UCRL-5335

UNIVERSITY OF CALIFORNIA
Radiation Laboratory
Livermore, California

Contract No. W-7405-eng-48

THE SELF-CONSISTENT FIELD OF SINGLE-TYPE ELECTRONS
IN A UNIFORM MAGNETIC FIELD

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September 1958

Printed for the U. S. Atomic Energy Commission

THE SELF-CONSISTENT FIELD OF SINGLE-TYPE ELECTRONS IN A UNIFORM MAGNETIC FIELD*

by
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The proposed thermonuclear machine, Astron, recently reported on at the Second Geneva International Conference on the Peaceful Uses of Atomic Energy has, as a vital element, a cylindrical layer of relativistic electrons which, circulating in a uniform impressed magnetic field, are able to reverse that field in the interior.

The electrons are initially injected tangentially around a circle with nearly identical energy, negligible radial velocity, and nearly identical axial velocity. Thus they all have the same canonical angular momentum. The resulting situation is severely complicated by:

1. The drag of plasma electrons which causes a progressive decrease in energy and progressive change in canonical angular momentum which is dependent upon trajectory.
2. The angular scattering by plasma ions and electrons which generates a statistical distribution in canonical angular momentum and in v -perpendicular, both dependent upon trajectory.
3. The contribution of the diamagnetism of the plasma.
4. The influence of possible electric fields.
5. Terminal conditions which confine the action to a finite length of axis.

The only hope of progress lies in solving simple problems first and adding complications as knowledge and concepts are developed along the way. Therefore, as a first step, all the complications were stripped away, leaving only a rotating cylinder of single-type electrons for which the self-consistent field configuration was to be found.

* Work was performed under auspices of the U.S. Atomic Energy Commission.

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This has its difficulties. In highly ionized plasmas the magnetic problem, where such self-consistency enters, has been largely approached by treating the radius of gyration of the particles as being ignorably small so that magnetohydrodynamic equations can be used.

The author, concerned about this simplification, has already reported¹ the computation of the transition between vacuum and fully developed plasma for cartesian geometry in a few very special cases. The analysis, but not the computations, included cylindrical geometry. The method of analyzing by trajectories appears to be less amenable to generalization than the present method.

Here the analysis concerns particles in cylindrical geometry all of which have the same charge e , the same mass m_A , the same speed v_A , perpendicular to the magnetic field, and the same canonical angular momentum p_A . Space charge neutralization is assumed and there is no electric field. The impressed magnetic field (i.e., field at infinity) is B_z in the direction of the z -axis and all quantities are independent of z . The particle distribution is uniform in θ .

Problems more complicated than this can be solved easily at the expense of applicability to physical situations. The method used here is based on a suggestion by E. G. Harris of the U. S. Naval Research Laboratory, who has pointed out that when the distribution of the particles is expressed as a function of the constants of the motion only, the Boltzmann Equation (which contains the magnetic force effects) is automatically satisfied. The vector potential and electric potential are then the solutions of a pair of coupled second-order differential equations found from the appropriate Maxwell equations. The essential difficulty is the setting up of the distribution function to correspond to a physical situation. There are ways to do this, but formidable complications arise.

The present analysis is concerned with a highly artificial situation but it is exact. It is of such simplicity that it serves well in orienting ideas, and the apparent anomalies are instructive.

1. L. Tonks, "Trajectory-wise Analysis of Cylindrical and Plane Plasmas in a Magnetic Field and without Collisions," UCRL-5302. (To be published in the Physical Review.)

With regard to the distribution function in space and velocity, we note that since v_A and also v_z are identical for all particles, this can be stated, with its attendant differential element as $f d\phi r dr d\theta$, where ϕ is the angle between trajectory and the radius vector to it, as will be clear from Fig. 1. To use Harris' method,

$$f d\phi r dr d\theta = a |\delta(p_0 - p_A)| d\phi r dr d\theta \quad (1)$$

with a a number related to the strength of the particle layer, and δ denoting the Dirac δ -function.

