A GENERALIZATION OF

NEWTON'S METHOD

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CHAPTER I

INTRODUCTION

It is our purpose here to investigate the method of solving equations for real roots by Newton's Method and to indicate a generalization arising from this method. We will confine our discussion here to single-value real functions of one real variable.

We will denote by $[a,b]$ the points x such that $a \leq x \leq b$; by (a,b) the points x such that $a < x < b$; by $[a,b)$ the points x such that $a \leq x < b$; by $(a,b]$ the points x such that $a < x \leq b$. $[a,b]$ will be called a closed interval (a,b) will be called an open interval, and $(a,b]$ and $[a,b)$ will be called half-open intervals.

A Dedekind Cut, written (A,B), in the domain of real numbers, is a separation of all real numbers into two classes A and B such that (1) neither set is empty, (2) all real numbers are contained in A or B, (3) every number in A is less than every number in B. To say y is a function of x, written $y = f(x)$, means that for any value of x, a unique value of y can be determined. A sequence, written $\{a_n\}$, is the association of all the integers 1, 2, 3, . . . with definite numbers a_1, a_2, a_3, . . . A monotone increasing sequence is such that $a_n \leq a_{n+1}$ for all values of n. A monotone decreasing sequence is such that $a_n \geq a_{n+1}$ for all values of n. A bounded sequence is such that there is a positive number K such that, for all a_n's of the sequence, $|a_n| \leq K$. Limit of a sequence: $\lim_{n\to\infty} a_n = a$ or $a_n \to a$ will mean that, for any positive number e, there exists another positive number N, such that when $n > N$ then $|a - a_n| < e$. We will say that f(x) is continuous at $x = a$ if for any positive number e, there exists a positive number δ, such that for any point a of an interval $|x - a| < \delta$, then $|f(x) - f(a)| < e$. f(x) continuous on an interval I means that f(x) is continuous at every point of I. Derivative of a function f(x) at $x = x_0$ means $\lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0}$ if this limit exists. The derivative of f'(x) is the second derivative of f(x), written f''(x). This would make it necessary for f'(x) to be continuous.

The following theorems will be stated without proof.[1]

Theorem 1.1 If f(x) is continuous on [a,b], then for any positive number e, there exists another positive number δ, depending on e, such that for x_1 and x_2 in [a,b] such that $|x_1 - x_2| < \delta$, $|f(x_1) - f(x_2)| < e$.

Theorem 1.2 A bounded monotone increasing (or decreasing) sequence is convergent.

Theorem 1.3 Taylor's Theorem. If f(x) and its first n derivatives are single valued and continuous in an

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[1] C. A. Stewart, Advanced Calculus.

interval $[a, a+h]$, then

$$f(a+h) = f(a) + h\, f'(a) + \frac{h^2}{2!} f''(a) + \cdots + \frac{h^n}{n!} f^{n}(a+\theta h)$$

where $0 < \theta < 1$.

Theorem 1.4 Fundamental Theorem of the Dedekind Theory. If (A,B) is any Dedekind Cut in the domain of real numbers, then there exists a real number ξ such that every real number greater than ξ is in B and every real number less than ξ is in A.

Theorem 1.5 If $f(x)$ is differentiable at $x = a$, then $f(x)$ is continuous at $x = a$.

Theorem 1.6 Theorem of the Mean. If $f(x)$ is differentiable on (a,b) and continuous at both a and b, then there exists a point c in (a,b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

CHAPTER II

NEWTON'S METHOD

One method of approximating a root of an algebraic or transcendental equation is Newton's Method. Suppose that from tables, graphing, substitution, or otherwise the equation $y = f(x)$, has been found to have a root in the interval (a,b). Then, according to Newton, by choosing a point a_1 in the interval $[a, a+h)$, where $a+h < b$ is the root of $y = f(x)$, a better approximation to the root is given by

$$a_2 = a_1 - \frac{f(a_1)}{f'(a_1)}.$$

Then, by repeating the above, a third approximation to the root is given by

$$a_3 = a_2 - \frac{f(a_2)}{f'(a_2)}.$$

After repeating this process sufficiently often, any degree of accuracy that is required may be obtained.

The method, as explained in most elementary books, either neglects to state that this "works" only when considering certain functions or dismisses the situation with a statement such as -- "curves for which Newton's Method fails will not be discussed here." But when is Newton's Method successful, that is, under what conditions will each new approximation be closer to the root?

