THE DYADIC OPERATOR APPROACH TO A STUDY IN COMICS,

WITH SOME EXTENSIONS TO HIGHER DIMENSIONS

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THE DYADIC OPERATOR APPROACH TO A STUDY IN CONICS,
WITH SOME EXTENSIONS TO HIGHER DIMENSIONS

THESIS

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THE DYADIC OPERATOR APPROACH TO A STUDY IN CONICS,
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1. Introduction

The discovery of a new truth in the older fields of mathematics is a rare event. Here an investigator may hope at best to secure greater elegance in method or notation, or to extend known results by some process of generalization.

It is our purpose, as the title indicates, to make a study of conic sections in the spirit of the above remark, using the symbolism developed by Gibbs.¹ Many of the facts we present were known to the Greek mathematicians, Menaechmus (c. 350 B. C.), Euclid (c. 300 B. C.), and Apollonius (c. 225 B. C.), and were developed further by the algebraic methods of Descartes (c. 1637 A. D.).² In fact our method of attack is suggested by Gibbs.³ We have found very meager use of it, however, in the literature available to us.

2. Assumed Relationships

In this section we state the preliminary assumptions, definitions,

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¹Josiah Willard Gibbs (1839–1903) developed a vector algebra based on the fundamental ideas of Grassmann and Hamilton. The treatment of linear vector functions, and the associated subject of dyadics constitute the most original part of his contribution to vector analysis; see A. P. Wills, Vector and Tensor Analysis, p. xxvi, xxvii.


³J. W. Gibbs and E. B. Wilson, Vector Analysis, p. 272.
symbols, and theorems that we shall use later in the development of our topic.

We assume familiarity with the following: line-vectors, hereafter called vectors: \( \mathbf{a}, \mathbf{b} \) etc.; their magnitude: \( \|a\|, \|b\| \) etc., respectively; their algebraic sum: \( \mathbf{a} + \mathbf{b} - \mathbf{c} \); the scalar, or direct, product of two vectors: \( \mathbf{a} \cdot \mathbf{b} \); non-coplanar vectors: \( \mathbf{a}, \mathbf{b}, \mathbf{c} \); the mutually perpendicular unit vectors: \( \mathbf{i}, \mathbf{j}, \mathbf{k} \); reciprocal vectors: \( \mathbf{a}, \mathbf{a}' \); vectors \( \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 \) called the reciprocal base system to non-coplanar base system \( \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 \); a vector function; a vector equation; the position vector \( \mathbf{r} \); the derivative of a variable vector \( \mathbf{r} \); and the summation notation: \( a_i \mathbf{r}_i \) for \( \sum a_i \mathbf{r}_i \).

We shall let D stand for definition, T for theorem and P for an assumed property. For example, T a.b. will stand for the bth theorem of the aeth section. T a.b.c. will be the cth corollary of T a.b. Also, we shall say \( \mathbf{r} \) when speaking of the point designated by the terminus of a position vector \( \mathbf{r} \).

D 2.1. The symbol \( \mathbf{a}_i b_i, i = 1 \) to \( n \), denoted by a Greek letter, say \( \phi \), is called a dyadic; if \( n = 1 \), a dyad. The vectors composing \( \phi \) shall be constant vectors.

D 2.2. The first vectors in each \( \mathbf{a}, \mathbf{b} \); are called the antecedents, the second, the consequents, of \( \phi \).

D 2.3. \( b, \mathbf{a} \), denoted by \( \phi_c \), is called the conjugate of the \( \phi \) of D 2.1.

D 2.4. \( \mathbf{r} \cdot \phi \) and \( \mathbf{r} \cdot \phi \) are defined to be \( \mathbf{a}_i b_i \cdot \mathbf{r} \) and \( \mathbf{r} \cdot \mathbf{a}, \mathbf{b} \).
respectively. In the first, \( \phi \) is called a prefactor, in the second, a postfactor.\(^4\)

D 2.5. \( \hat{\lambda}(a_i b_i) \), \( \hat{\lambda} \) a scalar, is defined to be equal
\[
i_i (m_i a_i)(n_i b_i) \quad i \text{ where the product } l_x m_x n_x a_x \cdot \hat{\lambda} \text{ for every } i.
\]

P 2.1. The combination of vectors in a dyad is distributive;
i.e.,
\[
(a + b)c = ac + bc,
\]
and
\[
a(b + c) = ab + ac.
\]
Likewise
\[
\hat{\lambda}, a b + \hat{\lambda}_2 a b = (\hat{\lambda}_1 + \hat{\lambda}_2) a b.
\]

These conform to the laws of ordinary algebra, except that the order of the factors must be maintained.

P 2.2. The direct product of dyadics obeys the distributive law;
i.e.,
\[
\phi \cdot (\psi + \psi') = \phi \cdot \psi + \phi \cdot \psi',
\]
\[
(\phi' + \phi) \cdot \psi = \phi' \cdot \psi + \phi \cdot \psi.
\]

P 2.3. These laws hold for the direct product of dyadics and vectors when, and only when, the vectors occur at either or both ends of the list of factors of the product.

T 2.1. Any dyadic may be reduced to the sum of three dyads of which either the antecedents or the consequents, but not both, may be arbitrarily chosen non-coplanar vectors.\(^5\)

D 2.5. If \( \phi \) can be reduced to the sum of three, two, or one dyad, it is said to be complete, planar, or linear, respectively. If

---

\(^4\)Incidentally, \( \phi \cdot r \) or \( r \cdot \phi \), where \( r \) is a variable vector, is a linear vector function of \( r \); thus the operator \( \phi \cdot \) operating on \( r \) accomplishes an affine transformation of space. We define \( \phi \cdot \psi, \phi = a_i b_i, \psi = c_j d_j \) to be \( a_i (b_i \cdot c_j) d_j = \Omega \).

\(^5\)Gibbs and Wilson, op. cit., p. 271.
in the second case, the plane of the antecedents and the plane of the
consequents coincide, \( \varphi \) is said to be uniplanar. If, in the third
case, the antecedent and the consequent are collinear, \( \varphi \) is said
to be unilinear.

The following are theorems on the operator \( \varphi \). We use note-
tions \( s = \varphi \cdot r \) and \( t = r \cdot \varphi \).

T 2.2. If \( \varphi \) is complete, \( s \) and \( t \) may be made to take on
any desired values by suitable choice of \( r \).\(^6\)

T 2.3. If \( \varphi \) is planar, the vector \( s \) may take on any value
in the plane of the antecedents, and \( t \) any value in the plane of the
consequents of \( \varphi \), but no values out of those planes, by suitable
choices of \( r \).\(^7\)

T 2.4. If \( \varphi \) is linear, the vector \( s \) may take on any value
collinear with the antecedents of \( \varphi \), and \( t \) any value collinear
with the consequents of \( \varphi \), but no other values.\(^8\)

T 2.5. It is always possible to reduce a complete dyadic to the
sum of three terms of which the antecedents among themselves and the
consequents among themselves are mutually perpendicular, written as
\[ \varphi = a i' i + b j' j + c k' k \]. This is called the normal form of \( \varphi \).\(^9\)

For uniplanar dyadics, the normal form of \( \varphi \) is \( a i' i + b j' j \).

