The Feynman-Schwinger representation in QCD

Yu. A. Simonov\textsuperscript{a,b}

\textsuperscript{a} Research Center, ITEP, Moscow, Russia
\textsuperscript{b} Jefferson Laboratory, Newport News, VA 23606, USA

and

J. A. Tjon\textsuperscript{c,d,e}

\textsuperscript{c} ITP, University of Utrecht, 3584 CC Utrecht, The Netherlands
\textsuperscript{d} KVI, University of Groningen, 9747 AA Groningen, The Netherlands
\textsuperscript{e} Department of Physics, University of Maryland, College Park, MD 20742, USA

Abstract

The proper time path integral representation is derived explicitly for Green’s functions in QCD. After an introductory analysis of perturbative properties, the total gluonic field is separated in a rigorous way into a nonperturbative background and valence gluon part. For nonperturbative contributions the background perturbation theory is used systematically, yielding two types of expansions, illustrated by direct physical applications. As an application, we discuss the collinear singularities in the Feynman-Schwinger representation formalism. Moreover, the generalization to nonzero temperature is made and expressions for partition functions in perturbation theory and nonperturbative background are explicitly written down.
1 Introduction

The present stage of development of field theory in general and of QCD in particular requires the exploitation of nonperturbative methods in addition to summing up perturbative series. This calls for specific methods where the dependence on vacuum fields can be made simple and explicit. A good example is provided by the so-called Fock-Feynman-Schwinger representation (FSR) based on the Fock-Schwinger proper time and Feynman path integral formalism[1,2]. For QED asymptotic estimates the FSR was exploited in Ref. [3]. Later on this formalism was used in Ref. [4] for QCD and rederived in the framework of the stochastic background method in Ref. [5]. (For a review see also Ref. [6].)

More recently some modification of the FSR was suggested in Ref. [7]. The one-loop perturbative amplitudes are especially convenient for the FSR. These amplitudes were extensively studied in Ref. [8] and a convenient connection to the string formalism was found in Ref. [9]. Important practical applications especially for the effective action in QED and QCD are contained in Ref. [10]. Moreover, the first extension of FSR to nonzero temperature field theory was done in Refs. [11,12]. This forms the basis of a systematic study of the role of nonperturbative (NP) configurations in the temperature phase transitions[11,13].

One of the most important advantages of the FSR is that it allows to reduce physical amplitudes to weighted integrals of averaged Wilson loops. Thus the
fields (both perturbative and NP) enter only through Wilson loops. For the latter case one can apply the cluster expansion method[14], which allows to sum up a series of approximations directly in the exponent. As a result we can avoid the summation of Feynman diagrams to get the asymptotics of form factors[15]. The role of FSR in the treatment of NP effects is more crucial. In this case one can develop a powerful method of background perturbation theory[16] treating the NP fields as a background[17].

In the present paper some of these problems will be treated systematically and in full detail, yielding a overall picture of the role of FSR in QCD. We will in particular focus on the relationship between the standard perturbative expansion and the FSR based expansion which clarify the important role of nonperturbative configurations in the vacuum. The previous publication of the authors on FSR in Ref. [18] was devoted to QED and $\varphi^3, \varphi^4$ theories, and many basic formulas of FSR are already contained there. The later development of FSR in the framework of the field theory can be found in Ref. [19]. It has in particular been used successfully to reconstruct exact solutions of the one and two-particle Greens’ function for $\varphi^3$ theory and scalar QED in the quenched approximation[20–25]. A review of the FSR applications to perturbation theory in QCD and a discussion of the connection between FSR and world-line formalism of Refs. [8–10] can be found in Ref. [26].

In the next section we describe how to derive the FSR formalism for the case of QCD. In section 3 we discuss the relationship between the usual perturbative expansion and the FSR. Section 4 deals with the study of two ways to determine the Green’s function, depending on the physical situation. One consists of an expansion in the perturbative fields and the other one is treating the nonperturbative fields as a correction. As applications we address in sections
5 and 6 the problems of collinear singularities and the finite temperature field theory in the FSR formalism, while some concluding remarks are made in the last section.

2 General form of FSR in QCD

Let us consider a scalar particle $\varphi$ (e.g. a Higgs boson) interacting with a nonabelian vector potential $A$, where the Euclidean Lagrangian is given by

$$L_\varphi = \frac{1}{2} |D_\mu \varphi|^2 + \frac{1}{2} m^2 |\varphi|^2 \equiv \frac{1}{2} |(\partial_\mu - igA_\mu)\varphi|^2 + \frac{1}{2} m^2 |\varphi|^2, \quad (1)$$

Using the Fock–Schwinger proper time representation the two-point Green’s function of $\varphi$ can be written in the quenched approximation as

$$G(x, y) = (m^2 - D_{\mu, xy}^2)^{-1} = \langle x | P \int_0^{\infty} ds e^{-s(m^2 - D_{\mu, xy}^2)} | y \rangle. \quad (2)$$

To obtain the FSR for $G$ a second step is needed. As in Ref. [1] the matrix element in Eq. (2) can be rewritten in the form of a path integral, treating $s$ as the ordering parameter. Note the difference of the integral (2) from the case of the Abelian QED treated in Refs. [1,3,8]: $A_\mu$ in our case is a matrix operator $A_\mu(x) = A_\mu^a(x)T^a$. It does not commute for different $x$. Hence the ordering operator $P$ in Eq. (2). The precise meaning of $P$ becomes more clear in the final form of a path integral

$$G(x, y) = \int_0^\infty ds (Dz)_{xy} e^{-K} P \exp \left( ig \int_0^x A_\mu(z) dz_\mu \right), \quad (3)$$
where \( K = m^2 s + \frac{1}{4} \int_0^s d\tau \left( \frac{dz_\mu}{d\tau} \right)^2 \). In Eq. (3) the functional integration measure can be written as

\[
(Dz)_{xy} \sim \lim_{N \to \infty} \prod_{n=1}^N \int d^4 z(n) \int d^4 p \exp(i \sum_{n=1}^N z(n) - (x-y))
\]

(4)

with \( N \varepsilon = s \). The last integral in Eq. (4) ensures that the path \( z_\mu(\tau), 0 \leq \tau \leq s \), starts at \( z_\mu(0) = y_\mu \) and ends at \( z_\mu(s) = x_\mu \). The form of Eq. (3) is the same as in the case of QED except for the ordering operator \( P \) which provides a precise meaning to the integral of the noncommuting matrices \( A_{\mu_1}(z_1), A_{\mu_2}(z_2) \) etc. In the case of QCD the forms (3) and (4) were introduced in Refs. [4,5].

The FSR, corresponding to a description in terms of particle dynamics is equivalent to field theory, when all the vacuum polarisation contributions are also included[4,5,10], i.e.

\[
\sum_{N=0}^\infty \frac{1}{N!} \prod_{i=1}^N \int \frac{ds_i}{s_i} \int (Dz_i)_{xx} \exp(-K) \exp \left( \int \frac{x}{y} A_\mu(z) dz_\mu \right) = \int D\varphi \exp \left( -\int d^4 x L_\varphi(x) \right).
\]

(5)

Both sides are equal to vacuum-vacuum transition amplitude in the presence of the external nonabelian vector field and hence to each other. For practical calculations proper regularization of the above equation has to be done. The field \( A_\mu \) in Eq. (1) can be considered as a classical external field or as a quantum one. In the latter case the Green’s functions \( \langle A..A \rangle \) induce nonlocal current-current interaction terms in the l.h.s. of Eq. (5). Such terms can also be generated by the presence of a \( \varphi \)-field potential, \( V(|\varphi|) \) in the r.h.s. of Eq. (5).
The advantage of the FSR in this case follows from the very clear space-time picture of the corresponding dynamics in terms of particle trajectories. This is especially important if the currents can be treated as classical or static (for example, in the heavy quark case). The mentioned remark on usefulness of the FSR (3) becomes clear when one considers the physical amplitude, e.g. the Green’s function of the white state \( tr(\phi^+(x)\phi(x)) \) or its nonlocal version \( tr[\phi^+(x)\Phi(x, y)\phi(y)] \), where \( \Phi(x, y) \) – to be widely used in what follows – is the parallel transporter along some arbitrary contour \( C(x, y) \)

\[
\Phi(x, y) = P \exp \left( ig \int_{y}^{x} A_{\mu}(z) dz_{\mu} \right). 
\]

One has by standard rules

\[
G_{\phi}(x, y) = \left< tr \left[ \phi^+(x)\phi(x) \right] tr \left[ \phi^+(y)\phi(y) \right] \right>_{A} \\
= \int_{0}^{\infty} ds_{1} \int_{0}^{\infty} ds_{2} (Dz)_{xy} (Dz')_{xy} e^{-K-K'} \left< W \right>_{A} + \ldots
\]

(7)

where dots stand for the disconnected part, \( \left< G_{\phi}(x, x)G_{\phi}(y, y) \right>_{A} \). We have used the fact that the propagator for the charge-conjugated field \( \phi^+ \) is proportional to \( \Phi^{\dagger}(x, y) = \Phi(y, x) \). Therefore the ordering \( P \) must be inverted, \( \Phi^{\dagger}(x, y) = P \exp(ig \int_{x}^{y} A_{\mu}(z) dz_{\mu}) \). Thus all dependence on \( A_{\mu} \) in \( G_{\phi} \) is reduced to the Wilson loop average

\[
\left< W \right>_{A} = \left< tr P_{C} \exp ig \int_{C} A_{\mu}(z) dz_{\mu} \right>_{A}.
\]

(8)

Here \( P_{C} \) is the ordering around the closed loop \( C \) passing through the points \( x \) and \( y \), the loop being made of the paths \( z_{\mu}(\tau), z'_{\mu}(\tau') \) and to be integrated over.
The FSR can also be used to describe the quark and gluon propagation. Similar to the QED case, the fermion (quark) Green’s function in the presence of an Euclidean external gluonic field can be written as

\[
G_q(x, y) = \langle \psi(x) \bar{\psi}(y) \rangle_q = \langle x | (m + \hat{D})^{-1} | y \rangle \\
= \langle x | (m - \hat{D})(m^2 - \hat{D}^2)^{-1} | y \rangle \\
= (m - \hat{D}) \int_0^\infty ds(Dz)_{xy} e^{-K} \Phi_\sigma(x, y),
\]  

(9)

where \( \Phi_\sigma \) is the same as was introduced in Ref. [1] except for the ordering operators \( P_A, P_F \)

\[
\Phi_\sigma(x, y) = P_A \exp \left( ig \int_y^x A_\mu dz_\mu \right) P_F \exp \left( g \int_0^s d\tau \sigma_{\mu\nu} F_{\mu\nu} \right)
\]  

(10)

with \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu] \) and \( \sigma_{\mu\nu} = \frac{1}{4i}(\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu) \), while \( K \) and \( (Dz)_{xy} \) are defined in Eqs. (3) and (4). Note that operators \( P_A, P_F \) in Eq. (10) preserve the proper ordering of matrices \( A_\mu \) and \( \sigma_{\mu\nu} F_{\mu\nu} \) respectively. Explicit examples are considered below.

Finally we turn to the case of FSR for the valence gluon propagating in the background nonabelian field. Here we only quote the result for the gluon Green’s function in the background Feynman gauge[5,17]. We have

\[
G_{\mu\nu}(x, y) = \langle x | (D_\lambda^2 \delta_{\mu\nu} - 2igF_{\mu\nu})^{-1} | y \rangle
\]  

(11)

Proceeding in the same way as for quarks, we obtain the FSR for the gluon Green’s function

\[
G_{\mu\nu}(x, y) = \int_0^\infty ds(Dz)_{xy} e^{-K_a} \Phi_{\mu\nu}(x, y),
\]  

(12)

where we have defined
\[ K_\theta = \frac{1}{4} \int_0^\infty \left( \frac{dz_\mu}{d\tau} \right)^2 d\tau, \]

\[ \Phi_{\mu\nu}(x, y) = \left[ P_A \exp \left( i g \int_y^z A_\lambda dz_\lambda \right) P_F \exp \left( 2g \int_0^s d\tau F_{\sigma\nu}(z(\tau)) \right) \right]_{\mu\nu}. \] (13)

Now in the same way as is done above for scalars in Eq. (7), we may consider a Green’s function, corresponding to the physical transition amplitude from a white state of \( q_1, \bar{q}_2 \) to another white state consisting of \( q_3, \bar{q}_4 \). It is given by

\[ G^F_{q\bar{q}}(x, y) = \langle G_q(x, y)G_{\bar{q}}(x, y)\Gamma - G_q(x, x)\Gamma G_q(y, y)\Gamma \rangle_A, \] (14)

where \( \Gamma \) describes the vertex part for the interaction between the \( q, \bar{q} \) pair in the meson. The first term on the r.h.s. of Eq. (14) can be reduced to the same form as in Eq. (7) but with the Wilson loop containing ordered insertions of the operators \( \sigma_{\mu\nu}F_{\mu\nu} \) (cf. Eq. (10)).

