Yu. Senichev
A. Budzko

Passing Through Half-Integer Resonance Due to Space Charge Under Different Initial Distributions

Superconducting Super Collider
Laboratory

Disclaimer Notice

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government or any agency thereof, nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise, does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.

Superconducting Super Collider Laboratory is an equal opportunity employer.

Passing Through Half-Integer Resonance Due to Space Charge Under Different Initial Distributions

Yu. Senichev*

Superconducting Super Collider Laboratory[†]
2550 Beckleymeade Avenue
Dallas, Texas 75237

and

A. Budzko*

TRIUMF
4004 Wesbrook Mall
Vancouver, BC V6T 2A3, Canada

September 1993

DISCLAIMER

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency thereof, nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.

---

* Permanent address: Institute for Nuclear Research, 60th October Anniversary Prospect 7a, Moscow 117312, Russia.

[†] Operated by the Universities Research Association, Inc., for the U.S. Department of Energy under Contract No. DE-AC35-89ER40486.

DISCLAIMER

Portions of this document may be illegible in electronic image products. Images are produced from the best available original document.

1.0 INTRODUCTION

The problem of space charge continues to be interesting in connection with the development of the Kaon and Neutron Facilities and the Superconducting Super Collider projects. In the former, the high average intensity of the proton beam can be reached by a super-high pick current in each acceleration cycle, giving rise to the painting procedure. In colliders, as is known, super luminosity can be reached by minimization of emittance growth under acceleration. In other words, the task consists of the beam acceleration with the maximum attainable density of current and with minimum achievable emittance growth. This is why this task arises when new accelerator projects are developed. There are at least two ways to investigate the space-charge effects on the designing stage: analytical and numerical simulation.¹⁻³ Each has its own advantages and disadvantages. The numerical simulation, with its use of macro particles approximation, is very well developed but it requires a lot of computer time and has limited flexibility in the variation of the initial conditions. As a result, the numerical methods are restricted in optimization possibility.

Analytical investigation gives a clearer idea about physical phenomena, but it must simplify the physical model and sometimes gives an incorrect interpretation of results. Analytical research can be divided into two types: the envelope equation method for self-consistent distribution, and the nonlinear equation for higher distribution moments. In our work we use numerical simulation as the base to achieve the correct results, but for interpretation of results the simplified analytical model is used as well.

2.0 PROBLEM STATEMENT

In almost any accelerator emittance growth is observed during the beam injection. One can explain this as the resonance crossing and beam mismatch with acceptance. But this growth depends on the method of distribution chosen under injection. We study in this paper the emittance growth under resonance crossing for different initial beam distributions. To eliminate the reasons for emittance growth, we consider the linear lattice without errors. In this case emittance should grow only due to the intrinsic resonances, or in other words, the envelope oscillation.

3.0 NUMERICAL CALCULATION

The problem is solved by the example of three different initial distributions: K-V, Waterbag, and Gaussian. To simulate self-consistent beam motion, the two-dimensional tracking program is used. Space-charge forces are found by solving the Poisson equation with zero boundary conditions. The longitudinal motion effects and beam bunching are not taken into consideration. The particle oscillation frequency is found from the rotating angle in the normalized phase space. As a test lattice, we use the racetrack structure for the

TRIUMF booster with two arcs and working point 7.65/5.6, although these results could be useful for the SSC Low Energy Booster and the Alternating Gradient Synchrotron as well.

The initial phase-space projections and the advanced phase diagram are shown in Figures 1(a), 1(b), and 1(c) for K-V, Waterbag, and Gaussian distributions, respectively. In all distributions we take the same rms emittance. One can see that all distributions have the same average point, and thus the same coherent tune shift.

Figure 1. Phase space and tune diagram for K-V (a), Waterbag (b), and Gaussian (c) distributions.

Figure 2 shows the emittance growth versus the number of turns for different distributions. The maximum emittance growth by a factor of 2.8 was observed in K-V and Waterbag, and a factor of 1.25 for Gaussian distribution.

Figures 3(a), 3(b), and 3(c) show the phase space for all observed distributions after five turns. It is obvious that the beam passes through half-integer and fourth-order resonance. In all cases the amplitude of the oscillation is finite.

Figures 4(a), 4(b), and 4(c) show the distribution after 45 turns. For Waterbag and K-V distribution we can observe half-integer resonance, which is stabilized by fourth-order resonance. For Gaussian distribution we observe the stabilized distribution by fourth-order resonance.

Figures 5(a), 5(b), and 5(c) show the final distribution for the same distributions. The instability does not grow. They are stabilized, but particles are still in half-integer resonance excited by envelope oscillation. The self-stabilization phenomenon is known in non-

linear systems, but in the space-charge problem we observe the nonlinear system, where nonlinearity changes in time.