D 2.7. If \( \varphi \cdot r = \varphi \cdot r \), for every \( r \), \( \varphi = \varphi \).

D 2.7.1. If \( r \cdot \varphi = r \) and \( \varphi \cdot r = r \), \( \varphi \) shall be called an
idemfactor, \( I \), which is unique.

\(^6\)Ibid., p. 285. \(^7\)Ibid., p. 285. \(^8\)Ibid., p. 286.

\(^9\)Ibid., p. 305.
D 2.8. When  \( \phi = a_{11} i i + a_{12} i j + a_{13} i k + \\
a_{21} j i + a_{22} j j + a_{23} j k + \\
a_{31} k i + a_{32} k j + a_{33} k k, \)
\( \phi \)

is said to be in monion form.

D 2.8.1. The idenfactor in monion form is
\( I = i i + j j + k k. \)

D 2.9. If \( \phi = d \bar{d}, \) and \( \psi = \bar{d}^{-1} \bar{a}^{-1}, \) then \( \psi = \phi^{-1} \) is called the reciprocal of \( \phi \) and \( \phi^{-1} \bar{f} = I. \)

D 2.10. If \( \phi = \phi c, \phi \) is said to be self-conjugate and if \( \phi = -\phi c, \phi \) is said to be anti-self-conjugate.

T 2.6. Any dyadic may be divided in one and only one way into the sum of two dyadics of which one is self-conjugate and the other anti-self-conjugate.\(^{10}\)

T 2.7. Any self-conjugate dyadic may be expressed in the form, \( \phi = a i i + b j j + c k k \) where \( a, b, c \) are scalars, positive or negative.\(^{11}\)

T 2.8. The equation \( r = a \cos \omega + b \sin \omega \) represents an ellipse with \( a \) and \( b \) as conjugate radii.\(^{12}\)

T 2.9. As \( \lambda \) varies \( \frac{x^2}{y^2 - \lambda} + \frac{y^2}{z^2 - \lambda} + \frac{z^2}{w^2 - \lambda} = 1 \) represents the equation of a family of confocal central quadrics.\(^{13}\)

T 2.10. If \( \phi' = \phi \cdot \phi \) \( d \nu = \phi \cdot d \phi \); and if \( \omega = \nu \cdot \phi \cdot \nu, \) \( d \omega = d \nu \cdot \phi \cdot \nu + \nu \cdot \phi \cdot d \nu. \)

\(^{10}\)Ibid., p. 297. \(^{11}\)Ibid., p. 305. \(^{12}\)Wills, op. cit., p. 43.

\(^{13}\)D. M. Y. Sommerville, Analytical Geometry of Three Dimensions, p. 235.

\(^{14}\)Gibbs and Wilson, op. cit., p. 404.
The two forms of dyadics used in the following discussion are known in the literature as tonic and special tonic.\textsuperscript{15}

3. The Parabola

T 5.1. If \( \phi \) is unilinear, \( \nu \) a variable vector, \( \nu \) any constant vector, and \( \lambda \) a constant scalar, then \( r \cdot \phi \cdot \nu = \nu \cdot I \cdot \nu + \lambda \) represents two parallel lines or a parabola.

Let \( \phi = n i j \), \( \nu = m i j \) and \( r = xi + y j \), then

\[
 r \cdot \phi \cdot \nu = m i j \cdot I \cdot r + \lambda \quad \text{gives} \quad ny^2 = my + \lambda ,
\]
two parallel lines.

Take \( \phi = ni j \), \( \nu = ai + bj \); then \( r \cdot \phi \cdot \nu = (ai + bj)I \cdot r \)
is equivalent to \( ny^2 = ax + by + \lambda \), a parabola.

We shall consider the parabola \( r \cdot \phi \cdot \nu = i \cdot I \cdot r \) where \( \phi = \frac{ij}{\lambda a} \).

T 5.2. The equation of the polar line\textsuperscript{16} of the point \( S \) is

\[
 S \cdot \phi \cdot a = i \cdot I \cdot \left( \frac{s + \lambda a}{2} \right) .
\]

Let \( S \) be a point in the polar line. A point on the line joining \( a \) and \( S \) is \( \frac{ys + xa}{x+y} \). If this point is on the parabola, \( \frac{ys + xa}{x+y} \cdot \phi \cdot \frac{ys + xa}{x+y} = i \cdot I \cdot \frac{ys + xa}{x+y} \).

This reduces to

\[
 (S \cdot \phi \cdot S - i \cdot I \cdot S) \left( \frac{y}{x} \right) + \left[ 2S \cdot \phi \cdot a - i \cdot I \cdot (s + a) \right] \frac{y^2}{x^2} + (a \cdot \phi \cdot a - i \cdot I \cdot a) = 0.
\]

\textsuperscript{15}Wills, op. cit., p. 154.

\textsuperscript{16}We assume the uniqueness of the harmonic division relationship of the four points determined by a line through the pole, intersecting the polar and the conic.
The two values of the ratio \( \gamma : \chi \) determined by this equation are equal in magnitude and opposite in sign, since \( S \) is on the polar line of \( a \). Consequently, the term in \( \gamma / \chi \) vanishes. Thus
\[
2S \cdot \phi \cdot a - i \cdot I \cdot (S + a) = 0
\]
reduces to
\[
S \cdot \phi \cdot a = i \cdot I \cdot S + a \frac{2}{2},
\]
the required equation.

**T 3.2.1.** The reciprocal relationship, if \( b \) is on the polar of \( a \), \( a \) is on the polar of \( b \), is now readily established.

The polar of \( b \) is
\[
S \cdot \phi \cdot b = i \cdot I \cdot (S + b). \tag{1}
\]
The polar of \( a \) is
\[
S' \cdot \phi \cdot a = i \cdot I \cdot (S + a). \tag{2}
\]
Since \( b \) is on the polar of \( a \), (2) is satisfied by \( S' = b \), or
\[
b \cdot \phi \cdot a = i \cdot I \cdot (b + a) \]
which is the same as (1) with \( a \) substituted for \( S \).

**T 3.2.2.** The polar of a point \( \gamma \) on the parabola is the tangent to the curve at that point.

\[
\gamma \cdot \phi \cdot \gamma = i \cdot I \cdot \gamma
\]
has for its total differential
\[
2 \, d\gamma \cdot \phi \cdot \gamma = i \cdot I \cdot d\gamma.
\]
The total differential of
\[
S \cdot \phi \cdot \gamma = i \cdot I \cdot (S + \gamma)
\]
is
\[
ds \cdot \phi \cdot \gamma = i \cdot I \cdot \frac{ds}{2},
\]
which equates to
\[
2 \, ds \cdot \phi \cdot \gamma = i \cdot I \cdot ds;
\]
but this equation is satisfied by \( ds = d\gamma \), which proves the theorem.

**T 3.4.** \( 2 \phi \cdot a - i \) is normal to the polar of \( a \).

\[
S \cdot \phi \cdot a = i \cdot I \cdot (S + a) \]
\[
ds \cdot (2 \phi \cdot a - i) = 0
\]
which proves the theorem.
T 3.5. The locus of the mid-points of a system of parallel chords of a parabola is a straight line parallel to the axis of the parabola.

