THE HISTORY OF THE CALCULUS

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THE HISTORY OF THE CALCULUS

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CHAPTER I

INTRODUCTION

The calculus is both an invention and the result of a long, slow process of development. Newton and Leibniz, working independently, invented the calculus in the latter part of the seventeenth century.

But the development of the concepts of the calculus began over two thousand years earlier. There is a logical consecutiveness in the growth of mathematical ideas and procedures that is lacking in the history of many other fields of study. Each generation of mathematicians stands upon the shoulders of all the mathematicians of all the preceding centuries. The modern tendency to arithmetize the whole of mathematics uses number concepts, such as positional notation and zero, that were developed by men whose names and dates we do not know. The logical rigor of mathematics today goes back for much of its inspiration and some of its techniques to Euclid.

The purpose of this essay is to trace the development of the concepts of the calculus from their first known appearance, through the formal invention of the method of the calculus in the second half of the seventeenth century, to our own day. The major part of the paper will deal with the history from the beginning through the seventeenth century. The developments of the eighteenth and nineteenth centuries will be sketched more briefly.
CHAPTER II

THE GREEK PERIOD

The history of Greek mathematics and of the calculus begins with Thales in the sixth century B.C. He is said to have been a great traveller, learning geometry from the Egyptians and astronomy from the Babylonians. To him tradition attributes these propositions:

1. A circle is bisected by any diameter.

2. The angles at the base of an isosceles triangle are equal.

3. If two straight lines cut one another, the vertically opposite angles are respectively equal.

4. If two triangles have two angles and one side in each respectively equal, the triangles are equal in all respects.

5. The angle in a semi-circle is a right angle.

This transition from rule-of-thumb mensuration to geometry as a science, the German philosopher Kant described as nothing less than "a revolution, brought about by the happy inspiration of one man."

Thales established mathematics as a deductive science. Pythagoras built upon this sure foundation. He, says Proclus, "transformed the study of geometry into a liberal education, examining the principles of the science from the beginning and probing the theorems through and through in a purely intellectual manner."

1T. L. Heath, History of Greek Mathematics, I, 140.
Favorinus says that he "used definitions on account of the mathematical nature of the subject." The Pythagorean motto, "a figure and a platform, not a figure and sixpence," means that each new theorem is a platform from which one ascends to the next.

Pythagoras insisted on the necessity of the utmost generality in reasoning. He discovered the importance of dealing with abstractions. He directed attention to number as characterizing the periodicities of notes of music. Thus, the abstract idea of periodicity was fortunately present at the beginning of mathematics and of European philosophy. He is said to have taught that mathematical entities, such as numbers and shapes, were the real stuff out of which the real entities of our perceptual experience are constructed. His practical counsel was to measure, to express quality in terms of quantity.

The Pythagorean brotherhood was the first secular association of scientists. The habit of ascribing the discoveries of all members of the society to "Him," Pythagoras, makes it difficult to name the results of the master's researches. Tradition has unanimously attributed to him the theorem which bears his name: "The square of the hypotenuse of a right triangle is equal to the sum of the squares of the other two sides." One of the theories of the Pythagoreans was that of the application of areas, which later led to the method of exhaustion. Perhaps through a study of the lengths of the sides of an isosceles right triangle they discovered the incommensurability of certain magnitudes.

The atomism of Democritus offered a solution of the problem of incommensurable lines. The universe is not composed of one substance, he taught, nor of a few substances. Matter and the soul are atomic in
structure. Out of Democritean atomism arose the theory of infinitesimals, which are infinitely small mathematical entities. Leibniz based his calculus on them. And there they remained until Weierstrass drove them out in the middle of the nineteenth century.

But infinitesimals troubled Democritus. If one takes a section of a cone infinitely near the base, is it equal to the base? Then the cone is a cylinder. If the section is not equal to the base, the cone is a terraced pyramid.

Infinitesimals troubled Zeno, too. Concerning them he propounded his four famous paradoxes. (1) The Dichotomy: One cannot traverse an infinite number of points in a finite time. Before he can traverse the whole, he must traverse its half. Before he can traverse the remainder, he must traverse its half. This goes on ad infinitum. If space is made up of points, one cannot traverse an infinity of them in a finite time. (2) The Achilles: Achilles cannot overtake a tortoise. By the time he reaches the place from which it started, the tortoise will have gone on a little way. By the time he reaches that point, it will again have moved forward. And so on. (3) The Arrow: An arrow in flight is at every instant of its flight at rest at some point. Since it is at rest, it cannot be in motion. (4) The Stade: Given three parallel rows of points in juxtaposition, as in Fig. 1. Let the points in B remain stationary, and the points in A and C move to the left and right respectively with equal velocity, so as to come into the position in Fig. 2.

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If an instant is the time required for a point to approach and pass another point, an instant is not indivisible. For a point in A will in a given time pass twice as many points in C as in B.

Zeno's paradoxes are concerned with infinitesimals, infinity, and continuity. Greek mathematicians, unable to resolve them, avoided the use of infinite processes. It was no shame to them that they were not able to deliver themselves from the dilemma into which Zerro had lead them. For the next twenty-four centuries mathematicians and philosophers pondered over these questions. Today it seems that Weierstrass, Dedekind, and Cantor have, with their theories of the infinite, driven the fixed infinitesimal into limbo.

Though Plato made no specific contribution to the development of the concepts of the calculus, he emphasized the abstract nature of mathematics and the importance to students of philosophy of a knowledge of geometry.

Eudoxus showed that it is not necessary to use infinitesimals. Instead he used the method of exhaustion. This method, which had been employed by Antiphon and Bryson before him, consisted in inscribing or circumscribing a polygon to a circle, or in both inscribing and circumscribing a polygon to a circle, and of repeatedly doubling the number of sides of the polygon, thereby making the difference between the area of the circle and that of the polygon as small as one pleases. The name, method of exhaustion, arises from the fact that as the number of sides of the polygon increases the difference between the areas of the circle
and of the polygon is progressively exhausted. Archimedes affirms that
Eudoxus used this method in proving that the volumes of a cone and of
a pyramid are respectively equal to one-third the volumes of a cylinder
and of a prism of equal altitude and equal base, and that he used this
method also in proving that the areas of two circles are to each other
as the squares of their diameters, and that the volumes of two spheres
are to each other as the cubes of their diameters.

By the application of a unit of length or of area or of volume,
the Pythagoreans had been able to express geometrical magnitudes as
integers. But with their discovery of the incommensurability of certain
magnitudes, they had reached an impasse. The Democritean belief in
infinitely small geometrical quantities offered a way of escape, only
to be blocked by the paradoxes of Zeno. Eudoxus used proportions in
which all the terms are geometric magnitudes. This is Euclid's state-
ment of Eudoxus' definition: "Magnitudes are said to be in the same ratio,
the first to the second and the third to the fourth, when, if any equi-
multiples whatever be taken of the first and third, and any equimultiples
whatever of the second and fourth, the former equimultiples alike exceed,
are alike equal to, or alike fall short of, the latter equimultiples
respectively taken in corresponding order." 3 This theory of proportion made
it unnecessary for Eudoxus to conceive of an irrational number and to
define it. Today we think of the ratio of the incommensurable magnitudes
as a number, but he did not.

3 Book V, Definition 5.
The son of a physician, Aristotle, was more of a biologist and less of a mathematician than Plato. From the days of the Greeks until recently, the techniques of the life sciences were descriptive and classificatory, rather than quantitative and mathematical. How much more the Middle Ages, which labored and thought under the imperial rule of Aristotle, would have learned and discovered if only they had measured instead of classifying.  

To Plato geometric figures are man made conceptions of the "ideas" of these figures. To Aristotle they are abstractions from forms in the natural world. Plato emphasized the theoretical and intellectual nature of knowledge. Aristotle emphasized the experiential and sensory character of knowledge.

Yet Aristotle sided with the mathematicians against the scientists in his rejection of the infinitely divisible that is neither zero nor a finite quantity. In the first two centuries of the history of the calculus, mathematicians believed in minimal quantities, and it is probably fortunate that they did. But during the last century they have denied their existence because they have not been able to give a logical definition of fixed infinitesimals.

Aristotle regarded the existence of the infinite as potential but not actual. In this wariness of the infinite he was in agreement with practically all other Greeks since the time of Zeno. Since geometry was preeminently the mathematical science of the Greeks, they thought of infinity in geometric, not in arithmetic, terms. "In point of fact

\[\text{A. N. Whitehead, Science and the Modern World, p. 43.}\]
they (the mathematicians)," says Aristotle, "do not need the infinite and do not use it. They postulate only that the finite straight line may be produced as far as they wish. . . Hence, for the purposes of truth, it will make no difference to them to have such an infinite instead, while its existence will be in the sphere of real magnitude."5 Today our notion of infinity is based upon the arithmetical continuum.

The fundamental idea of the differential calculus, the instantaneous rate of change of one variable with respect to another, Aristotle rejected, saying "nothing can be in motion in a present. . . nor can anything be at rest in a present." This dictum is in accord with our sensory experience which admits the reality of an average velocity during a given interval of time, but denies the mathematical determination of velocity at a particular instant of time. Thought, however, is not as narrowly confined as our senses. Later mathematicians have given a logical definition and elaboration of the idea of the rate of change at a given instant. Lacking this concept and the procedures which implement it, science could not have flourished in the last three centuries as it has. To break the chains which for 2,000 years had bound it to the great but ancient past, science needed the dynamism of a new approach to its problems. That new approach the calculus offered in its philosophy and science of a rate of change.

What has been said of Aristotle is not intended as a criticism. The philosophic ideas and the scientific theories of even the greatest men are a function of time. What is valid in one generation is not valid

5_5Physics, III, p. 207._
in another. Mathematical theories and techniques are also functions of time. To judge the validity of a geometric proof fairly one must know its date. A proof that was perfectly valid in the sixth century B.C. was not valid in the third century B.C. Other men had labored, and the latter century had entered into their labors.

In the Elements of Euclid, for twenty-two hundred years the textbook of the western world, geometry became a logical structure, consisting of definitions, postulates, hypotheses, constructions, proofs, and conclusions. Euclid gave to the propositions of the thirteen books, many of which had been intuitively established, a rigorous proof. True to Aristotle's statement that the infinite is unnecessary, Euclid avoided it, using instead the method of exhaustion. He followed Eudoxus in thinking of irrationals in geometric terms. The influence of Aristotle's philosophy of common sense upon the Elements is further seen in the conformity of its definitions and postulates to the data of sensory experience and of spatial intuition.