### 3 Perturbation theory in the framework of FSR. Identities and partial summation

In this section we discuss in detail how the usual results of perturbation theory follow from FSR. It is useful to establish such a general connection between the perturbation series (Feynman diagram technique) and FSR. At the same time the FSR presents a unique possibility to sum up Feynman diagrams in a very simple way, where the final result of the summation is written in an exponentiated way[15,17]. This method will be discussed in the next section.

Consider the FSR for the quark Green’s function. According to Eq. (9), the
2-nd order of perturbative expansion of Eq. (10) can be written as

\[ G_q(x, y) = (m - \hat{D}) \int_0^\infty ds \int_0^\infty d\tau_1 \int_0^\infty d\tau_2 e^{-K(Dz)_{xu} d^4u(Dz)_{uv} d^4v(Dz)_{vy}} \]

\[ \times (igA_\mu(u) \dot{u}_\mu + g\sigma_{\mu\nu} F_{\mu\nu}(u)) (igA_\nu(v) \dot{v}_\nu + g\sigma_{\lambda\sigma} F_{\lambda\sigma}(v)), \] (15)

where we have used the identities

\[ (Dz)_{xy} = (Dz)_{xu(\tau_1)} d^4u(\tau_1)(Dz)_{u(\tau_1)\nu(\tau_2)} d^4v(\tau_2)(Dz)_{\nu(\tau_2)y}, \] (16)

\[ \int_0^\infty ds \int_0^\infty d\tau_1 \int_0^\infty d\tau_2 f(s, \tau_1, \tau_2) = \int_0^\infty ds \int_0^\infty d\tau_1 \int_0^\infty d\tau_2 f(s + \tau_1 + \tau_2, \tau_1 + \tau_2, \tau_2). \] (17)

We can also expand only in the color magnetic moment interaction \((\sigma F)\). This is useful when the spin-dependent interaction can be treated perturbatively, as it is in most cases for mesons and baryons (exclusions are Goldstone bosons and nucleons, where the spin interaction is very important and interconnected with chiral dynamics). In this case we obtain to the second order in \((\sigma F)\)

\[ G_q^{(2)}(x, y) = \int_0^\infty ds \int_0^\infty d\tau_1 \int_0^\infty d\tau_2 e^{-m_2^2(s+\tau_1+\tau_2) - K_0 - K_1 - K_2} \]

\[ (Dz)_{xu} \Phi(x, u) g(\sigma F(u)) d^4u(Dz)_{uv} \Phi(u, v) g(\sigma F(v)) d^4v(Dz)_{vy}, \] (18)

In another way it can be written as

\[ G_q^{(2)}(x, y) = i(m - \hat{D})(m_0^2 - D_{\mu xu}^2)^{-1} d^4u g(\sigma F(u))(m_0^2 - D_{\mu uv}^2)^{-1} d^4v \]

\[ \times g(\sigma F(v))(m_0^2 - D_{\mu vy}^2)^{-1}. \] (19)

Here \((m_0^2 - D_{\mu xu}^2)^{-1}\) is the Green’s function of a scalar quark in the external gluonic field \(A_\mu\). This type of expansion is useful also for the study of small-\(x\) behavior of static potential, since the correlator \(\langle \sigma F(u)\sigma F(v) \rangle\) plays an important role there.
However, in establishing the general connection between perturbative expansion for Green’s functions in FSR and expansions of exponential $\Phi_\sigma$ in Eq. (10), one encounters a technical difficulty since the coupling constant $g$ enters in three different ways in FSR:

1. in the factor $(m - \hat{D})$ in front of the integral in Eq. (9)

2. in the parallel transporter (the first exponential in Eq. (10))

3. in the exponential of $g(\sigma F)$.

Therefore it is useful to compare the two expansions in the operator form

\[
(m + \hat{D})^{-1} = (m + \hat{\partial} - ig\hat{A})^{-1} = (m + \hat{\partial})^{-1} + (m + \hat{\partial})^{-1}ig\hat{A}(m + \hat{\partial})^{-1} + (m + \hat{\partial})^{-1}ig\hat{A}(m + \hat{\partial})^{-1} + \ldots \tag{20}
\]

and the FSR

\[
(m + \hat{D})^{-1} = (m - \hat{D})(m^2 - \partial^2)^{-1} \sum_{n=0}^{\infty} (\delta(m^2 - \partial^2)^{-1})^n, \tag{21}
\]

where we have introduced

\[
\delta = -ig(\hat{A}\hat{\partial} + \hat{\partial}\hat{A}) - g^2\hat{A}^2 \equiv \hat{D}^2 - \partial^2. \tag{22}
\]

To see how the expansion (21) works, using $\hat{D} = \hat{\partial} - ig\hat{A}$, Eq. (21) becomes

\[
(m + \hat{D})^{-1} = [(m + \hat{\partial})^{-1} + ig\hat{A}(m^2 - \partial^2)^{-1}] \sum_{n=0}^{\infty} [\delta(m^2 - \partial^2)^{-1}]^n
\]

Separating out the first term we may rewrite this as

\[
(m + \hat{D})^{-1} = (m + \hat{\partial})^{-1} + (m + \hat{\partial})^{-1}ig\hat{A}(m - \hat{D})(m^2 - \partial^2)^{-1} \sum_{n=0}^{\infty} [\delta(m^2 - \partial^2)^{-1}]^n. \tag{23}
\]
The last three factors in Eq. (23) are the same as occurring in Eq. (21). As a consequence the formal iteration of the resulting equation for the Greens’ function reproduces the same series as in Eq. (20), showing the equivalence of the two expansions.

It is important to note that each term in the expansion in powers of $\delta$, after transforming the operator form of Eq. (21) into the integral form of FSR, becomes an expansion of the exponential $\Phi_\sigma$ in Eq. (10) in powers of $g$. The second order term of this expansion was written down before in Eq. (15).

It is our purpose now to establish the connection between the expansion (21), (23) and the expansion of the exponential $\Phi_\sigma$ in Eq. (10) in the quark propagator (9). We can start with term linear in $\hat{A}$ and write (for the Abelian case see Appendix B of Ref. [18])

$$G_q^{(1)} = ig \int G_q^{(0)}(x, z(\tau_1)) d^4 z \frac{\xi \mu}(n) \hat{A}_\mu(\tau_1) G_q^{(0)}(z(\tau_1), y), \quad (24)$$

where the notation is clear from the general representation of $G_q$, given by Eq. (9)

$$G_q(x, y) = \int_0^\infty dse^{-\frac{sm^2}{8}} \prod_{n=1}^N \frac{d^4 \xi(n)}{(4\pi^2)^2} \exp \left[ -\sum_{n=1}^N \frac{\xi^2(n)}{4\varepsilon} \right] \Phi_\sigma(\hat{A}, \xi) \quad (25)$$

with $\xi(n) = z(n) - z(n - 1)$, $\hat{A}_\mu(n) = \frac{1}{2}[A_\mu(z(n)) + A_\mu(z(n - 1))]$ and

$$\Phi_\sigma(\hat{A}, \xi) = P \exp \{ ig \sum_{n=1}^N \tilde{A}_\mu(n) \xi_\mu(n) + g \sum_{n=1}^N \sigma_{\mu\nu} F_{\mu\nu}(z(n)) \varepsilon \}. \quad (26)$$

Representing $\xi(n)$ in Eq. (24) as $\frac{1}{2}(\xi_\mu(L) + \xi_\mu(R))$, where $\xi_\mu(L)$ refers to the integral over $\xi_\mu$ in $G_q^{(0)}$ to the left of $\xi_\mu$ in Eq. (24) and $\xi_\mu(R)$ to the integral...
in \( G_q^{(0)} \) standing to the right of \( \xi_\mu \), we obtain

\[
\int \xi_\mu(n) \frac{d^4 \xi(n)}{(4\pi \varepsilon)^2} e^{i p \xi - \frac{q^2}{4\varepsilon}} = -i \frac{\partial}{\partial p_\mu} e^{-ip^2 \varepsilon} = 2ip_\mu \varepsilon e^{-p^2 \varepsilon}.
\] (27)

Thus Eq. (24) in momentum space becomes

\[
G_q^{(1)} = -gG_q^{(0)}(q) \langle q| p_\mu A_\mu + A_\mu p_\mu |q' \rangle G_q^{(0)}(q')
\] (28)

In a similar way the second order term from the coinciding arguments yields

\[
G_q^{(2)}(\text{coinc}) = -g^2 \int G_q^{(0)}(x, z) A_\mu(z) d^4 z G_q^{(0)}(z, y).
\] (29)

Finally, the first order expansion of the term \( \sigma_{\mu\nu} F_{\mu\nu} \) in Eq. (10) yields the remaining missing component of the combination \( \delta \), Eq. (22), which can be rewritten as

\[
\delta = -ig(A_\mu \partial_\mu + \partial_\mu A_\mu) - g^2 A_\mu^2 + g\sigma_{\mu\nu} F_{\mu\nu}.
\] (30)

Hence the second term in the expansion (21)

\[
(m + \hat{D})^{-1} = (m - \hat{D})(m^2 - \partial^2)^{-1} + (m - \hat{D})(m^2 - \partial^2)^{-1}\delta(m^2 - \partial^2)^{-1} + ...
\] (31)

is exactly reproduced by the expansion of the FSR (9), where in the first exponential \( \Phi_\sigma \) in Eq. (9) one keeps terms of the first and second order, \( O(gA_\mu) \) and \( O((gA_\mu)^2) \), while in the second exponential one keeps only the first order term \( O(g\sigma_{\mu\nu} F_{\mu\nu}) \). It is easy to see that this rule can be generalized to higher orders of the expansion in \( \delta \) in Eq. (21) as well.
4 Perturbative vs nonperturbative: two types of expansion

As was discussed in Section 2, gluons can also be considered in FSR. To make this statement explicit and to prove Eqs. (11-13) written for the gluon Green’s functions, we can use the background perturbation theory [16]. As in Ref. [17] we combine the perturbative field $a_\mu$ and NP degrees of freedom $B_\mu$ in one gluonic field $A_\mu$, namely

$$ A_\mu = B_\mu + a_\mu $$

(32)

Under gauge transformations $A_\mu$ transforms as

$$ A_\mu \rightarrow A'_\mu = U^+(A_\mu(x) + \frac{i}{g} \partial_\mu)U $$

(33)

At this point we must distinguish two opposite physical situations, which require different types of expansions. Consider first systems of small size, e.g. heavy quarkonia, which are mostly governed by the color Coulomb interaction and have a radius of the order $(m\alpha_s(m))^{-1}$, where $m$ is the quark mass. For the ground state bottomonium this radius is around 0.2 fm and for charmonium 0.4 fm.

In this case we have the first type of expansion: at the zeroth order all gluon exchanges are taken into account (In practice the Coulomb contribution and the few first radiative corrections), while in first order one treats the nonperturbative contribution as a correction. This expansion is considered in detail at the end of this Section.

The second type of expansion takes into account NP interaction fully through NP vacuum correlators already in the zeroth order –this is the NP background
and in the next orders the usual background perturbation theory[16,17] is developed with necessary modifications.