<u>Theorem 2.1</u> If $f(x)$ is continuous on $[a,b]$ and if $f(a) > 0$ and $f(b) \leq 0$, then there exists a point ξ in $(a,b]$ such that (1) $f(\xi) = 0$ and (2) $f(x) > 0$, for all x such that $a < x < \xi$.

Let x belong to A if (1) $x \leq a$ or (2) $x > a$ and $f(x_o) > 0$ for all x_o in $[a,x]$: otherwise let x belong to B. Then a is contained in A and b is contained in B. Let x_1 be a real number in A and suppose $x_2 < x_1$. If $x_2 \leq a$, x_2 is in A. If $a < x_2 < x_1$, then $f(x_o) > 0$ for all x_o in $[a,x_2]$ since $f(x_o) > 0$ for all x_o in $[a,x_1]$. Hence x_2 is contained in A. Therefore, (A,B) is a Dedekind Cut in the real numbers and there exists a real number ξ such that every number less than ξ is in A while every number greater than ξ is in B.

Obviously $a \leq \xi \leq b$. Suppose first that $f(\xi) > 0$ and choose $0 < e < f(\xi)$. Since $f(x)$ is continuous at $x = \xi$, there exists a $\delta > 0$ such that, for x chosen so that $|x - \xi| < \delta$, $|f(x) - f(\xi)| < e$. Consider the point $x = \xi + \frac{\delta}{2}$. Then $f(x) > f(\xi) - e > 0$. Thus $\xi + \frac{\delta}{2}$ is a number in A which is greater than ξ. This is a contradiction so that $f(\xi) \not> 0$. Hence $a < \xi \leq b$.

Now suppose $f(\xi) < 0$ and choose $0 < e < |f(\xi)|$. Then, because of continuity, there exists a $\delta > 0$ such that, for all $|x - \xi| < \delta$, $|f(x) - f(\xi)| < e$. If $x = \xi - \frac{\delta}{2}$, $f(x) < f(\xi) + e < 0$. Thus $\xi - \frac{\delta}{2}$ is a number in B which is less than ξ. This is a contradiction so that $f(\xi) \not< 0$. Therefore $f(\xi) = 0$. If x be chosen such that $a < x < \xi$, x is in A and $f(x) > 0$.

<u>Corollary 2.11</u> If $f(x)$ is continuous on $[a,b]$ and if $f(a) < 0$ and $f(b) \geq 0$, then there exists a point ξ in $(a,b]$ such that (1) $f(\xi) = 0$ and (2) $f(x) < 0$, for all x such that $a < x < \xi$.

Let $F(x) = -f(x)$, then $F(a) > 0$, $F(b) \leq 0$. From the above theorem, there exists a point ξ in $(a,b]$ such that (1) $F(\xi) = 0$ and (2) $F(x) > 0$ for all x such that $a < x < \xi$. Then $f(\xi) = -F(\xi) = 0$ and, if $a < x < \xi$, $f(x) = -F(x) < 0$.

<u>Corollary 2.12</u> If $f(x)$ is continuous on the interval $[a,b]$ and if $f(a) \leq 0$ and $f(b) > 0$, then there exists a point ξ in $[a,b)$ such that (1) $f(\xi) = 0$ and (2) $f(x) > 0$, for x such that $\xi < x < b$.

Let $F(x) = f(-x)$, then $F(-a) \geq 0$, $F(-b) < 0$. From the above theorem, there exists a point ξ in $(-b,-a]$ such that $F(\eta) = 0$ and $F(x) > 0$, for $-b < -x < \eta$. Let $\xi = -\eta$ then $f(\xi) = f(-\eta) = F(\eta) = 0$. Choose x so that $\xi < x < b$. We have $-b < -x < \eta$ and consequently $f(x) = F(-x) > 0$.

<u>Corollary 2.13</u> If $f(x)$ is continuous on the interval $[a,b]$ and if $f(a) \geq 0$ and $f(b) < 0$, then there exists a point ξ in $[a,b)$ such that (1) $f(\xi) = 0$ and (2) $f(x) < 0$ for x such that $\xi < x < b$.

Let $F(x) = -f(-x)$, then $F(-b) > 0$ and $F(-a) \leq 0$. From the theorem, there exists a point η in $(-b,-a]$ such that $F(\eta) = 0$ and $F(x) > 0$ for all x such that $-b < -x < \eta$. Let $\xi = -\eta$, then $f(\xi) = f(-\eta) = -F(\eta) = 0$ and if $-b < -x < \eta$, $f(x) = -F(-x) < 0$.