Originating perhaps in our intuitive sense of variability and multiplicity, the derivative and the integral, which are the fundamental concepts of the calculus, required a relatively advanced stage of mathematical logic and abstraction for their formulation. That stage had not been reached in Euclid's day. For three centuries Greek geometers had been building a mathematical science. Into the somewhat negligent and confused results of their studies, Euclid brought an order, a logic, and a synthesis that were a model for over two thousand years. Similarly, the calculus waited for two centuries after its invention for rigorous definitions and proofs.
Perhaps the three greatest mathematicians of all time were Archimedes, Newton, and Gauss. Archimedes "gave birth to the calculus of the infinite, conceived and brought to perfection successively by Kepler, Cavalieri, Fermat, Leibniz, and Newton." In the invention of his theorems he used infinitesimals, considering a plane figure as made up of lines; and employing the concepts of center of gravity, lever, and fulcrum, he balanced the elements of a geometric figure of known area against those of the figure he was seeking to determine. Having computed the area by mechanics, he gave a rigorous geometric proof of the result. This proof was the Euclidean method of exhaustion with both inscribed and circumscribed polygons of an increasing, though finite, number of sides.

Heiberg's discovery of the Method, which Archimedes addressed to his friend Eratosthenes, in Constantinople in 1906 enables one to study his use of infinitesimals and of the principles of mechanics in solving problems in quadrature and cubature. One illustration will suffice. In finding the area of a segment of a parabola, the chord AC is bisected at D, both DE and AF are drawn parallel to the axis of the parabola, CF is a tangent, and CH bisects AF. The point H is chosen so that HK = KC. It is then shown geometrically and by the law of the lever that if

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that if K be taken as a fulcrum, any line OM drawn parallel to DE
is balanced by the line OP, provided that this segment is placed with
its center of gravity at N, for by a property of the parabola,
\[ \frac{OM}{AC} = \frac{NK}{KN}, \]
This will be true for all positions of O on AC. Since
\[ \frac{OP}{AO} = \frac{KN}{NK}, \]
the triangle AFG consists of all such lines as OM, and since the seg-
ment APC consists of all such lines as OP, the triangle APC in its
present position will balance the segment APC when the latter is so
placed that its center of gravity is at N when the fulcrum is at K.
Now the center of gravity of the triangle APC is on the median KC
and at a distance from K equal to one-third KC, or one-third KN.
Therefore, according to the principle of the lever, the segment APC
equals one-third the triangle AFG, or four-thirds triangle ALC. In
this way, by a combination of geometry and mechanics, Archimedes deter-
mined the area of a parabolic segment. In proving this proposition he
resorted only to geometry.

Even this brief discussion of his method shows that Archimedes
anticipated Stevin, Kepler, Cavalieri, Roberval, Pascal, Wallis, and
Huygens by many centuries in his use of indivisibles. Whether he re-
garded the number of indivisibles, such as lines composing an area,
as infinite or merely very great is not clear. A plane figure, he said,
is made up of all the elements in it. Nor is it clear how tinged his
thought was with Democritean atomism.

By this ingenious method of balancing a plane figure of known area
and a plane figure of unknown area or a solid of known volume and area
against a solid of unknown volume and area, he developed formulas for
the surface of a sphere and of its segments, for the volumes of hyperboloids of revolution, for the segment of a spheroid, and for the area under a spiral. Such results are achieved today by integration. But the definite integral is the limiting sum of an infinite series, not the summation of a very great number of geometric magnitudes, such as lines. The revival of interest in infinitesimals and indivisibles in the fourteenth, fifteenth, sixteenth, and seventeenth centuries prepared the way for the coming of the calculus of Newton and Leibniz, but that Archimedes recognized that his mechanistic methods of investigation had no rigorous basis is shown by the fact that he never used them in his proofs. In his demonstrations he was a pure mathematician.

In the Quadrature of the Parabola, Archimedes proves his formula for the area of a parabolic segment. In the segment he inscribed a triangle, having the same vertex and base as the segment. Then within each of the smaller arcs he inscribed triangles having the same bases and vertices as the arcs. By continuing to double the number of sides of the inscribed polygon, he approximated more and more closely the area of the segment. The area of the nth polygon in this series is \( A \left( 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}} \right) \), where \( A \) is the area of the original isosceles triangle. To this he added the remainder \( \frac{1}{2} \left( \frac{1}{2^{n-1}} \right) A \) to form the equation \( A \left( 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}} + \frac{1}{2^{n-1}} \cdot \frac{1}{2^{n-1}} \right) = \frac{3}{2} A \). By increasing the number of sides he could make the remainder as small as he pleased, i. e., exhaust the area of the segment. He did not make the number of terms infinite and find the limit. Instead he used the double

\[ \text{[Footnote: HI, II, 85-91.]} \]
reductio ad absurdum, proving that since the sum of \( n \) terms of the series can be neither greater nor less than \( \frac{2}{3} A \), it must be equal to \( \frac{2}{3} A \).

In *Conoids and Spheroids* Archimedes uses the method of exhaustion and series to compute the volumes of solids produced by the revolution of parabolas, hyperbolas, and ellipses. Perhaps the simplest case is that of a paraboloid of revolution. About the solid he circumscribed a cylinder having the same axis, BC, as the paraboloidal segment.

Through the \( n \) points of equal division of BD he passed planes parallel to the base. Let \( h \) equal the length of each part. On these \( n \) sections of the paraboloid, he constructed inscribed and circumscribed frustra of cylinders. He then obtains the equivalent of the proportions

\[
\frac{\text{Cylinder } ACFE}{\text{Inscribed figure}} = \frac{n^2 h}{h \not\div 2h \not\div \cdots \not\div (n-1)h}
\]

and

\[
\frac{\text{Cylinder } ACFE}{\text{Circumscribed figure}} = \frac{n^2 h}{h \div 2h \div \cdots \div nh}
\]

Previously he had proved the lemma that \( h \not\div 2h \not\div \cdots \not\div (n-1)h < \frac{1}{3} n^2 h \).

Therefore, \( \frac{\text{Cylinder } ACFE}{\text{Inscribed figure}} > 2 \)

and \( \frac{\text{Cylinder } ACFE}{\text{Circumscribed figure}} < \frac{2}{1} \).

Finally, by the method of exhaustion and the double reductio ad absurdum,

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he showed that the volume of the cylinder $\text{AOSF}$ equals twice the volume of the paraboloidal segment. Thus, by his long, indirect, geometrical method Archimedes obtained the determination of $\int_0^a x\,dx$.

In drawing a tangent to the spiral which bears his name, Archimedes used a method which was to be revived and extended in the seventeenth century by Toricelli, Roberval, Descartes, and Barrow. Euclid defined a tangent to a circle as a line which touches the circle at only one point. Again Archimedes called on mechanics to aid him, resolving the velocity of the moving point which generates his spiral into its two components by using a parallelogram of velocities. The direction of the resultant velocity is the tangent. That this method resembles a differentiation is evident, but there are these differences. Although Archimedes used it in drawing a tangent to one curve, he did not develop it into a general technique of drawing tangents to all curves. Nor apparently does he think of it as the instantaneous rate of change of one variable with respect to another. Greek geometry was static, not dynamic. It emphasised form, not variability. The functional idea, which underlies analytic geometry and the calculus, is not present in Greek geometry. Finally, the Greeks did not think of the tangent as the limiting position of a variable secant.

Unable to learn from the Method how Archimedes had achieved his results, amazing alike in their ingenuity and laboriousness, Toricelli and Wallis believed that he and the other great Greeks had concealed under the cloak of synthetic proofs the analytic devices by which they had made

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their discoveries. The infinitely large, the infinitely small, and instantaneous velocity, concepts with which the seventeenth century played gracefully, were forbidden to the Greeks. Zeno's arguments had not been answered. Until they were, no one dared lay hands on the infinite. Aristotle had denied the existence of instantaneous velocity.

After the death of Archimedes, Greek mathematics dealt chiefly with applications of principles already discovered. Diophantus is an exception. In his *Arithmetic* the earlier rhetorical algebra was succeeded by a syncopated algebra, in which words were replaced by abbreviations. But he did not introduce the ideas of variable and function, concepts indispensable to the calculus.
CHAPTER III

THE MIDDLE AGES

Arithmetic interested the Hindus more than geometry. Incommensurables, a problem in Greek geometry from Pythagoras on, were not a problem to them, for they saw no difference between rectilinear and curvilinear figures. Both were numbers to them. The discovery of incommensurable quantities, since they were rigorously logical and since they did not invent irrational numbers, forced the Greeks to dissociate number from geometrical magnitudes. Having no Zeno to baffle them with paradoxes, the Hindus were not stymied by the apparent contradiction between the discreteness of number and the continuity of geometric figures. But this was not all loss. The Hindus recognized negative numbers and the irrational roots of quadratic equations.

From India came also the numerals, which are known today as Arabic numerals, the number zero, and the principle of positional notation. Arabic mathematics is a fusion of Greek and Hindu elements. To the Hindus the Arabs were indebted for their arithmetic and trigonometry. Arabic algebra and geometry showed the influence of the Greeks though the former is rhetorical rather than syncopated. Their chief importance is as the transmitters to Western Europe of much of the Greek work which otherwise have been lost. In the twelfth century Latin translations from Arabic manuscripts began to appear. Knowledge of Euclid had until then been derived from Boethius' <i>Geometry</i>, which omitted many of the proofs. The works of Archimedes had fared no better.
Leonardo of Pisa in his great work, the *Liber Abaci*, in 1202, advocated the adoption of the Arabic notation. For centuries it was the storehouse from which writers on algebra and arithmetic got their materials. In his *Practica Geometriae*, Leonardo treats with skill and rigor all the geometry and trigonometry transmitted to him. After Leonardo the long wars of the fourteenth and fifteenth centuries so absorbed the energies of the people that mathematics was almost stationary.

Until the thirteenth century Aristotle, "The philosopher" to the Middle Ages, had been known chiefly through his logical treatises. Then his scientific works were translated. Proscribed in Paris and other places for a time, the whole of Aristotle was soon studied and taught. As already noted, the infinitesimal, the infinite, and continuity were discussed in the *Physics*. In the fourteenth century these topics were debated by scholastic philosophers. Thomas Bradwardine, Archbishop of Canterbury, opposed the atomic view of continuous magnitude. He asserted that though a continuous magnitude includes an infinite number of indivisibles, it is not composed of them. The ultimate elements of continuous quantities are themselves continuous, not discrete.

Contrary to the dictum of Aristotle, William of Occam insisted that a line is actually made up of points. But points, lines, and surfaces are mental constructs. They have not the reality of solids. It must be remembered that these discussions are the disputations of the schoolmen. Their chief concern is the reality of indivisibles. They are not mathematicians engaged in working out a self-consistent postulational system. Indivisibles, in spite of their rejection today, did
influence Leibniz, one of the two independent inventors of the calculus. The contempt of some writers today for these early believers in the theory of indivisibles ignores the history of mathematics. Intuition is always an early stage in the history of a mathematical science; rigor is always a late stage. The rigorous foundation of the calculus today was the work neither of the seventeenth nor of the twentieth century. It was the labor of the nineteenth. Sufficient unto the day is the rigor thereof. If one is to be fair, he must judge the seventeenth century by the light which it had. Macaulay's schoolboy knows more about calculus today than great mathematicians knew three centuries ago. But the end is not yet. Time has not stopped. Inventions in mathematics have not ceased.

