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Models and Stability Analysis

of

Boiling Water Reactors

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ABSTRACT

We have studied the nuclear-coupled thermal-hydraulic stability of boiling water reactors (BWRs) using a model that includes: space-time modal neutron kinetics based on spatial ω -modes; single- and two-phase flow in parallel boiling channels; fuel rod heat conduction dynamics; and a simple model of the recirculation loop. The BWR model is represented by a set of time-dependent nonlinear ordinary differential equations, and is studied as a dynamical system using the modern bifurcation theory and nonlinear dynamical systems analysis. We first determine the stability boundary (SB) — or Hopf bifurcation set — in the most relevant parameter plane, the inlet-subcooling-number/external-pressure-drop plane, for a fixed control rod induced external reactivity equal to the 100% rod line value; then we transform the SB to the practical power-flow map used by BWR operating engineers and regulatory agencies. Using this SB, we show that the normal operating point at 100% power is very stable, that stability of points on the 100% rod line decreases as the flow rate is reduced, and that operating points in the low-flow/high-power region are least stable. We also determine the SB that results when the modal kinetics is replaced by simple point reactor kinetics, and we thereby show that the first harmonic mode does not have a significant effect on the SB. However, we later show that it nevertheless has a significant effect on stability because it affects the basin of attraction of stable operating points. Using numerical simulations we show that, in the important low-flow/high-power region, the Hopf bifurcation that occurs as the SB is crossed is subcritical; hence, growing oscillations can result following small finite perturbations of *stable* steady-states on the 100% rod line at points in the low-flow/high-power region. Numerical simulations are also performed to calculate the decay

ratios (DRs) and frequencies of oscillations for various points on the 100% rod line. It is determined that the U.S. NRC requirement of $DR < 0.75 - 0.8$ is not rigorously satisfied in the low-flow/high-power region; hence, this region should be avoided during normal startup and shutdown operations. The frequency of oscillation is shown to decrease as the flow rate is reduced. Moreover, the simulation frequency of $0.5Hz$ determined in the low-flow/high-power region is consistent with those observed during actual instability incidents. Additional numerical simulations show that in the low-flow/high-power region, for the same initial conditions, the use of point kinetics leads to damped oscillations, whereas the model that includes the modal neutron kinetics equations results in growing nonlinear oscillations. Thus, we show that side-by-side out-of-phase growing power oscillations result due to the very important first harmonic mode effect, and that the use of point kinetics, which fails to predict these growing oscillations, leads to dramatically nonconservative results. Finally, the effect of a simple recirculation loop model that we have developed is studied by carrying out additional stability analyses and additional numerical simulations. We show that the loop has a stabilizing effect at certain points on the 100% rod line for time delays equal to integer multiples of the natural period of oscillation, whereas it has a destabilizing effect for half-integer multiples. However, for more practical time delays, it is determined that the overall effect generally is destabilizing. Inclusion of the recirculation loop also leads to *islands of instability* in the parameter plane and power-flow map, which do not exist in the absence of the recirculation loop.

I INTRODUCTION

Nuclear-coupled density-wave oscillation (DWO) instability can manifest itself under certain conditions in a BWR. Incidents involving this instability have occurred in several operating BWRs, and they also have been observed during stability tests performed at several operating BWR plants worldwide.^{1,2} Hence, it is important to study and completely understand this instability in order to develop confidence in the safe operation of existing BWRs. Moreover, such studies also are important for the safe operation of the next generation of BWRs, so-called advanced BWRs and simple BWRs. Interest in the study of BWR stability has been motivated by several instability events, including the out-of-phase power oscillation incident at the Caorso BWR² in 1984 and the in-phase power oscillation incident at the LaSalle-2 BWR in 1988. However, the many incidents that have occurred since then, including the one that took place recently at the Laguna Verde BWR³ (January 24, 1995), emphasize the continuing need to study and understand BWR stability.

Over the years extensive research has been carried out to study both side-by-side out-of-phase oscillations and core-wide in-phase oscillations in BWRs.^{2,4} Complex computer codes have been written, simple models have been developed and experimental tests have been carried out. Below we summarize some of this work.