<u>Theorem 2.2</u> In Theorem 2.1 let $f''(x)$ be assumed to exist and be continuous on an interval containing $[a,b]$ Then, if $f''(x) > 0$ on (a,ξ), we have also $f'(x) \neq 0$ for $a < x < \xi$.

Let $h = \xi - a$. Then $h > 0$. Let $\alpha = a + k$, $0 < k < h$, and suppose that $f'(\alpha) = 0$. By Taylor's Theorem, there exists a real number β in $(\alpha, a+h)$ such that

$$f[\alpha+(h-k)] = f(\alpha) + (h-k)f'(\alpha) + \tfrac{1}{2}(h-k)^2 f''(\beta).$$

By assumption $f[\alpha+(h-k)] = 0$ so that

$$0 = f(\alpha) + \tfrac{1}{2}(h-k)^2 f''(\beta).$$

Also $f(\alpha) > 0$. Thus $f''(\beta) < 0$ contrary to assumption. Therefore $f'(\alpha) \neq 0$.

<u>Corollary 2.21</u> In Corollary 2.11 let $f''(x)$ be assumed to exist and be continuous on an interval containing $[a,b]$. Then if $f''(x) < 0$ on (a, ξ), we have $f'(x) \neq 0$ for $a < x < \xi$.

Let $F(x) = -f(x)$. Then $F(a) > 0$ and $F''(x) > 0$ on (a, ξ), thus by Theorem 2.2 we have $F'(x) \neq 0$ for $a < x < \xi$. But $-F(x) = f(x)$ such that $-F'(x) = f'(x) \neq 0$.

<u>Corollary 2.22</u> In Corollary 2.12 let $f''(x)$ be assumed to exist and be continuous on an interval containing $[a,b]$. Then if $f''(x) > 0$ on (ξ, b), we have $f'(x) \neq 0$ for $\xi < x < b$.

Let $F(x) = f(-x)$. Then $F(-a) \leq 0$, $F(-b) > 0$ and $F'(-x) \neq 0$ for $-b < -x < \eta$, by Theorem 2.2. Then, if $\xi = -\eta$, $f'(x) = F'(-x) \neq 0$ for $\xi < x < b$.

<u>Corollary 2.23</u> In Corollary 2.13 let $f''(x)$ be assumed to exist and be continuous on an interval containing $[a,b]$. Then if $f''(x) < 0$ on (ξ, b) we have $f'(x) \neq 0$ for $\xi < x < b$.

Let $F(x) = -f(-x)$. Then $F(-b) > 0$, $F(-a) \leq 0$, and from the theorem $F'(x) \neq 0$ for $-b < -x < \eta$. If $\xi = -\eta$, $-F'(x) \neq 0$ for $\xi < x < b$ and thus $f'(x) \neq 0$ for $\xi < x < b$.

<u>Theorem 2.3</u> If $f(x)$ satisfies the hypotheses of Theorems 2.1 and 2.2, then the sequence $\{a_n\}$, where $a_1 = a$ and

$$a_{n+1} = a_n - \frac{f(a_n)}{f'(a_n)},$$

converges to ξ.

Again let $\xi = a + h$. From

$$f(a+h) = f(a) + h\, f'(a) + \tfrac{1}{2} h^2 f''(a + \theta h),$$

and $f(a+h) = 0$,

$$\frac{-hf'(a)}{f(a)} = \frac{\frac{1}{2} h^2 f''(a+\theta h)}{f(a)} + 1.$$

Since $h > 0$ and $\dfrac{f''(a+\theta h)}{f(a)} > 0$,

$$(-h)\,\frac{f'(a)}{f(a)} > 1.$$

Thus $f'(a) \neq 0$ and $\dfrac{f(a)}{f'(a)} < 0$, so that

$$-h < \frac{f(a)}{f'(a)} \quad .$$

Dividing by -1 we have

$$h > -\frac{f(a)}{f'(a)} > 0.$$

Then by adding a to both sides of the above inequality

$$a + h > a - \frac{f(a)}{f'(a)} > a.$$

Hence $a_2 = a - \dfrac{f(a)}{f'(a)}$ is in the interval $(a, a+h)$.

All the above arguments will again apply to the interval $(a_n, a+h)$ and will lead, by way of repetition, to

$$a_{n+1} = a_n - \frac{f(a_n)}{f'(a_n)} ,$$

with $a + h > a_{n+1} > a$, which defines a monotone increasing sequence in $[a, a+h)$.

