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# ENRICHMENT OF THE FINITE ELEMENT METHOD WITH REPRODUCING KERNEL PARTICLE METHOD

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## ABSTRACT

Based on the reproducing kernel particle method an enrichment procedure is introduced to enhance the effectiveness of the finite element method. The basic concepts for the reproducing kernel particle method are briefly reviewed. By adopting the well-known completeness requirements, a generalized form of the reproducing kernel particle method is developed. Through a combination of these two methods their unique advantages can be utilized. An alternative approach, the multiple field method is also introduced.

## INTRODUCTION

The finite element method has been the most widely used technique in the computational mechanics in the past two decades. However, recently the particle methods have been enjoying an increasing interest. Several different particle methods with unique advantages and disadvantages have been proposed, including smooth particle hydrodynamics (SPH) (Gingold and Monaghan, (1977), Lucy (1977), diffuse elements (Nayroles et al. (1992)), element free Galerkin (EFG) (Belytschko et al. (1994a,b,c)), particle in cell methods (PIC) (Sulsky et al. (1992)), reproducing kernel particle methods (Liu et al. (1993), Liu and Oberste-Brandenburg (1993), Liu et al. (1995a,b), Liu and Chen (1995)), and wavelet particle methods (WPM) (Liu and Oberste-Brandenburg (1993), Liu and Chen (1995)). Similar to other particle methods, the RKPM eliminates the need for a mesh, and additionally, is capable of treating the domain boundaries with a correction term. This paper is aimed at developing a procedure to generalize the RKPM. Through this procedure the traditional FEM can be treated as a special case of the RKPM establishing a natural way to blend the FEM and RKPM. The so-called p-enrichment or hp-enrichment in the FEM becomes easy to implement. Moreover, no compatibility problem along the element boundary and no restriction of choosing high order window function are required.

An alternative approach to enhancing the computational methods is the application of the concept of projection. Using multiple fields and their projections, the computational solution can be improved.

## PRELIMINARIES

In mathematical physics, the solution to a differential equation, or a set of differential equations can be expressed as

$$u^R(x)=\int_{-\infty}^{+\infty} u(y)\phi(x-y)dy \tag{1}$$

where $\phi$ is a kernel function, and acts like a projection operator, and $u^R$ is the "reproduced" solution of $u(x)$. This form is one of the fundamental developments in many interpolation methods, and will be referred to as the reproducing kernel methods. Widely used methods such as SPH, and wavelet methods also belong to this class of methods. One major drawback is the need for a special boundary treatment in finite domains. Through a proper construction of a boundary correction term, the artificial boundaries required by the SPH and wavelet methods can be eliminated, and the accuracy of the discrete solution is improved (Liu (1995)).

In a finite domain, Eq.(1) can be written as

$$u^R(x)=\int_{-\infty}^{+\infty} u(y)\overline{\phi}(x;x-y)dy \tag{2}$$

where

$$\overline{\phi}(x;x-y)=C(x;x-y)\phi(x-y) \tag{3}$$

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and $C(x;x-x_j)$ is the correction function.

In problems involving finite domains, the SPH solutions reveal amplitude and phase errors mainly due to the boundary effects. This phenomenon can be attributed to the failure of meeting of the completeness requirements. The correction function, $C(x;x-x_j)$, can be constructed in such a way to avoid the difficulties mentioned above. Since the integral defined in Eq. (2) is too complicated to be carried out analytically, it is generally discretized either by a grid (as in the Finite Element Method), or by particles (SPH methods). The class of methods involving mesh-free Lagrangian particles, and are concerned with the solution of Eq, (2) are referred to as reproducing kernel particle methods (RKPM). Before proceeding with the construction of a consistent correction function, a final word goes on the discrete convolution concept. By spatial/temporal discretization, high-frequency replicas (commonly referred to as aliasing) are introduced into the system response. In complex mechanical systems, these non-physical frequencies may interact with the physical frequencies. The elimination of aliasing is very important in representing the true frequency content of the system. A comprehensive discussion, and an error estimation procedure to separate physical and non-physical frequencies are given in Liu and Chen (1995).