We now require two integrals over velocity space, one for the particle density, $n(r)$, the other for the azimuthal current density which can be directly related to the vector potential, $A_\theta(r)$:

$$n(r) = \int f d\phi = a \int |\delta(p_0 - p_A)| d\phi \quad (2)$$

$$\frac{d}{dr} \left(\frac{r A_\theta}{r} \right) = - \frac{4\pi e}{c} \int r \phi f d\phi = - \frac{4\pi e a}{c} \int v_A \sin \phi |\delta(p_0 - p_A)| d\phi \quad (3)$$

Some care is required in the ϕ integration. Commonly the argument of a δ -function is dimensionless, but that is not so here. If for the δ -function

$$\int \delta(x - x_0) dx = 1$$

irrespective of the dimensions of x , then

$$\int \delta(g(x) - g(x_0)) dx = \int \frac{\delta(g(x) - g(x_0)) dg(x)}{dg(x)/dx} = \frac{1}{dg(x)/dx} \Big|_{x_0} \quad (4)$$

Thus to perform the last integration p_0 must be expressed in terms of velocity and position:

$$p_0 = m_A r v_A \sin \phi + (e/c) r A_\theta(r) \quad (5)$$

Taking note of Eqs. (4) and (5), Eq. (2) becomes

$$\begin{aligned} n(r) &= a \int |\delta(m_A v_A r \sin \phi + (e/c) r A_\theta - p_A)| d\phi \\ &= \frac{2a}{m_A v_A r |\cos \phi_0|} \end{aligned} \quad (6)$$

where

$$\sin \phi_0 = \frac{p_A}{m_A v_A r} - \frac{e A_\theta(r)}{m_A c v_A} \quad (7)$$

and the factor 2 appears because $\sin \phi$ traverses $\sin \phi_0$ twice in a complete 2π angle. We can now conveniently drop the distinction between ϕ and ϕ_0 .

We note that

$$\frac{d(r \sin \phi)}{dr} = - \frac{e}{m_A c v_A} \frac{d(r A_0)}{dr} \quad (8)$$

The same procedure that converted Eq. (2) to Eq. (6) then converts Eq. (3) to

$$s \frac{d}{ds} \left(\frac{d(s \sin \phi)}{s ds} \right) = \frac{\sin \phi}{\sqrt{1 - \sin^2 \phi}} \quad (9)$$

The dimensionless s being substituted for r :

$$s = \frac{8\pi a e^2}{m_A^2 c^2 v_A} r \quad (10)$$

It is convenient to use

$$R = - \sin \phi \quad (11)$$

so that Eq. (9) becomes

$$s \frac{d}{ds} \left(\frac{ds R}{s ds} \right) = \frac{R}{\sqrt{1 - R^2}} \quad (12)$$

From Eq. (6) it is readily found that the number of particles per unit length is

$$N = \frac{\gamma}{2 r_e} \int \frac{ds}{\sqrt{1 - R^2}} \quad (13)$$

Here $r_e = 2.80 \times 10^{-13}$ cm, the classical electron radius, γ is the ratio of relativistic to electron rest mass and the integration is carried over the annulus defined by the loci of perigees and apogees.

Obviously the integral is terminated by $|R| = 1$ at both inner limit, s_1 , and outer limit s_2 . The constant a , which bears some relation to particle cloud strength affects N through its implicit involvement in $R(s_2)$.

We think of a trajectory as starting at its apogee where $R = 1$, $s = s_2$, and the magnetic field is B_2 . This fixes all the necessary constants, but rather than B_2 it is more convenient to introduce a dimensionless "magnetic field"

$$h = \frac{d(sR)}{s ds} \quad (14)$$

which is consistent with the vector-potential nature of R as seen in Eq. (7).

It is easy to show that

$$h = \frac{m_A c B}{8\pi a e v_A} ; \quad h_2 = \frac{m_A c B_2}{8\pi a e v_A} \quad (15)$$

and

$$g_2 \equiv s_2 h_2 = \frac{e r_2 B_2}{m_A v_A c} = r_2/a \quad (16)$$

where r_2 is the radial coordinate of the apogees and a is the radius of gyration of a particle in the field B_2 . The quantity g_2 has such a simple physical meaning that it and s_2 at $R = 1$ have been consistently used as the trajectory-determining constants.

It quickly became apparent that Eq. 13 was unsuitable for machine calculation beginning at s_2 . The substitution

$$s \equiv s_2 - v^2 \quad (17)$$

converted it to

$$R'' = \frac{s_2 + v^2}{s_2 - v^2} \frac{R'}{v} + \frac{4v^2 R}{(s_2 - v^2)^2} + \frac{4v^2 R}{(s_2 - v^2) \sqrt{1 - R^2}} \quad (18)$$

where primes denote differentiation with respect to v . The series expansion near s_2 is

$$R = 1 - \frac{g_2 - 1}{s_2} v^2 + \frac{2\sqrt{2} v^3}{3\sqrt{s_2}(g_2 - 1)} + \dots \quad (19)$$

so that at $v = 0$;

$$R'(s_2)/v = -2(g_2 - 1)/s_2$$

$$v^2 R(s_2) = 0$$

$$v^2/\sqrt{1 - [R(s_2)]^2} = 0$$

$$R''(s_2) = -2(g_2 - 1)/s_2$$

Thus all terms are finite and the solution by calculating machine can be started. The Runge-Kutta-Gill method was used to start.²

In Eq. (19) $(g_2 - 1)/s_2$ must be positive because R can only decrease from unity. Accordingly, in choosing the constants, g_2 must always be greater than unity. If $g_2 < 1$ then by Eq. (16) the trajectory would lie outside r_2 so that this is really the condition for starting at the perigee and not the apogee.