Let \( S \) be the mid-point of one of the chords of the system parallel to \( a \). \( \gamma = S + \gamma a \) is the equation of the secant line upon which the chord lies. Solving for points of intersection of secant and parabola, we have, \(( S + \gamma a ) \cdot \phi \cdot ( S + \gamma a ) = \gamma \cdot ( S + \gamma a ) \), which gives 
\[ \gamma^2 S \cdot \phi \cdot a + \gamma (2 S \cdot \phi \cdot a - \gamma a ) + S \cdot \phi \cdot S - \gamma \cdot S = \phi . \]
Since \( S \) bisects the chosen chord the two solutions for \( \gamma \) are equal in magnitude and opposite in sign. Then 
\[ 2 S \cdot \phi \cdot a - \gamma \cdot a = \phi . \]

Let \( S = m_i - ma \); then
\[ 2m a \cdot \phi \cdot a = \gamma a . \]

\[ m = \frac{i \cdot a}{2a \cdot \phi \cdot a} \]

is constant, which establishes the theorem.

One might continue establishing the various properties of the parabola in this way. Theorems, such as the following, are readily established.

T 3.6. The polar of the focus is the directrix.

T 3.7. A tangent at \( \gamma \) cuts the axis at \( -\gamma a \).

T 3.8. The tangents at the ends of a chord through the focus meet on the directrix.

\[ S' \cdot \phi \cdot r = i \cdot \gamma \cdot ( S' + r ) \quad (1) \]

and

\[ S'' \cdot \phi \cdot [a i - m(r - ai)] = \gamma \cdot \gamma \cdot [ S'' + a i - m(r - ai)] \quad (2) \]

are the equations of the required tangents.

If \( S' = ai + \ell j \), \( \ell \) a variable scalar, satisfies these equations as values of \( S' \) and \( S'' \), the theorem follows. By substitution (1)
and (2) give
\[ L \frac{\mathbf{v}}{4a} - \frac{a \cdot \mathbf{v}}{2}, \quad L = -2a \frac{\mathbf{v}}{2} \] and
\[ \frac{b}{4a} \quad \text{which establishes the theorem.} \]

\[ \mathbf{r} \cdot \phi \cdot \mathbf{r} + \mathbf{v} \cdot \mathbf{I} \cdot \mathbf{v} + k = 0 \]
is only one type of an equation quadratic in the vector \( \mathbf{v} \). The most general type of such equation would contain terms such as \( \mathbf{r} \cdot \mathbf{r} \), \( (\mathbf{r} \cdot \mathbf{a}) \cdot (\mathbf{b} \cdot \mathbf{r}) \), \( \mathbf{r} \cdot \mathbf{c} \) and \( d \cdot \mathbf{e} \), where \( \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \) and \( \mathbf{e} \) are constant vectors. But \( \mathbf{r} \cdot \mathbf{r} = \mathbf{r} \cdot \mathbf{I} \cdot \mathbf{r} \) and \( (\mathbf{r} \cdot \mathbf{a}) \cdot (\mathbf{b} \cdot \mathbf{r}) = \mathbf{r} \cdot (\mathbf{a} \cdot \mathbf{b} \cdot \mathbf{r}) \). Then
\[ \mathbf{r} \cdot \mathbf{r} + (\mathbf{r} \cdot \mathbf{a}) \cdot (\mathbf{b} \cdot \mathbf{r}) = \mathbf{r} \cdot (\mathbf{I} + \mathbf{a} \cdot \mathbf{b}) \cdot \mathbf{r} = \mathbf{r} \cdot \phi \cdot \mathbf{r} \]
But \( \mathbf{r} \cdot \mathbf{c} = \mathbf{c} \cdot \mathbf{I} \cdot \mathbf{r} \) and \( d \cdot \mathbf{e} = \mathbf{k} \). Consequently,
\[ \mathbf{r} \cdot \mathbf{r} + (\mathbf{r} \cdot \mathbf{a}) \cdot (\mathbf{b} \cdot \mathbf{r}) + \mathbf{r} \cdot \mathbf{c} + d \cdot \mathbf{e} = \mathbf{r} \cdot \phi \cdot \mathbf{r} + c \cdot \mathbf{I} \cdot \mathbf{r} + \mathbf{k}. \]

If \( C = 0 \), (1) may be reduced to \( \mathbf{r} \cdot \phi \cdot \mathbf{r} = \text{constant} \). We now proceed to consider this type of equation.

4. Central Conics

Let \( \phi \) be a constant uniplanar dyadic in the equation
\[ \mathbf{r} \cdot \phi \cdot \mathbf{r} = \text{constant}. \]
If the constant is not zero it may be absorbed by the dyadic \( \phi \) so that the equation becomes \( \mathbf{r} \cdot \phi \cdot \mathbf{r} = \mathbf{r} \), or \( \mathbf{r} \cdot \phi \cdot \mathbf{r} = 0 \). In these equations, we take \( \phi \) to be self-conjugate, for

T 4.1. If \( \phi \) is an anti-self-conjugate dyadic, the product
\[ \mathbf{r} \cdot \phi \cdot \mathbf{r} \]
is identically zero for all values of \( \mathbf{r} \).

By T 2.6., \( \phi \) = \( \frac{1}{2} (\phi - \phi^t) \). Let \( \mathbf{r} = xi + yj \).

By T 2.5., \( \phi = ai'i + bj'j \) can be expressed as
\[ \phi = \frac{1}{2} [ai'i + bj'j - ai'i - bj'j]. \]
Then \((x'i + y') \cdot \left[ a \dot{i} + b \dot{j} - a \dot{i} - b \dot{j} \right] \cdot (x'i + y')\) gives
\[ a x'i + b y'j - a x'i - b y'j = 0.\]

**T 4.2.** If \(\gamma' = \phi \cdot r\) and \(r\) is allowed to generate a circle, \(\gamma'\) generates a central conic.

\[ \gamma' = \phi \cdot r; \quad r = \phi^{-1} \cdot r; \quad r = \phi^{-1} \cdot \phi^{-1} \cdot r. \]

Let \(r = x'i + y'j\); then \((x'i + y'j) \cdot (a \dot{i} + b \dot{j}) \cdot (x'i + y'j) = \)
reduces to \(a x'^2 + b y'^2 = 1\), a central conic.

Let \(\phi = \frac{t \dot{i} + u \dot{j}}{a \dot{i} + b \dot{j}}\), then \(r \cdot \phi \cdot r = 0\), gives two lines, real or imaginary; \(r \cdot \phi \cdot r = 1\) gives \(\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1\), an ellipse, real or imaginary, or an hyperbola.

**T 4.3.** \(\phi \cdot r\) is normal to the central conic at \(r\).

The total differential of \(r \cdot \phi \cdot r = 1\) is \(dr \cdot \phi \cdot r + r \cdot \phi \cdot dr = 0\);
\(2 dr \cdot \phi \cdot r = 0\); \(dr \cdot (\phi \cdot r) = 0\). Since \(dr\) is along the tangent, the theorem is proved.

**T 4.3.1.** The magnitude of the normal, i.e., \(\phi \cdot r\), at \(r\), is equal to the reciprocal of the distance from the origin to the tangent at \(r\).