Belblidia et al.⁵ adopted a nodal approach to describe the neutron kinetics of a BWR core. They subdivided the core into radial zones and developed nodal kinetics equations by introducing internodal coupling. Using this model they showed that the point kinetics model representation of BWR neutron kinetics yields conservative results, but that for better

assessment of core stability, radial coupling effects should be included. However, we shall show that the point kinetics model representation can yield dramatically nonconservative results by failing to capture the important effect of the first harmonic mode, which leads to out-of-phase growing power oscillations, and only a modal kinetics representation will predict those growing oscillations. Nevertheless, the mass-flux/pressure-drop results predicted by their model compared well with production codes such as FIBWR⁶ and COBRA-III.⁷

Peng et al.⁸ developed a linear frequency domain computer code (NUFREQ-NP) for BWR stability analysis under conditions of either forced or natural circulation. That code is based on a one-dimensional drift flux model for the two-phase flow and takes into account subcooled boiling, arbitrary nonuniform axial and radial power shapes, distributed local losses, detailed fuel element dynamics, and system pressure perturbation. Peng et al.⁸ compared stability results they obtained using NUFREQ-NP with experimental data from Peach Bottom-2 stability tests and found good agreement.

March-Leuba et al. developed a phenomenological model to simulate the “qualitative” behavior of BWRs.⁹ They also developed a detailed nodalized (numerically discretized) physical model to simulate the dynamic behavior of the Vermont Yankee BWR over a broad operating range.¹⁰ Their analysis led to the conclusion that for a wide range of oscillation amplitudes, no significant effect on the integrity of the fuel is expected. However, in order to keep their model simple, they assumed that the coolant enters the core at saturation enthalpy (not as a subcooled liquid), i.e. that the boiling boundary always is at the bottom of the core channels.

Ward and Lee¹¹ developed a simplified BWR core model. They described the neutron

kinetics by a point kinetics equation without any delayed precursor group, used a single node representation of the fuel rod temperature dynamics, and simplified the thermal hydraulics representation of the two phases in the flow channel by omitting the temporal acceleration term in the momentum equations. They reported extensive results on limit-cycle behavior in nuclear-coupled DWOs that they obtained using singular perturbation analyses.

Bergdhal et al.¹² conducted a series of noise measurements at Forsmark-1 and 2 (Swedish reactors of ABB-ATOM design) to investigate the oscillations that occur at low-flow/high-power operating conditions during reactor startup. The LPRM signals in the measurements on Forsmark-1 indicated that these oscillations, which occurred at 0.5 *Hz*, were generally in phase throughout the core. However, the oscillations varied in strength at different radial positions of the core. Furthermore, the upper LPRM signals were influenced by the lower ones in the same probe due to void transport upwards in the core. They also noted that the decay ratio (DR) — that characterizes the stability margin and is a measure of how rapidly ($DR \approx 0$) or how slowly ($DR \approx 1$) a disturbance is damped — ranged above 0.7, instead of 0.6 expected based on a large computer code calculation. This indicates that the U.S. Nuclear Regulatory Commission (NRC) requirement of $DR < 0.75 - 0.8$ may not be rigorously satisfied in ABB-ATOM BWRs.

Valtonen¹³ validated the RAMONA-3B three-dimensional BWR transient analysis code developed by the Brookhaven National Laboratory, and the TRAB one-dimensional BWR transient analysis code, using DWO data from an oscillation incident that occurred at the TVO-1 BWR. This work showed that both spatially out-of-phase and spatially in-phase oscillations are possible in BWRs operating in the low-flow/high-power region of the power-

flow map. Sensitivities to the inlet orifice of the core, the fuel gap conductance, the axial power distribution and the fuel type were studied. It was shown that decreasing the fuel gap conductance, which leads to increasing the time constant of the fuel heat transfer, has a destabilizing effect.

Rizwan-uddin and Dorning¹⁴ studied the effects of unheated riser sections that are added to enhance natural circulation in the next generation of BWRs. They found that, for a fixed flow rate, the addition of the riser sections makes the system less stable. They also showed that the feedback recirculation loop plays an important role in reactor stability, and if omitted from the model, can lead to nonconservative conclusions.