4.1 Expansion in perturbative fields

We start this Section with the second type of expansion with some modifications due to the independent integral over the background field, as in the 't Hooft’s identity[17]. It is convenient to impose on $a_\mu$ the background gauge condition[16]

$$D_\mu a_\mu = \partial_\mu a_\mu^a + gf^{abc}B_\mu^b a_\mu^c = 0.$$ (34)

In this case a ghost field has to be introduced. Defining $D_\lambda^a = \partial_\lambda \cdot \delta_{ca} + g f^{cba}B_\lambda^b \equiv \hat{D}_\lambda$, We can write the resulting partition function as

$$Z = \frac{1}{N!} \int DB e^{\int J_\mu B_\mu d^4 x Z(J, B)},$$ (35)

where

$$Z(J, B) = \int Da \ det(\frac{\delta G^a}{\delta \omega^b}) \ exp \int (L_0 + L(a) - \frac{1}{2\xi} (G^a)^2 + J_\mu a_\mu^a].$$ (36)

In Eq. (36) we have

$$L_0 = -\frac{1}{4} (F_{\mu\nu}^a (B))^2$$ (37)

and

$$L(a) = L_1(a) + L_2(a) + L_{int}(a)$$ (38)

with
\[ L_1(a) = a^c_\mu D^a_\mu (B) F^a_{\mu \nu} \]
\[ L_2(a) = +\frac{1}{2} a_\nu (\hat{D}_\lambda^2 \delta_{\mu \nu} - \hat{D}_\mu \hat{D}_\nu + ig \hat{F}_{\mu \nu}) a_\mu = \]
\[ = \frac{1}{2} a^c_\nu [D^a_\lambda D^d_\nu \delta_{\mu \nu} - D^a_\mu D^{ad}_\nu - g  f^{cad} F^a_{\mu \nu}] a^d_\mu , \]
\[ L_{int} = -\frac{1}{2} g (D_\mu (B)a_\nu - D_\nu (B)a_\mu)^a f^{abc} a^b_\mu a^c_\nu - \frac{1}{4} g^2 f^{abc} a^b_\mu a^c_\nu f^{def} a^e_\mu a^f_\nu. \]

\(G^a\) in Eq. (36) is the background gauge condition

\[ G^a = \partial_\mu a^a_\mu + g f^{abc} B^b_\mu a^c_\mu = (D_\mu a_\mu)^a. \]  

(40)

The ghost vertex is obtained from \( \frac{4 G^a}{a^a_\mu} = (D_\mu (B)D_\mu (B + a))^{ab} \) [16] to be

\[ L_{ghost} = -\theta^+_a (D_\mu (B)D_\mu (B + a))^{ab} \theta_b , \]  

(41)

\( \theta \) being the ghost field. The linear part of the Lagrangian \( L_1 \) disappears if \( B_\mu \) satisfies the classical equations of motion. Here we do not impose this condition on \( B_\mu \). However, it was shown in Ref. [17] that \( L_1 \) gives no important contribution

We now can identify the propagator of \( a_\mu \) from the quadratic terms in the Lagrangian \( L_2(a) - \frac{1}{12} (G^a)^2 \). We get

\[ G^{ab}_{\nu \mu} = [\hat{D}_\lambda^2 \delta_{\mu \nu} - \hat{D}_\mu \hat{D}_\nu + ig \hat{F}_{\mu \nu} + \frac{1}{\xi} \hat{D}_\nu \hat{D}_\mu]^{-1} . \]  

(42)

It will be convenient sometimes to choose \( \xi = 1 \) and end up with the well-known form of the propagator in – what one would call – the background Feynman gauge

\[ G^{ab}_{\nu \mu} = (\hat{D}_\lambda^2 \delta_{\mu \nu} - 2ig \hat{F}_{\mu \nu})^{-1} \]  

(43)

This is exactly the form of gluon propagator used in Eq. (11). Integration over the ghost and gluon degrees of freedom in Eq. (36) yields
\[ Z(J, B) = \text{const} (\det W(B))^{-1/2} [\det (-D_\mu(B) D_\mu(B + a))]_{a=sJ} \]
\[ \times \{ 1 + \sum_{l=1}^{\infty} \frac{S_{\text{int}}(a = \frac{sJ}{l})}{l!} exp \left( -\frac{1}{2} JGJ \right) \bigg|_{J_\nu = D_\nu(B) F_{\mu\nu}(B)} \} , \]  

where \( S_{\text{int}} \) is the action corresponding to \( L(a) \) and \( G \) is defined in Eq. (43).

Let us mention the convenient gauge prescription for gauge transformations of the fields \( a_\mu, B_\mu \). Under the gauge transformations the fields transform as

\[ a_\mu \rightarrow U^+ a_\mu U, \]  
\[ B_\mu \rightarrow U^+ (B_\mu + \frac{i}{g} \partial_\mu) U. \]

All the terms in Eq. (36), including the gauge fixing one \( \frac{1}{2} (G^a)^2 \) are gauge invariant. That was actually one of the aims put forward by ’t Hooft in Ref. [16]. It has important consequences:

(i) Any amplitude in the perturbative expansion in \( g a_\mu \) of Eqs. (36) and (44) corresponding to a generalized Feynman diagram, is separately gauge invariant (for colorless initial and final states of course).

(ii) Due to gauge invariance of all terms, the renormalization is specifically simple in the background field formalism[16], since the counterterms enter only in gauge–invariant combinations, e.g. \( F_{\mu\nu}^2 \). The Z–factors \( Z_g \) and \( Z_B \) are connected: \( Z_g Z_B^{1/2} = 1. \)

As a consequence, the quantities like \( gB_\mu, gF_{\mu\nu}(B) \) are renormalization-group (RG) invariant. Consequently all background field correlators are also RG invariant and they can be considered on the same footing as the external momenta in the amplitudes. This leads to a new form of solutions of RG equations, where \( \alpha_s = \alpha_s(M_B), M_B \) being the (difference) of the hybrid ex-
citations, typically $M_B \approx 1GeV$. As a result a new phenomenon appears, freezing or saturation of $\alpha_s$ at large Euclidean distances. For more discussion see Ref. [17] and recent explicit extraction of the freezing $\alpha_s$ from the spectra of heavy quarkonia[31]. In the rest of this subsection we demonstrate how the background perturbation series (44) works for the meson Green’s function. To this end we use Eq. (14) and consider the flavour nonsinglet case to disregard the second term in Eq. (14).

Let us start with the meson Green’s function and use the FSR for both quark and antiquark.

$$G_M(x,y) = \langle \text{tr}\Gamma^{(f)}(m - \hat{D}) \int_0^\infty ds \int_0^\infty dse^{-K-K} (Dz)_{xy}(D\bar{z})_{xy} \Gamma^{(i)}(\bar{m} - \hat{D})W_F \rangle$$

(47)

Here the barred symbols refer to the antiquark and

$$W_F = P_A P_F \exp(ig \int dz_\mu A_\mu) \exp(g \int \bar{s} \sigma^{(1)}_{\mu\nu} F_{\mu\nu}) \exp(-g \int \bar{s} \sigma^{(2)}_{\mu\nu} F_{\mu\nu} d\bar{\tau}).$$

(48)

In Eq. (47) integrations over proper times $s, \bar{s}$ and $\tau, \bar{\tau}$ occur, which also play the role of an ordering parameter along the trajectory, $z_\mu = z_\mu(\tau), \quad \bar{z}_\mu = \bar{z}_\mu(\bar{\tau})$.

It is convenient to go over to the actual time $t \equiv z_4$ of the quark (or antiquark), defining the new quantity $\mu(t)$, which will play a very important role in what follows

$$2\mu(t) = \frac{dt}{d\tau}, \quad t \equiv z_4(\tau).$$

(49)
For each quark (or antiquark and gluon) we can rewrite the path integral (47) as (see Refs. [38,42] for details)

\[
\int_0^\infty ds (D^4z)_{xy} \ldots = \text{const} \int D\mu(t) (D^3z)_{xy} \ldots
\]  

(50)

where \((D^3z)_{xy}\) has the same form as in Eq. (4) but with all 4-vectors replaced by 3-vectors. The path integral \(D\mu(t)\) is supplied with the proper integration measure, which is derived from the free motion Lagrangian.

In general \(\mu(t)\) can be a strongly oscillating function of \(t\) due to the Zitterbewegung. In what follows we shall use the stationary point method for the evaluation of the integral over \(D\mu(t)\), with the extremal \(\mu_0(t)\) playing the role of an effective or constituent quark mass. We shall see that in all cases, where spin terms can be considered as a small perturbation, i.e. for the majority of mesons, \(\mu_0\) is positive and rather large even for vanishing quark current masses \(m, \bar{m}\), and the role of the Zitterbewegung is small (less than 10\% from the comparison to the light-cone Hamiltonian eigenvalues, see Refs. [38,39] for details).

Now the kinetic terms can be rewritten using Eq. (49) as

\[
K + \bar{K} = \int_0^T dt \left\{ \frac{m^2}{2\mu(t)} + \frac{\mu(t)}{2} [\dot{z}_i(t)]^2 + 1 \right\} + \frac{\bar{m}^2}{2\bar{\mu}(t)} + \frac{\bar{\mu}(t)}{2} [\dot{\bar{z}}_i(t)]^2 + 1 \right\},
\]  

(51)

where \(T = x_4 - y_4\). In the spin-dependent factors the corresponding changes are

\[
\int_0^s d\tau \sigma_{\mu \nu} F_{\mu \nu} = \int_0^T dt \frac{\sigma_{\mu \nu} F_{\mu \nu}(z(t))}{2\mu(t)}.
\]  

(52)
In what follows in this section we may systematically do a perturbation expansion of the spin terms. They contribute to the total mass corrections of the order of 10-15% for lowest mass mesons, while they are much smaller for the high excited states. This perturbative approach fails however for pions (and kaons) where the chiral degrees of freedom should be taken into account. In this case another equation should be considered[40,41].

Therefore as a starting approximation we may use the Green’s functions of mesons made of spinless quarks. This amounts to neglecting in Eqs. (47,48) the terms \((m - \hat{D}), (\bar{m} - \hat{D})\) and \(\sigma_{\mu\nu} F_{\mu\nu}\). As a result, we have

\[
G_M^{(0)}(x, y) = \text{const} \int D\mu(t) D\bar{\mu}(t) (D^3 z)_{xy} (D^3 \bar{z})_{xy} e^{-\sqrt{\mathcal{K} - \mathcal{R}}(W)}. \tag{53}
\]

The Wilson loop in Eq. (53) contains both perturbative and NP fields. It can be expanded as

\[
W(B + a) = W(B) + \sum_{n=1}^{\infty} (ig)^n W^{(n)}(B; x(1)\ldots x(n)) a_\mu dx_\mu 1(1)\ldots dx_\mu n(n). \tag{54}
\]

After averaging over \(a_\mu, B_\mu\) we obtain, keeping the lowest correction term:

\[
\langle W(B + a) \rangle_{B,a} = \langle W(B) \rangle_B - g^2 \langle W^{(2)}(B; x, y) \rangle dx dy + \ldots, \tag{55}
\]

where the second term in Eq. (55) can be written as

\[
-g^2 W^{(2)} dx dy = -g^2 \int \Phi^{\alpha\beta}(x, y, B) \\
\times t_a^{\delta\alpha} t_b^{\gamma\beta} G_{\mu\nu}^{ab}(x, y, B) \Phi^{\gamma\delta}(y, x, B) dx_\mu dy_\nu, \tag{56}
\]

\(G_{\mu\nu}^{ab}(x, y, B)\) being the gluon propagator in the background field (11).
We can easily see that \( W^{(2)} \) contains 3 pieces, two of them are perturbative self-energy quark terms. The third one, assuming that \( x \) and \( y \) refer to the quark and antiquark trajectory respectively, is the color Coulomb term, modified by the confining background. As argued in Ref. [17], this term represents the gluon propagating inside the world-sheet of the string between \( q \) and \( \bar{q} \). When the time \( T \) is large, the long film of this world-sheet does not influence the motion of the gluon, reducing it to the free OGE term. Hence \( \langle W^{(2)} \rangle \) factorizes into the film term (\( \langle W(B) \rangle_B \)) and gluon propagator \( dx_\mu dy_\nu G_{\mu\nu} \), yielding finally the color Coulomb term in the potential. (This is however only true for the lowest order term \( W^{(2)} \) and only at large distances \( |x - y| \lesssim T_g \). Otherwise perturbative-nonperturbative interference comes into play[28].

In what follows we restrict our attention to the first term, \( \langle W(B) \rangle_B \). Our next approximation is the neglect of perturbative exchanges in \( \langle W \rangle \) (they will be restored in the final expression for Hamiltonian). This yields for large Wilson loops, i.e. \( R, T \gg T_g \),

\[
\langle W \rangle_B = \text{const} \exp(-\sigma S_{\text{min}}) \tag{57}
\]

where \( S_{\text{min}} \) is the minimal area inside the given trajectories \( z(t), \bar{z}(t) \) of the quark and antiquark,

\[
S_{\text{min}} = \int_0^T dt \int_0^1 d\beta \sqrt{\det g}, \quad g_{ab} = \partial_a w_\mu \partial_b w^\mu, \quad a, b = t, \beta. \tag{58}
\]

Here a point \( w \) on the surface is parameterized by \( w_\mu = \beta z_\mu(t) + (1 - \beta) \bar{z}_\mu(t) \).

The Nambu-Goto form of \( S_{\text{min}} \) cannot be quantized due to the square root. To get rid of the square root we may use the auxiliary field approach[36] with functions \( \nu(\beta, t) \) and \( \eta(\beta, t) \) as is usually done in string theories. As a result
the total Euclidean action becomes\[42\]

\[ A = K + \tilde{K} + \sigma S_{\text{min}} = \int_0^T dt \int_0^1 d\beta \left\{ \frac{1}{2} \left( \frac{m^2}{\mu(t)} + \frac{\tilde{m}^2}{\tilde{\mu}(t)} \right) + \frac{\mu_+(t)}{2} \tilde{R}^2 \right\} \]

\[ + \frac{\tilde{\mu}(t)}{2} r^2 + \frac{\nu}{2} [\tilde{w}^2 + \left( \frac{\sigma}{\nu} \right)^2 r^2 - 2\eta(\tilde{w}r) + \eta^2 r^2] \}. \] (59)

Here \( \mu_+ = \mu + \tilde{\mu}, \quad \tilde{\mu} = \frac{\mu \tilde{\mu}}{\mu + \tilde{\mu}}, \quad R_i = \frac{\mu z_i \tilde{\mu} z_i}{\mu + \tilde{\mu}}, \quad r_i = z_i - \tilde{z}_i. \) Performing the Gaussian integrations over \( R_\mu \) and \( \eta \) we arrive in the standard way at the Hamiltonian (we take \( m = \tilde{m} \) for simplicity)

\[ H = \frac{p_r^2 + m^2}{\mu(\tau)} + \mu(\tau) \]

\[ + \frac{\hat{L}^2}{r^2} \frac{r^2}{\mu + 2 \int_0^1 (\beta - \frac{1}{2})^2 \nu(\beta) d\beta} + \frac{\sigma^2 r^2}{2} \frac{1}{\nu(\beta)} \int_0^1 d\beta + \frac{1}{2} \frac{\nu(\beta)}{\nu(\beta)} \int_0^1 d\beta, \] (60)

where \( p_r^2 = (pr)^2/r^2 \) and \( L \) is the angular momentum, \( \hat{L} = (r \times p) \).