Let's try to explain the phenomenon using a simplified analytical model that describes the resonance interaction.

Figure 2. RMS emittance growth vs. turn number for K-V, Waterbag, and Gaussian distributions.

Figure 3. Phase space for K-V (a), Waterbag (b), and Gaussian (c) distributions after 5 turns.

Figure 4. Phase space for K-V (a), Waterbag (b), and Gaussian (c) distributions after 45 turns.

Figure 5. Stabilized distribution for initial K-V (a), Waterbag (b), and Gaussian distributions.

4.0 ANALYTICAL DESCRIPTION

Taking into account that we disconnected two plates in the numerical experiment, the analytical approach has been done for the symmetrical cylindrical beam. Let's write Maxwell's equation for the symmetrical beam with the distribution of density ρ :

$$\frac{1}{r} \frac{\partial(r E_r)}{\partial r} = \frac{1}{\epsilon_0} \rho(r). \quad (1)$$

We choose the m -order binomial distribution, which includes in itself three considered types of distribution: Kapchinskij-Vladimirskij ($m = 0$), the Waterbag ($m = 1$), and the Gaussian ($m \rightarrow \infty$):

$$\rho(r) = \frac{m}{\pi b^2} \left[1 - \frac{r^2}{b^2} \right]^{m-1}, \quad (2)$$

where b is the maximum size of the beam. Then the dispersion function σ is defined as $\sigma = \frac{b}{\sqrt{2m+2}}$ and the rms emittance $\varepsilon_{rms} = \sigma^2/\beta$. The motion equation for any particle will be:

$$\frac{d^2 r}{ds^2} + \mathcal{K}(s)^2 r - \frac{r_0 N}{\pi R \beta_0^2 \gamma^3} \frac{r}{2\sigma^2} \mathcal{F}(r) = 0, \quad (3)$$

or using the smooth approximation $r = \eta\sqrt{\beta}$ and the new longitudinal coordinate $d\theta = \nu_0 \beta ds$:

$$\eta'' + \nu_0^2 \eta = 2\nu_0 \frac{r_0 N}{4\pi \beta_0^2 \gamma^3 \varepsilon_{rms}} \eta \left[1 + \sum_p b_p \cos p\theta \right] \mathcal{F}(\eta), \quad (4)$$

where we use the Fourier's expansion for $\beta = \bar{\beta} \left(1 + \sum_p b_p \cos p\theta \right)$ and the average mean of β -function $\bar{\beta} = \frac{R}{\nu_0}$, R is the average radius of the accelerator.

$$\mathcal{F}(r) = \begin{cases} 1/2 & \text{for K-V} \\ \frac{2}{3} - \frac{2}{9} \frac{1}{2!} \frac{\eta^2}{2\varepsilon_{rms}} & \text{for Waterbag} \\ 1 - \frac{1}{2!} \frac{\eta^2}{2\varepsilon_{rms}} + \frac{1}{3!} \left(\frac{\eta^2}{2\varepsilon_{rms}} \right)^2 - \dots & \text{for Gaussian.} \end{cases} \quad (5)$$

Usually one denotes the value $r_0 N / 4\pi \beta^2 \gamma^3 \varepsilon_{rms}$, as the tune shift $\delta\nu$. Bogolubov's method can be used to solve this nonlinear equation:

$$\begin{aligned} \eta &= \sqrt{\varepsilon} \cos \Phi \\ \eta' &= -\nu_0 \sqrt{\varepsilon} \sin \Phi, \end{aligned} \quad (6)$$

or passing to new the variables,

$$\begin{aligned} \varepsilon &= \eta^2 + \left(\frac{\eta'}{\nu_0} \right)^2 \\ \Phi &= -\arctan(\eta'/\nu_0 \eta). \end{aligned} \quad (7)$$

Differentiating new variables ε and Φ with respect to θ and substituting Eq. (5), we get:

$$\begin{aligned} \frac{d\varepsilon}{d\theta} &= -2\delta\nu_0 \varepsilon \sin 2\Phi \left[1 + \sum_p b_p \cos p\theta \right] \mathcal{F}(\varepsilon) \\ \frac{d\Phi}{d\theta} &= \nu_0 - 2\delta\nu \cos^2 \Phi \left[1 + \sum_p b_p \cos p\theta \right] \mathcal{F}(\varepsilon). \end{aligned} \quad (8)$$

The function $\mathcal{F}(\varepsilon)$ has the same view as Eq. (5), but with the new argument $\frac{\varepsilon}{2\varepsilon_{rms}} \cos^2 \Phi$. In the absence of any resonances we can average Eq. (8) over the whole cross section of the beam:

$$\begin{aligned}\overline{\frac{d\varepsilon}{d\theta}} &= 0 \\ \overline{\frac{\Phi}{d\theta}} &= \nu_0 - \Delta\nu,\end{aligned}\tag{9}$$

where $\Delta\nu$ is the averaged tune shift. It may be equal to $\frac{5}{8}\delta\nu$ for Gaussian, $\frac{5}{9}\delta\nu$ for Waterbag, and $\frac{1}{2}\delta\nu$ for K-V. It means that in the resonanceless system ($b_p = 0$), the emittance doesn't grow and the coherent tune shift approximately doesn't depend on the distribution with the same rms emittance. On other hand, the tune spread is maximum in Gaussian and equals zero in K-V.