Interest in the infinite came naturally to the scholastic philosophers. They believed in a God, infinite in wisdom, power, and love. Their philosophy was a centuries-long attempt to reconcile reason and faith; and the "master of those who know" had spoken of infinity.

It was, therefore, a part of the canon of knowledge and of thought. The statement of Aristotle was that the infinite was potential, but not actual. Potential magnitude, he admitted in infinitely small, continuous magnitudes and in infinitely large numbers.

Toward the end of the thirteenth century there was a reaction against Aristotelianism. Aristotle had regarded motion as a quality, not a quantity. Probably because they considered number as discrete and geometric forms as continuous, the Greeks did not develop a science of dynamics. Greek astronomy conceived of the heavenly bodies as moving with uniform velocity in circular orbits. Aristotle had said that
mathematics studies objects which are continuous, physics studies objects which are moving, and philosophy studies objects which are becoming. Form rather than variability was the subject matter of Greek mathematicians.

In the fourteenth century appeared a large number of tractates dealing with latitude of forms. Form here refers to a quality, such as velocity; and latitude refers to a variation, either an increase or a decrease in that quality. Latitude of form was the degree, whether large or small, of a quality. Today we express changes in density, heat, and acceleration by using the symbols of algebra and the calculus, but the fourteenth century had to use the dialectic of scholasticism.

Richard Suiseth, usually known as the Calculator, discusses the latitude of forms in Liber Calculatiomn. The average intensity of a form which increases or decreases uniformly throughout a given interval, he asserts, is the mean of its maximum and minimum intensities. If the maximum and minimum intensities of a form approach and reach the mean at a uniform rate, the total intensity remains unchanged.

Being a scholastic philosopher and not a Greek mathematician, Suiseth was not balked by the paradoxes of Zeno and the consequent interdiction against resort to the infinite. In the Liber he frankly uses an infinite series. What is the average intensity for a given interval of time if a variation continues at a certain intensity throughout half of the given interval, at double this intensity throughout the next quarter of the interval, at three times this intensity throughout the next eighth of the interval, and so ad infinitum? He answers that the average intensity is one. This is equivalent to halving the sum of
the infinite series \( 1/2 + 2/4 + 3/8 + 4/16 + \ldots + n/2^n + \ldots = 2. \)

That Archimedes used series in problems of quadrature and cubature
has already been noted. But he did not use infinite series. He said
that by taking \( n \) sufficiently large the difference between \( 4/3 \) and the
sum of the series \( 1 + 1/4 + 1/16 + 1/64 + \ldots + 1/4n-1 \) could be made
as small as one pleased. Suiseth, though he lacked our concept of the
limit, did formulate a series of an infinite number of terms and,
summing it, arrived at the correct result. Really there are two
infinities in his series. Both the given interval of time and the
given intensity are infinitely subdivided. Had the Calculator been
primarily interested in the mathematics of his argument, the develop-
ment of the concepts of the calculus would have made a great advance.

Most famous of the fourteenth century philosophers who discussed
the latitude of forms was Nicole Oresme. In his arguments the Calcula-
tor had used only dialectic and arithmetic. Oresme employed geometrical
diagrams. The quality, or the time in case of time rates of change,
he represented by a horizontal line, the longitude; the intensity or
remission of the quality, i.e., the increase or decrease in the quality,
he represented by a vertical line or lines. This is not the first use
of coordinates, for Greek geographers had employed them. But it is
their first employment in studying rates of change. Nor can Oresme's
latitude and longitude be regarded as an early analytic geometry, for
it is not underlaid by the assumption that every algebraic equation
may be represented by a curve and that every curve has its equation.

Oresme illustrated the distance covered by a body which moves at a
constant velocity by a rectangle and the distance transversed by a body which starts from rest and accelerates uniformly by a right triangle. Graphically he shows that the distance moved over by a body starting from rest and increasing its velocity uniformly equals the distance it would have covered had its rate of movement been the mean between its initial and terminal velocities and had the interval of time been the same in the two cases. Here is the diagram.

![Diagram](image-url)

The distance transversed at uniform velocity is the rectangle ABCD. The distance transversed by the object starting from rest is the right triangle AEG. But these areas are equal, for, since E is given the midpoint of EG, the right triangles BCE and DEG are congruent, and the rectangle ADEF is common to the two figures. This may be the first time that the area under a curve was regarded as a physical quantity, but that practice is a commonplace in the calculus.

His study of the latitude of forms led Oresme, as it had Suiseth before him, to use infinite series. In fact, one of his series the Calculator had already employed. A body moves with uniform velocity for half a given interval of time, at double that during the next quarter sub-interval, and at triple the original velocity during the next eighth interval, and so ad infinitum. Oresme decides that the total distance transversed is four times that passed over in the first half of the interval.
Two other trends in the later Middle Ages, one of them mathematical and the other philosophic, were important in the history of the calculus. They were a revival of Archimedes and a revival of Plato and Pythagoras. Really there was a fusion of the idea of variation, which had been developed by dialectic and arithmetico in the treatises on latitude of forms, and the geometry of Archimedes. The revival of Platonism and of Pythagoreanism further freed the fifteenth century from the bondage to the finite inherited from the Greeks. Zeno by his then insoluble paradoxes and Aristotle by his explicit statement had forbidden the use of the two infinities, large and small. This liberation from the finite permitted mathematicians the free use of the infinitesimal and the infinitely large. For Nicholas of Cusa, as for Pythagoras, number was the basis of philosophy and mathematics offered the only rational interpretation of the universe. He emphasized the deductive character of mathematics, insisting that its teachings are to be tested by logic, not by their conformity to the evidence of the senses. Thus released, mathematics entered upon four centuries of free and easy invention and experiment. The rigor of a Euclid was partly forgotten. No one could rigorously define instantaneous rate of change, but it became the central concept of differential calculus. No one could rigorously define limit, but it became the fundamental idea of the calculus. Rigor was to have its day in the nineteenth century. During the preceding two hundred years, men of mathematics were fully employed in building the calculus.

Nicholas of Cusa defined the infinitely large as that which cannot be made larger and the infinitely small as that which cannot be made
smaller. The triangle and the circle he regarded as the polygons with
the smallest and the largest number of sides. Zero and infinity were to
him the upper and lower bounds of operations with finite numbers. Since
he thought that finite intelligence can approach truth only asymptotically,
he considered infinity as the unattainable goal of all knowledge. The
area of a circle, he argued, can be found by dividing it into an infinite
number of triangles and multiplying half the perimeter by the apothegm.
Michael Stifel followed Nicholas in classifying a circle as a polygon
of an infinite number of sides. But his influence is most clearly dis-
cernible in the writings of Kepler, who refers to Nicholas as the divine
Cusa. Nicholas was so busy with his duties as cardinal that he had little
time for mathematics. Kepler, though he studied divinity for a time,
was a mathematician. Both men showed the influence of Platonism and
Pythagoreanism in their ideas of the infinitesimal and the infinite.
CHAPTER IV

THE CENTURY BEFORE NEWTON AND LEIBNIZ

The sixteenth century witnessed important advances in algebra. The Hindus and Arabs did not clearly distinguish between rational and irrational numbers. Sixteenth-century mathematicians, following the Hindus and the Arabs, recognized irrationals as numbers, calling them surds as Leonardo of Pisa had; and following the Greeks, they interpreted irrationals as ratios of lines. Negative numbers, characterized as fictitious or false numbers, were adopted in this century. Even more important in the development of the calculus than these extensions of the number system was the introduction of symbols into algebra.

Francis Viète used consonants to represent known quantities and vowels to represent unknown quantities. Between arithmetic and algebra he drew this distinction: arithmetic studies computations with numbers; algebra studies calculations both with numbers and with letters, used as general symbols for numbers. Symbols are as important in mathematics as instruments in music. How important they are is shown by the fact that Englishmen, stubbornly clinging to the cumbersome notation which Newton used in his calculus, made no progress in analysis during the century following Newton while continental mathematics, employing the dy notation of Leibniz, pushed steadily ahead. Symbols are the tools of the mathematician. The more concrete a symbol, the more highly

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1Cajori, History of Mathematics, p. 139.
specialized its use. The cruder the tool, such as an awkward notation, the rougher the work. For the representation of variability and functionality, essential concepts in the calculus, mathematics required such general symbols as letters. Even the Arabic-Hindu numerals, excellent as the Middle Ages saw they were, are too specific to express a general idea like functionality. Symbols are half of mathematics.

Now the sixteenth century had a symbolic algebra, lineal descendant of the rhetorical algebra of the earlier Greeks and of the syncopated algebra of Diophantus; the philosophical, and at times mathematical, speculations of medieval schoolmen on variability and on the nature of the infinite, the infinitesimal, and the continuous; and a series of printed editions of the works of Archimedes, showing his astounding accuracy, though infinitely tedious and laborious, results in computing the areas and volumes of figures bounded by curved lines and curved surfaces. So equipped mathematicians stepped up their rate of progress. After a hundred years of necessary preparation came the calculus, and with it came inevitably modern mathematics and modern science. Without the calculus there would be neither modern mathematics nor modern science.

Since the calculus is a method, the preparation for its advent consisted of a series of improvements in the methodology of mathematics. Omitting the reductio ad absurdum of Archimedes, Simon Steven furnished only direct proofs of the method of exhaustion. Nor did he use both circumscribed and inscribed figures, considering the use of either sufficient as does the calculus. To prove that the center of gravity of a triangle ABC lies on the median AD, he inscribed a number of
parallelograms. The center of gravity of the inscribed figure lies on the median because the bilaterally symmetrical figures of which it consists are in equilibrium. The centers of gravity of all such inscribed parallelograms lie on the median, and their number can be increased to infinity. The greater the number of inscribed parallelograms, the smaller the difference between the inscribed figure and the triangle ABC. If the triangles ABD and BCD are unequal in weight, they differ by a fixed amount. But since the parallelograms inscribed within these triangles are equal, and since the difference between each of these triangles and the sum of the parallelograms inscribed within it can be made less than the given, fixed difference between the triangles ABD and BCD, they cannot be unequal. Therefore, they are equal, and the center of gravity of ABC lies on the median. Archimedes used only n inscribed figures; Stevin lets the number go to a potential infinity. Archimedes says that the difference between the original figure and the sum of the inscribed figures can be made as small as one made as small as one pleases; Stevin, that the difference can be made less than any given quantity. With further arithmetization of his concepts and with greater precision in terminology, this procedure of Stevin will be the method of limits.