A reasonable approximation to the integrations over \( \mu \) and \( \nu \) is to replace them by their corresponding extremum values[42]. For these values the terms \( \mu(t) \) and \( \nu(\beta) \) have a simple physical meaning. E.g. when \( \sigma = 0 \) and \( L = 0 \), we find from Eq. (60)

\[ H_0 = 2\sqrt{p^2 + m^2}, \quad \mu_0 = \sqrt{p^2 + m^2}, \] (61)

so that \( \mu_0 \) corresponds to the energy of the quark. Similarly in the limiting case \( L \to \infty \) the extremum over \( \nu(\beta) \) yields

\[ \nu_0(\beta) = \frac{\sigma r}{\sqrt{1 - 4y^2(\beta - \frac{1}{2})^2}}, \quad H_0^2 = 2\pi \sigma \sqrt{L(L+1)} \]. (62)

Hence \( \nu_0 \) is the energy density along the string with \( \beta \) playing the role of the coordinate along the string.
4.2 Nonperturbative fields as a correction

Now we turn to the first type of expansion mentioned above, i.e. when the NP contribution is considered to be a (small) correction to a basically perturbative result. As an example let us consider the spectrum of heavy quarkonia. We can calculate the NP shift of the Coulombic levels of heavy quark–antiquark ($q\bar{q}$) system, following Ref. [30]. When the quark mass $m$ is large, the spatial and temporal extensions of the n-th bound state are

$$ r_n \approx \frac{n}{m\alpha_s}, \quad t_n \approx \frac{n^2}{m\alpha_s^2}. \quad (63) $$

For low $n \sim 1$ these may be small enough to disregard the NP interaction in first approximation. So for the spin–averaged spectrum we can write

$$ M(n, l) = 2m \{ 1 - \frac{C_F\alpha_s^2}{8n^2} + O(\alpha_s^3) + \Delta_{NP} \} \quad (64) $$

where $\Delta_{NP}$ is the expected nonperturbative correction, which should be small for states of small spatial extension. This conclusion can be drawn from the lattice (and phenomenological) parameterization of the static $q\bar{q}$ potential

$$ V(r) = -\frac{4\alpha_s(r)}{3r} + \sigma r + \text{const} \quad (65) $$

Using the empirical values found for $\sigma = 0.2 GeV^2$ and $\alpha_s(r) \sim 0.3$ (at $r \approx 0.2 fm$) we may deduce that the first term on the l.h.s. of Eq. (65) is at $r \approx 0.3 fm$ comparable in magnitude to the second term, the NP contribution. Hence this suggests that the states with a radius $r \ll 0.3 fm$ are mainly governed by the (color) Coulomb dynamics, while those with $r \gg 0.3 fm$ are mostly NP states. So we may expect e.g. the $n = 1$ bottomonium state to be largely Coulombic.
Let us now consider the general path–integral formalism for the $q\bar{q}$ system interacting via perturbative gluon exchanges and nonperturbative correlators. We start with the quark Green’s function in the FSR form (cf. Eqs. (9,10))

$$S(x,y) = i(m - \hat{D}) \int_0^\infty ds(Dz)_{xy}e^{-K}\Phi_\sigma(x,y), \quad (66)$$

where

$$K = m^2s + \frac{1}{4}\int_0^s z_\mu^2d\tau$$

and $\Phi_\sigma$ contains spin insertions into the parallel transporter

$$\Phi_\sigma(x,y) = P_AP_F\exp[ig\int_x^y A_\mu dz_\mu + g\int_0^s d\tau\sigma_{\mu\nu}F_{\mu\nu}(z(\tau))]. \quad (67)$$

Double ordering in $A_\mu$ and $F_{\mu\nu}$ is implied by the operators $P_A, P_F$. We have also introduced the $4 \times 4$ matrix in Dirac space

$$\sigma_{\mu\nu}F_{\mu\nu} \equiv \bar{\sigma}_i \left( \begin{array}{cc} \vec{B}_i & \vec{E}_i \\ \vec{E}_i & \vec{B}_i \end{array} \right). \quad (68)$$

Neglecting spins, we have instead of Eq. (67)

$$\Phi_\sigma(x,y) \rightarrow \Phi(x,y) \equiv P_A\exp(ig\int_y^x A_\mu dz_\mu). \quad (69)$$

In terms of the single quark Green’s functions (66) and initial and final state matrices $\Gamma_i, \Gamma_f$ such that $\bar{q}(x)\Gamma_f\Phi(x, \bar{x})q(x)$ is the final $q\bar{q}$ state) the total relativistic gauge–invariant $q\bar{q}$ Green’s function in the quenched approximation is similar to Eq. (14)

$$G(x, \bar{x}; y, \bar{y}) = <tr(\Gamma_fS_1(x, y)\Gamma_i\Phi(y, \bar{y})S_2(\bar{y}, \bar{x})\Phi(\bar{x}, x))> - <tr(\Gamma_fS_1(x, \bar{x})\Phi(\bar{x}, x))tr(\Gamma_iS_2(\bar{y}, y)\Phi(y, \bar{y}))>. \quad (70)$$
The angular brackets in Eq. (70) imply averaging over the gluonic field $A_\mu$ and the trace is taken over Dirac space.

Since we are interested in this case primarily in heavy quarkonia, it is reasonable to do a systematic nonrelativistic approximation. To this end we introduce as in Refs. [34,42] the real evolution (time) parameter $t$ instead of the proper time $\tau$ in $K, (\bar{\tau}$ in $\bar{K}$) and the dynamical mass parameters $\mu, \bar{\mu}$ as in Eq. (49)

$$\frac{dt}{d\tau} = 2\mu_1, \frac{dt}{d\bar{\tau}} = 2\mu_2; \quad \int_0^\tau \dot{z}_\mu^2(\tau)d\tau = \int_0^T 2\mu_1 dt (\frac{dz_\mu(t)}{dt})^2. \quad (71)$$

Here we have denoted

$$T \equiv \frac{1}{2}(x_4 + \bar{x}_4). \quad (72)$$

The nonrelativistic approximation is obtained, when we write for $z_4(t), \bar{z}_4(t)$

$$z_4(t) = t + \zeta(t); \quad \bar{z}_4(t) = t + \bar{\zeta}(t) \quad (73)$$

and expands in the fluctuations $\zeta, \bar{\zeta}$, which are $0(1/m)$. Note that the integration in $ds_1ds_2$ goes over into $d\mu_1d\mu_2$. Physically the expansion (73) means that we neglect trajectories with backtracking of $z_4, \bar{z}_4$, i.e. dropping the so-called $Z$ graphs. We can persuade ourself that the insertion of Eq. (73) into $K, \bar{K}$ allows to determine $\mu_1, \mu_2$ from the extremum in $K, \bar{K}$. We get

$$\mu_1 = m_1 + 0(1/m_1), \quad \mu_2 = m_2 + 0(1/m_2). \quad (74)$$

We can further make a systematic expansion in powers of $1/m_i[34]$). At least to lowest orders in $1/m_i$ this procedure is equivalent to the standard (gauge–noninvariant) nonrelativistic expansion[41].
Let us keep the leading term of this expansion

\[ G(x\bar{x}, y\bar{y}) = 4m_1m_2e^{-(m_1+m_2)^T} \int D^3 z D^3 \bar{z} e^{-K_1-K_2} < W(C) >, \] (75)

where \( K_1 = \frac{m_1}{2} \int_0^T \dot{z}_i^2(t) dt \), \( K_2 = \frac{m_2}{2} \int_0^T \dot{\bar{z}}_i^2(t) dt \). Furthermore, \( < W(C) > \) is the Wilson loop operator with a closed contour \( C \) comprising the \( q \) and \( \bar{q} \) paths, and the initial and final state parallel transporters \( \Phi(x, \bar{x}) \) and \( \Phi(y, \bar{y}) \).

The representation (75) will be our main object of study in the remaining part of this Section. For the heavy \( q\bar{q} \) system the perturbative interaction contains an expansion in powers of \( \frac{a_s}{v} \) (\( v \) being the velocity in the c.m. system). This should be kept entirely, while the nonperturbative interaction can be treated up to the lowest order approximation. For the total gluonic field \( A_\mu \) we may write

\[ A_\mu = B_\mu + a_\mu, \] (76)

where \( B_\mu \) is the NP background, while \( a_\mu \) is the perturbative fluctuation.

It is convenient in this first type of expansion to split the gauge transformation as

\[ B_\mu \to V^+ B_\mu V, \quad a_\mu \to V^+(a_\mu + \frac{i}{g} \partial_\mu) V, \] (77)

so that the parallel transporter

\[ \Phi(a; x, y) \equiv P \exp(i \int_x^y a_\mu dz_\mu) \] (78)

transforms as

\[ \Phi(a; x, y) \to V^+(x)\Phi(a; x, y)V(y). \] (79)
Using Eq. (75) we may now determine the effects of the NP contribution as a correction. The Wilson loop average in Eq. (75) can be written, using Eq. (76), as

\[
< W (C') > = < tr P exp(i g \int_C A_\mu dz_\mu) > = < tr P exp(i g \int_C a_\mu dz_\mu) > \\
+ \frac{(i g)^2}{2!} \int_C dz_\mu \int_C dz'_\mu < tr P \Phi(a; z, z') B_\mu(z') \Phi(a; z', z) B_\mu(z) > + ... \\
= W_0 + W_2 + ..., \tag{80}
\]

where we have omitted the term linear in \( B_\mu \) since it vanishes when averaged over the field \( B_\mu \). The dots imply terms of higher power in \( B_\mu \). The contour and points \( z, z' \) are schematically shown in Fig. 1.

![Fig. 1. The contour C, characterized by the quark trajectories z and z' in the Wilson loop with possible ladder-type gluon exchanges between the quarks.](image)

Let us discuss the first term on the r.h.s. of Eq. (80). It is the Wilson loop average of the usual perturbative fields, discussed extensively in Ref. [27]. We can use for \( W_0 \) the cluster expansion to obtain

\[
W_0 = Z \exp(\varphi_2 + \varphi_4 + \varphi_6 + ...), \tag{81}
\]
\[ \varphi_2 \equiv -\frac{g^2}{8\pi^2} \int \int \frac{d\zeta' d\zeta}{(\zeta - \zeta')^2} C_2, \quad C_2 = \frac{N_c^2 - 1}{2 N_c} \]  

where regularization is implied in the integral \( \varphi_2 \) to be absorbed into the \( Z \) factor. Note that \( \varphi_2 \) contains all ladder-type exchanges. In addition also the "Abelian-crossed" diagrams — those where times of the vertices can not be ordered while color generators \( t^a \) are always kept in the same order, as in the ladder diagrams. Therefore all crossed diagrams (minus "Abelian-crossed") are contained in \( \varphi_4 \) and contribute \( 0(1/N_c) \) as compared with ladder ones (cf. the discussion in Ref. [27]). In addition \( \varphi_4 \) contains "Mercedes-Benz diagrams", again repeated infinitely many times. It is interesting to note that each term \( \varphi_2, \varphi_4 \) etc. in Eq. (81) sums up to an infinite series of diagrams.