The sequence $\{a_n\}$ is bounded and therefore convergent. Suppose $\{a_n\}$ converges to γ. Then since $f''(x)$ exists and

$f'(x)$ does not vanish in $(a, a+h)$, $\frac{f(x)}{f'(x)}$ is differentiable and continuous in $(a, a+h)$. Then, by passing to the limit,

$$\gamma = \gamma - \frac{f(\gamma)}{f'(\gamma)} .$$

Thus $\frac{f(\gamma)}{f'(\gamma)} = 0$, but $f'(\gamma) \neq 0$ so that $f(\gamma) = 0$. But according to assumption $a+h$ is the only zero of $f(x)$ in $[a, a+h]$. Hence $\gamma = a+h$. Thus $\{a_n\}$ converges to $\xi = a+h$.[2]

Corollaries that will correspond to those of Theorems 2.1 and 2.2 may be added to the above theorem but they result in the following general statement. Let $f(x)$ be a function with a continuous second derivative on an interval containing $[a,b]$ and let there be a point ξ contained in $[a,b]$ such that $f(\xi) = 0$. If $[a,b]$ has a subinterval $[a, \xi)$ or $(\xi, b]$ at every point of which $f(x) \cdot f''(x) > 0$, then the sequence $\{a_n\}$, where a_1 is a point of the subinterval and

$$a_{n+1} = a_n - \frac{f(a_n)}{f'(a_n)} ,$$

converges to ξ.

Thus, if the above conditions are satisfied by a given function, then Newton's Method is applicable.

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[2] Theodore Chaundy, The Differential Calculus, p. 190.

Suppose now we have a function such that the second derivative $f''(x)$ is of the opposite sign of $f(x)$ at both a and b. Certain conditions sufficient for the applicability of Newton's Method are discussed in the following theorem.

Lemma: Let $f(x)$ be a function having a continuous second derivative $f''(x)$ on an interval containing $[a,b]$ and suppose there exists a point ξ in $[a,b]$ such that $f(\xi) = 0$. If $[a,b]$ has a subinterval $[a, \xi)$ or $(\xi, b]$ at every point of which $f(x) \cdot f''(x) < 0$, then

$$f(x)\, f'(\xi) \cdot (x - \xi) > 0$$

for every x in the subinterval.

Let $f(x)$ be a function satisfying the above hypotheses, and let x_0 be any point of the subinterval mentioned. Then there exists a point η between x_0 and ξ such that

$$f(x_0) = f(\xi) + f'(\xi) \cdot (x_0 - \xi) + f''(\eta) \cdot \frac{(x_0 - \xi)^2}{2}.$$

We know that $f(\xi) = 0$. Futhermore, $f(x) \neq 0$ at every point of the subinterval so that $f(x)$ has the same sign at all point of the subinterval. Thus $f(\eta) \cdot f''(\eta) < 0$ implies $f(x_0) \cdot f''(\eta) < 0$. We have

$$[f(x_0)]^2 = f(x_0) f'(\xi) \cdot (x_0 - \xi) + f(x_0)\, f''(\eta) \frac{(x_0 - \xi)^2}{2}$$

and it follows that

$$f(x_0) f'(\xi) \cdot (x_0 - \xi) > 0$$

This concludes the proof of the lemma.

Theorem 2.4 If there exists a positive number δ such that $f(x)$ satisfies the hypotheses of the lemma using the interval $[\xi-\delta, \xi+\delta]$ and if $f''(\xi) \neq 0$, then the sequence $\{a_n\}$ converges to ξ, provided a_1 is any point in $[\xi-\delta, \xi+\delta]$ but not in the subinterval mentioned and $a_{n+1} = a_n - \dfrac{f(a_n)}{f'(a_n)}$.

For simplicity, suppose $[\xi-\delta, \xi)$ to be the subinterval in which $f(x)\cdot f''(x) < 0$ and suppose $f(\xi-\delta) > 0$. Then, for every x in $[\xi-\delta, \xi)$,

$$f(x)\cdot f'(\xi)\cdot(x-\xi) > 0.$$

It follows that $f'(\xi) < 0$. Hence there exists a positive number δ_1 such that $f(x) < 0$ for $\xi < x < \xi+\delta_1$. If $f''(\xi) < 0$, we know from the continuity of $f''(x)$ that there exists a positive number δ_2 such that, for $|x-\xi| < \delta_2$, $f''(x) < 0$. Choosing $\Delta = \min.[\delta_1, \delta_2]$, we obtain $f(x)\cdot f''(x) > 0$ for $\xi < x < \xi+\delta$. But $f''(\xi) > 0$ is impossible because of the continuity of $f''(x)$. Therefore $\{a_n\}$ converges to ξ provided $\xi < a_1 < \xi+\Delta$ and $a_{n+1} = a_n - \dfrac{f(a_n)}{f'(a_n)}$. The other possible cases are similar.