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Similarly Stevin proved propositions on the parabolic and the conoidal segment. But from the point of view of the calculus, the most interesting thing he did was to find the pressure of water against a dam. The mathematical demonstration that the average pressure against the dam equals the pressure at its midpoint he supplemented with a proof of numbers. The surface of the dam, which he assumes to be one square foot, he divides into four rectangular strips of equal area. Then he rotates the strips so that they lie in a horizontal plane. If the first strip is at the top of the dam, the pressure of water against it will be zero. If it is one-fourth of a foot below the top of the dam, the pressure will be one-sixteenth of the weight of a cubic foot of water. The average pressure against the first strip lies, therefore, between 0 and one-sixteenth. Similarly, he computes the average pressure against the second strip to lie between $1/16$ and $2/16$; that against the third, between $2/16$ and $3/16$; and that against the fourth, between $3/16$ and $4/16$. Hence, the total pressure against the dam is between $6/16$ and $10/16$ of the weight of a cubic foot of water. Increasing the numbers of strips, he shows that the smaller sequence tends toward $1/2 - 1/2n$ and that the larger sequence tends toward $1/2 + 1/2n$. Hence the two sequences tend toward $1/2$ as $n$ is made sufficiently large. The pressure against the dam equals the weight of a prism of water of base one square foot and of height one-half foot.

Had Stevin used only one sequence, and that one of an infinite number of terms with a limit one-half, he would have employed the method

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of the calculus. But geometry, the apotheosis of Greek mathematics, rather than arithmetic was first in the thought of the sixteenth and seventeenth centuries. The availability and prestige of Archimedes saw to that. Out of geometric conceptions came the calculus, which today is based on arithmetic, on the numbers which are the terms in an infinite sequence or series.

Luca Valerio generalized Archimedes' method of computing areas under a curve by showing that if the distance between points on the curve which are on opposite sides of the diameter can be made to vanish, the difference between the areas of the inscribed and circumscribed rectangles can be made smaller than any given area. By making the number of rectangles sufficiently large, the excess of the circumscribed figure over the inscribed figure, the excess being the area of the largest circumscribed rectangle, becomes less than any given area. If the difference between the areas of the inscribed and circumscribed figures is less than any given area, the difference between the area of either of the polygons and that under the curve can be made smaller than any given quantity. Valerio did not, however, say that the area of the curve is the limit of that of either polygon as the number of rectangles increases indefinitely.

Educated for the Lutheran ministry, Johann Kepler became a teacher of mathematics. His reverence for Cardinal Nicholas of Cusa has already been noted. From his divinity training, his profound respect for Cusa, and his deep interest in Platonism and Pythagoreanism stems Kepler's strain of mysticism. His speculations on the nature of the
infinite and his belief that the universe is an ordered mathematical
harmony fuse in his work with Archimedean exactness.

Kepler's interest in the determination of the volumes of solids
of revolution arose out of the crude methods of estimating the contents
of wine casks. The result was the Nova Stereometria, which certainly
is one bit of mathematical progress that grew out of the social context.
The work begins with the quadrature of the circle. Regarding the circle
as a polygon with an infinite number of sides, he found its area by
summing the areas of the infinitesimal triangles which have their bases
in the circumference and their center at the origin. That summation
equals one-half the radius times the circumference. He found the area
of a sphere by summing the infinitesimal cones, infinite in number, which
have their bases on the surface of the sphere and their vertices at its
center. The cone and the cylinder were composed, he thought, either of
infinitely numerous, circular laminae or of an infinity of wedges radiating
from the axis. By infinitesimal methods he calculated their volumes.
Similarly he computed the volumes of ninety-two solids not treated by
Archimedes.4

The difference between curvilinear and rectilinear figures,
which had greatly troubled the Greeks was no handicap to Kepler. He
accepted the curve as the symbol of God and the straight line as the
symbol of his creatures. He saw no difference between a polygon and a
circle. In fact, he defined a circle as a special case of the polygon.

4A. Wolf, A History of Science, Technology, and Philosophy in
16th and 17th Centuries, p. 204–205.
In his search for the best proportions for a wine barrel, Kepler listed the volumes for given sets of dimensions. His tables showed him an important thing: the change in volume for a given change in dimensions decreased as the maximum volume was approached. Centuries earlier Oresme had noted that the rate of change of a semi-circle is least at its maximum. Oresme and Kepler did not, however, approach the problem of maxima from the same angle. A Scholastic philosopher, Oresme was interested in the graphical representation of rates of change. Kepler approached the problem through numerical data on the increases or decreases in volume as the maximum volume was neared.

Before discussing the indivisibles of Cavalieri, it will be necessary to study his teacher, Galileo. Both Galileo and Cavalieri were influenced by Plato, Archimedes, and the Scholastics as was Kepler. Calculator had shown that the average velocity of a body moving with uniform acceleration is its velocity at the mid-point of the given time interval. Oresme had graphically represented the distance covered by the area under the line which represented the velocity. Galileo’s diagram is very similar to that of Oresme.

Let $AB$ be the distance covered by a body moving with uniform acceleration in the time $CD$, and let $EC$ be the final velocity. The parallels to $EC$ are the successive speeds of the body. They may also be representations of the increments of distance covered. The area of the right triangle $CDE$, which represents the distance traversed by a body which starts from rest and accelerates uniformly throughout the given time interval, is composed of all lines parallel to $EC$ and terminated by $DE$ and $CD$. The area of the rectangle $CDHG$ represents the distance
covered by a body which moves throughout the given time interval at
a velocity equal to the mean of those initial and final velocities.
The area of the rectangle is the sum of all lines parallel to EC and
terminated by CD and CH. But since F and G are taken as the mid-points
of BD and CE respectively, the areas of the triangle and rectangle are
equal. Thus, Galileo proves that the distance covered is the same
under these two conditions. This type of proof Cavalieri was to
develop into the method of indivisibles.

The similarity between the proofs of Oresme and Galileo is
apparent. They differ in that Galileo makes explicit the implicit
assumption of Oresme, namely, that both the triangle and the rectangle
are made up of the parallel lines which illustrate the incremental
distances traversed in instantaneous times. It will be noted that
these parallels now represent velocities; and now, increments of dis-
tance.

In the Two New Sciences Galileo discusses that trio of mathe-
matical concepts, dear to the later Scholastics, the infinite, the
the infinitesimal, and the continuum. To Calculator's statement that a finite quantity can have no ratio to an infinite one, he added that larger, smaller, and equal cannot be used in describing the relative sizes of two infinite magnitudes, or of a finite quantity and an infinite one. Turning away from Aristotle, he followed Plato in thinking of the infinite, not as magnitude, but as multitude or aggregation. Furthermore, he indicated that a part of an infinite set can be put in one-one correspondence with the whole of the set, for example, the perfect squares can be put into one-to-one correspondence with the infinite class of all the positive integers.

Believing that the infinite and the infinitesimal are intuitively incomprehensible to us, he tried to bridge the gap between them, suggesting that there might be an intermediate aggregate between the finite and the infinite. Continuous magnitudes, he said, were composed of indivisibles. Being an aggregate of an infinite number of indivisibles, the continuous resembles a fluid more than a very fine powder.

Galileo regarded rest as infinitely slow motion. In ascending, a body passes through an infinite number of grades of slowness, each of smaller degree than any that has preceded it, until it finally comes to rest. When an object starts to move, it passes through a like number of stages, but their order is reversed. Galileo seems not to have realized that only the limit concept can give meaning to the last term and the sum of an infinite series.

The indivisibles which Galileo had used in explaining his discoveries in physics Cavalieri made the basis of his mathematics. But he did not define them. He thought of a surface as composed of an indefinite
number of parallel lines, and of a solid as composed of an indefinite number of parallel planes. How great the number of such lines and planes he did not say. Galileo had spoken confidently of the nature of infinity. On this point Cavalieri was an agnostic. He centered his attention upon the correspondence between the constituent indivisibles of two geometric figures, not upon the aggregate of indivisibles in each.

An illustration of this method. A parallelogram is double either into which a diagonal divides the parallelogram. Take CEZAF. Draw FH

\[\text{and } \overline{CE} \parallel \overline{AB}. \text{ In the congruent triangles } \triangle APH \text{ and } \triangle CGE, \overline{FH} = \overline{GE}.\]

Similarly all the lines drawn through the points of \(\overline{AD}\) and \(\overline{BC}\), equally distant from \(A\) and \(B\), are equal; and the triangles which are the totality of these lines are equal. The parallelogram is, therefore, twice either triangle. Another of his propositions is that the sum of the squares of the lines which make up a parallelogram is three times the sum of the squares of the lines which compose either of its constituent triangles. This theorem enabled Cavalieri to prove that the volume of a cone is one-third that of the circumscribed cylinder, and that the area of a parabolic
segment is two-thirds that of the circumscribed triangle. Archimedes had proved these last propositions by the much more tedious method of exhaustion.

Continuing this work on the indivisibles (lines) of a parallelogram and those of each of its component triangles, he found that the cubes of the lines of the former is four times the cubes of those of the latter, and that the fourth power of the lines of the first is five times the fourth power of those of the second. By analogy he concluded that the ratio for the fifth powers would be six to one, and, finally, that the ratio for the n-th powers would be $n^5/1$ to 1. These results would today be written as

Open to the criticism that it does not give a logical or clear definition of indivisibles, Cavalieri's method has a generality that made it broadly applicable and very fruitful. Even today his influence has not vanished. His theorem still appears in solid geometry textbooks: If two solids have equal altitudes, and if sections made by planes parallel to the bases and at equal distances from them are always in a given ratio, the volumes of the solids are also in this ratio.

Paul Guldin, who was also a Jesuit, criticized the method of indivisibles. To the contention that since lines lacked breadth, one could not by summing them produce an area, Cavalieri replied that one could substitute for them small elements of area. Similarly in a problem involving solids, he could replace the indivisibles by small elements of volume. To the criticism that one could not compare two infinites of indivisibles, Cavalieri declared that he had used the
method only when the figures compared had equal altitudes and that it
was permissible to assume that the number of lines or sections and also
the number of indivisibles between lines or sections were the same in
the two cases. In order to make the concept clearer, Cavalieri compared
the indivisibles of a surface to the threads of a piece of cloth, and
the indivisibles of a solid to the pages of a book. Though he did not
elaborate the idea, he regarded a line as generated by a moving point,
a surface as generated by a moving line. The important thing here is
that he had a dynamic conception of an indivisible. A flowing infinitesimal
links him with Newton and Leibniz. Newton based his calculus on flowing
quantities; fluxions he called them. The differential is basic in the
calculus of Leibniz. Cauchy later made the derivative the central con-
cept.