In particular \( exp(\varphi_2) \) contains all terms with powers of \( \frac{\alpha_s}{\pi} \), as we shall see below. For heavy (and slow) quarks we can write:

\[ \varphi_2 = \frac{g^2}{4\pi^2} \int_0^T \int_0^T dt dt' C_2(1 + \frac{\hat{z}_i \hat{z}'_j}{\vec{r}^2 + (t - t')^2}) \]

\[ \approx \int_0^T \frac{C_2 \alpha_s}{|\vec{r}|} (1 + 0(v^2/c^2)) dt = \varphi_2^{(0)} + 0(v^2/c^2) \]  

where \( \vec{r} = \vec{z} - \vec{z}' \). In this way we obtain a singlet one-gluon-exchange (OGE) potential, the effective time difference being \( \Delta t = |t - t'| \sim |\vec{r}| \). In addition also the radiative corrections due to the transverse gluon exchange can be obtained in this way. For the \( q\bar{q} \) mass \( \varphi_2 \) leads to a correction of order \( 0(\alpha_s) \), being the Coulombic energy. It should however be noted that in the wavefunction the Coulomb potential has to be kept to all orders because of its singular character. In fact we shall not expand \( exp(\varphi_2^{(0)}) \), while this is done for the other contributions like the radiative corrections.
Turning to $W_2$ we may determine the leading (in $N_c$) set of diagrams. They consist of the diagrams, where the gluons propagate between the $q$ and $\bar{q}$ lines with the same time coordinates, i.e. diagrams with Coulombic or instantaneous gluon exchanges. We have for the gluon propagator

$$<a^a(x)a^b(y)> = \frac{\delta_{ab}\delta_{\mu\nu}}{4\pi^2(x - y)^2}. \quad (84)$$

Here we have chosen for simplicity the Feynman gauge, since $W_0$ and $W_2$ are gauge invariant. Expanding $W_2$ in Eq. (80) in powers of $a_\mu$ we have in view of Eq. (84) terms typically of the form

$$tr(t^{b_k}t^{b_{k-1}}...t^{b_1}t^{b_2}...t^{b_1}t^{a_1}...t^{a_n}t^a B^a_{\mu}t^{a_n}...t^{a_1}B^b_{\mu}t^b...)) \rightarrow (C_2)^k tr(t^{a_1}...t^{a_n}t^a B^a_{\mu}t^{a_n}...t^{a_1}B^b_{\mu}t^b...)) \quad (85)$$

Now due to the equality

$$t^c t^a t^c = -\frac{1}{2N_c} t^a, \quad (86)$$

We obtain for all exchanges in the time interval between times of $B_\mu(z)$ and $B_\nu(z')$, a factor $(-\frac{1}{2N_c})$ instead of the factor $C_2$ for all the other exchanges.

The correction term $W_2$ can be worked out explicitly. In particular, we now derive the lowest order corrections to the energy levels and wave–functions due to the NP field correlators. As shown in Fig. 2 we may divide the total time interval $T$ into three parts

1. $0 \leq t \leq w'_4$
2. $w'_4 \leq t \leq w_4$
3. $w_4 \leq t \leq T$

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Fig. 2. Contribution of the Gaussian correlator to the q̅q Green’s function. The points \( w \) and \( w' \) of the correlator are connected by the parallel transporter, shown by the solid line going through the point \( x_0 \). This makes the correlator gauge invariant.

where \( t \) is the c.m. time. In \( K + \bar{K} \) we may separate the c.m. and relative coordinates

\[
R_i = \frac{m_1 z_i + m_2 \bar{z}_i}{m_1 + m_2}; \quad r_i(t) = z_i(t) - \bar{z}_i(t),
\]

so that

\[
K + \bar{K} = \int_0^T \frac{MR^2}{2} dt + \frac{\tilde{m}}{2} \int_0^T r^2(t) dt,
\]

with \( \tilde{m} = \frac{m_1 m_2}{m_1 + m_2}, \quad M = m_1 + m_2. \)
Separating out the trivial c.m. motion, we have in the parts (1) and (3) the path integrals, representing actually the singlet Coulomb Green’s function

\[ G^1_C(r(t_1), r(t_2); t_1 - t_2) = \int \mathcal{D}\bar{t} D\bar{r} e^{-i \int_{t_1}^{t_2} \bar{r}^2 dt + \alpha_s \int_{t_1}^{t_2} \frac{d\bar{r}}{|\bar{r}|}}. \]  

(89)

In the part (2) instead we have an octet Coulomb Green’s function

\[ G^8_C(r(w_4), r(w'_4); w_4 - w'_4) = \int \mathcal{D}\bar{t} D\bar{r} e^{-i \int_{w_4}^{w'_4} \bar{r}^2 dt - \frac{C_2}{2N_c} \int_{w_4}^{w'_4} \frac{d\bar{r}}{|\bar{r}|}}. \]  

(90)

As a result \( W_2 \) can be written as

\[ W_2 = \frac{(ig)^2}{2} \int \mathcal{D}z \int \mathcal{D}z' < tr B_v B_\mu > e^{\int_{t_1}^{t_2} dt v_C + \int_{w_4}^{w'_4} dt v_0} \]  

(91)

where

\[ V^0_C = C_2 \frac{\alpha_s}{r}, \quad V^8_C = -\frac{1}{2N_c} \frac{\alpha_s}{r}. \]  

(92)

We can easily identify \(-V^0_C\) and \(-V^8_C\) as a singlet and octet \(q\bar{q}\) potential, considered in Ref. [30]. For \(< tr B_v B_\mu >\) we can use the modified Fock–Schwinger gauge to obtain:

\[ B_\mu(z) = \int_{x_0}^{z} dw \alpha(w) F_\mu(w) \]  

(93)

and

\[ \int d\mu dz' < B_\mu(z) B_{\mu'}(z') > = \int d\sigma d\sigma_{\mu\lambda} < F_{\mu\rho}(w) F_{\nu\lambda}(w') >, \]  

(94)

where we have introduced a surface element \(d\sigma_{\mu\rho} = dz dw \alpha(w)\). To make Eq. (94) fully gauge–invariant we can introduce in the integral in Eq. (93) factors \(\Phi(x_0, w)\) identically equal to unity in the Fock–Schwinger gauge. As a result we get
\[ <\text{tr} B_{\nu} B_{\mu}> = \int d\sigma_{\mu\nu} d\sigma_{\nu\lambda} \times \text{tr}\{\Phi(x_0, w) F_{\mu\rho}(w) \Phi(w, x_0) \Phi(x_0, w') F_{\rho\lambda}(w') \Phi(w', x_0)\} \]

which is fully gauge invariant.

Using Eq. (91) we find for the correction to the total \(q\bar{q}\) Green’s function
\[ G = G^{(0)} + \Delta G, \]

\[ \Delta G = -\frac{g^2}{2} \int d\sigma_{\mu\nu}(w) \int d\sigma_{\mu'\nu'}(w') d^3 r(w_4) d^3 r(w'_4) \]
\[ G_c^{(1)}(r(T), r(w_4); T - w_4) G_c^{(8)}(r(w_4), r(w'_4), w_4 - w'_4) \]
\[ \times <F_{\mu\nu}(w) F_{\mu'\nu'}(w') > G_c^{(0)}(r(w'_4), r(0); w'_4), \quad (96) \]

where the integrals over \(d\sigma_{\mu\nu}, d\sigma_{\mu'\nu'}\) are taken over the surface \(\Sigma_{\mu\nu}\). It is convenient to identify \(\Sigma_{\mu\nu}\) with the minimal surface inside the contour \(C\), formed between the trajectories \(z(t)\) and \(\bar{z}(t)\). Introducing the straight line between \(z(t)\) and \(\bar{z}(t)\)

\[ w_\mu(t, \beta) = z_\mu(t) + \bar{z}_\mu(t)(1 - \beta) = R_\mu + r_\mu(\beta - \frac{m_1}{m_1 + m_2}) \]

with

\[ w_4 = z_4 = \bar{z}_4 = t \]

we may write the surface element as

\[ d\sigma_{\mu\nu}(w) = (w'_\mu \dot{w}_\nu - \dot{w}_\mu w'_\nu) dtd\beta \equiv a_{\mu\nu} dtd\beta \]

(98)

with \(w'_\mu = \frac{\partial w_\mu}{\partial \beta} = r_\mu, \dot{w}_\mu = \frac{\partial w_\mu}{\partial t}\). We also have in the c.m. system

\[ a_{i4} = r_i; \quad a_{ij} = e_{ijk} L_k \frac{1}{\tilde{m}} (\beta - \frac{m_1}{m_1 + m_2}), \]

(99)
while the Minkowskian angular momentum $L$ is given by

$$L_i = e_{ikl} r_k \cdot \frac{1}{r} \frac{\partial}{\partial r_l}.$$ (100)

In the nonrelativistic approximation we expand in powers of $\frac{1}{m_1}, \frac{1}{m_2}$. Hence $a_{ij}$ can be neglected in lowest order and we are left with only $a_{i4}$, i.e. in Eq. (96) only the electric field correlators should be kept. The field correlators have the following representation in terms of the two Lorentz invariants $D$ and $D_1$[29]

$$g^2 tr < E_i(w)E_k(w')> = \frac{1}{12} [\delta_{ik}(D(w-w') + D_1(w-w') + h_i^2 \frac{\partial D_1}{\partial h^2}) + h_i h_k \frac{\partial D_1}{\partial h^2}],$$ (101)

where $h_i = w_i - w'_i$. $D$ and $D_1$ are normalized as

$$D(0) + D_1(0) = g^2 < tr F_{\mu\nu}(0) > = \frac{1}{2} 4\pi^2 G_2.$$ (102)

$G_2$ is the standard definition of the gluonic condensate[32]

$$G_2 = \frac{\alpha_s}{\pi} < F^{a}_{\mu\nu} F^a_{\mu\nu} > = 0.012 GeV^4.$$ (103)

Inserting Eq. (101) into Eq. (96) and neglecting the terms $h_i h_k \sim O(\frac{1}{m^2})$ we get

$$\Delta G = -\frac{1}{24} G^{(1)}_{C}(r(T), r) d^3 r G^{(8)}_{C}(r, r')d^3 r'$$

$$+ r_i d\beta r'_i d\beta' d\beta'' \Delta(w - w') G^{(1)}_{C}(r', r(0)),$$ (104)

where we have defined

$$\Delta(w - w') = D(w - w') + D_1(w - w') + h_i^2 \frac{\partial D_1}{\partial h^2}.$$ (105)
Using the spectral decomposition for $G_c$

$$G_c^{(1,8)}(r, r', t) = \langle r | e^{-H_c^{(1,8)} t} | r' \rangle = \sum_n \psi_n^{(1,8)}(r) e^{-E_n^{(1,8)} t} \psi_n^{(1,8)*}(r'),$$

(106)

we can rewrite Eq. (96) for the matrix element of $\Delta G$ between singlet Coulomb wave functions

$$< n | \Delta G | n > = -\frac{e^{-E_n^{(1,8)} T}}{24} \int \frac{dp_4 dp}{(2\pi)^4} \Delta(p) d\beta d\beta' \sum_{k=0,1,...} \frac{< n | r_i e^{i(\beta - \frac{m_1}{m_1 + m_2}) r} | k > < k | r_i' e^{-i(\beta - \frac{m_1}{m_1 + m_2}) r'} | n >}{E_k^{(8)} - E_n - ip_4},$$

(107)

where $\Delta(p)$ is the Fourier transform of Eq. (105). The set of states $| k >$ in Eq. (107) with eigenvalues $E_k^{(8)}$ refer to the octet Hamiltonian piece in Eq. (90)

$$H^{(8)} = \frac{\vec{p}^2}{2m} + \frac{C_2 \alpha_s}{2N_c |\vec{r}|}.$$  

(108)

The correlator $\Delta(x)$ depends on $x$ as $\Delta(x) = f\left(\frac{t}{T_g}\right)$, and decays exponentially at large $|x|[42]$. For what follows it is crucial to compare the two parameters, $T_g$ and the Coulombic size of the $n$-th state of the $q\bar{q}$ system, $R_n = \frac{n}{m_c \alpha_s}$. In the Voloshin-Leutwyler case[34] it is assumed explicitly or implicitly that

Case (i)  $T_g \gg R_n$

In the opposite case

Case (ii)  $T_g \ll R_n$

as we shall see completely different dynamics occurs.
Writing \( G_0 + \Delta G = const \ e^{-(E_n^{(1)} + \Delta E_n)T} \approx const \ e^{-E_n^{(1)}T(1 - \Delta E_n T)} \) we finally obtain in case (i) for \( \Delta E_n \)

\[
\Delta E_n = \frac{\pi^2 G_2}{18} \frac{n |r_i| k < k |r_i| n >}{E_k^{(8)} - E_n^n}.
\] (109)

Results of the calculations [30] using the Voloshin-Leutwyler approximation (VLA) (Eq. (109) for charmonium and bottomonium) and using the standard value of \( G_2 \) [32] are given in Table 1. One can see that a rough agreement exists only for the lowest bottomonium state. Consider now the opposite case, \( T_g \ll R_n \). Since lattice measurements yield \( T_g \approx 0.2 fm \), we may expect that this case is generally applicable to all \( b \bar{b} \) and \( c \bar{c} \) states. However in this case it is not enough to keep only the \( w_2 \), but also sum up all NP terms, which amounts to the exponentiation of the NP contribution. The NP local potential appears in addition to the Coulomb term. This has been done in the framework of the local potential picture in Ref. [35], where the NP potential is expressed via correlators \( D(x) \) and \( D_1(x) \).

Results of calculations of Ref. [35] yield a very consistent picture both for levels and wave functions of bottomonium and quarkonium. To compare with VLA and our results here the results of Ref. [35] are listed in the middle column of the Table, demonstrating a much better agreement with experiment than results for the VLA. Note, that in Ref. [35] the NP interaction was not treated as a perturbation, but nonperturbatively by including the NP part in the potential. Consequently this explains why the predictions have improved. To summarize, because of the small \( T_g \approx 0.2 fm \), the potential picture is more adequate for quarkonia than the VLA formalism or QCD sum rules, including even the bottomonium case.
Table 1.