Let \(N = \phi \cdot r\), \(r \cdot \phi \cdot N = 1\); \(r \cdot N = 1\); \(r \cdot N_o = 1\),
where \(N_o\) is a unit vector along \(N\). \(r \cdot N_o = \frac{1}{a}, \frac{1}{fa} = n\).

**T 4.4.** The vector radius of a circle is perpendicular to the vector tangent.
Let \( \rho \) be the vector from the center of the circle perpendicular to the vector tangent. Then \( r \cdot \rho = p \cdot \rho \), since \( r \) can not vary in magnitude, \( r = \rho \).

**T 4.4.1.** For a circle \( \phi = \frac{i^2}{a^2} + \frac{jj}{a^2} \).

Since \( r = \rho \) and \( \rho \) is parallel to \( N \),
\[
\phi \cdot r = k r, \quad \left( \frac{i^2}{a^2} + \frac{jj}{b^2} \right) (xi + yj) = k xi + kyj \text{ then } a = c.
\]

**T 4.5.** The equation \( S \cdot \phi \cdot \alpha = 1 \) is that of the polar line of the point \( \alpha \) with respect to the central conic determined by \( \phi \).

The proof of this theorem is like that for T 3.2.

**T 4.6.** The locus of the mid-points of a system of parallel chords of an ellipse is a straight line passing through the center of the ellipse.

Let \( S \) be a vector to the mid-points of a system of parallel chords parallel to \( \alpha \), then \( r = s + x \alpha \) and \( (s + x \alpha) \cdot \phi \cdot (s + x \alpha) = 1 \) gives \( S \cdot \phi \cdot s + 2x S \cdot \phi \cdot \alpha + x^2 \alpha \cdot \phi \cdot \alpha = 1 \), a quadratic in \( x \).

The stated conditions make \( S \cdot \phi \cdot \alpha = 0 \) for any one, and thus, with \( S \) a variable, all of the chords parallel to \( \alpha \). This is the equation of a straight line passing through the origin.

**T 4.8.1.** The locus of the mid-points of the chords parallel to \( S \) falls upon that member of the \( \alpha \) system which passes through the origin.
The new locus is \( S' \cdot \phi \cdot S' = 0 \) with \( S' \) variable. This is satisfied by \( S' = a \).

We thus have a reciprocal relationship of the two systems of parallel lines.

D 4.2. The \( \gamma_i \) and \( \gamma_2 \) members of these respective systems, are called conjugate radii, and thus we have,

\[
T 4.6.2. \quad \gamma_i \cdot \phi \cdot \gamma_2 = 0.
\]

\( \gamma_i \) is perpendicular to \( \phi \cdot \gamma_2 \) and \( \gamma_2 \) is perpendicular to \( \gamma_i \cdot \phi \). By \( T 4.5 \), \( \gamma_i \cdot \phi \) and \( \phi \cdot \gamma_2 \) are perpendicular to the tangents at \( \gamma_i \) and \( \gamma_2 \), respectively. Consequently \( \gamma_2 \) and \( \gamma_i \) are parallel to the tangents at \( \gamma_i \) and \( \gamma_2 \) respectively.

\[
T 4.7. \quad \text{The equation of the line through two conjugate radii } \gamma_i \text{ and } \gamma_2 \text{ of an ellipse is } S \cdot \phi \cdot (\gamma_i + \gamma_2) = 1.
\]

The equations of the tangents at \( \gamma_i \) and \( \gamma_2 \) are \( S \cdot \phi \cdot \gamma_i = 1 \) and \( S_2 \cdot \phi \cdot \gamma_2 = 1 \), respectively. Since the polar line of any point on the tangents passes through the point of tangency, the point of intersection of the two tangents is the pole of the line through \( \gamma_2 \). The equation of the line is \( S \cdot \phi \cdot a' = 1 \) where \( a' \) is the vector to the pole and \( S \) the vector to any point in the line. But \( a' = \gamma_i + \gamma_2 \), since \( \gamma_i \) and \( \gamma_2 \) and the two tangents form a parallelogram. Then \( S \cdot \phi \cdot (\gamma_i + \gamma_2) = 1 \) becomes the equation of the said line.

\[
T 4.8. \quad \text{If } \gamma' = \phi \cdot \gamma, \text{ where } \gamma \cdot \phi \cdot \gamma = 1 \text{ is an ellipse or hyperbola,}
\]
\( \mathbf{r}' \), as a position vector, describes an ellipse or hyperbola, respectively.

\[ \mathbf{r}' = \phi \cdot \mathbf{r}, \quad \text{then} \quad \mathbf{r} = \mathbf{r}' \cdot \phi^{-1} \quad \text{and} \quad \mathbf{r}' \cdot \phi^{-1} \cdot \phi \cdot \mathbf{r}' \cdot \phi^{-1} = 1; \]
but \( \mathbf{r}' \cdot \phi^{-1} \cdot \phi \cdot \phi^{-1} \cdot \mathbf{r}' = 1 \) gives \( \mathbf{r}' \cdot \phi^{-1} \cdot \mathbf{r}' = 1 \).

**T 4.9.** The polar lines of all points on a line through the center of a central conic are parallel.

\[ \phi \cdot \mathbf{r}_o = l \left( \phi \cdot \mathbf{r}_o \right) \]

\[ 5 \cdot \phi \cdot \mathbf{r}_o = l \quad \text{gives lines perpendicular to } \phi \cdot \mathbf{r}_o \text{ for all values of } l, \text{ cutting } \phi \cdot \mathbf{r}_o \text{ at a distance } \frac{1}{l} \text{ from the origin. Thus } \]
\[ 5 \cdot \phi \cdot \mathbf{r}_o = l \quad \text{is a family of parallel lines, with parameter } x. \]

We state two special cases of T 4.9, the proofs of which are obvious.

The polar line of a pole is parallel to the tangent which is drawn at the point where the vector from the center of the conic to the pole intersects the conic.

When \( \mathbf{a} = \mathbf{r} \), the polar line becomes identical with the vector tangent at the terminus of \( \mathbf{a} \).

**T 4.10.** Any secant line \( AB \) is the polar of the point \( A \) determined by the tangent at \( A \) and \( B \).

We shall designate \( A \) and \( B \) by \( \mathbf{r}' \) and \( \mathbf{r}'' \), respectively. Then

(1) \( S' \cdot \phi \cdot \mathbf{r}' = l \); (2) \( S'' \cdot \phi \cdot \mathbf{r}'' = l \); and (3) \( S \cdot \phi \cdot \mathbf{a} = 1 \).

