|) - K'_m(|kr|) I'_m(ka) \right], \quad (72)$$

where c is a normalization factor. Choosing c so that $\xi_1(r_{0i}) = 1$, we obtain

$$\xi'_1(r_{0i}) = \frac{1}{r_{0i}} L(r_{0i}, a), \quad (73)$$

where

$$L(r_1, r_2) = \frac{mB_{\theta} - krB_z}{mB_{\theta} + krB_z} \bigg|_{r=r_1} + \frac{k^2 r_1^2 + m^2}{k^2 r_1^2} \frac{kr_1 I'_m(kr_1) K'_m(|kr_2|) - |kr_1| K'_m(|kr_1|) I'_m(kr_2)}{I'_m(kr_1) K'_m(|kr_2|) - K'_m(|kr_1|) I'_m(kr_2)}. \quad (74)$$

It can easily be shown from elementary properties of the Bessel functions that $\xi_1(r)$ never vanishes between a and r_{0i} .

To find $\xi_2(r)$, the solution in the current layer, let ϵ be some length such that $\delta^2/r_0 \ll \epsilon \ll \delta$; and start by considering the interval $r_s - \epsilon < r < r_s$. The approximations $f \cong a(r - r_s)^2$ and $g \cong \beta$ are good throughout this interval, and since $\xi_2(r)$ is small at r_s , we obtain

$$\xi_2(r) \cong A \left(\frac{r_s - r}{\epsilon} \right)^{-n_1}, \quad (75)$$

where A is some constant of order unity and n_1 is the small root of Eq. (30). One can easily see from Eqs. (32) that α and β are of order δ^{-2} and δ^{-1} respectively. Equation (30) therefore reduces to $n_1 \cong -\beta/\alpha$, which is of order δ .

Let us next consider the interval $r_{01} < r < r_s - \epsilon$, for which the right-hand boundary condition is that $\xi_2(r)$ and its derivative should both be continuous:

$$\xi_2(r_s - \epsilon) \cong A, \quad (76a)$$

$$\xi_2'(r_s - \epsilon) \cong An_1/\epsilon. \quad (76b)$$

We assert that the approximation $\xi_2(r) \cong A$ is valid throughout the interval now under consideration, and to verify this assertion we write

$$\frac{d}{dr} \left(f \frac{d\xi_2}{dr} \right) = g\xi_2 \cong Ag. \quad (77)$$

It follows immediately that

$$\frac{1}{A} f \frac{d\xi_2}{dr} \cong n_1 \alpha \epsilon - \int_r^{r_s - \epsilon} g dr. \quad (78)$$

Now n_1 , α and g are of order δ , δ^{-2} and δ^{-1} respectively. We can therefore neglect the term $n_1 \alpha \epsilon$ and set the upper limit of integration equal to r_s with errors of order ϵ/δ ,²⁸ which is small because the main term is of order unity. The result is

$$\frac{1}{A} f \frac{d\xi_2}{dr} \cong - \int_r^{r_s} g dr. \quad (79)$$

²⁸ As a matter of fact, these two errors cancel to lowest order, so that the remaining error is even smaller than ϵ/δ . This cancellation, however, is not needed in the proof.

In the neighborhood of r_s we have $f \approx a(r - r_s)^2 \sim (r - r_s)^2/\delta^2$; hence the integration of Eq. (79) leads to

$$\xi_2(r) - A \sim \int_{r_s-\epsilon}^{r_s} \frac{\delta^2 dr}{(r - r_s)^2} \sim \frac{\delta^2}{\epsilon}. \quad (80)$$

Since δ^2/ϵ is small, the solution $\xi_2(r)$ is indeed close to A , and Eq. (79) is approximately valid everywhere between r_{0i} and $r_s - \epsilon$. Choosing $A = 1$ to satisfy the second part of Eq. (65b), we obtain

$$f(r_{0i}) \xi'_2(r_{0i}) \approx - \int_{r_{0i}}^{r_s} g dr. \quad (81)$$

This equation is exact in the limit as ϵ/δ and δ^2/ϵ both approach zero.