²Mr. Thomas Haratani of UCRL was most helpful in coding this problem and seeing the machine calculations through.

The starting difficulty was overcome by a reformulation of the differential equation but no similarly simple stratagem can cope with the termination at $R = \pm 1$. The derivative d^2R/ds^2 , or d^2R/dv^2 , still goes to infinity, and the location of s_1 , about which an expansion could be made, is one of the quantities to be determined. By using the fact that in the neighborhood of s_1

$$R = R_1 \left[1 - \frac{1 - g_1}{s_1} u^2 + \frac{2\sqrt{2} u^3}{3\sqrt{s_1(1 - g_1)}} + \dots \right]$$

where

$$u^2 = s - s_1 \quad \text{and} \quad g_1 < 1$$

one finds that

$$h_1 = 1/s - \frac{1}{2v} \left| \frac{dR}{dv} \right| \left[1 + \frac{4\sqrt{2(1 - |R|)} v^2}{s(dR/dv)^2} \right] + O(u^2) \quad (20)$$

and all the quantities needed for the coefficients appear directly in the numerical calculation based on Eq. (18). For the most part, it was felt that the last value of h given by the machine corresponding to $|R| = 0.9999$ was good enough for present purposes.

In general the value of s_1 is different from zero, but there is a singular solution for which $R = 0$ at $s = 0$. This corresponds to a trajectory passing through the axis where its motion must be purely radial. The series expansion near the origin is

$$R = (h_0/2) s + (h_0/6) s^2 + \dots \quad (21)$$

where h_0 is the dimensionless "magnetic field", Eq. (14), on the axis. The paired values of g_2 and s_2 found by solving Eq. (12) in this way locate the discontinuities in the solutions of that equation.

A qualitative survey of the field and trajectory relations serves for preliminary orientation. Consider the case that $g_2 < 2$, and that we proceed from very lightly populated to heavily populated electron clouds. When the electrons are sparse they have almost circular orbits, as indicated in Fig. 2a. At radial distances less than r_1 , the field is slightly less than at r_2 so that $h_1/h_2 < 1$. Increasing the number of electrons per axial cm, N , reduces the curvature of the orbit for all radial distances less than r_2 , so that r_1 increases and the cloud forms a thinner

and thinner layer in, what can be expected to be, a smooth sequence of configurations.

Consider now the case that $g_2 > 2$, and that we again proceed from weak to strong electron clouds. Figure 2b shows in Curve 1 a typical orbit (schematic) in the weak cloud. A reduction in curvature of path by strengthening the cloud here causes r_1 to decrease and cross the axis in the progression: orbit 1, orbit 2, orbit 3. When the envelope of the perigees crosses the axis, the circulating electric current in this envelope reverses its sense of rotation with a consequent sharp decrease of magnet field. Mathematically, it is increasing s_2 that causes this progression. The details of the relation to N for all values of g_2 can only be determined by the numerical solution of Eq. (12) and the integration in Eq. (13).

It is noteworthy that although g_2 has the clear physics meaning shown by Eq. (16), s_2 must be looked upon as being only of mathematical usefulness. It is true that s/s_2 is the radial position relative to the apogee but s_2 itself is not interpretable simply. The second important physics parameter is N , or $(2r_e/\gamma)N$, and this only appears in the mathematics as a result of the integration of Eq. (13). But results will be clearer if they are shown in terms of the parameter s_2 .

Figure 3 with $g_2 = 2^{2/3} = 1.587$ shows a progression of field configurations corresponding to Fig. 2a as s_2 varies. The field strength relative to B_2 , h/h_2 , is plotted against relative radial distance s/s_2 . The inner terminus of each curve lies at s_1/s_2 , h_1/h_2 . These three curves encompass the range of maximum field depression and, as will appear, maximum N . As s_2 approaches zero, h_1/h_2 approaches unity (in this and all other cases) and s_1/s_2 approaches 0.260.