Evangelista Toricelli, friend of Cavalieri and pupil of Galileo,
believed that the ancients had some method similar to indivisibles,
but that they concealed it by casting their proofs in another form.
Heiberg's discovery and publication of Archimedes' Method show that he
was right. That Toricelli was not wholly satisfied by the method of
indivisibles is evident from his proving ten of the twenty-one proposi-
tions on the quadrature of the parabola in his De Dimensione Parabolae
by the method of Greek geometry, and eleven by the use of indivisibles.
He made explicit the implied assumption of Valerio, namely, that since
the difference between the arcs of the parallelograms circumscribed and
inscribed to a parabolic segment can be made less than any given area,
the difference between the area of the parabolic segment and that of
either of the polygons can be made less than any given area.
By indivisibles he proved that the volume of a solid generated by the revolution of a portion of an equilateral hyperbola about its asymptote is finite. Oresme had previously shown that a figure could have one infinite dimension and still be finite.

Let ED be a fixed horizontal line, and NL be any parallel to ED. Resolve the hyperbola about AB. Cavalieri then shows that the lateral area of the cylinder NLIO equals the area of the section IM of the right cylinder ACON with altitude AG and diameter AH. But the lateral area of the infinitely long solid of revolution FEBDC is made up of all such lateral areas as NLIO, and the cylinder ACON is made up of all such circular sections as IM. Therefore, FEBDC is finite. This is an extension of the work of Cavalieri, who had used only plane indivisibles. The similarities and the differences between determining volumes by indivisibles and by the calculus are obvious. In the calculus the circular and cylindrical
elements have a thickness which approaches zero as their number increases without bound and the limit of their sum is determined.

Galileo had incorporated in his dynamics the fact, known since the fourteenth century, that the motion of a freely falling body is uniformly accelerated. Calculator and Oresme had shown that the distance through which the object falls equals the distance traversed by a body that moves throughout the given interval of time at half its final velocity. Galileo asserted that the velocity varied as the square of the time. Toricelli saw that this would mean that the distance covered would equal the sum of the squares of the lines in the triangle ABE.

Since Cavalieri had proved that the sum of the squares of the lines in the parallelogram ABHE is three times the sum of the squares of the lines in the triangle ABE, the body would have traversed three times the distance covered if it had moved throughout the time-interval AB at its final velocity.

These considerations lead Toricelli to a method of determining tangents. Let ABC be a segment of a cubical parabola, and BE a tangent to it. If the arc is regarded as the path of a projectile which moves
so that it has a uniform horizontal velocity and a vertical velocity which varies as the square of the time, \( ED = 2AD \). The instantaneous direction \( BE \) has \( ED \) and \( ED \) as its components directions. \( BE \) is, therefore, the tangent. By determining the value of \( \frac{ED}{AD} \), one ascertain the degree of the parabola. Toricelli knew that he had achieved the same sort of results by the method of indivisibles that the ancients had secured by the method of exhaustion, but he did not know that the two procedures could be logically connected by the introduction of the limit concept into both.

Gregory of St. Vincent carried to infinity the subdividing of areas which he was seeking to determine. He used infinitely many, infinitely thin rectangles instead of the parallelograms of Stevin. Following Nicholas of Cusa, he inscribed a polygon of an infinite number of sides instead of the \( n \)-sided polygon of Archimedes. He thought of indivisibles as having a thickness which varied inversely as their number. Thus, he was able literally to exhaust the area or volume of a figure as the Greeks had never been able to do even though we describe their procedure as the method of exhaustion. Obviously the area of a parabolic segment is not exhausted, using the word literally, by inscribing an \( n \)-sided polygon. He knew that an infinite series has a limit. "The terminus of a progression is the end of the series to which the progression does not attain, even if continued to infinity, but to which it can approach more closely than by any given interval."\(^5\)

Assuming that motion is a quantity, Gregory calculated by geometric

progressions the points at which Achilles would overtake the tortoise. But he did not answer the central inquiry of the paradox: How can Achilles catch up with the tortoise?

Instead of using the word indivisibles, Andre Tacquet employed the term homogenea, explaining that they are elements of like dimension. He rejected the idea that plane figures are composed of lines and solids of sections. Nor did he believe that a sphere is generated by the revolution of a circle about a diameter. He regarded a plane figure, such as a segment of a parabola, as being composed of other and much smaller plane figures, such as rectangles of minute breadth. Like his teacher, Gregory of St. Vincent, he employed geometrical progressions in his attempt to resolve the paradox of Achilles, showing in another connection that in an infinite progression with \( r \) less than one the last term vanishes. Thus, step by step mathematicians approach the notion of a limit, without which there can be no calculus.

Giles Persone de Roberval also differed from Cavalieri in his interpretation of indivisibles. He did not regard the indivisibles of areas and solids as lines and surfaces respectively, but as elemental areas and elemental solids. In the Traité des Indivisibles he says that by the points which constitute a line he means the infinitely small segments which compose the line, and that by the lines which make up a surface he refers to the infinitely small elements of a surface which taken together are the whole surface. For the finite number of terms in the method of exhaustion which he had attentively studied in the works of his "divine Archimedes," Roberval substituted an infinite number of
terms. As he divided a figure into sections, he increased their number and decreased their size, getting the area or volume by summing the terms of an infinite series. Cavalieri had regarded indivisibles as elements of fixed size.

One illustration of his method will suffice. Let the number of indivisibles in each of the legs of an isosceles right triangle be four, then the area of the triangle is \(\frac{1}{2} (4)^2 \neq \frac{1}{2} (4) = 10\). If the number of indivisibles in each leg is five, the area is \(\frac{1}{2} (5)^2 \neq \frac{1}{2} (5) = 15\). The second term in each leg number is one-half one of the equal sides, the amount by which Roberval considered the area of the triangle to exceed the area of half a square of the same base and altitude as the triangle. As the number of indivisibles in a leg increases, the value of the second term in comparison with the first term decreases correspondingly. The second term may, therefore, be disregarded when the number of indivisibles goes to infinity. This proves that the area of a triangle equals one-half its base times its altitude. By similar methods Roberval calculated the areas under the parabola, the hyperbola, the cycloid, and the sine curve. Roberval was also ingenious in the invention of indivisibles of various types, such as triangles, parallelograms, cylinders, and concentric cylindrical shells. That the idea of limits is implied in these demonstrations appears in the proof of the area of a curve to which polygons have been inscribed and circumscribed. The area under the curve is less than that of the circumscribed figure and greater than that of the inscribed, which can be made to differ by less than any given amount. The ratio of the area under the curve to that of the circumscribed
figure can be made less than a given ratio, and the ratio of the area under the curve to that of the inscribed figure can be made greater than the given ratio. "If there is a true ratio $R:S$ and two quantities $A$ and $B$ such that for a small (quantity) added to $A$, then this sum has to $B$ a greater ratio than $R:S$ and for a small (quantity) subtracted from $A$, the remainder has to $B$ a ratio less than $R:S$, then I say that $A:B$ as $R:S$." This is equivalent to saying that the limit of the quotient of two variables equals the quotient of their limits.

Roberval was also concerned with the problem of drawing a tangent to a curve. He considered a curve as the path of a moving point, and the tangent as the instantaneous direction of the tracing point at the point of tangency. Considering the motion of a point as composed of two component movements, he regarded its instantaneous direction as the tangent, that is, as the resultant of its two components. Since a parabola is the locus of a point equidistant from the directrix and the focus, its tangent is the resultant of equal forces operating away from the focus and from the directrix. The bisector of the angle formed at a point by the focal radius and the perpendicular to the directrix is the tangent. The difficulty inherent in this method is obvious. One must discover the quantitative relationship between the forces acting upon the point before he can draw the tangent.

Through his close friendship with Étienne Pascal, Roberval may have influenced the brilliant son, Édouard Pascal. His two periods devoted to geometry were separated by an interval during which Édouard Pascal devoted

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himself to theology. This fact is important in understanding the mysticism with which he explains indivisibles.

His dominant mathematical interests, geometry and the theory of numbers, unite in his triangle, 7

\[
\begin{align*}
1 & \\
1 & 1 & 1 \\
1 & 2 & 3 & 4 \\
1 & 3 & 6 & 10 & 15 \\
1 & 4 & 10 & 20 & 35 & 56 \\
1 & 5 & 15 & 35 & 70 & 126 & 210 \\
\end{align*}
\]

The numbers in the first row or column, representing indivisibles in a line, give the arithmetical triangle a geometric interpretation. The numbers in the second row, the sums of those in the first, are lines. Likewise the numbers in the third row, the sums of those in the second, are triangles. Similarly the numbers in the fourth row denote pyramids.

One of the characteristics of Pascal’s computation with indivisibles in his dropping of terms of lower order because of their relative insignificance in comparison with terms of higher order. Pascal compared the infinitesimals of geometry with the zero of arithmetic. Seeking further to clarify its nature and magnitude, he said that the indivisible is to a geometric figure as the justice of man is to the justice of God. Though admitting the potential existence of the infinitely great, Aristotle had denied the existence of the infinitely small. Pascal maintained that an infinitely great number implies an infinitely small number, for the reciprocal of a large number is a small number. 7

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time and space, number is subject to the two infinities of greatness and of smallness. The two infinities are not given to man to understand but to admire. They are of the order of mysteries.

When Leibniz said that Pascal seemed sometimes to have a bandage over his eyes, he may have been thinking of the nearness with which Pascal came to the differential triangle.

In determining the area under a portion of a curve, Pascal observed that AD is to DI as EK is to RK, and that for small intervals the arc may be substituted for the tangent. Had he been more interested in tangents and less occupied in determining an area by the method of indivisibles, he might have discovered the importance of the limit of a quotient both for the drawing of tangents and for quadratures. But he was far more interested in classical than in analytic geometry.

A friend of Pascal's, Pierre Fermat, perhaps the greatest French mathematician of the seventeenth century, is one of the two inventors of analytic geometry, Descartes being the other. Viète had discovered that geometrical problems yield more readily to solution if they are reduced to algebraic equations. Oresme, two centuries earlier, had represented variables by graphs. Fermat and Descartes went beyond these

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5Tbid., pp. 339-402.
anticipations in associating with every curve an equation which implies all its properties. Fermat called the equation the "specific property" of the curve.

Fermat also studied maxima and minima. If a line a units long is divided by the point P into two segments of length x and a - x, there are generally two different values of the variables x and a - x which will give the rectangle that has these segments as its dimensions an area of A. But in the case of the maximum area, P must be the mid-point and the rectangle, accordingly, a square. These considerations lead Fermat to his process of determining maxima and minima. As above, let P divide the line into the segments x and a - x. The area of the rectangle which has these dimensions is: \( A = x(a-x) \). Increase the first dimension by the small amount \( E \), and decrease the second by the same quantity. The area of the latter rectangle is: \( A = (x + E)(a-x-E) \). If the area of the rectangles is a maximum, the right members of the two equations are equal. If they are equated, all terms containing \( E \), an arithmetic infinitesimal, vanish, and \( x = \frac{a}{2} \). Up to this time indivisible has meant an infinitely small geometric magnitude.