Wulff et al.¹⁵ simulated the instability that occurred at the LaSalle-2 power plant (and several other BWR transients) using the Brookhaven National Laboratory Engineering Plant Analyzer (EPA) in order to determine the causes that led to the observed magnitudes of power, flow and temperature oscillations. They found it to be a powerful tool for scoping calculations and for supporting accident management. Although very valuable in many contexts, production codes such as EPA are not very useful for thorough stability analyses or extensive parametric studies in general, because of their complexity and large computer running times.

While all the works cited above on stability analyses were performed either using a time-domain (nonlinear, numerical) approach or a frequency-domain (linear) approach, Tsuji et al.¹⁶ developed a simple BWR core model as a dynamical system comprised of a set of nonlinear ordinary differential equations and used bifurcation theory to investigate the dependencies of the steady-state on parameters — such as the inlet pressure loss coefficient,

the inlet subcooling, the gap conductivity between the fuel and cladding, the system pressure, and the void reactivity feedback coefficient — all of which have large influences on stability. They showed that loss of stability took place at low-flow/high-power conditions, and concluded that the sustained in-phase limit-cycle power oscillations observed in some instabilities¹ were not directly excited by the subcritical Hopf bifurcation that resulted from their model.

To study the mechanisms for out-of-phase (regional) instability, March-Leuba and Blake-man¹⁷ developed a model by modifying the LAPUR code.¹⁸ They studied the effect of the first harmonic neutron kinetics mode on the out-of-phase instability. They showed that it has a very important influence on this instability and that because of its effect, there is a region in the power-flow map where an out-of-phase instability mode is likely even though the core-wide mode is stable.

Munoz-Cobo et al.¹⁹ extended the phenomenological model developed by March-Leuba et al.⁹ to study both in-phase and out-of-phase instability. They replaced the point kinetics equations⁹ by modal kinetics equations which they developed based on λ -modes. Using this model they showed that in-phase oscillations only appear when the first harmonic mode does not have enough thermal-hydraulic feedback to overcome the eigenvalue separation. Further, they showed that if the modal reactivities have appropriate feedback gains, self-sustained out-of-phase oscillations could arise due to the different thermal-hydraulic properties of the reactor planar halves.

Turso et al.²⁰ developed a modal-based reduced order model using the fundamental and first harmonic modes of the neutron diffusion equation. Using this model they carried out an

out-of-phase simulation of the LaSalle-2 BWR and illustrated the ability of the modal-based model to effectively reproduce the phenomenon that occurred at LaSalle-2.

Van Bragt and Van der Hagen²¹ developed a six dimensional model to study the out-of-phase oscillations. In this model they retained only the neutron kinetics equation corresponding to the fundamental mode with one delayed precursor group. Using this model they demonstrated the important effect that the fuel time constant has on the overall stability.

From the work summarized above we see that the computational and mathematical analyses are in general agreement with the actual BWR incidents and the experimental BWR tests. This has led to a better understanding of the instability which has helped the U.S. NRC to put forth new regulations for safer operation of BWRs. However, the out-of-phase power oscillations, where the power in a planer half of the reactor oscillates out-of-phase with respect to the power in the other half, remain of major interest because they can diverge without any action by current BWR protection systems (1992).²² The purpose of this paper is thus to study the BWR nuclear-coupled thermal-hydraulic stability in general, and to focus on the effect of the first harmonic, which is responsible for the out-of-phase power oscillations. Also, in this paper we study the effect of the BWR recirculation loop by developing and adding a simple recirculation loop model to the core model. Using this extended model, we determine the important effects of the time delays associated with the loop dynamics on overall BWR stability.

We previously have developed a simple model to study BWR stability.⁴ The neutronics in that model was represented by the point kinetics equations. Hence, only the effect of the fundamental mode was studied. By determining the stability boundaries (SBs) in various

operating-parameter/phase-variable spaces, we showed that the stability of operating points on the 100% rod line decreases as the flow rate is reduced and that the least stable operating points on the 100% rod line are in the low-flow/high-power region of the power-flow map. Subsequently we improved the thermal-hydraulics modeling to represent more accurately the enthalpy profile along the channel.²³ In that improved model however, the neutronics was still represented by the simple point kinetics equations.