It is clear that the trial function $\xi_0(r)$ given by Eq. (66) does not vanish anywhere between a and r_s . We can therefore substitute Eqs. (73) and (81) into the inequality (67) to obtain the stability criterion

$$\frac{1}{r_{0i}} f(r_{0i}) L(r_{0i}, a) + \int_{r_{0i}}^{r_s} g dr > 0. \quad (82)$$

Now suppose that there is a singular point $r_s^{(0)}$ between a and r_{0i} . (There cannot be more than one because $k + m\mu$ is monotonic.) Since the Euler-Lagrange solution (72) never vanishes in the independent subinterval $a < r < r_s^{(0)}$, the pinch is stable in that interval according to Theorems 10 and 11. In the independent subinterval $r_s^{(0)} < r < r_s$, we must use the Euler-Lagrange solution that is small at $r_s^{(0)}$; the stability criterion is therefore given by (82) with $L(r_{0i}, r_s^{(0)})$ in the place of $L(r_{0i}, a)$. This means that the singularity at $r_s^{(0)}$ exerts the same stabilizing influence as would a conducting wall at the same point. Let us write

$$a_{\text{eff}} = \begin{cases} r_s, & \text{if } r_s \text{ is a singular point between } a \text{ and } r_{0i}, \\ a, & \text{if there is no singular point between } a \text{ and } r_{0i}. \end{cases} \quad (83)$$

The stability criterion is then given in both cases by (82) with a_{eff} in the place of a .

If r_s now designates the last singular point in the current layer, the stability criterion for the interval $r_s < r < b$ is completely analogous to (82). It is

$$-\frac{1}{r_{0e}} f(r_{0e}) L(r_{0e}, b_{\text{eff}}) + \int_{r_s}^{r_{0e}} g dr > 0, \quad (84)$$

where b_{eff} is defined by analogy with a_{eff} . If there is more than one singular point in the layer, let $r_s^{(1)}$ and $r_s^{(2)}$ be two successive ones, and pick r_0 somewhere between $r_s^{(1)}$ and $r_s^{(2)}$.

By an argument similar to the one leading to Eq. (81) we obtain

$$f(r_0) \xi_1'(r_0) \approx \int_{r_s^{(1)}}^{r_0} g dr, \quad (85a)$$

$$f(r_0) \xi_2'(r_0) \approx - \int_{r_0}^{r_s^{(2)}} g dr. \quad (85b)$$

As before, the trial function $\xi_0(r)$ never vanishes. The stability criterion is therefore given by the inequality (67), which now reduces to

$$\int_{r_s^{(1)}}^{r_s^{(2)}} g dr > 0. \quad (86)$$

Finally, suppose there are no singular points in the layer. We must then apply Theorem 12 to the independent subinterval $a_{\text{eff}} < r < b_{\text{eff}}$, taking r_0 at some interior point of the layer. Using the fact that $\xi_1(r)$ and its derivative are continuous at the inner edge of the layer, and proceeding just as in the derivation of Eq. (81), we obtain

$$f(r_0) \xi_1'(r_0) \approx \frac{1}{r_{0i}} f(r_{0i}) L(r_{0i}, a_{\text{eff}}) + \int_{r_{0i}}^{r_0} g dr. \quad (87a)$$

Similarly,

$$f(r_0) \xi_2'(r_0) \approx \frac{1}{r_{0e}} f(r_{0e}) L(r_{0e}, b_{\text{eff}}) - \int_{r_0}^{r_{0e}} g dr. \quad (87b)$$

We again find that $\xi_0(r)$ never vanishes, and the stability criterion reduces to

$$\frac{1}{r_{0i}} f(r_{0i}) L(r_{0i}, a_{\text{eff}}) - \frac{1}{r_{0e}} f(r_{0e}) L(r_{0e}, b_{\text{eff}}) + \int_{r_{0i}}^{r_{0e}} g dr > 0. \quad (88)$$

The following expressions for f and g are valid to lowest order in δ throughout the layer:

$$f = \frac{r_0 (k r_0 B_z + m B_\theta)^2}{k^2 r_0^2 + m^2}, \quad (89)$$

$$g = \frac{2k^2 r_0^2}{k^2 r_0^2 + m^2} \frac{dP}{dr}. \quad (90)$$

(We need not distinguish here between r_{0i} and r_{0e} because their difference is only of order δ .) The integrations in the inequalities (82), (84), (86), and (88) are therefore easily carried out. We obtain from (86), for example,

$$P(r_s^{(2)}) > P(r_s^{(1)}), \quad (91)$$

which leads to part (2) of the following theorem. Similarly, the other inequalities lead to parts (3) and (4).