Really, for any lattice the Fourier's expansion of the envelope β -function involves in itself all harmonics that could give the resonance condition for an emittance growth. Let's consider the case when $\mathcal{F}$ involves two first terms and can be represented as $\mathcal{F} = f_0 - \varepsilon f_1 \cos^2 \Phi$. Then

$$\begin{aligned}\overline{\frac{d\varepsilon}{d\theta}} &= \delta\nu\varepsilon b_m f_0 \sin(2\Phi - m\theta) - \delta\nu\varepsilon^2 f_1 b_m \left[\frac{1}{2} \sin(2\Phi - m\theta) - \frac{1}{4} \sin(4\Phi - m\theta) \right] \\ \overline{\frac{\Phi}{d\theta}} &= \nu_0 - \delta\nu f_0 + \frac{3}{4}\delta\nu\varepsilon f_1 + \frac{1}{4}\delta\nu f_0 b_m \cos(2\Phi - m\theta) \\ &\quad - \delta\nu\varepsilon f_1 b_m \left[\frac{1}{8} \cos(4\Phi - m\theta) + \frac{1}{2} \cos(2\Phi - m\theta) \right],\end{aligned}\tag{10}$$

where m is the resonant harmonic number with amplitude b_m . One can see from these equations that any square nonlinearity in distribution gives the half-integer and the fourth-order resonances simultaneously. They can exist only together. The main question is whether the fourth-order resonance stabilizes the half-integer resonance. The trajectories of particles on the phase plane are described by the equation:

$$\begin{aligned}\frac{1}{2}\delta\nu b_m f_0 \varepsilon \cos 2\Psi - \frac{1}{4}\delta\nu b_m f_1 \varepsilon^2 \cos 2\Psi - (\nu_0 - \delta\nu f_0)\varepsilon - \frac{3}{8}\delta\nu f_1 \varepsilon^2 \\ - \frac{1}{4}\delta\nu f_0 b_m \varepsilon \cos 2\Psi + \frac{1}{4}\delta\nu f_1 b_m \varepsilon^2 \cos 2\Psi = C(\varepsilon, \Psi),\end{aligned}\tag{11}$$

where $\Psi = \Phi - m\theta/n$ is the slow phase in n -th order resonance. The terms with ε^2 give stabilization of parametric resonance. So for the distribution, where the square term is absent, the half-integer resonance will give sufficient growth. Of course we should mention here that our analytical model does not take into account the self-consistent redistribution,

which could create the term with ϵ^2 for any distribution after passing through the resonance. At least this explanation answers why the emittance grows so strongly in resonance for K-V distribution.

Using more high-order terms of the binomial distribution, it is possible to show that any order resonance will be stabilized by the next order distribution. Since the envelope β -function has all harmonics, we will always have the resonant condition for one or more harmonics. So maximum stabilization will be observed for Gaussian, where any perturbation will be distributed over an unlimited number of harmonics.

5.0 CONCLUSION

We have studied in this paper the beam passing through the grid of intrinsic-envelope resonances, where the half-integer resonance is maximum. At the initial stage, the beam with any kind of distribution is very sensitive to the half-integer and fourth resonances. During 60–100 turns the distribution becomes self-stabilized and very similar to the Gaussian. Maximum growth is observed for maximum uniform distribution in real space. For intrinsic resonances the self-stabilization comes when most particles remain in resonance.

ACKNOWLEDGEMENTS

We would like to thank Dr. A. Iliev for helpful discussions during this work.

REFERENCES

1. S. Machida, "Space-Charge Effects in Low-energy Proton Synchrotron," *Nuclear Instruments and Methods in Physics Research*, **A309** (1991).
2. I. Hofmann, "Simulation of Space Charge and Integer Resonance Crossing in Storage Rings and Its Application to Heavy Ion Fusion," *Particle Accelerator*, 1992, v. **39**, pp. 169-184.
3. A. Budzko and Yu. Senichev, "Study of Space Charge Effects Close by Half-integer Resonances," HEACC-92, Hamburg.