If every neighborhood of ξ contains at least one point x different from ξ such that $f(x)\cdot f''(x) = 0$, we would have an infinite sequence converging to ξ at each point of which $f(x) = 0$ or an infinite sequence converging to ξ at each point of which $f''(x) = 0$ (or both). Intuitively this indicates infinite oscillation and we should not expect Newton's Method to work. Ruling out this case, then, we demand existence of a positive number δ such that $f(x)\cdot f''(x) \neq 0$

for all x in $[\xi-\delta, \xi+\delta]$ save $x = \xi$. We shall conclude our discussion with the proof of the next theorem in which, it should be noted, the hypotheses imply $f''(\xi) = 0$.

<u>Theorem 2.5</u> Let $f(x)$ be a function having a continuous monctone second derivative $f''(x)$ on $[\xi-\delta, \xi+\delta]$, where is a positive number and $f(\xi) = 0$. If $f(x)\cdot f''(x) < 0$ and $|f(x)|\cdot|f''(x)| < [f'(x)]^2$ on $[\xi-\delta, \xi)$ and $(\xi, \xi+\delta]$, then a in $[\xi-\delta, \xi)$ and

$$a_{n+1} = a_n - \frac{f(a_n)}{f'(a_n)}$$

defines a sequence $\{a_n\}$ converging to ξ.

Let $f(x)$ be a function as above and suppose $f(x) > 0$ at $\xi-\delta$. Now choose $\xi-\delta \le a_1 < \xi$. We have $f'(x) \neq 0$ on $[a, \xi)$. From the Theorem of the Mean there exists a point x_1 between a_1 and ξ such that $f'(x_1) = \dfrac{-f(a_1)}{\xi - a_1} < 0$. Suppose there exists a point x_2 in $[a_1, \xi)$ such that $f'(x_2) > 0$. Since $f'(x)$ is differentiable, it is continuous and there would exist a point x_3 between x_1 and x_2 such that $f'(x_3) = 0$, contrary to fact. It follows that $f'(x) < 0$ for every x in $[a, \xi)$. Then

$$0 = f(\xi) = f(a_1) + f'(a_1)\cdot(\xi - a_1) + \frac{f''(\eta)}{2!}\cdot(\xi - a_1)^2$$

where $a_1 < \eta < \xi$ and the tangent to the curve at a_1 is given by

$$y = f(a_1) + f'(a_1)\cdot(x - a_1).$$

Then the height of the tangent at $x = \xi$ is $-\dfrac{f''(\eta)}{2!}\cdot(\xi - a_1)^2$.

Since this is a positive number, $a_2 > \xi$. Suppose

$$-f''(\eta)\ \frac{(\xi - a\ )^2}{2\,!} \geq 1/2\, f(a_1).$$

Then

$$(\xi - a_2)^2 \cdot |f''(\eta)| \geq f(a_1)$$

and

$$|f(a_1)| \cdot |f''(\eta)| \geq \left[\frac{f(a_1)}{(\xi - a_1)}\right]^2.$$

By the lemma, $f'(\xi) < 0$ and $f(x)$ must change sign at $x = \xi$. Thus $f''(x)$ must do likewise. It follows that $f''(x)$ is monotone increasing and $|f''(a_1)| > |f''(\eta)|$ and, substituting the larger value $(a_2 - a_1)$ for $(\xi - a_1)$, we have

$$|f(a_1)| \cdot |f''(a_1)| > \left[\frac{f(a_1)}{(a_2 - a_1)}\right]^2 = [f'(a_1)]^2$$

contrary to assumption. Therefore

$$\frac{-f''(\eta)}{2\,!} \cdot (\xi - a\ )^2 < 1/2\, f(a_1)$$

and hence, from similar triangles,

$$a_2 - \xi < 1/2\,(a_2 - a_1).$$

Therefore

$$a_2 - \xi < \xi - a_1.$$

A similar argument may now be used with a_2 as the chosen point and we thus obtain a_3 such that $\xi - a_3 < a_2 - \xi$. Therefore, we have