Theorem 13. A sharp linear pinch is stable if and only if: (1) The quantity $d\phi/dr$ does not vanish at any point in the current layer where dP/dr is negative. (2) If r_1 and r_2 are any two points in the current layer such that $\phi(r_2) - \phi(r_1)$ is an integral multiple of π , then $P(r_2) > P(r_1)$ if $r_2 > r_1$. (3) For all values of m and k such that there are no singular points in the current layer, we have

$$\begin{aligned}
 & (kr_0 B_{zi} + mB_{\theta i})^2 L(r_{0i}, a_{\text{eff}}) - (kr_0 B_{ze} + mB_{\theta e})^2 L(r_{0e}, b_{\text{eff}}) \\
 & + 2k^2 r_0^2 (P_e - P_i) > 0.
 \end{aligned} \tag{92}$$

(4) For all values of m and k such that there is at least one singular point in the current layer, we have

$$(kr_0 B_{zi} + mB_{\theta i})^2 L(r_{0i}, a_{\text{eff}}) + 2k^2 r_0^2 [P(r_s^{(1)}) - P_i] > 0 \tag{93a}$$

and

$$-(kr_0 B_{ze} + mB_{\theta e})^2 L(r_{0e}, b_{\text{eff}}) + 2k^2 r_0^2 [P_e - P(r_s^{(2)})] > 0, \tag{93b}$$

where $r_s^{(1)}$ and $r_s^{(2)}$ are respectively the first and last singular points in the layer.

As was pointed out in the introduction, the stability criterion given by this theorem depends critically on the detailed structure of the current layer; it obviously cannot be obtained by simply treating the current-density components as delta functions.

Let us now suppose that the plasma pressure vanishes in one or both of the constant-pressure regions. What would be the effect of replacing the infinitely conductive pressureless plasma with a vacuum? The energy integral for a hydromagnetic system with a vacuum region is given by

$$W = W_f + W_s + W_v, \tag{94}$$

in which the three terms represent contributions from the plasma, the interface, and the vacuum respectively (1). The plasma contribution is given by Eq. (3) with the integration restricted to the region occupied by the plasma; it reduces as before to Eq. (14). The interface contribution is

$$W_s = \frac{1}{2} \int d\sigma (\underline{n} \cdot \underline{\hat{t}})^2 \underline{n} \cdot \left[\underline{\nabla} \left(P + \frac{1}{2} B^2 \right) \right], \tag{95}$$

where dr is the element of surface area, $\underline{n}$ is the unit normal pointing into the vacuum, and the bracketed expression is the jump in $\nabla(P + \frac{1}{2} B^2)$ as we pass through the interface into the vacuum. Using Eq. (2), we obtain

$$\left[\frac{d}{dr} \left(P + \frac{1}{2} B^2 \right) \right] = - \left[B_{\theta}^2 / r \right], \quad (96)$$

which vanishes because B_{θ} is continuous at the edges of the current layer; hence $W_s = 0$. The vacuum contribution is

$$W_v = \frac{1}{2} \int d^3x (\nabla \times \underline{A})^2, \quad (97)$$

where $\underline{A}$ is the vector potential of the perturbed vacuum magnetic field. We are to minimize with respect to $\underline{A}$ and $\underline{\xi}$ independently, except that the boundary condition

$$\underline{n} \times \underline{A} = - (\underline{n} \cdot \underline{\xi}) \underline{B} \quad (98)$$

must be satisfied.

Let us extend the definition of $\underline{\xi}$ into the vacuum region by writing $\underline{A} = \underline{\xi} \times \underline{B}$. Then W_v has the same form as W_f , and the boundary condition (98) reduces to the continuity of ξ_r at the interface. The expression for the complete energy integral is therefore unchanged if the pressureless plasma is replaced by a vacuum. The only difference is that $\underline{\xi}$, since it is no longer represents the displacement of a real fluid, can be singular as long as the perturbed magnetic field $\underline{Q} = \nabla \times \underline{A} = \nabla \times (\underline{\xi} \times \underline{B})$ is well-behaved. It follows that the stability criterion is the same for a vacuum as it is for a pressureless plasma if there are no singular points in the vacuum region. If there is a singular point, the Euler-Lagrange solution given by Eq. (72) (or by its analogue for the outer region $r_0 < r < b$) becomes infinite at that point. We have seen in the case of a pressureless plasma that this requires us to use a_{eff} instead of a in the stability criterion. With a vacuum, however, the

singular behavior of ξ is immaterial;²⁹ what we must examine is the behavior of $\underline{Q}$. Because of Eqs. (6c), (10), and (13b), it is easy to express $\underline{Q}$ as a functional of ξ alone. Using the relations

$$\frac{dB_z}{dr} = \frac{d}{dr}(rB_\theta) = 0, \quad (99)$$

we obtain

$$Q_r = i(krB_z + mB_\theta) \xi / r, \quad (100a)$$

$$Q_\theta = \frac{-m}{k^2 r^2 + m^2} \frac{d}{dr} [\xi (krB_z + mB_\theta)], \quad (100b)$$

$$Q_z = \frac{kr}{m} Q_\theta, \quad (100c)$$

which remains finite at the singular point. It follows that the singular point has no effect on the stability criterion for a vacuum, and we have