In this report we introduce space-time modal kinetics equations and representative parallel channels in each region of the reactor, and combine them with extensions of our earlier channel two-phase flow model²⁴ and our earlier fuel rod heat conduction model⁴ to study the effect of the most important harmonic, the first azimuthal harmonic, on out-of-phase oscillations in BWRs. First, we analytically study the effect on the SBs — or Hopf bifurcation sets^{25,26} — in the operating parameter space and in the power-flow plane. Then, we carry out numerical simulations to determine the nature of the Hopf bifurcation that occurs as the SB is crossed. Next, we perform numerical simulations to study the decay ratios and frequencies of oscillations at various operating points. Then, we use numerical simulations to find conditions under which out-of-phase growing oscillations are possible, and relate them to the effect of the first harmonic mode. Finally, we develop and include a simple recirculation loop model in which the time-delayed effects of the phase variables at the exit of the channel are related to the channel inlet subcooling. The effects of the time delays on the overall stability are studied by carrying out additional stability analyses and numerical simulations.

In the next section, we derive the ordinary differential equations (ODEs) that represent the BWR neutron kinetics. Then, in Section III, we develop the ODEs that model the single-

and two-phase flow in BWR channels, and those that model the heat conduction in the fuel rods.

II NEUTRON KINETICS

It is well known that the modal equations for space-time neutron kinetics can be developed from the full energy-dependent and full multi-group neutron transport equations for a multi-region reactor. However, it is not our intention here to start from the most general neutron migration description; rather, our objective is to develop a simple modal kinetics model for the space-time kinetics and then introduce into that model empirical values and correlations, based on operating BWR data, for parameters and expressions that appear in the final equations, e.g. the neutron generation time, the effective delayed neutron fraction, the void reactivity feedback coefficient and the Doppler feedback coefficient. Thus, we begin the development from the simple one-speed neutron diffusion equation coupled to one effective delayed neutron precursor group. Further, we use the equations for a one-region reactor with the effect of the reflector being included formally through the reflector savings added to the extrapolation length. This is not inappropriate since the final modal kinetics equations developed will include, not only the spectral effects due to the energy dependence of the neutron flux, but also the multi-region and general heterogeneous reactor effects, because the parameters and expressions used in them will be taken from BWR operating data. Therefore, the final simple modal equations obtained will represent the overall space-time behavior of the neutron flux in a typical BWR.

Modal kinetics equations can easily be developed in terms of either λ -modes or ω -modes.²⁷

In fact, those based on the λ -modes have very recently been developed from the two-group neutron diffusion equations for a one-region (bare core) reactor to study out-of-phase oscillations in BWRs.¹⁹ However, when λ -modes are used, the final ODEs for the time-dependent modal coefficients are coupled not only in the mode coefficients, but also in their time derivatives for two-group or two-region problems. Thus, approximations have been made to decouple the time-derivatives in these equations.¹⁹ Actually, the ω -modes are more natural and appropriate, which is evidenced by the fact that the final equations that result from using them are not plagued by this difficult additional coupling in the time-derivatives. Hence, they have been used in the development here.

We begin from the one-speed diffusion equation and one effective delayed neutron precursor group equation for finite cylindrical geometry

$$\dot{\underline{Y}} = \underline{\underline{A}}\underline{Y}, \quad (1)$$

where

$$\underline{Y}(\underline{r}, t) = (N(\underline{r}, t), C(\underline{r}, t))^T, \quad (2)$$

$$\underline{\underline{A}}(\underline{r}, t) = \begin{bmatrix} v\nabla \cdot D(\underline{r}, t)\nabla - v\Sigma_a(\underline{r}, t) + (1 - \beta)v\nu\Sigma_f(\underline{r}, t) & \lambda \\ v\beta\nu\Sigma_f(\underline{r}, t) & -\lambda \end{bmatrix}, \quad (3)$$

and $N(\underline{r}, t)$ is the neutron number density, $C(\underline{r}, t)$ is the averaged delayed neutron precursor group concentration, and $\underline{\underline{A}}(\underline{r}, t)$ is the matrix representing the coupled neutron diffusion

and precursor concentration operators.