## REPRODUCING CONDITIONS

In a Galerkin finite element approximation method, the admissible function space, $\mathcal{V}^h$, consists of all linear combinations of shape functions $N_A$, A=1,....,neq (neq is the number of equations, i.e. the total number of unknowns):

$$\mathcal{V}^h = \{N_1, \cdots, N_{neq}\} \tag{4}$$

In order to guarantee the convergence of the approximate Galerkin solution with successive mesh refinements, the shape functions have to be complete, i.e. are capable of spanning the linear field exactly (Hughes (1987)).

A set of arbitrary window functions generated from the translations of a single function can be defined as

$$\mathcal{V}^h = \left\{\phi_j(x) \middle| \phi_j(x) = \phi(x-x_j)\right\} \tag{5}$$

These window functions may not satisfy the completeness requirement. Thus, it is necessary to introduce the following "reproducing conditions" (Liu (1995)):

$$\bar{m}_0(x) = 1 \text{ and } \bar{m}_k(x) = 0 \quad \text{for } k > 1 \tag{6}$$

where

$$\bar{m}_k(x) = \sum_{j=1}^{np} (x-x_j)^k \phi(x-x_j)\Delta x_j \tag{7}$$

and np designates the number of particles.

If the selected window functions do not automatically satisfy the reproducing conditions, Eq. (6), then the correction function in Eq. (3) is constructed such that

$$\bar{\phi}(x;x-y) = C(x;x-y)\phi(x-y) = \sum_{k=0}^{n} \beta_k(x)(x-y)^k \phi(x-y) \tag{8}$$

The unknown functions $\beta's$ are determined by imposing the reproducing conditions, Eq. (6) which results in the following matrix equation

$$\begin{bmatrix} m_0(x) & m_1(x) & \cdots & m_n(x) \\ m_1(x) & m_2(x) & \cdots & m_{n+1}(x) \\ \vdots & & & \vdots \\ m_n(x) & m_{n+1}(x) & \cdots & m_{2n}(x) \end{bmatrix} \begin{bmatrix} \beta_0(x) \\ \beta_1(x) \\ \vdots \\ \beta_n(x) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{9}$$

## GENERALIZED REPRODUCING CONDITIONS

### Arbitrary Window Function

The reproducing conditions can be extended to a set of generalized linearly independent window functions including finite element shape functions as the window functions. As indicated earlier, this arbitrary set of window functions

$$\mathcal{V}^h = \{\phi_1, \phi_2, \cdots, \phi_{np}\} \tag{10}$$

may not satisfy the reproducing conditions, Eq. (6). Hence, the correction function $C(x;x-x_j)$ of Eq.(6) is constructed such that

$$\bar{\phi}_j(x) = \left[\sum_{k=0}^{n} \beta_k(x)(x-x_j)^k\right]\phi_j(x) = C(x;x-x_j)\phi_j(x) \tag{11}$$

With this modification of window functions, the generalized reproducing conditions become

$$\bar{m}_0(x) = \sum_{j=1}^{np} 1 \cdot \bar{\phi}_j(x)\Delta x_j = 1 \tag{12}$$

$$\bar{m}_k(x) = \sum_{j=1}^{np} (x-x_j)^k \bar{\phi}_j(x)\Delta x_j = 0 \tag{13}$$

and the $\beta's$ are to be determined by Eq. (9).

**FEM to RKPM**

The FEM shape functions introduced in Eq. (4) belong to a function space which satisfies the completeness requirement. Therefore, the reproducing conditions need not to be enforced in the FEM mesh. Since the RKPM is shown to be very effective in handling high gradients, large variations etc. (Liu and Oberste-Brandenburg (1993), Liu and Chen (1995), Liu et al. (1995b)), the partitioning of the domain into FEM and RKPM regions in these types of problems may be a good choice. These ideas are elaborated below.