Figure 4 with $g_2 = 2.0$ is a similar plot for the case which just falls short of reversal. For $s_2 = 4$, h_1/h_2 lies close to zero, and a more accurate treatment for calculating h_1 might show that it lies even closer. The analytical solution which might show it actually to be zero has not been attempted. It is noteworthy that this minimum field occurs with a very thin layer.

Figure 5 with $g_2 = 2^{4/3} = 2.520$ is the first progression shown corresponding to Fig. 2b. As s_2 approaches zero s_1/s_2 approaches 0.413.

As s_2 increases toward 0.8076, s_1/s_2 approaches zero and h_1/h_2 increases above unity, as will be clearer in Fig. 6. This inequality of internal and external magnetic fields is a clear demonstration of the inapplicability of the pressure balance equation

$$B_2^2 - B_1^2 = p_1^2 - p_2^2 = 0 \quad (22)$$

since B_2 and B_1 are both in vacuum. The failure of this equation lies in the tensor nature of the pressure within the electron swarm. (In the comparable cartesian case there is no inside and outside, and the tensor nature of the pressure does not enter the expression for $\text{grad } B^2$ so that Eq. (22) is valid.) The electron trajectory can be visualized as essentially egg-shaped with the small end at the perigee and the large end cut and pulled slightly apart before joining the next egg shape so that the egg, which is described in a clockwise sense, itself progresses counterclockwise thus furnishing the circulating current about the axis which enhances the field there.

At $s_2 = 0.8076$, the orbit passes through the axis and the internal field decreases discontinuously to the neighborhood of zero. It is definitely negative in the range $s_2 = 1$ to 8 and probably remains so for larger values.

Figures 6 and 7 are for $g_2 = 4$ and 8 respectively and show the same behavior as Fig. 5 except for the definite overshoot in the discontinuity of h .

Figure 8 shows how the swarm strength, $(2r_e/\gamma)N$, varies with s_2 in the 5 cases already discussed. This should be considered simultaneously with Fig. 9 showing the relative internal magnetic field h_1/h_2 and Fig. 10 of the relative perigee distance s_1/s_2 . Here, for the sake of continuity in the representation the values of s_1 associated with a reversal in mechanical angular momentum have been assigned negative values. Points associated with the singular solution of Eq. (12) are identified by crosses.

The first notable feature of Fig. 8 is the limit to the number of electrons which a set of physics conditions will support. This maximum can always be exceeded by making g_2 larger, so that the attempt to crowd in too many electrons would cause the cloud to expand beyond r_2 thereby allowing it to find the larger g_2 for itself.

A second striking feature is the ambiguity associated with the layer strength. For $g_2 \leq 2$ and again for $g_2 = 8$ and undoubtedly for a range beginning somewhat below 8 and extending on upward two distinct configurations are consistent with a single strength in a certain strength range. In the low- g_2 range the same number of electrons can lie in either a thin or a thick layer. In the high- g_2 range the same number are consistent with either a reversed or non-reversed field.

The whole situation is highly hypothetical and very unrealistic but it is interesting to speculate on how with a single-type-electron layer available one could achieve field reversal. Apparently by setting experimental conditions - electron energy, injection radius and impressed magnetic field - for g_2 in the range 2.52 to 4.0 it is simply a matter of feeding in enough electrons to push $(2r_e/\gamma)N$ above the crosses on the corresponding $(2r_e/\gamma)N$ -versus- s_2 curve. For $g_2 = 2.52$ the value is 1.159. For $g_2 = 4.0$ it is 5.705, but there is the definite danger of overcharging the layer. If it is now desired to increase the reversal from the 50 per cent (Fig. 9) possible at $g_2 = 4$ to the somewhat more than 70% apparently possible at $g_2 = 8$, the programming of N with g_2 to accomplish this is not clear.

It is to be emphasized that the present analysis is but a first step in a rather complicated program. A second step which is now in the process of being carried out involves electrons distributed in a range of canonical angular momentum. Later further degrees of complexity approaching the physical circumstances more closely will be included.

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Fig. 1. Spatial relations.

(a)

(b)

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Fig. 2. Types of trajectory evolution. (a) $g_2 < 2$; (b) $g_2 > 2$

Fig. 3. Magnetic field distributions for $g_2 = r_2/a = 1.587$.

Fig. 4. Magnetic field distributions for $g_2 = r_2/a = 2.0$.

Fig. 5. Magnetic field distributions for $g_2 = r_2/a = 2.52$.

Fig. 6. Magnetic field distributions for $g_2 = r_2/a = 4$.

Fig. 8. Strength of layer of circulating electrons.

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Fig. 9, 10. Internal vacuum magnetic field relative to external vacuum field.

Radius of internal uniform-field region relative to external radius of electron layer.