At a reference steady-state, which here is the steady-state of a reference BWR operating at 100% full power, the two eigenvalues ω_{ik} and the corresponding two eigenvectors $\underline{\phi}_{ik}(\underline{r})$, $i = 0, 1$, of the matrix $\underline{A}_o(\underline{r})$, for each spatial mode $\psi_k(\underline{r})$, $k = 0, 1, \dots$, satisfy

$$\underline{A}_o(\underline{r})\underline{\phi}_{ik}(\underline{r}) = \omega_{ik}\underline{\phi}_{ik}(\underline{r}) \quad (4)$$

$$\underline{\phi}_{ik}(\underline{r}) = \begin{bmatrix} 1 \\ \frac{v\beta\nu\Sigma_f(\underline{r})}{\omega_{ik} + \lambda} \end{bmatrix} \psi_k(\underline{r}), \quad i = 0, 1, \quad k = 0, 1, \dots \quad (5)$$

where $A_o(\underline{r})$ is given by Eq. (3) with only space-dependent material properties, and $\psi_k(\underline{r})$ are the spatial ω -modes. [For the general case of six rather than one (averaged) delayed neutron group, the index i runs from 0 to 6.] Of the infinite number of spatial ω -modes $\psi_k(\underline{r})$, $k = 0, 1, \dots$, we retain here only the fundamental $\psi_0(\underline{r})$ and the first azimuthal harmonic $\psi_1(\underline{r})$ which has been shown to be the most important harmonic with respect to BWR oscillations.¹⁷ For a one-region finite cylinder geometric model these two modes are

$$\psi_0(r, z, \theta) = J_0(2.40483r/R) \sin(\pi z/L) \quad (6)$$

$$\psi_1(r, z, \theta) = J_1(3.83171r/R) \sin(\pi z/L) \sin(\theta). \quad (7)$$

For each $k = 0, 1$, the expressions for the eigenvalues ω_{ik} and the corresponding eigenvectors $\underline{\phi}_{ik}(\underline{r})$, $i = 0, 1$ are straightforwardly determined from the $\psi_k(r_{extrap}) = 0$ boundary condition on the spatial mode where r_{extrap} includes the reflector savings. For the two spatial

modes included in the model, Eq. (4) becomes

$$[v\nabla \cdot D\nabla - v\Sigma_a + (1 - \beta)v\nu\Sigma_f]\psi_k(\underline{r}) + \left[\frac{\lambda\beta v\nu\Sigma_f}{\omega_{ik} + \lambda} - \omega_{ik} \right] \psi_k(\underline{r}) = 0, \quad i = 0, 1, \quad k = 0, 1. \quad (8)$$

The material properties actually used in this equation are those for an equivalent one-region critical reactor operating at the reference steady-state of 100% power. The eigenvalue ω_{00} is zero and the other three are real and negative. The numerical values of the reactor properties used, and the eigenvalues that were calculated based on these properties, are listed in Appendix A.

We expand the unknown vector $\underline{Y}(\underline{r}, t)$ in the four eigenvectors $\underline{\phi}_{ik}(\underline{r})$, with the terms in the sum on $i = 0, 1$ written explicitly

$$\underline{Y}(\underline{r}, t) = \sum_{k=0}^1 U_{0k}(t) \underline{\phi}_{0k}(\underline{r}) + U_{1k}(t) \underline{\phi}_{1k}(\underline{r}). \quad (9)$$

We then substitute this expansion into Eq. (1) and write $\underline{\underline{A}}(\underline{r}, t)$ as

$$\underline{\underline{A}}(\underline{r}, t) = \underline{\underline{A}}_o(\underline{r}) + \delta\underline{\underline{A}}(\underline{r}, t) \quad (10)$$

where

$$\delta\underline{\underline{A}}(\underline{r}, t) = \begin{bmatrix} v\nabla \cdot \delta D(\underline{r}, t)\nabla - v\delta\Sigma_a(\underline{r}, t) + (1 - \beta)v\nu\delta\Sigma_f(\underline{r}, t) & 0 \\ v\beta\nu\delta\Sigma_f(\underline{r}, t) & 0 \end{bmatrix}. \quad (11)$$