**p-FE with RKPM Enrichment**

Considering these useful properties associated with an approach utilizing a window function, the function space $\mathcal{V}^h$ can be modified by replacing one of the shape functions with a higher order window function

$$\mathcal{V}^h = \{N_1, \cdots, N_{p-1}, \phi_p, N_{p+1}, \cdots, N_{neq}\}$$

This concept is described in Fig. 1(a). The completeness requirement is satisfied everywhere in the domain except under the support of $\phi_p$. This type of enrichment of the finite element method with the RKPM will referred to as p-FE method. This development can be extended by substituting several shape functions by window functions. This p-FE fails to satisfy the completeness requirement unless the functions in $\mathcal{V}^h$ are selected in a specific form. The completeness conditions have to be imposed on an arbitrary window function to construct the new functions. If we select $\mathcal{V}^h$ as a set of linear independent window functions as illustrated in Fig. 1a, then the consistency conditions are satisfied in the whole domain except under the support of $\phi_p(x)$, $\overline{\Omega}$. The p-FE with RKPM enrichment is achieved by enforcing the reproducing conditions only in region $\overline{\Omega}$.

**hp-FE with RKPM Enrichment**

A different enrichment can be achieved by inserting an additional node in the domain of interest and a higher order $\phi_{neq+1}(x)$ window function in $\mathcal{V}^h$ as shown in Fig. 1b. Similar to the p-FE with RKPM enrichment, reproducing conditions are required in $\overline{\Omega}$. However, unlike the traditional hp finite elements, the additional window $\phi_{neq+1}(x)$ can cover some of the finite element nodes and no special adjustment of the higher order window function is needed along the inter-element boundary.

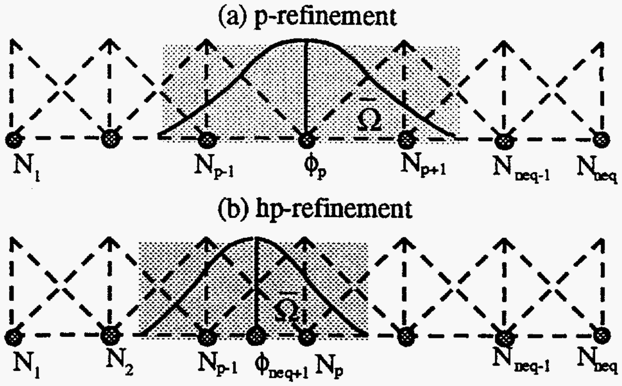


Figure 1 p-FE and hp-FE refinement with RKPM

For a better visualization, the use of the FEM shape functions, and the RKPM window functions is demonstrated in Figures 2. Figure 2a shows the shape and window functions in their original form. The reproducing conditions are applied to obtain the functions in Figure 2b.

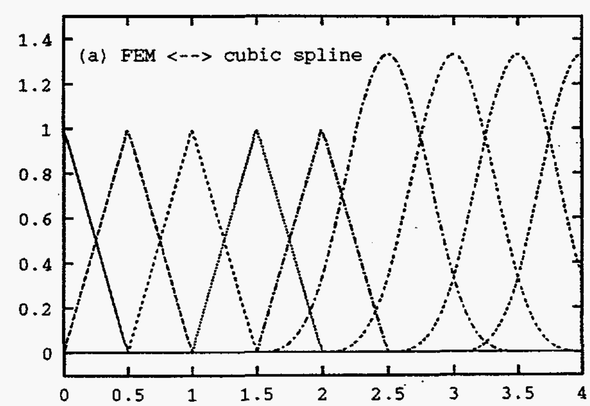


Figure 2a. The FEM shape functions, and the RKPM window functions

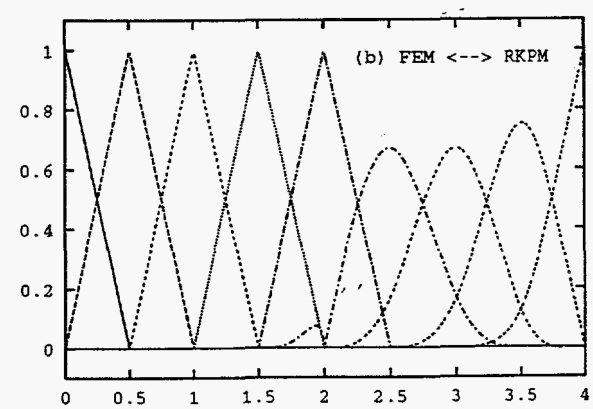


Figure 2b. The shape functions after the application of the reproducing conditions