\( S' = S'' = \mathbf{a} \) satisfies (1) and (2), but \( S = \mathbf{r}' \) and \( S = \mathbf{r}'' \) satisfies (3) which makes the secant through \( \mathbf{r}' \) and \( \mathbf{r}'' \) the polar of \( \mathbf{a} \).
T 4.11. The limit of the polar of \( \mathbf{a} = k \mathbf{i} + m \mathbf{j} \); when \( m \) becomes large and \( \phi \) remains some fixed constant is \( y = 0 \), or \( S = xi \).

\[
S \cdot \phi \cdot \mathbf{a} = 1 ;
\]
\[
(x \mathbf{i} + y \mathbf{j}) \cdot \phi \cdot (k \mathbf{i} + m \mathbf{j}) = 1 ;
\]
\[
\lim_{m \to \infty} \left( \frac{k x}{a^2} + \frac{m y}{b^2} = 1 \right) = \lim_{m \to \infty} \left( \frac{k x}{m a^2} + \frac{y}{b^2} = \frac{1}{m} \right);
\]
\[
\frac{m y}{b^2} = 0 ; \quad y = 0 ; \quad S = xi .
\]

T 4.12. If \( \phi = \frac{ii}{a^2} + \frac{ij}{b^2} \), \( \psi = \frac{ji}{a^2} - \frac{aj}{b^2} \), and \( r \cdot \phi \cdot r^{-1} \), \( r \cdot \psi \) is perpendicular to \( \phi \cdot r \).

\[
r \cdot \psi = -\frac{ay}{b} + \frac{xb}{a} ; \quad \phi \cdot r = \frac{xi}{a^2} + \frac{yj}{b^2} ;
\]
\[
(r \cdot \psi) \cdot (\phi \cdot r) = -\frac{xy}{a^2} + \frac{xy}{b^2} = 0 .
\]

T 4.12.1. \( r \psi \) is the radius conjugate to \( r \) in the ellipse \( r \cdot \phi \cdot r = 1 \).

\( \phi \cdot r \) is perpendicular to the radius vector conjugate to since it is perpendicular to the tangent at the terminus of \( \mathbf{a} \) which is parallel to the conjugate of \( \mathbf{a} \). We need, then, only to check for length.

\[
(r \cdot \psi) \cdot \phi \cdot (r \cdot \psi) \geq 1,
\]
\[
\left( -\frac{ay}{b} + \frac{xb}{a} \right) \cdot \phi \cdot \left( -\frac{ay}{b} + \frac{xb}{a} \right) \geq 1 ; \quad \frac{y^2}{a^2} + \frac{x^2}{b^2} = 1 .
\]

T 4.13. The polar line of a point at infinity in the direction \( a \) is the same as the diameter conjugate with \( \mathbf{a} \).
In \( S \cdot \phi \cdot a = 1 \) let us substitute \( \psi a \) in place of \( a \), where
\( \psi \)

is a scalar constant. Then \( S \cdot \phi \cdot \psi a = 1 \); or \( S \cdot \phi \cdot \bar{a} = \frac{1}{\psi} \). When
\( \psi \to \infty \), \( S \cdot \phi \cdot \bar{a} = 0 \). By T 4.8.1, this is the equation of a radius
\( S \) conjugate with \( a \).

**T 4.14.** If \( \psi \cdot \phi \cdot r = 1 \) is a circle, i.e., \( \phi = \frac{ii}{\alpha^2} + \frac{jj}{\epsilon^2} \), the locus
of the terminus of the normal vector \( \phi \cdot r \) is a circle concentric with
\( r \cdot \phi \cdot r = 1 \).

By T 4.4. \( \phi \cdot r \) and \( r \) are collinear; or \( \phi \cdot r + r = m r \). But
\[
\sqrt{\phi \cdot r \cdot \phi \cdot r} = \frac{1}{\mu} - \frac{\sqrt{\alpha}}{\alpha} \quad \text{and} \quad r = a.
\]
Consequently the locus of \( (a + \frac{1}{\mu}) r \),
as \( \alpha \psi \) generates a circle, is a circle concentric with \( \psi \cdot \phi \cdot r = 1 \), and
with equation \( r' \cdot \phi \cdot r' = 1 \), where
\[
\psi = \frac{ii}{(a^2 - \beta^2)^2} + \frac{jj}{(a - \beta)^2}, \quad \text{and}
\]
\[
r' = [(a + \frac{\beta}{\mu}) r].
\]

**T 4.15.** The normal at any point on an ellipse, \( r \cdot \phi \cdot r = 1 \),
makes equal angles with the lines joining that point to the foci.

The position vectors of the foci are \( F = a e i \) and \( F' = -a e i \).

The vectors joining the foci to the point in question are, \( r + a e i \)
and \( r - a e i \). Unit vectors along these are
\[
\frac{r + a e i}{\sqrt{(r + a e i) \cdot (r + a e i)}} \quad \text{and} \quad \frac{r - a e i}{\sqrt{(r - a e i) \cdot (r - a e i)}}
\]
respectively.

But \( (a + e \chi) = \sqrt{(r + a e i) \cdot (r + a e i)} \), and \( (a - e \chi) = \sqrt{(r - a e i) \cdot (r - a e i)} \)
by the focus-directrix-eccentricity definition of the ellipse.

The equality of \( \frac{r + a e i}{a + e \chi} \cdot \phi \cdot r = \frac{r - a e i}{a - e \chi} \cdot \phi \cdot r \),
\[
1 + \frac{e \chi}{a} = \frac{1 - \frac{e \chi}{a}}{a - e \chi} \quad \text{and} \quad \frac{1}{a} = \frac{1}{a}
\]
establishes the truth of the theorem.
T 4.16. The normal at any point of the hyperbola \( \nu \cdot \phi \cdot r = 1 \) makes equal angles with the lines joining the point to the foci of the hyperbola.

We must merely test the equality of

\[
\frac{(r + ae_1) \cdot \phi \cdot r}{\sqrt{(r + ae_1)(r + ae_1)}} = \frac{(ae_1 - r) \cdot \phi \cdot r}{\sqrt{(ae_1 - r)(ae_1 - r)}}.
\]

Substituting \( \lambda r + a = \sqrt{(r + ae_1)(r + ae_1}) \) and \( \lambda r - a = \sqrt{(ae_1 - r)(ae_1 - r)} \), we have

\[
\frac{(\lambda r + ae_1) \cdot \phi \cdot r}{\lambda r + a} = \frac{(ae_1 - r) \cdot \phi \cdot r}{\lambda r - a};
\]

\[
\frac{1}{\lambda(1 + \lambda r)} = \frac{\lambda r - a}{a(\lambda r - a)} ; \quad \frac{1}{\lambda} = \frac{1}{a},
\]

which proves the theorem.

T 4.17. The necessary and sufficient conditions that two conics \( \nu \cdot \phi \cdot r = 1 \) and \( \nu \cdot \psi \cdot r = 1 \) be confocal is that \( \phi' \) and \( \psi' \) shall differ only by a scalar multiple of the idemfactor.

Let \( \phi = \frac{i i}{a^2} + \frac{j j}{b^2} \) and \( \psi = \frac{i i}{\nu^2} + \frac{j j}{\mu^2} \). Then

\[
\phi' = \lambda^2 i i + \lambda b^2 j j ; \quad \psi' = \lambda^2 i i + \lambda \nu^2 j j .
\]

\[
\phi' + \lambda I = \psi' ; \quad \phi' = -\lambda I + \psi'.
\]

where \( \lambda \) is a scalar constant.