The experimental values and the predicted splittings in \( MeV \) of various states in bottomium and charmonium in the Voloshin-Leutwyler approximation and Ref. [35].

<table>
<thead>
<tr>
<th>splitting</th>
<th>VLA</th>
<th>[35]</th>
<th>exp.</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 2S - 1S(b\bar{b}) )</td>
<td>479</td>
<td>554</td>
<td>558</td>
</tr>
<tr>
<td>( 2S - 2P(b\bar{b}) )</td>
<td>181</td>
<td>112</td>
<td>123</td>
</tr>
<tr>
<td>( 3S - 2S(b\bar{b}) )</td>
<td>4570</td>
<td>342</td>
<td>332</td>
</tr>
<tr>
<td>( 2S - 1S(c\bar{c}) )</td>
<td>9733</td>
<td>582</td>
<td>670</td>
</tr>
</tbody>
</table>

5 IR and collinear singularities in FSR

It is known that in QED some matrix elements and partial cross-sections display singularities[43], which are of two general types; a) due to soft photon exchange (IR singularities) b) due to the collinear motion of an exchanged and emitted photon (collinear singularities). A similar situation exists in perturbative QCD, where the same lowest order amplitudes contain both IR and collinear singularities, see Ref. [44].

Examples of Feynman graphs which produce both types of singularities are given in Fig. 3. These graphs refer to the process \( e^+e^- \rightarrow \bar{q}q \) and the total cross section for the sum of three graphs where an additional gluon can be
Fig. 3. *Feynman graphs to order $\alpha_s$ with singularities, which are cancelled in the crosssection.*

generated is finite to the order $O(\alpha_s)$ and is equal to

$$
\sigma^{(1)} = \sigma^{(0)} \left\{ 1 + \frac{3C_F \alpha_s}{4\pi} \right\}.
$$

(110)

This is in line with the KLN theorem[45] derived and proved in the framework of QED.

Fig. 4. *The gauge-invariant $\gamma - \gamma$ amplitude, where all singularities cancel, while they are separately present in the parts of graphs obtained by cutting along the dash-dotted lines.*

We now argue that the situation is different in QCD when the nonperturbative
confining vacuum is taken into account. This in particular can be demonstrated in the framework of the FSR. We start with the QCD contribution to the photon self-energy part, which is shown graphically in Fig. 4. It has the form

$$\Pi_{\mu\nu}(Q) = (Q_{\mu}Q_{\nu} - Q^{2}\delta_{\mu\nu})\Pi(Q^{2}).$$  \hspace{1cm} (111)

In particular, the $O(\alpha_{s})$ part of $\Pi_{\mu\nu}(Q^{2})$, shown in Fig. 4b, yields an analytic function of $Q^{2}$, with the imaginary part (discontinuity across the cut $Q^{2} \geq 0$) given by the sum of the three graphs of Fig. 4a,b multiplied by the complementary parts as shown in by the dash-dotted lines in Fig. 4b. It is clear, that both the whole function $\Pi(Q^{2})$ and its total absorptive part is free from IR and collinear singularities, while each piece in the imaginary part, yielding partial crossections $\sigma_{(a)}$, $\sigma_{(b,c)}$ corresponding to the graphs Fig. 3a-c are IR divergent.

At this point the difference between QED and QCD can be felt even on the purely perturbative level. Namely, in QED the process $e^{+}e^{-} \rightarrow e^{+}e^{-} + n\gamma$ cannot be associated with the imaginary part of photon self-energy $\Pi_{\gamma}(Q^{2})$, since photons can be emitted in any amount off the $\Pi_{\gamma}(Q^{2})$ and hence should be summed up separately.

This fact is formulated as a notion of a physical electron, containing a bare electron plus any amount of additional soft photons (see the Bloch-Nordsiek method discussed for example in Ref. [43]). In QCD the situation is different, since separate gluons cannot escape the internal space of $\Pi_{\mu\nu}(Q^{2})$ (cannot be emitted), except when they create (pairwise, triplewise etc.) massive glueballs, or else when they are accompanied by the sea quark pairs forming hybrid states.

To illustrate our ideas we shall use the FSR, introducing the background
confining field, and using the method of Ref. [47]. We define the photon vacuum polarization function $\Pi(q^2)$ as a correlator of electromagnetic currents for the process $e^+ e^- \rightarrow \text{hadrons}$ in the usual way

$$-i \int d^4 x e^{iqx} <0|T(j_\mu(x)j_\nu(0))|0> = (q_\mu q_\nu - g_{\omega \nu} q^2)\Pi(q^2),$$  

where the imaginary part of $\Pi$ is related to the total hadronic ratio $R$ as

$$R(q^2) = \frac{\sigma(e^+ e^- \rightarrow \text{hadrons})}{\sigma(e^+ e^- \rightarrow \mu^+ \mu^-)} = 12\pi Im\Pi(\frac{q^2}{\mu^2}, \alpha_s(\mu)).$$

There are two usual approaches to calculate $\Pi(q^2)$. The first one is based on a purely perturbative expansion, which is now known to the order $0(\alpha_s^3)$[50]. The second one is the OPE approach[32], which includes the NP contributions in the form of local condensates. For two light quarks of equal masses ($m_u = m_d = m$) it yields for $\Pi(Q^2)$

$$\Pi(Q^2) = -\frac{1}{4\pi^2}(1 + \frac{\alpha_s}{\pi})ln\frac{Q^2}{\mu^2} + \frac{6m^2}{Q^2} + \frac{2m < q\bar{q}>}{Q^4} + \frac{\alpha_s < FF >}{12\pi Q^4} + ...$$

In what follows we shall include the NP fields as they enter into Green’s functions, i.e. nonlocally. Moreover, we shall be mostly interested in the large distance behaviour, where the role of NP fields is important. To this end we first of all write the exact expression for $\Pi(Q^2)$ in the presence of the nonperturbative background and formulate some general properties of the perturbative series. More specifically, the e.m. current correlator can be written in the form[17,47]

$$\Pi(Q^2) = \frac{1}{N} \int e^{iQx} d^4 x \int DB \int Da e^{-S_E(B+a)} \times tr(\gamma_\mu G_q(x,0)\gamma_\mu G_q(0,x))det(m + \hat{D}(B + a))$$
Here $G_q$ is the single quark Green’s function in the total field $B_\mu + a_\mu$,

$$G_q^{(B+a)}(x, y) = <x| (m + \partial - ig(\bar{B} + \bar{a}))^{-1}|y> . \quad (116)$$

The background quark propagator is conveniently written using the FSR as

$$G_q^{(B)}(x, y) = \int_0^\infty dse^{-K} DzP \exp [i g \int_x^y (B_\mu) dz_\mu] \exp [g \int_0^s \sigma_{\mu\nu} F_{\mu\nu} d\tau] \quad (117)$$

with $K = \frac{1}{4} \int_0^s \tilde{z}^2(\tau) d\tau$.

To simplify our analysis we take the limit $N_c \to \infty$ and drop the det term in Eq. (115). We are then left with only planar diagrams containing gluon exchanges $G_q^{(B)}$ in the external background field. Moreover, all gluon lines $G_q^{(B)}$ in the limit $N_c \to \infty$ are replaced by double fundamental lines[51] and we are left only with diagrams, where the area $S$ between the quark lines in $\Pi(Q^2)$ is divided into a number of pieces $\Delta S_k$, shown schematically in Fig. 5.

**Fig. 5.** A generic Feynman diagram for the photon self-energy in high order of background perturbation theory at large $N_c$. All areas $\Delta S_k$ between double gluon lines are covered by the confining film, yielding the area law in Eq. (118).

The infrared behaviour of $\alpha_s(Q^2)$ at small $Q^2$ is connected to the limit of large areas of $S$. In this limit the product of all phase integrals $\Phi(x_i, y_i) =$
\( \exp(ig \int_{x_i}^{y_i} B_{\mu} dz_{\mu}) \) from all Green’s functions can be averaged using the area law, i.e. we have (modulo spin insertions \( \sigma F \), which are unimportant for large distances\[17\])

\[
< \Pi_{i=1}^{n} \Phi(x_{i}, y_{i}) >_{B} = \Pi_{k=1}^{n} < W(\Delta S_{k}) > \\
\approx \exp(-\sigma \sum_{k=1}^{n} \Delta S_{k}) \text{ for } N_{c} \to \infty. \tag{118}
\]

This last factor serves as the IR regularizing factor in the Feynman integral, preventing any type of IR divergence. Using representations (115) and (117), we can formulate the following theorem.

**Theorem:**

Any term in the perturbative expansion of \( \Pi(Q^2) \), Eq. (115), in powers of \( ga_{\mu} \) at large \( N_{c} \) can be written as a configuration space Feynman diagram with an additional weight \( < W_s(C_{i}) > \) for each closed contour. Here \( W_s(C_{i}) \) is the Wilson loop with spin insertions, as in Eq. (117). Brackets denote averaging over background fields.

This theorem is easily proved by expanding Eq. (115) in powers of \( ga_{\mu} \) and using double quark lines, Eq. (117), for gluon lines. Since at large \( N_{c} \) we can replace adjoint color indices by doubled fundamental ones, then in each planar diagram the whole surface is divided into a set of closed fundamental contours, for which Eq. (118) holds true, again due to large \( N_{c} \). Thus to each contour is assigned the Wilson loop \( < W(C_{i}) > \). The rest is the usual free propagators written in the FSR.

Looking now at the large distance behaviour of the resulting configuration space planar Feynman diagram, we may derive from the above theorem the following corollary.
Corollary:

Any planar diagram for $\Pi(Q^2)$ is convergent at large distances in the Euclidean space–time in the confining phase, when $< W(C_i) > \sim exp(-\sigma S_i)$.

The proof is trivial, since the free planar diagram may diverge at large distances at most logarithmically. The kernel $< W(C_i) >$ makes all integrals convergent at large distances. At small distances (i.e. for small area, $S_i \to 0$) the kernel $< W(C_i) >$ behaves as

$$< W(C_i) > \sim \exp\left(-\frac{g^2 < F_{\mu\nu}^a F_{\mu\nu}^a(0) > S_i^2}{24N_c}\right)$$

(see second reference in Ref. [29]). Hence the structure of small-distance perturbative divergencies is the same for the planar diagram whether the NP background is present or not. Therefore the usual renormalization technique (e.g. the dimensional renormalization) is applicable. As a result the planar Feynman diagram contributions to $\alpha_s^n \Pi^{(n)}(Q^2)$ in the background is made finite also at small distances. The consequence of this is that any renormalized term $\alpha_s^n \Pi^{(n)}(Q^2)$ in the perturbative expansion of $\Pi(Q^2)$ is finite at all finite Euclidean $Q^2$, including $Q^2 = 0$.

Then it follows that we can choose the renormalization scheme for $\alpha_s$, which renders $\alpha_s$ finite for all $0 \leq Q^2 < \infty$ and the Landau ghost pole will be absent. To make explicit this renormalization of $\alpha_s$, we can write the perturbative expansion of the function $\Pi(Q^2)$, Eq. (115), as

$$\Pi(Q^2) = \Pi^{(0)}(Q^2) + \alpha_s \Pi^{(1)}(Q^2) + \alpha_s^2 \Pi^{(2)}(Q^2) + ...$$

(119)

We now again use the large $N_c$ approximation, in which case $\Pi^{(0)}$ contains
only simple poles in $Q^2[51]$:

$$\Pi^{(0)}(Q^2) = \frac{1}{12\pi^2} \sum_{n=0}^{\infty} \frac{C_n}{Q^2 + M_n^2},$$  \hspace{1cm} (120)$$

where the mass $M_n$ is an eigenvalue of the Hamiltonian $H^{(0)}$. It contains only quarks and background field $B_\mu$,

$$H^{(0)}\Psi_n = M_n \Psi_n,$$  \hspace{1cm} (121)$$

while the constant $C_n$ is connected to the eigenfunctions of $H^{(0)}[47]$. We have

$$C_n = \frac{N_c Q^2 f_n^2 \lambda_n^2}{M_n},$$  \hspace{1cm} (122)$$

where

$$f_n = \frac{1}{2\pi^2} \int_0^\infty u_n(k) k dk \frac{E_k + m}{E_k} \left(1 + \frac{1}{3} \frac{(E_k - m)}{(E_k + m)}\right),$$

$$\lambda_n^2 = 2\pi^2 \int_0^\infty d\nu_n^2(k) \nu_n(k)^{-1}; \quad E_k \equiv (k^2 + m^2)^{1/2}.$$

In what follows we are mostly interested in the long-distance effective Hamiltonian. It can be obtained from $G_{q\bar{q}}$ for large distances, $r \gg T_g$, where $T_g$ is the gluonic correlation length of the vacuum, $T_g \approx 0.2 fm[33]$:

$$G_{q\bar{q}}(x,0) = \int DB\eta(B)tr(\gamma_\mu G_q(x,0)\gamma_\mu G_q(0,x)) = \langle x| e^{-H^{(0)}|x||0} \rangle .$$  \hspace{1cm} (123)$$

At these distances we can neglect in Eq. (117) the quark spin insertions $\sigma_{\mu\nu} F_{\mu\nu}$ and use the area law:

$$\langle W_C \rangle \rightarrow \exp(-\sigma S_{\text{min}}),$$  \hspace{1cm} (124)$$

where $S_{\text{min}}$ is the minimal area inside the loop $C$.  