## MULTIPLE FIELD RKPM

A multiple scale method based on WPM and RKPM was introduced in Liu and Oberste-Brandenburg (1993), and Liu and Chen (1995). Consider a two scale decomposition of $u(x)$:

$$u(x) = v(x) + w(x) \tag{14}$$

where $v(x)$ and $w(x)$ are the solutions to different scales or different fields. A projection operator for $v$ is defined such that $Pv = v$

$$u(x) = Pu(x) - Pw(x) + w(x) \tag{15}$$

where $Pu$ is the projected solution and $Pw$ is the interaction term, and Eq. (15) is the general expression for multiple scale analysis. In particular, if there is no overlapping of scales between $v(x)$ and $w(x)$, then by the property of the projection operator, the interaction term is zero. By repeated applications of Eq. (15), multiple scale methods can be developed within the framework of multilevel data structure.

Two sample cases are presented to clarify the concept of projection:

*Example 1: Enriched projection method*

If $Pu$ is the projection operator for the reproducing kernel domain, and $w(x)=cf(x)$, where c is a coefficient and $f(x)$ is a given analytical function. Equation (15) can be rewritten as

$$u(x) = \int_\Omega \bar{\phi}_a(x; x-y) u(y) dy + c\left[ f(x) - \int_\Omega \bar{\phi}_a(x; x-y) f(y) dy \right] \tag{16}$$

*Example 2: FEM enriched with RKM*

If we choose $u^{FEM}(x)=Pu(x)$ in $\Omega_{2a}$ (FEM domain) and $w(x)=Qw(x)$ is defined in $\Omega_a$ (RKPM domain), then the multiple scale solution becomes

$$u(x) = \sum_{j=1}^{NP} N_j(x) u(x_j) + \int_{\Omega_a} \left[ \bar{\phi}(x; x-y) - \sum_{j=1}^{NP} N_j(x) \bar{\phi}(x_j; x_j - y) \right] w(y) dy \tag{17}$$

where $N_j(x)$ is the FEM shape function and NP is the number of FEM nodes. To take full advantage of the adaptive hp-finite element method with wavelet enrichment, $w(x)$ should not intersect with any essential boundary conditions. A similar approach in enforcing the essential boundary conditions in meshless approximations is presented by Krongauz and Belytschko (1995).

## NUMERICAL EXAMPLES

### a. Coupling of the RKPM with the FEM

A simple one-dimensional case with the following equilibrium equation is considered to demonstrate the effect of coupling between the RKPM and the FEM:

$$u_{,xx}(x) + f(x) = 0$$

with boundary conditions

$$u(0) = 0$$
$$u_{,x}(1) = 0$$

and the forcing term

$$f(x) = 1$$

The exact solution is given as

$$u(x) = x - \frac{1}{2}x^2$$

The $L_2$-Norm results for FEM, RKPM and FEM (only on the essential boundary end) with RKPM are presented in Figure 3.

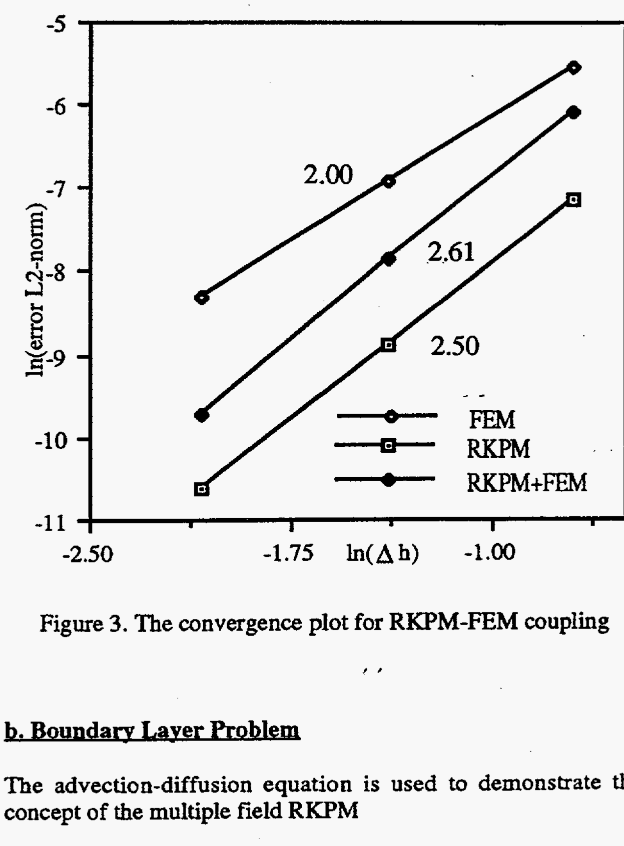


Figure 3. The convergence plot for RKPM-FEM coupling

### b. Boundary Layer Problem

The advection-diffusion equation is used to demonstrate the concept of the multiple field RKPM