From now on it comes to mean an infinitely small number. Perhaps the preeminence of geometry for over two thousand years was due to its concreteness, to its pictures. With certain changes this is the method employed today. Instead of using \( E \) to represent the small change in the variable, mathematics now uses \( \Delta x \). And the limit of \( \Delta x \) as \( x \to 0 \) is introduced. It seems that Fermat was thinking, not of limits, but of

\[ ^9 \text{Ibid., pp. 610-612.} \]
two expressions that were almost equal when $E$ differed from zero, but which were equal when $E$ vanished.

In quadratures Fermat divided the area under the curve into small rectangles, indefinitely increased their number, and then summed the infinitesimal areas. Furthermore, he seems partly to have realized that the problems of drawing a tangent and of determining an area under a curve are inverses.

Rene Descartes was an acrimonious critic of Fermat's work. This quarrel, largely one sided, maintained his interest in tangents, not only "the most useful and general problem that I know but even that I have ever desired to know in geometry." His method of drawing a tangent to a curve was to pass a circle, with its center on the $x$-axis, through the given point and through some other point on the curve and to let the second point coincide with the given point. The center of the circle, which is the intersection of the $x$-axis and the normal to the curve at the given point, enabled him to draw the tangent. In his work on tangents Descartes used the conceptions of algebra; Fermat employed infinitesimals.

In England John Wallis, mathematician and theologian, applied the analytic geometry of Descartes and Fermat to the quadrature of conic sections. Thinking his goal easily attainable, he attempted the complete liberation of arithmetic from geometry. This transition from the geometric infinitesimal to the arithmetic indivisible is well illustrated by his proof that the area of a triangle equals one-half the product of its base by its altitude. He assumed that the triangle is made
up of an infinite number of parallelograms, which have altitudes equal to \( \frac{1}{\infty} \). This is the first use of the symbol \( \infty \) for infinity. Fermat had merely implied that \( E \) is an infinitesimal; Wallis explicitly states that \( \frac{1}{\infty} \) is infinitely small. Though a parallelogram with altitude \( \frac{1}{\infty} \) is scarcely more than a line, an infinite multiplication of them yields the triangle. The area of any one of these parallelograms is its altitude, \( \frac{1}{\infty} \cdot a \), where \( a \) is the altitude of the triangle, multiplied by its base. The areas of these parallelograms form an arithmetic progression \( 0, 1, 2, \ldots \). Now the sum of such a sequence equals one-half its last term times the number of terms. Therefore, the area of the last parallelogram equals \( \frac{1}{\infty} \cdot a \times b \), where \( b \) is the base of the triangle; and the area of the triangle is \( \frac{1}{\infty} \cdot a \times b \cdot \frac{\infty}{2} = \frac{1}{2} ab \).

For proof of his equation, Wallis cited the ancient and honorable method of inscribed and circumscribed polygons.

Since \( \frac{0}{1/2} = \frac{1}{2} \) and since \( \frac{0}{2/2} = \frac{1}{2} \), Wallis concluded that the ratio will be \( \frac{1}{2} \) whether the number of terms is finite or infinite.

He also noted that in the equalities

\[
\begin{align*}
0 \times \frac{1}{2} &= \frac{1}{2} - \frac{1}{3} = \frac{1}{6}; & 0 \times \frac{1}{2} \times \frac{1}{4} &= \frac{1}{12} = \frac{1}{3} \times \frac{1}{4}; & 0 \times \frac{1}{2} \times \frac{1}{4} \times \frac{1}{9} &= \frac{1}{36} = \frac{1}{3} \times \frac{1}{4} \times \frac{1}{9}
\end{align*}
\]

the greater the number of terms, the more closely the ratio approaches \( \frac{1}{3} \). "At length it (the sum) differs from it \( \frac{1}{3} \) by less than any assignable magnitude." If the number of terms becomes infinite, the difference "will be about to vanish completely." The ratio for an infinite number of terms is, therefore, \( \frac{1}{3} \). The ratio for the third powers he found to be \( \frac{1}{4} \); that for the fourth powers, \( \frac{1}{5} \); and that for the fifth powers, \( \frac{1}{6} \). He then extended the rule to apply to irrational as well as rational powers, thereby
breaking the Pythagorean geometry, which held that irrationals are magnitudes but not numbers.

Wallis applied his rule concerning the powers of numbers to the solution of cubatures and quadratures, virtually determining areas and volumes as the limits of infinite series. In this work Wallis went far toward the arithmetization of the method of indivisibles. Using the ancient process of inscribed and circumscribed polygons, James Gregory continued the arithmetizing of the determination of areas and volumes by employing converging infinite series.

In drawing tangents Gregory replaced the $E$ of Fermat by an $o$, Newton adopting the latter notation. The arithmetizing tendency of Fermat, Wallis, and Gregory was opposed by Thomas Hobbes and Isaac Barrow. Hobbes also objected to the application of algebra to geometry. Newton illustrates the seventeenth century preference for geometry by using it instead of his fluxional calculus in his demonstrations in the *Principia*. Hobbes had a further reason for his depreciation of algebra. He regarded mathematics as an idealization of sensory experience rather than as a branch of abstract formal logic. For him a line was not wholly without breadth, nor was a surface entirely devoid of thickness. The infinitely small was for him the smallest possible. He followed the Pythagoreans in considering a number as a collection of units.

For Aristotle's conception of motion as the fulfillment of the potential, which is a metaphysical rather than a mathematical view, fourteenth century philosophy had substituted the doctrine of impetus or inertia, the tendency of a body in motion to remain in motion. This
idea of impetus centered attention on the motion itself, rather than on change of position, and made the concept of motion at a point intelligible. Accordingly Calculator and Oresme made quantitative studies of instantaneous velocity. Toricelli, Roberval, and Descartes successfully applied the concept to geometry. Hobbes added the notion of the conatus as the beginning of motion corresponding to the point as the beginning of a line. Obviously Hobbes is emphasizing here motion at a point, not change of position. He did not realize as had Aristotle that instantaneous motion is purely an intellectual concept. Man can perceive motion through a distance, but not through a point. Under the influence of Hobbes, mathematicians neglected the attempt to base instantaneous velocity on number, preferring to ground it in intuition. This gave to the calculus a freedom and an inventiveness that it would otherwise have lacked.

Isaac Barrow also opposed analytic geometry and the arithmetization of mathematics. He felt that arithmetic belongs to geometry, that numbers have no independent existence. Believing that algebra is a branch of logic, he denied that it is a mathematical science. Barrow may have influenced his distinguished student, Sir Isaac Newton, to found his calculus on the continuous variation found in motion and geometry. He thought of time as a continuous magnitude, measurable by motion. In regard to the continuum and instantaneous velocity, he said: "To every instant of time, or indefinitely small particle of time, (I say instant or indefinite particle, for it makes no difference whether we suppose a line to be composed of points or of indefinitely small linelets; and so
in the same manner, whether we suppose time to be made up of instants of indefinitely minute intervals); to every instant of time, I say there corresponds some degree of velocity, which the moving body is considered to possess at the instant.10 Here Barrow is obviously thinking of infinitesimals, but his definition of them is not clear. His method of calculating the distance a body traverses in a given time at a given velocity is the summation of the ordinates. Barrow compares time to a line, saying that the former may be regarded as generated by the flowing of an instant and the latter as the path of a moving point, or they may both be considered as aggregates of instants and points respectively.

Perhaps Barrow understood more clearly than had his predecessors the inverse relationship of tangents and quadratures. But unfortunately he did not use the analytics of Descartes and Fermat. His preference for synthetic geometry and for indivisibles makes him a reactionary in some of his techniques.

The differential triangle, previously employed by Torricelli, Roberval, Pascal, and Fermat, appears also in the work of Barrow. Fermat had used only one infinitely small quantity $E$; Barrow used two $a$ and $e$, which are designated today as $\Delta y$ and $\Delta x$, respectively. Though Barrow acknowledges his indebtedness to Descartes, Huygens, Galileo, Cavalieri, Gregory of St. Vincent, James Gregory, and Wallis, he makes no reference to Fermat. But one must not think that the ideas back of his notation, $a$ and $e$, are those of $\Delta y$ and $\Delta x$. Barrow is concerned with the geometry of infinitesimals, rather than with functions and

continuous variables. Fermat invented analytic methods of procedure equivalent to integration and differentiation though he did not recognize the inverse relationship of the processes. Barrow apparently understood that the procedures are inverses, but he used synthetic instead of analytic methods.
CHAPTER V

NEWTON AND LEIBNIZ

This essay purposes to show that the calculus is both an invention and an evolution. Preceding chapters have traced the long, slow development of the fundamental ideas and of the techniques of drawing tangents, which is the central problem of differential calculus, and of determining the areas under curves, which is the central problem of integral calculus. The calculus neither began nor ended with Newton and Leibniz. They had predecessors, centuries of them, to whom they were indebted. They have already had two and one-half centuries of successors who have extended the method of the calculus, improved it, and given it a rigorously logical foundation. The history of a mathematical concept or process is like that of a great cathedral. Both grow slowly to meet the needs of those who use them. Both begin modestly and expand somewhat uncertainly. Both require constant work on the foundations and on the superstructure. A cathedral gets an adequate foundation earlier than a mathematical method, such as the calculus. In spite of the fact that mathematics is a rigorously logical science, its methods usually grow out of intuitive conceptions. For a time workers are so busy improving and extending the procedure that they neglect rigor. Eventually, with the hind sight of generations of progress, some men of genius who give the processes an adequate foundation in irrefragable logic and a synthesis.

Into the unfortunate controversy between the friends of Newton and
those of Leibniz as to who had priority in the invention of the calculus, this paper will not go. This is not the only such controversy in the history of mathematics though it is the most famous. It is a little surprising that men who practice a logical science should be so jealous, envious, and quarrelsome. It seems that the spirit of mathematics like the spirit of religion does not permeate the whole of the lives of those who profess it. Certainly mathematicians who know the history of their science, and all of them do to a greater or lesser degree, should thoroughly understand that all the credit for a discovery does not belong to one or two men. The glory belongs to all who work on the problem, whether predecessors, contemporaries, or successors. Not even a Newton works in a vacuum. Both he and his work are a part of the past, the present, and the future—events in a ceaseless flow of time, thought, and effort.

This quarrel over the calculus was long and bitter because of the greatness of the two protagonists and because the combatants on both sides were partly actuated by nationalistic feeling. The result of the controversy was lamentable. For a hundred years English mathematicians sealed themselves hermetically in their tight, little isle. For a century they held no intercourse of ideas with the men of mathematics in Europe. For three generations they stagnated, desiccated, angry because the continent would not acclaim their great Newton the inventor of the calculus. While Europe made great progress, England marked time.