$$(1)\quad a_{2n+1} = a_{2n} - \frac{f(a_{2n})}{f'(a_{2n})}$$

which defines a monotone increasing sequence $\{a_{2n+1}\}$ in and

$$(2) \quad a_{2n} = a_{2n-1} - \frac{f(a_{2n-1})}{f'(a_{2n-1})}$$

which defines a monotone decreasing sequence $\{a_{2n}\}$ in $(\xi, \xi+\delta]$. Since the above sequence $\{a_{2n}\}$ is monotone decreasing and bounded and the sequence $\{a_{2n+1}\}$ is monotone increasing and bounded, both are convergent. Suppose $\{a_{2n+1}\}$ converges to γ and $\{a_{2n}\}$ converges to ρ. Now since $f''(x)$ and $f'(x)$ do not vanish in (a_1, ξ), $\frac{f(x)}{f'(x)}$ is differentiable and continuous in (a_1, ξ). Then, by passing the limits

$$\gamma = \gamma - \frac{f(\gamma)}{f'(\gamma)}.$$

Thus $\frac{f(\gamma)}{f'(\gamma)} = 0$ in $[a, \xi)$ but $f'(\gamma) \neq 0$, therefore $f(\gamma) = 0$. But by assumption $x = \xi$ is the only point of $[a_1, \xi)$ where $f(x) = 0$. Hence $\gamma = \xi$ and $\{a_{2n+1}\}$ converges to ξ. By the same argument $\{a_{2n}\}$ converges to $\rho = \xi$. Thus the sequence $\{a_n\}$ converges to ξ.

Here again corollaries may be written to cover all possible cases but to be more general we make the following statement. <u>Let</u> $f(x)$ <u>be a function having a continuous monotone second derivative on an interval</u> $[\xi-\delta, \xi+\delta]$, where δ <u>is a positive number and</u> $f(\xi) = 0$. <u>Then if</u> $f(x)\cdot f''(x) < 0$ <u>and</u> $|f(x)|\cdot|f''(x)| < [f'(x)]^2$ <u>in</u> $[\xi-\delta, \xi)$ <u>and</u> $(\xi, \xi+\delta]$, <u>the sequence</u> $\{a_n\}$, <u>where</u> a_1 <u>is in the interval</u> $[\xi-\delta, \xi)$ <u>or</u>

$(\xi; \xi+\delta]$ <u>and</u>

$$a_{n+1} = a_n - \frac{f(a_n)}{f'(a_n)},$$

<u>converges to</u> .

Thus we have a second condition for which Newton's Method is applicable.

CHAPTER III

A GENERALIZATION OF NEWTON'S METHOD

J. S. Frame[3] and H. S. Wall[4] have used the approximation formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n) + \frac{f''(x_n) \cdot f(x_n)}{2f'(x_n)}}$$

rather than the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

of Newton. The new formula uses the second derivative of the function and the second approximation thus obtained often is sufficiently accurate so that repeated application is eliminated. Both, J. S. Frame and H. S. Wall have given examples, such as the following, for which convergence to the required zero of $f(x)$ is more rapid than by Newton's Method.

<u>Example 1</u>. To compute $\sqrt{2}$, starting with $x_0 = 1$.

Newton's	Generalization
$x_1 = 1.50000$	$x_1 = 1.4$
$x_2 = 1.41667$	$x_2 = 1.414213$
$x_3 = 1.41422$	
$x_4 = 1.414213$	

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[3] J. S. Frame, "A Variation of Newton's Method," <u>The American Mathematical Monthly</u>, LI (January, 1944), 36.

[4] H. S. Wall, "A Modification of Newton's Method," <u>ibid</u>., LV (February, 1948), 90.

The value of x_2 found by the generalization formula is found correct to five decimal places, but the value of x_2 found by Newton's Method is correct to only one decimal place.

Example 2. To evaluate sin 50° by solving $8x^3 - 6x + 1 = 0$ of which it is a root. Since sin 50° is between 1/2 and 1, take $x_1 = 3/4$.

Newton's	Generalization
$x_1 = .75$	$x_1 = .75$
$x_2 = .766667$	$x_2 = .766026$
$x_3 = .766050$	

The value of x_2 found by Newton's Method is correct to two decimal places, while the value of x_2 found by the generalization formula is correct to four decimal places.

The two writers mentioned above have not given sufficient conditions under which their method is applicable. In fact, not even a concise and rigorous development of Newton's Method has been located in the mathematical literature by this writer.

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