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Then the Hamiltonian in Eq. (123) is readily obtained by the method of Ref. [36]. In the c.m. system for the orbital momentum \( l = 0 \) it has the familiar form:

\[
H^{(0)} = 2\sqrt{\hat{p}^2 + m^2} + \sigma r + \text{const},
\]

where a constant appears due to the perimeter term in \( <W_C> \). For \( l = 2 \) a small correction from the rotating string appears[42], which we neglect in first approximation.

Now we can use the results of the quasiclassical analysis of \( H^{(0)} \)[52], where the values of \( M_n, C_n \) have already been found. They can be represented as follows \((n = n_r + l/2, n_r = 0, 1, 2, ..., l = 0, 2)\)

\[
M_n^2 = 2\pi\sigma(2n_r + l) + M_0^2,
\]

where \( M_0^2 \) is a weak function of the quantum numbers \( n_r, l \) separately, comprising the constant term of Eq. (125). In what follows we shall put it equal to the \( \rho \)-meson mass, \( M_0^2 \approx m_\rho^2 \). For \( C_n \) one obtains quasiclassically[52]

\[
C_n(l = 0) = \frac{2}{3} Q_f^2 N_c m_0^2, \quad m_0^2 \equiv 4\pi\sigma
\]

\[
C_n(l = 2) = \frac{1}{3} Q_f^2 N_c m_0^2.
\]

Using the asymptotic expressions Eqs. (126-127) for \( M_n, C_n \) and starting with \( n = n_0 \), we can write

\[
\Pi^{(0)}(Q^2) = \frac{1}{12\pi^2} \sum_{n=0}^{n_0-1} \frac{C_n}{M_n^2 + Q^2} - \frac{Q_f^2 N_c}{12\pi^2} \psi \left( \frac{Q^2 + M_0^2 + n_0 m_0^2}{m_0^2} \right) + \text{divergent constant}.
\]
Here we have used the equality
\[
\sum_{n=n_0}^{\infty} \frac{1}{M_n^2 + Q^2} = -\frac{1}{m_0^2} \psi\left(\frac{Q^2 + M_0^2 + n_0m_0^2}{m_0^2}\right) + \text{divergent constant} \quad (129)
\]
and \(\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}\).

In Eq. (128) we have separated the first \(n_0\) terms to treat them nonquasiclassically, while keeping for the other states with \(n \geq n_0\) the quasiclassical expressions (126-127). In what follows, however, we shall put \(n_0 = 1\) for simplicity. We shall show below that even in this case our results will reproduce \(+e\) experimental data with good accuracy (see Ref. [47] for details).

Consider the asymptotics of \(\Pi^{(0)}(Q^2)\) at large \(Q^2\). Using the asymptotics of \(\psi(z)\):
\[
\psi(z)_{z \to \infty} = \ln z - \frac{1}{2z} - \sum_{k=1}^{\infty} \frac{B_{2k}}{2^k z^{2k}},
\]
where \(B_n\) are Bernoulli numbers, we obtain from Eq. (128)
\[
\Pi^{(0)}(Q^2) = -\frac{Q^2 N_c}{12\pi^2} \ln \frac{Q^2 + M_0^2}{\mu^2} + O\left(\frac{m_0^2}{Q^2}\right). \quad (131)
\]

We can easily see that this term coincides at \(Q^2 \gg M_0^2\) with the first term in the OPE (114) – the logarithmic one. Taking the imaginary part of Eq. (131) at \(Q^2 \to -s\) we find
\[
R(q^2) = 12\pi Im \Pi^{(0)}(-s) = N_c Q_f^2, \quad (132)
\]
i.e. it means that we have obtained for \(\Pi^{(0)}\) the same result as for free quarks.

This fact is the explicit manifestation of the quark–hadron duality.

The analysis of more complicated planar graphs as in Fig. 5 can be carried
out as in Ref. [47], yielding in this way a new perturbative series with $\alpha_s$
renormalized in the background fields and having therefore no Landau ghost
poles. We refer to Refs. [17] and [47] for more details.

Now instead we return to the diagrams in Fig. 3, which give the lowest order
perturbative amplitudes associated with the production of 2 and 3 jets. The
corresponding photon self-energy part is depicted in Fig. 4b. When the non-
perturbative interaction is disregarded, even in the hadronization process, so
that a gluon emitted with the moment $k$ is directly associated with a gluon jet,
then the singularities of the partial cross sections of Fig. 3 are transmitted into
the singularities of the jet cross sections. These are cured by the introduction
of the jet thickness, as in Sterman-Weinberg method[48] or introducing finite
angular resolution $\eta_0$ to distinguish 2-jet and 3-jet events (see Refs. [44],[53]
for details). In what follows we show, that in the FSR with account of back-
ground fields, all IR and collinear singularities disappear. Therefore the cross
sections for 2-jet and 3-jet events remain finite.

As was discussed above, in the leading $1/N_c$ approximation we have only
planar graphs for $\Pi(Q^2)$, describing ”1-jet events”, which actually create a
constant behaviour of $R(q^2)$ (apart from new opening thresholds), exactly
reproducing the hadronic ratio, see Eq. (132). Speaking of 2-jet events we
may actually consider the next approximation in $1/N_c$, since we need an extra
quark loop in the $\Pi(Q^2)$. This can be easily derived from Eq. (115), where the
determinant can be expanded in the FSR

$$
\ln \det(m + \hat{D}) = \frac{1}{2} \ln[\det(m^2 - \hat{D}^2)] = \frac{1}{2} tr \ln(m^2 - \hat{D}^2),
$$

(133)
where we have used the symmetry property of the spectrum of $\hat{D}$. Hence

$$\det(m + \hat{D}) = \exp \left\{ -\frac{1}{2} t r \int \frac{ds}{s} \xi(s) e^{-s m^2 - K} D z_{xx} W_\sigma(A, F) \right\}$$

with $\xi(s)$ a regularizing factor. For this we may take $\xi(t) = \lim_{s \to 0} \frac{d}{ds} M^2 s t^s |_{s=0}$ or we can use the Pauli-Villars form for $\xi(t)$. Furthermore in Eq. (134) we have

$$W_\sigma(A, F) = P_A P_F \exp i g \int \int A_\mu dz_\mu \cdot \exp g \int_0^s \sigma_{\mu \nu} F_{\mu \nu} d\tau.$$

It is clear that $\det(m + \hat{D})$ allows an expansion in the number of quark loops, which is done by expanding the exponential in Eq. (134). Keeping only one quark loop for the 2-jet events, we obtain the graph, shown in Fig. 6 with an internal quark loop from the determinant. It is essential, that the whole region between the loops is covered by the NP correlators, creating a kind of "film" - the world surface of the string, with perturbative (i.e. generated by $a_\mu$) exchanges.

![Fig. 6. Photon self-energy graph corresponding to the 2-jet cross-section with one dynamical quark loop.](image)

It is clear that in this situation quarks are never on the mass shell (in contrast to the purely perturbative case) and therefore both IR and collinear singularities are absent. We can consider also the amplitude for the 3-jet event, with one gluon jet. Its perturbative amplitude corresponds to Fig. 3b,c. The
perturbative situation is discussed in detail in Ref. [53], and the 3-jet cross
section for the process $e^+e^- \rightarrow q\bar{q}g$ is given by

$$
\frac{1}{\sigma} \frac{d^2\sigma}{dx_1dx_2} = C_F \frac{\alpha_s}{2\pi} \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)},
$$

where the integration region is $0 \leq x_1, x_2 \leq 1$, $x_1 + x_2 \geq 1$. The integral
is divergent both due to collinear and IR effects, since $1 - x_1 = x_2 E_g(1 - \cos \theta_{2g})/\sqrt{s}$ and $1 - x_2 = x_1 E_g(1 - \cos \theta_{1g})/\sqrt{s}$, where $E_g$ is gluon energy and
$\theta_{ig}$ is angle between the gluon and i-th quark. To handle these divergencies
one can use the so-called JADE algorithm[54], where the minimum invariant
mass of a parton pair is larger than $y_s$, i.e. $\text{min}(p_i + p_j)^2 > y_s$. With this
condition the energy region for 3-jet events looks like

$$
0 < x_1, x_2 < 1 - y, \quad x_1 + x_2 > 1 + y.
$$

The nonperturbative counterpart is obtained in two ways: a) the emitted gluon
is accompanied by another gluon, forming together a two-gluon glueball (as
was calculated in the framework of FSR in Ref. [49]), or b) a hybrid formation,
when the emitted gluon is accompanied by a sea quark-antiquark pair. Both
possibilities are depicted in Fig. 7. For the hybrid case we should expand
the determinant term (the exponential in Eq. (134)) to the second power,
producing in this way the two quark loops.

It is clear in this case, that all particles, including the gluon, are off-shell
and IR and collinear singularities are absent. Moreover, assuming as usual
almost collinear hadronisation[53] we should replace the momenta of quarks
and gluons by the corresponding momenta of hadrons. We can easily see that
the factor $\frac{1}{2p_1k}$ is singular in case of $q\bar{q}g$ system becomes $\frac{1}{2p_1k + \Delta M^2}$, where
Fig. 7. Graphs corresponding to the 3-jet cross-section with the gluon hadronized into a glueball or accompanied by a sea-quark pair forming a hybrid.

\[ \Delta M^2 = M_1^2 - M_2^2 + M_g^2 \] and \( M_g \) is the hybrid (glueball) mass, so that \( \Delta M^2 \) is in the GeV region. It effectively cuts off the singularity at small \( y_{cut} \), as is clearly seen in the experimental data[55]. The same reasoning applies for higher jet events. The only issue which exists is of experimental character. It amounts to the precise definition of the number of jets, i.e. to classify the hadrons between the several jets.

Thus experiment as well as background perturbation theory do not exhibit collinear and IR singularities pertinent to the standard perturbation theory.

6 FSR at nonzero temperature

Within the framework of the FSR the problem of the various Green’s functions at finite temperature can be studied[11,12]. We first discuss the basic formalism for \( T > 0 \) and then turn to the calculation of the gluon and quark Green’s functions.
6.1 Basic equations

We start with standard formulae of the background field formalism[16,17] generalized to the case of nonzero temperature. We assume that the gluonic field $A_\mu$ can be split into the background field $B_\mu$ and the quantum field $a_\mu$

$$A_\mu = B_\mu + a_\mu,$$

(138)

both satisfying the periodic boundary conditions

$$B_\mu(z_4, z_i) = B_\mu(z_4 + n\beta, z_i); \quad a_\mu(z_4, z_i) = a_\mu(z_4 + n\beta, z_i),$$

(139)

where $n$ is an integer and $\beta = 1/T$. The partition function can be written as

$$Z(V, T) = \langle Z(B) \rangle_B$$

(140)

with

$$Z(B) = N\int D\phi \exp \left(-\int_0^\beta d\tau \int d^3x L_{tot}(x, \tau)\right)$$

(141)

and where $\phi$ denotes all set of fields $a_\mu, \Psi, \Psi^+$. $L_{tot}$ is the same as $L(a)$ defined in Eq. (38) and $N$ is a normalization constant. Furthermore, in Eq. (140) $\langle \rangle_B$ means some averaging over (nonperturbative) background fields $B_\mu$. The precise form of this averaging is not needed for our purpose.

Integration over the ghost and gluon degrees of freedom in Eq. (140) yields the same answer as Eq. (44), but where now all fields are subject to the periodic boundary conditions (139).

$$Z(B) = N'(\det W(B))^{-1/2}[\det(-D_\mu(B)D_\mu(B + a))]_{\alpha = \frac{\mu}{T}}$$
\[
\times \left\{ 1 + \sum_{l=1}^{\infty} \frac{S_{nL}(a = \frac{g}{\pi T})}{l!} \right\} \exp \left( -\frac{1}{2} J G J \right) \bigg|_{J_{\mu} = D_{\mu}(B)F_{\mu\nu}(B)}.
\]

(142)

We can consider strong background fields, so that \( g B_{\mu} \) is large (as compared to \( \Lambda_{QCD}^2 \)), while \( \alpha_s = g^2/4\pi \) in that strong background is small at all distances. Moreover, it was shown that \( \alpha_s \) is frozen at large distances[17]. In this case Eq. (142) is a perturbative sum in powers of \( g^n \), arising from the expansion in \( (g a_{\mu})^n \).