$$u_{,xx}(x) - au_{,x}(x) = b(x)$$

with boundary conditions

$$u(0)=0$$
$$u(4)=1$$

and body force

$$b(x)=\left[\frac{1}{2}\delta(x)-\delta(x-2)\right]-a\left[\frac{1}{2}\langle x\rangle^0-\langle x-2\rangle^0\right]$$

where <•> is the Macaulay bracket and it is defined as

$$\langle x-y\rangle^n=\begin{cases}0, & x\le y\\ (x-y)^n, & x>y\end{cases}$$

and $\delta(x)$ is the Dirac delta function. The advection parameter, a, is chosen as 75 to ensure a boundary layer formation at the boundary.

The exact solution is given as

$$u(x)=\frac{1}{1-e^{4a}}\left(1-e^{ax}\right)+\frac{1}{2}\langle x\rangle^1-\langle x-2\rangle^1$$

The analytical form to be coupled with the RKPM is the boundary layer term:

$$w(x)=cf(x)=c\left(1-e^{ax}\right)$$

where $c$ is the extra unknown to be solved for.

As it can be seen in Figure 4 and Table 1, the absolute error in $L_2$-Norm for the multiple field RKPM is approximately half of that for the regular RKPM.

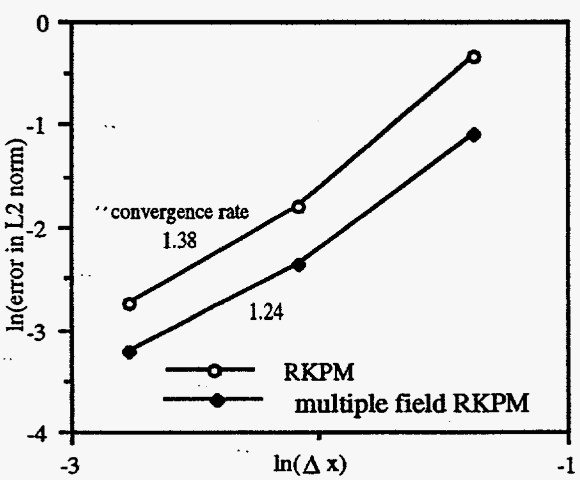


Figure 4. Error plot of the boundary layer problem

| | RKPM | multiple field RKPM |
|---|---|---|
| 17 nodes | 0.72062 | 0.33436 |
| 33 nodes | 0.16797 | 9.41136E-2 |
| 65 nodes | 6.45532E-2 | 3.97927E-2 |

Table 1. Errors of the boundary layer problem

## DISCUSSION AND CONCLUSIONS

In order to combine the merits of two computational methods, the finite element method and the reproducing kernel particle method, enrichment procedures are introduced. One approach is based on constructing the appropriate boundary correction function to satisfy the boundary restraints, and the completeness requirements. A general set of window functions can be converted into a set of approximation functions by imposing the reproducing conditions. A p-enrichment in FEM is achieved by replacing some of the standard FEM shape functions by window functions, and applying the reproducing conditions. In addition to the FEM shape functions additional window functions can be introduced, a so-called, hp-enrichment is obtained. A one-dimensional example with two finite elements to represent the essential boundary is considered. The rest of the domain is handled by the RKPM. The convergence characteristics are studied.

The second approach considers the enhancement of the FEM or a given field with RKPM through projection operators. The one-dimensional advection-diffusion equation is studied. An analytical boundary layer field is added on the RKPM solutions. The convergence characteristics are analyzed.

The enrichment provides stable and accurate solutions in coarse meshes. The absolute error in the $L_2$-Norm is smaller in the multiple field RKPM.

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