Now, using the orthogonality of the eigenvectors

$$(\phi_{jm}^\dagger, \phi_{ik}) = c_{jm} \delta_{ji} \delta_{mk}, \quad (12)$$

where the adjoint eigenvectors

$$\phi_{jm}^\dagger(\underline{r}) = \begin{bmatrix} 1 \\ \frac{\lambda}{\omega_{jm} + \lambda} \end{bmatrix} \psi_m(\underline{r}), \quad j = 0, 1, \quad m = 0, 1 \quad (13)$$

satisfy

$$\underline{A}_o^\dagger(\underline{r}) \phi_{jm}^\dagger(\underline{r}) = \omega_{jm} \phi_{jm}^\dagger(\underline{r}), \quad j = 0, 1, \quad m = 0, 1, \quad (14)$$

and introducing

$$\begin{aligned} n_k(t) &= U_{0k}(t) + U_{1k}(t), \\ u_k(t) &= U_{1k}(t), \end{aligned} \quad (15)$$

where $n_k(t)$ is the number of neutrons in the k -th spatial mode, we arrive at the modal kinetics equations based on the ω -modes

$$\frac{dn_k(t)}{dt} = \omega_{0k} n_k(t) + (\omega_{1k} - \omega_{0k}) u_k(t) + \sum_{m=0}^1 \frac{\rho_{km}(t)}{\Lambda_k} n_m(t) \quad (16)$$

$$\frac{du_k(t)}{dt} = \omega_{1k} u_k(t) + \sum_{m=0}^1 \frac{\rho_{km}(t)}{\Lambda_{1k}} n_m(t), \quad k = 0, 1. \quad (17)$$

The $\Lambda_k = 1 / \sum_{i=0}^1 (1 / \Lambda_{ik})$ and the Λ_{1k} in these equations, which are the ω -modal analogs of

the generation time that arises in point kinetics, are given by

$$\frac{1}{\Lambda_{ik}} = \frac{(\psi_k(\underline{r}), \nu \nu \Sigma_f(\underline{r}) \psi_k(\underline{r}))}{(\phi_{ik}^\dagger(\underline{r}), \phi_{ik}(\underline{r}))}, \quad i = 0, 1, \quad k = 0, 1, \quad (18)$$

and the reactivities, which transmit the effect of the m -th mode to the k -th mode, are

$$\rho_{km}(t) = \frac{(\phi_{ik}^\dagger(\underline{r}), \delta \underline{A}(\underline{r}, t) \phi_{jm}(\underline{r}))}{(\psi_k(\underline{r}), \nu \nu \Sigma_f(\underline{r}) \psi_m(\underline{r}))}, \quad k = 0, 1, \quad m = 0, 1. \quad (19)$$

In these expressions the inner products that involve the eigenvectors (the $\underline{\phi}$'s) imply summation over the vector components and integration over the entire reactor volume, and those that involve the modes (the ψ 's) imply integration over the entire reactor volume. Because of the zeros in the last column of $\delta \underline{A}(\underline{r}, t)$ (see Eq. (11)), the expressions given by Eq. (19) become independent of the index j when the inner products are taken. They also are made independent of the index i by neglecting the term $(\lambda\beta)/(\omega_{ik} + \lambda)$ which appears in the numerators; this term is very small, and it is indeed negligible when compared to the terms to which it is added.