At Cambridge, Isaac Newton was a student of Barrow, whose Geometrical Lectures he helped prepare for publication. He was, therefore, familiar with his professor's views that a geometrical magnitude may be generated either by indivisibles or by flowing quantities, and that a tangent may
be regarded either as the prolongation of one of the infinitely many lineal elements of which the curve may be assumed to be composed or as the instantaneous direction of the generating point. Newton acknowledged his indebtedness to Wallis, who considered a number not as a collection of units, but as the abstract ratio of one quantity to another. Wallis, it will be recalled, wished to arithmetize the whole of mathematics. In his use of infinite series, Newton was under further obligation to Wallis.

In 1665-66, Newton conceived the idea of his fluxionary calculus. In 1669, in his De Analysis Per Equations Modarum Terminorum Infinitas, published in 1711, he gave the first notice of his invention. In this monograph he determined first the rate of change in the area at the point in question and then found the area from this. In other words, he took the derivative first and then integrated it. This is the first explicit recognition of the inverse relationship between derivative and integral. For the E of Fermat, Newton substituted the letter O. Much more important than this substitution is Newton’s adoption of his method as a universal algorithm. Fermat and his adapters, Sluse and Huyse, had considered their process as available only in the case of rational algebraic functions. As Newton admitted, his method "is shortly explained rather than accurately demonstrated." For instance, he did not justify the dropping of powers of O. By the ordinate Newton seems to represent the velocity; by the abscissa, the time. The moment, or increment, of area equals the product of the ordinate times a small portion of the abscissa. The sum of the moments is the distance.

Newton made the notion of instantaneous motion the basis of his
Methodus Fluxionum Et Serierum Infinitarum, written about 1672 but not published until 1736. In this book he regards his variable quantities as generated by the continuous motion of points, lines, and planes, not as the sum of infinitesimal elements as he had in *De analysi*. Designating the rate of generation, which he called a fluxion, by a "pricked" letter, i.e., by a letter with a dot over it, he called the generated quantity a fluent. Calculator had used these terms for these ideas. If \( x \) is a fluent, \( \dot{x} \) is its fluxion. If the fluxion \( \dot{x} \) becomes a fluent, its fluxion is \( x \). If \( x \) is a fluxion, its fluent is \( \frac{1}{x} \). If \( \frac{1}{x} \) is a fluxion, \( x \) is its fluent.

Given the infinitely small interval of time \( 0 \) and the fluents \( x \) and \( y \), to determine the fluxion of \( y = x^n \). The moments, or indefinitely small increments, of \( x \) and \( y \) are \( x_0 \) and \( y_0 \). Substitute \( x \neq x_0 \) for \( x \) and \( y \neq y_0 \) for \( y \) in the equation, expand by the binomial theorem, cancel the terms which do not contain \( 0 \), and divide both members by \( 0 \). Neglect the terms containing \( 0 \), for \( 0 \) was assumed to be infinitely small. The result is \( y = mx^{n-1} \). That Newton felt a need for the limit concept is shown by his statement that fluxions are never considered alone, but always in ratios.

In *De Quadrature Curvarum*, written in 1676 but not published until 1704, he considers mathematical quantities not as made up of very small quantities, but as described by continuous motion. To determine the fluxion of \( x^n \), he replaced \( x \) by \( (x + \delta) \), expanded by the binomial theorem, subtracted \( x^n \). The remainder was the change in \( x^n \) corresponding to the

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change 0 in x. Instead of neglecting terms in 0, he wrote the ratio of the change in \( x^n \) as 1 to \( nx^{n-1} \neq n \frac{(\frac{dx}{x})}{2^n} \cdot nx^{n-2} \ldots \). Then he allowed 0 to approach zero, leaving the ultimate ratio, as he called it, 1 to \( mx^{n-1} \). Today the calculus uses the phrase, limiting ratio, instead of ultimate ratio, for there is no last ratio. The infinite sequence has a limit, but no last term. The limiting ratio is now regarded as a single number instead of the quotient of two rates of change. Nor is time any longer used as an auxiliary independent variable.

Becoming aware that Leibniz was working on similar problems, Newton wrote him, through Oldenburg, a letter, proposing, in an anagram, the fundamental problem of the calculus: "Given in an equation the fluxents of any number of quantities, to find the fluxions, and vice versa." He also stated his obligation to Wallis, James Gregory, and Sluse.

Principia Mathematica Philosophiae Naturalis, 1687, contains the first published account of his calculus. Though the problems in this book deal with velocity, acceleration, tangents, and curvatures, they are solved by synthetic geometry, not by calculus. Nor does Newton refer frequently to his new method. However, this famous book contains his clearest statement as to the nature of ultimate ratios. "Ultimate ratios in which quantities vanish, are not, strictly speaking, ratios of ultimate quantities, but limits to which the ratios of these quantities, decreasing without limit, approach, and which, though they come nearer than any given difference whatever, they can neither pass over nor attain before the quantities have diminished indefinitely."

\(^2\)Opera Omnia, I, 251.
In 1672 Gottfried Wilhelm von Leibniz met Huygens, who urged him to study mathematics. The next year he visited London, met a number of mathematicians, studied infinite series, and bought a copy of Barrow’s Lectures. Later that year he returned to Paris, where he studied the works of Cavalieri, Toricelli, Gregory of St. Vincent, Roberval, Pascal, Descartes, Wren, James Gregory, Sluse, and Hudde. To this broad reading in geometry, he added an earlier interest in arithmetic. Soon he was working on tangents and quadratures, using the differential triangle, which Toricelli, Fermat, and Barrow had previously employed. The inspiration for his use of the characteristic triangle, he wrote thirty years later, was a figure he found, in 1673, in Pascal’s Traité des Sujets du Quart de Cercle. Suddenly he realized that the tangent to a curve depends upon the ratio of the differences in the ordinate to those in the abscissa as these differences become infinitely small, and that the area under the curve is the sum of the rectangles which have ordinates and infinitely small portions of the abscissa as their dimension. He realized also the inverse relationship of these operations.

By 1675 Leibniz had developed a notion for the differences and the sum of the infinitely small changes in the ordinate and the abscissa. For the sum of all the x’s, or the integral as he called it later, he used \( x \) at first, but improved it later by employing \( \int \). For the differences in \( x \) he used \( I \) at first; later he invented the present notation \( \frac{d}{dx} \). The calculus has adopted this symbolism, preferring the d-ism of Leibniz to the dot-age of Newton.
Leibniz found the "difference" of $xy$ to be $x dy + y dx$.\(^3\) The "difference of the quotient $x$ by $y$" he found to be $\frac{\frac{dx}{y} - \frac{dy}{x}}{\frac{x^2}{y^2}}$. The "difference" of $x^n$ he determined to be $mx^{n-1}$, and its integral to be $\frac{x^{n+1}}{n+1}$. The facility gained by the calculus, whether the calculus of fluxions of Newton or the calculus of differences of Leibniz, to be appreciated should be compared with the longer and more laborious quadratures of Cavalieri, Toricelli, Roberval, Pascal, Fermat, and Wallis. The result of the summation by the calculus is no more accurate than that obtained, for example, by indivisibles, but it is much more quickly and easily obtained.

The calculus has the further advantage of being an universal algorithm. Elementary calculus today follows Newton and Leibniz in regarding differentiation as the fundamental operation, and integration as its inverse. The indefinite integral is sometimes referred to as the integral in the sense of Newton; and the definite integral, as the integral in the sense of Leibniz. This is because Newton regarded the fluent as the inverse of the fluxion and because Leibniz thought of the integral as the sum of infinitely many, infinitely small rectangles.

In 1684, three years earlier than Newton's first account, Leibniz published the first treatise on the calculus, limiting it to a crabbed exposition of the method of finding "differences." Two years later in another article in the same journal, Acta Eruditorum, he discussed quadratures.

From the first he realized that he was creating a new and universal method, applicable to integral, fractional, and irrational quantities.

He insisted that his method was as different from that of Archimedes as the geometry of Descartes was different from that of Euclid. In order to popularize it, he gave all the rules and explained them. He emphasized that "sums" and "differences" are inverses just as powers and roots are. Apparently regarding the calculus as only a mathematical method, Leibniz did not attempt a clarification of the concepts underlying it. He contented himself with making the mechanics clear.

But his contemporaries were not so easily satisfied as Leibniz. They wanted an exposition of the logical and philosophical bases of the new method. Leibniz regarded the differential, instead of the derivative, as fundamental. Although he disregarded differentials of higher order in his operations because he considered them as infinitely small in comparison with differentials of the first order, he thought that the number of orders was infinitely great. Unable to define a differential satisfactorily, he described it as infinitely small or, if one objected to the phrase infinitely small, as incomparably small. He affirmed that one could neither prove nor disprove the existence of infinitely small quantities. However, Leibniz realized that it was the ratio between two differentials which was important just as Newton used the ratio between two fluxions in his calculations.

Newton gradually abandoned infinitesimals for a somewhat vague concept of limits. Leibniz gave to the calculus its characteristic notation. Newton's ultimate ratios and Leibniz's differentials, whether infinitely or incomparably small, laid the calculus open to attacks, which were not long in coming.
CHAPTER VI

EIGHTEENTH AND NINETEENTH CENTURIES

Primarily interested in showing that religion is no more characterized by mysteries than is mathematics, George Berkeley published, in 1734, The Analyst; or a Discourse Addressed to an Infidel Mathematician (Newton's friend, Edmund Halley) therein It Is Examined Whether the Object, Principles, and Inferences of the Modern Analysis Are More Distinctly Conceived, or More Evidently Deduced, Than Religious Mysteries and Points of Faith. "First Cast the Beam Out of Thine Own Eye; and Then Shalt Thou See Clearly to Cast Out the Motu Out of Thy Brother's Eye." Berkely does not deny the utility of the new method or the validity of its results, but he does deny that it is based on logic and philosophy. He criticizes Newton's method of computing the moment of a product. Let $AB$ represent a rectangle, and let the dimensions $A$ and $B$ be decreased by $\frac{1}{2}a$ and $\frac{1}{2}b$. The diminished area will be $(A-\frac{1}{2}a)(B-\frac{1}{2}b) = AB - \frac{1}{2}AB - \frac{1}{2}BA + \frac{1}{2}ab$. Now let the dimensions be increased by $\frac{1}{2}a$ and $\frac{1}{2}b$. The area of the enlarged rectangle will be $AB + \frac{1}{2}AB + \frac{1}{2}BA + \frac{1}{2}ab$. Subtract the area of the smallest rectangle from that of the largest. The remainder, $\frac{1}{2}ab$, is the moment of the original rectangle, corresponding to the moments a $\frac{1}{2}b$ of $A$ and $B$. Berkeley argued that since $a$ is the increment of $A$ and $b$ the increment of $B$, Newton should have used $a$ instead of $\frac{1}{2}a$ as the moment of $A$ and $b$ instead of $\frac{1}{2}b$ as the moment of $B$ in computing the moment.