Theorem 14. The stability criterion for a sharp linear pinch with a vacuum in the region $a < r < r_0$ (or $r_0 < r < b$) is given by Theorem 13 with $P_i = 0$ (or $P_e = 0$) and a in the place of a_{eff} (or b in the place of b_{eff}).

Thus a vacuum and a pressureless plasma are equally stable if there is no singular point in the pressureless region, but the pressureless plasma is more stable if there is such a point. This result is consistent with the comparison theorem in Section 3b of Reference (1). Also, the stability criterion given by Theorem 14 agrees with that of Rosenbluth (21) for the sharp columnar pinch if P_e , $B_{\theta i}$ and a are set equal to zero and with that of Kaufman and Newcomb (22) for the sharp tubular pinch if P_i and P_e are set equal to zero. We note that the results of this section provide a rigorous justification for the heuristic method used by Kaufman and Newcomb, which was based on the assumption that the minimizing displacement $\xi_0(r)$ is constant in the

²⁹ It does, however, prevent us from integrating Eq. (14) by parts to obtain Eq. (15); the simple form of the energy integral given by Eq. (15) is therefore invalid for a vacuum region containing a singular point.

current layer. It has been shown that $\xi_0(r)$ is indeed constant except in the small intervals $r_s - \epsilon < r < r_s + \epsilon$, and one can easily verify that these intervals do not contribute appreciably to $W(\xi)$. Thus $W(\xi)$ is not changed in lowest order if $\xi_0(r)$ is replaced by a constant.

We have obtained the stability criterion for a sharp pinch from Theorem 12, which involves two Euler-Lagrange solutions, ξ_1 and ξ_2 , in each independent subinterval. It may also be instructive to show how it can be obtained directly from Theorem 10, which involves only one Euler-Lagrange solution.³⁰ Assuming, for example, that there is at least one singular point in the current layer but none between a and r_{0i} , let us consider the independent subinterval $a < r < r_s$, where r_s is the first singular point. The relevant Euler-Lagrange solution is given by Eq. (72) between a and r_{0i} . Requiring $\xi(r)$ and its derivative to be continuous at r_{0i} , we can find the solution in the interval $r_{0i} < r < r_s - \epsilon$ by an argument similar to the one leading to Eq. (79). The result is

$$\xi \cong 1, \quad (101a)$$

$$f \frac{d\xi}{dr} \cong \frac{1}{r_{0i}} f(r_{0i}) L(r_{0i}, a) + \int_{r_{0i}}^r g dr. \quad (101b)$$

In the interval $r_s - \epsilon < r < r_s$ we have $f \cong \alpha(r_s - r)^2$, $g \cong \beta$, and for the general solution of the Euler-Lagrange equation:

$$\xi(r) \cong c_1(r_s - r)^{-n_1} + c_2(r_s - r)^{-n_2}, \quad (102)$$

where n_1 and n_2 are determined by Eq. (30). The relevant orders of magnitude are $\alpha \sim \delta^{-2}$, $\beta \sim \delta^{-1}$, and $n_1 = 1 - n_2 \sim \delta$. The constants c_1 and c_2 are determined by the boundary conditions

³⁰ This will be equivalent in its essentials to Rosenbluth's treatment of the sharp columnar pinch (21).

$$\xi(r_s - \epsilon) \approx 1, \quad (103a)$$

$$\xi'(r_s - \epsilon) \approx A/a\epsilon^2, \quad (103b)$$

where

$$A = \frac{1}{r_{0i}} f(r_{0i}) L(r_{0i}, a) + \int_{r_{0i}}^{r_s} g dr. \quad (104)$$

Since the second term of Eq. (102) dominates when r is sufficiently close to r_s , the solution $\xi(r)$ goes through zero if and only if c_2 is negative. Substituting (102) into the boundary conditions (103), and solving for c_2 to lowest order in ϵ/δ and δ^2/ϵ , we obtain $c_2 \approx A/a$. Since a is necessarily positive, the stability criterion for the independent subinterval $a < r < r_s$ is that A should be positive, which is the same as our previous result, the inequality (82).