Then \( r \cdot \phi \cdot r = 1 \) gives

\[
\frac{\lambda^2}{\nu^2 - \lambda} + \frac{\lambda b^2}{\mu^2 - \lambda} = 1.
\]

Now we shall test any two of the set of conics produced as \( \lambda \) varies, say \( \lambda = \lambda_1 \); and \( \lambda = \lambda_2 \).
\[ \varphi = \frac{ix}{n^2 - \lambda_1} + \frac{jj}{m^2 - \lambda_1}; \quad \psi = \frac{ix}{n^2 - \lambda_2} + \frac{jj}{m^2 - \lambda_2}; \]

\[ \Phi^{-1} = (n^2 - \lambda_1)ii + (m^2 - \lambda_1)jj; \quad \Psi^{-1} = (n^2 - \lambda_2)ii + (m^2 - \lambda_2)jj; \]

\[ \Phi^{-1} \cdot \Psi^{-1} = (\lambda_2 - \lambda_1)ii + (\lambda_1 - \lambda_2)jj; \]

\[ \Phi^{-1} \cdot \Psi^{-1} = (\lambda_2 - \lambda_1)(\bar{u} + jj). \]

**T 4.18.** The sum of the squares of a pair of conjugate radii and \( b' \) of an ellipse, \( \gamma \cdot \varphi \cdot \gamma = 1 \), is constant.

We wish to prove \( a' \cdot a' + b' \cdot b' = \) constant.

\[ a' \cdot a' + b' \cdot b' = \gamma \cdot \gamma + (\gamma \cdot \psi) \cdot (\gamma \cdot \psi); \]

\[ a' = \gamma \quad \text{and} \quad b' = \gamma \cdot \psi; \]

\[ \gamma \cdot \gamma = x^2 + y^2 \quad \text{and} \quad \gamma \cdot \varphi \cdot \gamma = \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1; \]

\[ y^2 = \frac{b^2}{a^2} (a^2 - x^2); \]

\[ \gamma \cdot \gamma = \frac{a^2 - b^2}{a^2} x^2 = b^2 + \frac{a^2 - b^2}{a^2} x^2; \]

\[ (\gamma \cdot \psi) \cdot (\gamma \cdot \psi) = \frac{a^2}{b^2} y^2 + \frac{b^2}{a^2} x^2; \]

But

\[ \frac{a^2}{b^2} y^2 = a^2 - x^2; \]

hence, \( (\gamma \cdot \psi) \cdot (\gamma \cdot \psi) = a^2 - x^2 + \frac{a^2 - b^2}{a^2} x^2 = a^2 - (1 - \frac{b^2}{a^2}) x^2; \]

\[ = a^2 - \left( \frac{a^2 - b^2}{a^2} \right) x^2; \]

\[ (\gamma \cdot \psi) \cdot (\gamma \cdot \psi) = a^2 - x^2 \cdot x^2. \]

Thus,

\[ \gamma \cdot \gamma + (\gamma \cdot \psi) \cdot (\gamma \cdot \psi) = a^2 + b^2. \]

An alternate proof for T 4.18.

We wish to prove \( a' \cdot a' + b' \cdot b' \) constant.

By theorem 2.20, we may express \( \gamma \) in terms of the sine and cosine of an angle \( \beta \);
\[ r = a \cos s + b \sin s; \]
\[ \frac{dr}{ds} = -a \sin s + b \cos s; \]
\[ b' = m \frac{dr}{ds}; \quad b = m(a \sin s + b \cos s); \]
\[ m(-a \sin s + b \cos s) \cdot \phi \cdot m(a \sin s + b \cos s) = 1; \]
\[ m^2(\sin^2 s + \cos^2 s) = 1; \quad m^2 = 1; \quad m = \pm 1; \]
\[ a' \cdot a' = a^2 \cos^2 s + b^2 \sin^2 s; \quad (1) \]
\[ b \cdot b = a^2 \sin^2 s + b^2 \cos^2 s; \quad (2) \]

Adding (1) and (2) we have \[ a' \cdot a' + b' \cdot b' = a^2 + b^2. \]

**T 4.19.** The length of the latus rectum of an ellipse is \( \frac{2b}{a}. \)

\[ r = ai + nj \text{ makes the latus rectum equal } 2n. \]
\[ (ai + nj) \cdot \phi \cdot (ai + nj) = 1; \]
\[ r^2 = \frac{n^2}{b^2} = 1; \quad n^2 = b^2(1 - l^2); \]
\[ n = b \sqrt{1 - l^2}; \]
\[ 2n = 2b \sqrt{1 - l^2}; \]

but \[ b^2 = a^2(1 - l^2); \quad \frac{b}{a} = \sqrt{1 - l^2}; \]
\[ 2n = \frac{2b^2}{a}. \]

**T 4.20.** If two confocal central conics intersect they do so at right angles.

Let the conics be \( r \cdot \phi \cdot r = 1 \) and \( r \cdot \psi \cdot r = 1. \) Let \( N = \phi \cdot r, \) and \( N = \psi \cdot r. \) Then, \( r = \phi^{-1}N \) and \( r = \psi^{-1}N. \) Since the conics are confocal, \( \phi^{-1} \cdot \psi^{-1} \equiv \gamma I. \)
Then

\[ \mathbf{r} \cdot \mathbf{r} = \mathbf{\phi}^{-1} \mathbf{N} \cdot \mathbf{\phi} \cdot \mathbf{\phi}^{-1} \mathbf{N} = 1 ; \mathbf{N} \cdot \mathbf{\phi}^{-1} \mathbf{N} = 1 ; \]

\[ \mathbf{r} \cdot \mathbf{r} = \mathbf{\psi}^{-1} \mathbf{N} \cdot \mathbf{\psi} \cdot \mathbf{\psi}^{-1} \mathbf{N} = 1 ; \]

\[ \mathbf{N} \cdot \mathbf{\psi}^{-1} \mathbf{N} = 1 ; \]

\[ \mathbf{\gamma} = \mathbf{\psi}^{-1} \mathbf{N} = \mathbf{\phi}^{-1} \mathbf{N} = (\mathbf{\psi}^{-1} + \lambda \mathbf{I}) \cdot \mathbf{N} = \mathbf{\psi}^{-1} \mathbf{N} + \mathbf{\psi} \mathbf{N} ; \]

\[ \lambda \mathbf{N} = \mathbf{\psi}^{-1} \mathbf{N} - \mathbf{\psi}^{-1} \mathbf{N} ; \]

\[ \lambda \mathbf{N} \cdot \mathbf{N} = 1 - \mathbf{N} \cdot \mathbf{\psi}^{-1} \mathbf{N} , \quad (1) \]

Or

\[ \mathbf{\gamma} = \mathbf{\phi}^{-1} \mathbf{N} = \mathbf{\psi}^{-1} \mathbf{N} = (\mathbf{\phi}^{-1} - \lambda \mathbf{I}) \cdot \mathbf{N} = \mathbf{\phi}^{-1} \mathbf{N} - \lambda \mathbf{N} ; \]

\[ -\lambda \mathbf{N} = \mathbf{\phi}^{-1} \mathbf{N} - \mathbf{\phi}^{-1} \mathbf{N} ; \]

\[ \lambda \mathbf{N} \cdot \mathbf{N} = \mathbf{N} \cdot \mathbf{\phi}^{-1} \mathbf{N} - \mathbf{N} \cdot \mathbf{\phi}^{-1} \mathbf{N} ; \]

\[ \lambda \mathbf{N} \cdot \mathbf{N} = \mathbf{N} \cdot \mathbf{\phi}^{-1} \mathbf{N} - 1 . \quad (2) \]

(1) + (2) gives \( \lambda \mathbf{N} \cdot \mathbf{N} = 0 \), which proves the theorem.

\[ T 4.81. \text{ If } \mathbf{r} \cdot \mathbf{\phi} \cdot \mathbf{r} = 1 \text{ is the equation of an hyperbola, the} \]

\[ \text{equation of the tangents at infinity, i.e., the asymptotes, are} \]

\[ \text{represented by } \mathbf{r} \cdot \mathbf{\phi} \cdot \mathbf{r} = 0 . \]

Let \( \mathbf{r} = m \mathbf{r}_o \) be a point on the hyperbola, where \( \mathbf{r}_o \) is a unit vector.