In what follows we shall discuss the Feynman graphs for the free energy \( F(T) \), connected to \( Z(B) \) via

\[
F(T) = -T \ln \langle Z(B) \rangle_B.
\]

(143)

As will be seen, the lowest order graphs already contain a nontrivial dynamical mechanism for the deconfinement transition, and those will be considered in the next subsection.

6.2 The lowest order gluon contribution

To lowest order in \( g a_{\mu} \) (keeping all dependence on \( g B_{\mu} \) explicit) we have

\[
Z_0 = e^{-F_0(T)/T} = N' \langle \exp(-F_0(B)/T) \rangle_B,
\]

(144)

where using Eq. (142) \( F_0(B) \) can be written as

\[
\frac{1}{T} F_0(B) = \frac{1}{2} \ln \det G^{-1} - \ln \det(-D^2(B)) =
\]

\[
= Sp \left\{ -\frac{1}{2} \int_0^\infty \xi(t) \frac{dt}{t} e^{-t G^{-1}} + \int_0^\infty \xi(t) \frac{dt}{t} e^{t D^2(B)} \right\}.
\]

(145)
In Eq. (145) $Sp$ implies summing over all variables (Lorentz and color indices and coordinates) and $\xi(t)$ is a regularization factor as in Eq. (134). Graphically, the first term on the r.h.s. of Eq. (145) is a gluon loop in the background field, while the second term is a ghost loop.

Let us turn now to the averaging procedure in Eq. (144). With the notation $\varphi = -F_0(B)/T$, we can exploit in Eq. (144) the cluster expansion\[14\]

$$
\langle \exp \varphi \rangle_B = \exp \left( \sum_{n=1}^{\infty} \langle \varphi^n \rangle \frac{1}{n!} \right) 
= \exp \{ \langle \varphi \rangle_B + \frac{1}{2} \langle \varphi^2 \rangle_B - \langle \varphi^2 \rangle_B + O(\varphi^3) \}. 
$$

To get a closer look at $\langle \varphi \rangle_B$ we first should discuss the thermal propagators of the gluon and ghost in the background field. We start with the thermal ghost propagator and write the FSR for it\[11\]

$$
(-D^2)^{-1}_{xy} = \langle x | \int_0^\infty dt e^{tD^2(B)} | y \rangle = \int_0^\infty dt (Dz)^w_{xy} e^{-\hat{K}\hat{\Phi}(x,y)}. 
$$

Here $\hat{\Phi}$ is the parallel transporter in the adjoint representation along the trajectory of the ghost:

$$
\hat{\Phi}(x,y) = P \exp(ig \int \bar{B}_\mu(z)dz_\mu) \tag{148}
$$

and $(Dz)^w_{xy}$ is a path integration with boundary conditions imbedded (denoted by the subscript $(xy)$) and with all possible windings in the Euclidean temporal direction (denoted by the superscript $w$). We can write it explicitly as

$$
(Dz)^w_{xy} = \lim_{N \to \infty} \prod_{m=1}^{N} \frac{d^4\xi(m)}{(4\pi^2)^2} \sum_{n=0,\pm,\ldots} \frac{d^4p}{(2\pi)^4} \exp \left[ ip \left( \sum_{m=1}^{N} \zeta(m) - (x-y) - n\beta\delta_{\mu4} \right) \right]. \tag{149}
$$
Here, $\zeta(k) = z(k) - z(k - 1)$, $N \varepsilon = t$. We can readily verify that in the free case, $\hat{B}_\mu = 0$, Eq. (147) reduces to the well-known form of the free propagator

$$(-\partial^2)^{-1}_{xy} = \int_0^\infty dt \exp \left[ -\sum_{m=1}^N \frac{\zeta^2(m)}{4 \varepsilon} \right] \prod_m d\zeta(m) \sum_n \frac{d^4 p}{(2\pi)^4} \times \exp \left[ ip \left( \sum_m \zeta(m) - (x - y) - n_\beta \delta_{m4} \right) \right] = \sum_n \int_0^\infty \exp \left[ -p^2 t - ip(x - z) - ip_4 n_\beta \right] dt \frac{d^4 p}{(2\pi)^4}$$

with

$$d\zeta(m) \equiv \frac{d\zeta(m)}{(4\pi \varepsilon)^2}.$$

Using the Poisson summation formula

$$\frac{1}{2\pi} \sum_{n=0,\pm 1,\pm 2...} \exp(ip_4 n_\beta) = \sum_{k=0,\pm 1,...} \delta(p_4\beta - 2\pi k)$$

we finally obtain the standard form

$$(-\partial^2)^{-1}_{xy} = \sum_{k=0,\pm 1,...} \int \frac{T d^3 p}{(2\pi)^3} \frac{\exp[-ip(x - y) - i2\pi kT(x_4 - y_4)]}{p_4^2 + (2\pi kT)^2}.$$  \hspace{1cm} (152)

Note that, as expected, the propagators (147) and (152) correspond to a sum of ghost paths with all possible windings around the torus. The momentum integration in Eq. (149) asserts that the sum of all infinitesimal ”walks” $\zeta(m)$ should be equal to the distance $(x - y)$ modulo $N$ windings in the compactified fourth coordinate. For the gluon propagator in the background gauge we obtain similarly to Eq. (147)

$$G_{xy} = \int_0^\infty dt(Dz)^w_{xy} e^{-K \hat{F}_F(x, y)},$$

where

$$\hat{F}_F(x, y) = P_F P \exp \left( -2ig \int_0^t \hat{F}(z(\tau)) d\tau \right) \exp \left( ig \int_y^x \hat{B}_\mu dz_\mu \right).$$

$$52$$
The operators $P_P P$ are used to order insertions of $F$ on the trajectory of the gluon.

Now we come back to the first term in Eq. (146), $\langle \varphi \rangle_B$, which can be represented with the help of Eqs. (147) and (153) as

$$\langle \varphi \rangle_B = \int \frac{dt}{t} \zeta(t) d^4 x (Dz)^w_{xx} e^{-K} \left[ \frac{1}{2} tr \langle \Phi_F(x, x) \rangle_B - \langle tr \Phi(x, x) \rangle_B \right], \quad (155)$$

where $tr$ implies summation over Lorentz and color indices. We can easily show[11] that Eq. (155) yields for $B_\mu = 0$ the usual result for the free gluon gas:

$$F_0(B = 0) = -T \varphi(B = 0) = -(N_c^2 - 1) V_3 \frac{T^4 \pi^2}{45}. \quad (156)$$

### 6.3 The lowest order quark contribution

Integrating over the quark fields in Eq. (140) leads to the following additional factor in Eq. (142)

$$\det(m + \dot{D}(B + a)) = [\det(m^2 - \ddot{D}^2(B + a))]^{1/2}. \quad (157)$$

In the lowest approximation, we may omit $a_\mu$ in Eq. (157). As a result we get a contribution from the quark fields to the free energy

$$\frac{1}{T} F_0^q(B) = -\frac{1}{2} \ln \det(m^2 - \ddot{D}^2(B)) = -\frac{1}{2} Sp \int_0^\infty \frac{dt}{t} e^{-tm^2 + t\dot{D}^2(B)}, \quad (158)$$

where $Sp$ has the same meaning as in Eq. (145) and

$$\dot{D}^2 = (D_\mu \gamma_\mu)^2 = D_\mu^2(B) - g F_{\mu\nu} \sigma_{\mu\nu} \equiv D^2 - g\sigma F; \quad (159)$$

$$\sigma_{\mu\nu} = \frac{i}{4} (\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu).$$
Our aim now is to exploit the FSR to represent Eq. (158) in a form of the path integral, as was done for gluons in Eq. (147). The equivalent form for quarks must implement the antisymmetric boundary conditions pertinent to fermions. We find

\[
\frac{1}{T} F_0^q(B) = -\frac{1}{2} t r \int_0^\infty \xi(t) \frac{d t}{t} d^4 x (Dz)_{xx}^w e^{-K-m^2} W_\sigma(C_n),
\]

(160)

where

\[
W_\sigma(C_n) = P_F P_A \exp \left( ig \int A_\mu dz_\mu \right) \exp \left( g (\sigma F) \right),
\]

and

\[
(Dz)_{xy}^w = \prod_{m=1}^N \frac{d^4 \zeta(m)}{(4\pi\varepsilon)^2} \sum_{n=0,\pm1,\pm2,\ldots} \frac{d^4 p}{(2\pi)^4} \exp \left[ ip \left( \sum_{m=1}^N \zeta(m) - (x-y) - n\beta \delta_{4} \right) \right].
\]

(161)

It can readily be checked that in the case \( B_\mu = 0 \) the well known expression for the free quark gas is recovered, i.e.

\[
F_0^q(\text{free quark}) = -\frac{7\pi^2}{180} N_c V_3 T^4 \cdot n_f,
\]

(162)

where \( n_f \) is the number of flavors. The derivation of Eq. (162) starting from the path-integral form (160) is done similarly to the gluon case given in the Appendix of the last reference in Ref. [11].

The loop \( C_n \) in Eq. (160) corresponds to \( n \) windings in the fourth direction. Above the deconfinement transition temperature \( T_c \) one sees in Eq. (160) the appearance of the factor

\[
\Omega = P \exp \left[ ig \int_0^\beta B_4(z) dz_4 \right].
\]

(163)
For the constant field $B_4$ and $B_i = 0, i = 1, 2, 3$, we obtain

$$
\langle F \rangle = -\frac{V_3}{\pi^2} \text{tr}_c \sum_{n=1}^{\infty} \frac{\Omega^n + \Omega^{-n}}{n^4} (-1)^{n+1}.
$$

(164)

This result coincides with the one obtained in the literature[36].

7 Discussion and conclusions

Three basic approaches to QCD which are largely used till now are: i) lattice simulations ii) standard perturbation theory, and iii) OPE and QCD sum rules[41]. The two latter methods are analytic and have given enormous amount of theoretical information about the high-energy domain, where perturbative methods are applicable, and about nonperturbative effects both in the high and low energy regions.

These methods have their own limitations. In particular, the standard perturbation theory is plagued by the Landau ghost pole and IR renormalons and slow convergence, which necessitates the introduction of methods, where summation of perturbative subseries can be done automatically and the Landau ghost pole is absent. The QCD sum rules are limited by the use of only a few OPE terms, while the OPE series is known to be badly convergent and at best asymptotic.

One of the great challenges of QCD is to have a tractable analytic treatment of it. In particular, the improvement of the standard perturbation theory and the search for a systematic approach to nonperturbative phenomena are important objectives. The methods presented here in the present paper, commonly entitled The Fock-Feynman-Schwinger Representation, are meant to exactly
do this. The main advantage of the FSR is that it allows to treat both perturbative and nonperturbative configurations of the gluonic fields.

In case of purely perturbative fields the FSR yields a simple method of summation and exponentiation of perturbative diagrams[15]. Nonperturbative fields are introduced in the FSR naturally via the Field Correlator Method[29,56]. A recent discovery on the lattice of the Gaussian correlator dominance (Gaussian Stochastic Model) (see Ref. [46] for discussion and further references.) makes this method accurate (up to a few percent). There is another very important result of taking into account nonperturbative fields in the QCD vacuum: this fact allows to develop perturbation theory in the nonperturbative background – which is realistic unlike the standard perturbation theory. It contains no Landau ghost poles and IR renormalons[17].

As two applications of the FSR we have considered the problem of collinear singularities and finite temperature QCD. As an important special feature we should stress the absence in the background perturbation theory of all IR and collinear singularities pertinent to standard perturbation theory. This feature discussed here, opens new perspectives to the application of FSR to high-energy QCD processes. Although not discussed here, the FSR can readily be extended to treat deep inelastic scattering, Drell-Yan and other processes, including the fundamental problem of the connection between constituent quark-gluon model and parton model.

Moreover, as shown here the FSR can be used to describe QCD at nonzero temperature at above and around phase transition point. As was shown before[11–13] the vacuum is predominantly magnetic and nonperturbative above $T_c$. Therefore methods based on FSR are working well in this region. In conclu-
sion, since the FSR is replacing field degrees of freedom by corresponding quantum mechanical ones it has the advantage, that the results can often be interpreted in a simple and transparant way. It has been applied with success to both Abelian and non-Abelian situations. We have found that the FSR is a powerful approach for studying problems in QCD.

8 Acknowledgements

This work was started while one of the authors (Yu.S) was a guest of the Institute for Theoretical Physics of Utrecht University. The kind hospitality of the Institute and all persons involved and useful discussions with N. van Kampen, Th. Ruijgrok and G. ’t Hooft are gratefully acknowledged. The authors have been partially supported by the grant INTAS 00-110. One of the authors (J.T) would like to thank the TQHN group at the University of Maryland and the theory group at TJNAF for their kind hospitality. Yu. S. was partially supported by the RFFI grants 00-02-17836 and 00-15-96786 and also by the DOE contract DE-AC05-84ER40150 under which SURA operates the TJNAF.

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