Acknowledgement. The author would like to thank Dr. A. N. Kaufman for several interesting discussions and for his critical reading of the manuscript.

REFERENCES

1. I. B. Bernstein, E. A. Frieman, M. D. Kruskal, and R. M. Kulsrud, Proc. Roy. Soc. A244, 17 (1958).
2. T. G. Cowling, Magnetohydrodynamics, Interscience Publishers, Inc., New York (1957).
3. W. A. Newcomb, Ann. Phys. 3, 347 (1958).
4. S. Lundquist, Phys. Rev. 83, 307 (1951).
5. A. Baños and K. Schwartz, Phys. Fluids (to be published).
6. A. N. Kaufman, T. G. Northrop, and K. M. Watson, Ann. Phys. (to be published).
7. M. D. Kruskal and C. R. Oberman, Phys. Fluids 1, 275 (1958).

REFERENCES (Contd.)

8. M. N. Rosenbluth and N. Rostoker, *Phys. Fluids* 2, 23 (1959).
9. W. A. Newcomb, *Ann. Phys.* (to be published).
10. J. Dawson and I. B. Bernstein, Hydromagnetic Instabilities Caused by Runaway Electrons, TID-7558 (Papers presented at the Controlled Thermonuclear Conference Held at Washington, D.C., Feb. 3-5, 1958.).
11. W. A. Newcomb, Self-Excitation of Ion Oscillations in a Plasma, WASH-289 (Proceedings of the Conference on Thermonuclear Reactions, University of California Radiation Laboratory, Livermore, California, February 7, 8, and 9, 1955).
12. K. Wilson and M. N. Rosenbluth, (to be published).
13. B. R. Suydam, Stability of a Linear Pinch with Continuous B-Field, TID-7558 (See Reference 10).
14. M. N. Rosenbluth, Los Alamos Scientific Laboratory Report LA-2030, April 1956.
15. M. D. Kruskal and M. Schwarzschild, *Proc. Roy. Soc. A* 223, 348 (1954).
16. M. D. Kruskal and J. L. Tuck, *Proc. Roy. Soc.* (to be published); also Los Alamos Scientific Laboratory Report LA-1716, November 1953.
17. R. J. Tayler, *Proc. Phys. Soc. B* 70, 31 (1957).
18. R. J. Tayler, *Proc. Phys. Soc. B* 70, 1049 (1957).
19. M. A. Leontovich and V. D. Shafranov, Physics of Plasmas and Problems of Controlled Thermonuclear Reactions, Vol. 1, p. 207, Publishing House of the Academy of Sciences of the U.S.S.R. (1958). (An English translation of this series of papers, hereafter referred to as PPPCTR, will be published by the Pergamon Press. The page numbers refer to the Russian edition.)

REFERENCES (Contd.)

20. V. D. Shafranov, *Atomnaya Energia* 5, 38 (1956); also PPPCTR, Vol. 2, p. 130.
21. M. N. Rosenbluth, Paper 347, Proceedings of the Second United Nations International Conference on the Peaceful Uses of Atomic Energy held in Geneva September 1 - 13, 1958.
22. A. N. Kaufman and W. A. Newcomb, *Phys. Fluids* (to be published); also designated as UCRL-5434.
23. V. D. Shafranov, PPPCTR, Vol. 4, p. 61.
24. M. N. Rosenbluth, private communication through A. N. Kaufman.
25. J. W. Dungey and R. E. Loughhead, *Aust. J. Physics* 7, 5 (1954).
26. R. J. Tayler, *Phil. Mag.* 2, 33 (1957).
27. R. P. Agnew, Differential Equations, McGraw-Hill Book Co., Inc., New York and London (1942).
28. G. A. Bliss, Calculus of Variations, The Open Court Publishing Co., LaSalle, Illinois (1925).
29. H. B. Dwight, Tables of Integrals and Other Mathematical Data, The MacMillan Co., New York (1947).
30. K. Hain and K. U. von Hagenow, private communication.

ADDENDUM

Corollary 11-2. A necessary condition for the stability of a columnar pinch is that the plasma pressure should have a minimum at $r = 0$ (30). (Because the ohmic heating is greatest at a finite distance from the axis, this condition is generally fulfilled in practice, even though the density has a maximum at $r = 0$.)

Proof: For small but nonvanishing values of r Suydam's condition reduces to $dP/dr > 0$, since this term is of order r while the other term is only of order r^3 .