The equation \( S \cdot \mathbf{\phi} \cdot m \mathbf{r}_o = 1 \) is that of the tangent line of \( m \mathbf{r}_o \).

\[ \lim_{m \to \infty} (S \cdot \mathbf{\phi} \cdot m \mathbf{r}_o = 1) \text{ is the } \lim_{m \to \infty} (S \cdot \mathbf{\phi} \cdot \mathbf{r}_o = \frac{1}{m}) \text{, or} \]

\[ S \cdot \mathbf{\phi} \cdot \mathbf{r}_o = 0 . \]

Since \( \mathbf{r}_o \) has the direction of \( \mathbf{r} \), and \( S \) is perpendicular

to \( \mathbf{\phi} \cdot \mathbf{r} \), \( S \cdot \mathbf{\phi} \cdot \mathbf{r} = 0 \), which is a line through the origin. In this line

\[ S = x \mathbf{r} \], hence \( \mathbf{r} \cdot \mathbf{\phi} \cdot \mathbf{r} = 0 \), or \( \mathbf{r} \cdot \mathbf{\phi} \cdot \mathbf{r} = 0 \), which is \( \mathbf{r}^2 - \frac{x^2}{a^2} = 0 \).
T 2.22. The two segments of a secant line parallel to \( a \), intercepted between the asymptotes and the hyperbola are equal.

Let \( b \) be a vector from the origin to the mid-point of that part of \( a \) which is the chord falling on the given secant,

\[
\gamma = b + \lambda a
\]

is the equation of the secant,

\[
\gamma \cdot \phi \cdot \gamma = 1
\]

is that of the hyperbola,

\[
(b + \lambda a) \cdot \phi \cdot (b + \lambda a) = 1
\]

gives

\[
b \cdot \phi \cdot b + 2 \lambda b \cdot \phi \cdot a + \lambda^2 a \cdot \phi \cdot a = 1
\]

whose solutions give the intersections of the secant with the hyperbola.

Since the two values of \( \lambda \) are equal and opposite in sign,

\[
b \cdot \phi \cdot a = 0.
\]

The common solution of the secant, \( \gamma = b + \lambda a \), and the asymptotes, \( \gamma \cdot \phi \cdot \gamma = 0 \), are involved in \((b + \lambda a) \cdot \phi \cdot (b + \lambda a) = 0\), i.e.,

\[
b \cdot \phi \cdot b + 2 \lambda' b \cdot \phi \cdot a + \lambda^2 a \cdot \phi \cdot a = 0
\]

But \( b \cdot \phi \cdot a = 0 \), therefore the coefficient of \( \lambda \) vanishes, which means the two roots of this equation in \( \lambda' \) are equal in magnitude and opposite in sign.

T 2.23. The polar \( a' \) of the secant through the radii \( a \) and \( b \) is \( k(a + b) \).

Let \( \gamma = ma + nb \) be the pole of the line through \( a \) and \( b \).

Then, \( S \cdot \phi \cdot (ma + nb) = 1 \), and \( a \cdot \phi \cdot (ma + nb) = 1 \) gives \( m + n a \cdot \phi \cdot b = 1 \). Similarly \( b \cdot \phi \cdot (ma + nb) = 1 \) gives \( n + m b \cdot \phi \cdot a = 1 \). Subtracting, \((m-n) - b \cdot \phi \cdot a (n-m) = 0 \)

\[(m-n)(1-b \cdot \phi \cdot a) = 0 \]

Either \( m-n = 0 \) or \( b \cdot \phi \cdot a = 1 \).

The latter is false unless \( a = b \). Thus \( m = n \), and \( a' = k(a + b) \).
T 2.23.1. \((a+b) \cdot \varphi \cdot (s-a) = 0\) is the equation of the secant determined by radii \(a\) and \(b\).

\[ s \cdot \varphi \cdot k(a+b) = 1, \]
\[ ks \cdot \varphi \cdot (a+b) = 0, \]
\[ ds \cdot \varphi \cdot (a+b) = 0. \]

Thus \(ds\) is perpendicular to \(\varphi \cdot (a+b)\). Since \(ds\) is along the secant we may substitute \(s-a\) for \(ds\). Thus, \((a+b) \cdot \varphi \cdot (s-a) = 0\).

T 2.23.2. The equation of the tangent at \(a\) is \(s \cdot \varphi \cdot a = 1\).

Let \(b\) approach \(a\) as a limit; then

\[ (a+a) \cdot \varphi \cdot (s-a) = 0; \]
\[ 2s \cdot \varphi \cdot a - 2a \cdot \varphi \cdot a = 0; \]
\[ s \cdot \varphi \cdot a = 1. \]

5. Quadric Surfaces

The proofs of the theorems on conics can be modified to establish the analogous theorems in space of three dimensions. We need to make uniplanar \(\varphi\) for the parabola, planar, planar \(\varphi\) of central conics, complete, and to set \(r = x_i + y_j + z_k\). The following theorems will be numbered to correspond with the similar theorems of sections three and four. Thus, T 5.3.1. will correspond to T 3.1., and T 5.4.1. will correspond to T 4.1.

Proofs will be given only when they differ sufficiently from the proofs in two dimensions to be of especial interest.

T 3.5.1. If \(\varphi\) is uniplanar, \(r\) a variable vector, \(V\) any constant vector, and \(k\) any scalar, then \(r \cdot \varphi \cdot r = r \cdot k\).
\[ a x^2 + a z = a x + b y + c z + k, \]
a non-central quadric.

In the next theorem, let \( \phi = \frac{ij}{a^2} + \frac{jj}{b^2} \), \( \nu = \nu \kappa \) and \( k = 0 \).

**T 5.3.2.** The equation of the polar plane of the point determined by the vector \( \alpha \) is
\[ S \cdot \phi \cdot \alpha = \nu \kappa \cdot I \cdot (s + a), \]

If the signs of the two dyads of \( \phi \) are different and \( \kappa \) is positive, \( \nu \phi \cdot \nu = \nu \kappa \cdot I \cdot \nu \) gives an hyperbolic paraboloid; if the signs of \( \phi \) are alike, an elliptic paraboloid.