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1 David Eugene Smith, A Source Book in Mathematics, pp. 627-34.
of $AB$. Then the infinitely small quantity $ab$ would have been neglected. Berkeley cited, in this connection, a passage in *De Quadratura* in which Newton said that he neglected no quantity, however small.

He also criticized Newton's determination of the fluxion of $x^n$. Let $0$ be the increment of $x$, substitute $(x \neq 0)$ for $x$, expand $(x \neq 0)^n$ by the binomial theorem, subtract $x^n$ from the expansion, divide the remainder by $0$ to get the ratio of the increment of $x^n$ to that of $x$, let $0$ become evanescent, thus determining the ultimate ratios of the fluxions. Berkeley insists that in this process Newton has violated the law of contradiction by first assuming that $x$ has a moment and then, by letting it vanish, denying that it has a moment. Inasmuch as the terms "evanescent quantity" and "prime and ultimate ratio" had not been sufficiently defined, the argument is valid. An infinite sequence may have a limit, but it has no last term.

Berkeley pays his respects to the method of differentials, too. In finding a tangent by the method of differentials, one assumes increments, which determine the secant, not the tangent. Then one neglects the higher differentials. Thus "by virtue of a two fold mistake you arrive, though not at science, yet at truth." So far Berkeley has been posing problems which cannot be solved until the concepts of the calculus are given a rigorous formulation in the mid-nineteenth century. But he falls into error when he objects to certain of Newton's terms, such as fluxions and evanescent augments, as having no warrant in experience or intuition. Had Newton been able to give logical definitions

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of all his terms, they would not have been objectionable even if they had no validation in human experience. Berkeley argued that instantaneous velocity has no physical reality. That is true. But now that it has been given a logical definition, it is a perfectly good mathematical abstraction. Since mathematics deals with relations rather than with physical existence, its criterion of truth is inner consistency.

The criticisms of Berkeley did not receive a satisfactory answer for a century. Though the notion of a limit had developed gradually out of the Greek method of exhaustion and though Newton had included it in the Principia, it lacked logical formulation. This slow development of the limit concept was due to its being based on geometrical intuition. In the work of A.L. Cauchy the notion of limit became purely arithmetical instead of geometrical. He said: "When the successive values attributed to a variable approach indefinitely a fixed value so as to end by differing from it as little as one wishes, this last is called the limit of all the others." This definition is grounded upon the ideas of number, variable, and function. It contains no appeal to geometry.

Using this arithmetical definition of limit, Cauchy defines infinitesimal, which had eluded definition for centuries. Ever since the days of the Greeks it had been given a geometric interpretation, being regarded as a more or less fixed minimum of extension. The concept had rarely appeared in arithmetic because one was considered the smallest number; the fractions were defined as the ratios of two numbers. Since the union of algebra and geometry in analysis, the infinitesimal had been studied by both algebraists and geometers. Cauchy defines an
infinitesimal as a variable with zero as its limit. He defines the
infinite as a variable the successive values of which increase beyond
any given number.

Let \( y = f(x) \). To the variable \( x \) give the increment \( \Delta x = i \). Write
the ratio \( \frac{\Delta y}{\Delta x} = \frac{f(x + i) - f(x)}{i} \). If this ratio has a limit as \( i \)
approaches 0, it is the derivative of \( y \) with respect to \( x \), that is
\( f'(x) \). Leibniz had made the differential the fundamental concept;
Cauchy made the derivative the basic notion.

The idea of function, which had been developing during the eighteenth
century, came about this time to mean any relationship between variables.
Cauchy uses this new development in his definition of continuity. The
function \( f(x) \) is continuous within an interval if between these limits
to every infinitely small increment \( i \) in the variable \( x \) there corresponds
an infinitely small increment, \( f(x + i) - f(x) \), in the variable function.
Preoccupied with the inverse relationship between the problem of tangents
and that of quadratures, mathematicians of the eighteenth century had
neglected the definite integral. Cauchy based the definite integral on
the limit concept, defining it as the limit of a sum, not as a sum.

This conception of the definite integral made a study of the con-
vergence of infinite series imperative. Cauchy defined such a series
as convergent if for increasing values of \( n \), the sum \( S \) approaches in-
definitely a limit \( S \), the limit \( S \) is the sum of the series. Thus, Cauchy
gave to the concepts of the calculus their present general form, based
upon the limit concept.

Karl Weierstrass continued the arithmetization of analysis. In 1872
he read a paper which showed that a function which is continuous throughout
an interval may not have a derivative in that interval. It had previously been held that a continuous function necessarily has a tangent and, consequently, a derivative. This discrediting of geometrical intuition lead Weierstrass to establish the calculus upon the concept of number alone. He defined an irrational number such as $\sqrt{2}$, not as the limit of a sequence, such as $1, 1.4, 1.41, \ldots$ but as the aggregate of $1.q, q. B, 1 \ Y, \ldots$ where $q$ is the principal unit and $B, Y, \ldots$ are certain of its aliquot parts.

To a variable Weierstrass gave a static instead of a dynamic interpretation. Cauchy had spoken of a variable’s approaching a limit. Weierstrass defined a variable as a letter possessing any one of a set of numerical values. The latter statement avoids the vague notions of mobility implied by the former. He also made clearer the meaning of continuous function. $f(x)$ is continuous within an interval if for any $x_0$ within this interval and for an arbitrarily small value $\varepsilon$, it is possible to find an interval about $x_0$ such that for all values in the interval the absolute value of $f(x) - f(x_0)$ is less than $\varepsilon$. Similarly he defines the limit of a variable of function. The number $\Delta$ is the limit of the function $f(x)$ for $x = x_0$ if, given any arbitrarily small number $\varepsilon$, another number $\delta$ can be found such that for all values of $x$ differing from $x_0$ by less than $\delta$, the value of $f(x)$ will differ from that of $\Delta$ by less than $\varepsilon$. Since this famous statement contained no reference to fixed, infinitely small quantities, it effectively banished the fixed infinitesimal from the calculus. With the Weierstrassian static theory of a variable, the old, intuitive idea of a variable’s moving
consecutively through all the values in an interval was also abandoned. Dynamic variables had served their day, having prepared the way for the coming of the calculus and having served it faithfully during the first century of its life.

Richard Dedekind continued the search for a definition of irrational number which would be independent of the limit concept. He observed that the points of a line form a continuum, but the rational numbers do not. If one selects any point on a line, he divides the points of the line into two classes. All members of the first class are to the left of the point of division; all members of the second class are to the right. The points of a line can be put in one-to-one correspondence with the real numbers. If all the real numbers are now divided into two classes, A and B, such that all numbers whose squares are less than two are in A and all numbers whose squares are greater than two are in B, the cut defines 2. The real numbers have, therefore, been shown to be everywhere dense as are the points of a line. Number does not differ from a geometrical magnitude in that the former is discrete; and the latter, continuous. Both are continuous. Arithmetic became the basis of the calculus.

As his contribution to the theory of infinite aggregates, Dedekind said: "A system \( S \) is said to be infinite when it is similar to a proper part of itself; in the contrary case \( S \) is said to be a finite system."\(^3\) Since \( \infty \) had been used variously to represent the largest positive integer and the sum of the positive integers, George Cantor chose a new symbol \( \infty \) to represent an infinite aggregate instead of an infinite magnitude. He

added the idea of the power of an infinite set of elements. The rational numbers and the positive integers are of the same power because they can be put into one-to-one correspondence. But there are transfinite numbers higher than $\aleph_0$. 
CHAPTER VII

CONCLUSIONS

The thesis of this essay is that the calculus is both an evolution and an invention. As an invention its story can be quickly told. At the end of the seventeenth century, two men, an Englishman, Sir Isaac Newton, and a German, Gottfried Wilhelm Leibniz, apparently working independently, invented a new mathematical method—the calculus. After a time their friends and fellow nationals waged a ruthless, though fortunately bloodless, war to decide which of the two had first conceived the idea of the indisputably great algorithm. Like many other wars before and since, this war of pamphlets was indecisive. Upon the completion of the longest and bitterest battle in the belligerent history of mathematics, the mathematicians of Europe went busily about their proper task of extending and strengthening the procedures of their new subject. For over a century, until 1816, the British mathematicians sulked in their tents, fighting all over again the issues of the disastrous war.

As an evolution the calculus is an organism, living and growing in time. Its two chief problems, the drawing of a tangent at any point of a curve and the calculation of the area under a curve, had been subjects of study by mathematicians since the days of the great Greeks. Newton and Leibniz did not invent either of these problems. Certainly they were not the first to do creditable work on them. Considering the mathematical
tools, such as concepts, symbols, procedures, he had, no one has sur-
passed Archimedes, whose solutions are marvels of ingenuity, patience,
insight, imagination, and industry. Nor were the Middle Ages a thousand
years of darkness and ignorance mathematically. Oresme, Calculator,
and others made valuable contributions both to the conceptions and to
the techniques necessary in the solution of the two problems. The
-calculus neither began nor ended with Newton and Leibniz. Not until
the nineteenth century did the calculus receive rigorous formulation
and rigorous proofs. The concepts of infinity, limit, continuity,
variable, instantaneous rate of change, and function were not given
their first and final definition by Newton and Leibniz. Mathematicians
and philosophers of all centuries have considered and discussed some,
or all, of these ideas. The nature of the finite and of the infinite
and their relation to each other, problems posed by the paradoxes of
Zeno, have been subjects of debate for twenty-four centuries, perhaps
longer. Almost simultaneously two men invented the mathematical method
that is the calculus, but they neither invented nor perfected the con-
cepts which give that method meaning, significance, and rigor. Perhaps
mathematics is freer from the ceaseless ebb and flow of the human life
about it than any other art or science. But no mathematical concept or
method is static. No student could confuse the calculus of the seventeenth
century with that of the late nineteenth century.

The fixed infinitesimal that is neither finite nor zero, usually
given a geometrical interpretation, is a very important notion in the
history of the calculus. In his early discussions of fluxions, Newton
uses indivisibles though he relies more on limits in his later writings. Apparently Leibniz never abandoned infinitesimal considerations. These minimal quantities had existed more or less continuously since Democritus. They served the calculus faithfully for two centuries before they were banished because no one could give them a logical definition.

Like other branches of mathematics, the calculus is not a stable subject. What it will be in the twenty-first century is unpredictable. That it will change is certain. For twenty-two centuries Euclid was the final word on geometry. Today there are several non-Euclidean geometries. For three centuries the West knew but one algebra. Today there are very many algebras. In spite of the contrary belief of the ordinary man, mathematics is not the same yesterday, today, and forever.
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