If \( \phi = \frac{ij}{Ha} + \frac{jj}{wa} \), then \( \nu \phi \cdot \nu = \kappa \cdot I \cdot \nu \) becomes
\[ \gamma^2 + \gamma^2 = \frac{a}{4} \], a paraboloid of revolution.

**T 5.3.2.1.** The reciprocal relationship; if \( \beta \) is on the polar of \( \alpha \), \( \alpha \) is on the polar of \( \beta \), may be readily established.

**T 5.3.2.2.** The polar of a point \( \alpha \) on the paraboloid is the tangent plane to the paraboloid at that point.

**T 5.3.6.** The locus of the mid-points of a system of parallel chords of a paraboloid is a plane parallel to the axis.

**T 5.4.1.** If \( \phi \) is an anti-self-conjugate dyadic, the product of \( \nu \phi \cdot \nu \) is identically zero for all values of \( \nu \).

**T 5.4.2.** If \( \nu' = \phi \cdot \nu \) and \( \nu \) is allowed to generate a sphere, \( \nu' \) generates a central quadric.

If \( \phi = \frac{ij}{a^2} + \frac{jj}{b^2} + \frac{k\kappa}{c^2} \), where any combinations of
signs may occur, \( \gamma \cdot \phi \cdot \gamma = 1 \) and \( \gamma \cdot \phi \cdot \gamma = 0 \) give rise to the central quadric surfaces; the real ellipsoid, the hyperboloid of one sheet, an hyperboloid of two sheets, the imaginary ellipsoid, and the real or imaginary cone.

T 5.4.3. \( \phi \cdot \gamma \) is normal to the tangent plane to the central quadric.

T 5.4.3.1. The magnitude of the normal \( \mathbf{N} \), at \( \gamma \), i.e., \( \phi \cdot \gamma \), is equal to the reciprocal of the distance from the origin to the tangent plane at \( \gamma \).

T 5.4.4. The radius vector of a sphere is perpendicular to the tangent plane.

T 5.4.5. In any central quadric, \( 5 \cdot \phi \cdot \mathbf{a} = 1 \) is the equation of the polar plane of the pole \( \mathbf{a} \) with respect to that quadric.

T 5.4.6. The locus of the mid-points of a system of parallel chords of any quadric is a plane. For central quadrics this plane passes through the center.

This plane is called the diametral plane conjugate with the system of chords. It is parallel to the plane drawn tangent to the ellipsoid at the extremity of that one of the chords which passes through the center. This theorem leads to a system of three radii, conjugate in pairs.
T 5.4.7. The equation of the plane determined by three conjugate radii, \(a\), \(b\), and \(c\) is \(\gamma \cdot \phi \cdot (a + b + c) = \). From the pole and polar relationships involved, \(a\), \(b\), and \(c\) determine six planes, three diametral and three tangent, which determine a parallelepiped with a vertex \(a + b + c\). This vertex has for its polar, \(\gamma \cdot \phi \cdot (a + b + c) = \), which passes through \(a\), \(b\), and \(c\).

T 5.4.8. If \(\gamma' = \phi \cdot \gamma\), where \(\gamma \cdot \phi = 1\) is an ellipsoid or hyperboloid, \(\gamma'\), as a position vector describes an ellipsoid or hyperboloid, respectively.

T 5.4.9. The polar planes of all points on a line through the center of a central quadric are parallel.

T 5.4.10. Any secant plane is the polar of the point \(\alpha\), determined by the tangent cone of the conic of intersection.

T 5.4.11. The polar of any point in a plane passes through the pole of that plane.

T 5.4.15. The normal at any point on a central quadric, \(\gamma \cdot \phi \cdot \gamma = \), makes equal angles with the lines joining that point to the foci of the quadric.

T 5.4.20. The necessary and sufficient condition that two quadrics whose equations are, \(\gamma \cdot \phi \cdot \gamma = 1\) and \(\gamma \cdot \phi \cdot \gamma = 1\),
shall be confocal is that \( \phi' \) and \( \psi' \) shall differ by a scalar multiple of the idemfactor.\(^{17}\)

The proof is as follows:

Let 
\[
\psi = \frac{ii'}{n^2} + \frac{jj'}{m^2} + \frac{kk'}{p^2} ;
\]
then 
\[
\psi' = n^2ii + m^2jj + p^2kk ; \quad \phi'^{-1} = -\lambda \mathbf{I} + \psi'^{-1} ;
\]
where \( \lambda \) is a scalar constant.

\[
\phi = \left( \frac{1}{n^2-\lambda} \right) ii + \left( \frac{1}{m^2-\lambda} \right) jj + \left( \frac{1}{p^2-\lambda} \right) kk ,
\]
\[\gamma \cdot \phi \cdot \gamma^{-1} \text{ gives } \frac{x^2}{n^2-\lambda} + \frac{y^2}{m^2-\lambda} + \frac{z^2}{p^2-\lambda} = 1.\]

By T 3.9, as \( \lambda \) varies \( \gamma \cdot \phi \cdot \gamma^{-1} \) gives a system of confocal quadrics. Let
\[
\phi = \frac{ii}{n^2-\lambda_1} + \frac{jj}{m^2-\lambda_1} + \frac{kk}{p^2-\lambda_1} ;
\]
and 
\[
\psi = \frac{ii'}{n^2-\lambda_2} + \frac{jj'}{m^2-\lambda_2} + \frac{kk'}{p^2-\lambda_2} ;
\]
then
\[
\phi'^{-1} = (n^2-\lambda_1)ii + (m^2-\lambda_1)jj + (p^2-\lambda_1)kk ;
\]
\[
\psi'^{-1} = (n^2-\lambda_2)ii + (m^2-\lambda_2)jj + (p^2-\lambda_2)kk ;
\]
\[
\phi'^{-1} \cdot \psi'^{-1} = (\lambda_2 - \lambda_1) \mathbf{I} , \text{ which establishes the theorem.}
\]

**T 5.4.19.** The sum of the squares of three conjugate radii \( a', b', \) and \( c' \) of an ellipsoid, \( \gamma \cdot \phi \cdot \gamma^{-1} \), is constant.

**T 5.4.21.** If \( \gamma \cdot \phi \cdot \gamma^{-1} \) is the equation of an hyperboloid, the equation of the tangent cone at infinity, is \( \gamma \cdot \phi \cdot \gamma^{-1} = 0.\)

**T 5.2.22.** The two segments of a secant line parallel to \( a \) intercepted between the cone \( \gamma \cdot \phi \cdot \gamma^{-1} = 0 \) and the hyperboloid \( \gamma \cdot \phi \cdot \gamma^{-1} = 1 \) are equal.

\(^{17}\text{Wills, op. cit., p. 165.}\)
6. Concluding Remarks

We refrained in the title of this paper from saying a study of conics; we are investigating a method and not conics. And now, as we come to the end of the discussion, we refrain from using the word conclusion. New and enticing topics present themselves and commencement might be a more appropriate word. Some topics for further study with the aid of the dyadic operator would be: the linear function in plane and solid analytic geometry, functions in four and higher dimensions, poles and polars, the principle of duality, and other topics of projective geometry. Here there is opportunity for further investigation and generalization in the spirit and method of the attack used in this paper.
BIBLIOGRAPHY