Finally, we relate the reactivities $\rho_{km}(t)$, which give the effect of the m -th mode on the k -th mode, to the control rod induced external reactivities $\rho_{ext, kml}$, $l = 0, 1$ (operating parameters of the system), and to the void and Doppler feedback reactivities

$$\rho_{km}(t) = \sum_{l=0}^1 \rho_{ext, kml} + c_{1, kml} [\alpha_l(t) - \alpha_{o, l}] + c_{2, kml} [T_{avg, l}(t) - T_{avg, o, l}], \quad k = 0, 1, \quad m = 0, 1. \quad (20)$$

The summation index l corresponds to a representative flow channel. Since here we consider

only two representative channels, one for each planar half of the reactor, this index also corresponds to a half of the reactor. The quantity $\alpha_l(t)$ is the void fraction spatially averaged over the two-phase region, and $T_{avg,l}(t)$ is the fuel temperature averaged over the rod cross-sectional area and the rod length (both the part adjacent to the single-phase region and the part adjacent to the two-phase region). They are intermediate variables which depend on the phase variables, operating parameters and the design parameters. The detailed expression of $\alpha_l(t)$ will be given later [in Eq. (50)], after the fluid flow equations in the channels are developed. The detailed expression for $T_{avg,l}(t)$ also will follow [in Eq. (98)], after the fuel rod heat conduction model is developed. The quantities $\alpha_{o,l}$ and $T_{avg,o,l}$ are respectively, the steady-state values of $\alpha_l(t)$ and $T_{avg,l}(t)$ at normal operating conditions of a reference BWR at 100% power. The coefficients $c_{1,kl}$ and $c_{2,kl}$ are respectively, the void feedback coefficient and the Doppler feedback coefficient from the m -th spatial mode to the k -th spatial mode for the l -th region of the reactor (the l -th half of the reactor in the present work). These coefficients, and the Λ_{ik} , were obtained from typical BWR operating data (see Appendix A) by calculating the theoretical values for the fundamental mode quantities Λ_{00} , $c_{1,00l}$ and $c_{2,00l}$, $l = 0, 1$, and then scaling them by the appropriate scale factors to make them equal to the typical BWR values. Because the theoretical expressions for the higher mode quantities Λ_{ik} , $c_{1,kl}$ and $c_{2,kl}$ differ from their fundamental mode analogs by known geometric factors that result from the spatial integrations of various combinations of the modes, they also then were determined from the empirically scaled fundamental mode values by the appropriate replacement of the geometric factors. The details of how the numerical values were determined for Λ_{ik} , $c_{1,kl}$ and $c_{2,kl}$ are given in Appendix B. Since $c_{1,kl}$ (and $c_{2,kl}$) were not taken as identical for $l = 0$ and $l = 1$ in the studies reported

here, the dynamics that result are in general not symmetric in the two halves of the reactor. However because $\alpha_{o,l}$ and $T_{avg,o,l}$ are the values of the void fraction and the average fuel temperature at the 100% power critical steady-state, the factors that multiply the $c_{1,kml}$ and $c_{2,kml}$ are zero for this case, and the steady-state solution, is symmetric.

III THERMAL-HYDRAULICS

A BWR lattice cell, shown in Fig. 1, is represented as a vertical fuel-centered boiling flow channel. The fuel rod consists of a UO_2 fuel pellet of radius r_p^* surrounded by zircaloy clad of thickness $(r_c^* - r_g^*)$, where r_c^* is the clad outer radius and r_g^* is the clad inner radius. The gap between the pellet and the clad $(r_g^* - r_p^*)$ has a conductance h_g^* . The flow channel is of length L^* , and it has a square cross-section with a side of width p_c^* equal to the pitch of the BWR lattice cell. Here and everywhere below, an asterisk on a variable indicates the original dimensional quantity, and any quantity without an asterisk is dimensionless. The details of the development of the dimensionless variables are given in Appendix C.

The heat generated in the fuel rod by nuclear fission is conducted through the fuel rod and convected to the surrounding coolant in the flow channel. This heat causes the single-phase liquid coolant entering the bottom of the channel with an inlet velocity of $v_{inlet}^*(t^*)$ to start boiling at a certain point downstream (i.e. above the inlet). This point is the boiling boundary $\mu^*(t^*)$, beyond which the coolant continues as a two-phase mixture of liquid and its vapor. The coolant enters the flow channel at an inlet temperature T_{inlet}^* , and the temperature increases in the single-phase region $[0 < z^* < \mu^*(t^*)]$, reaching the

saturation temperature T_{sat}^* at the boiling boundary — beyond which the temperature of the coolant remains the same (although its quality increases). During any transient analysis, the system pressure p^* (at which fluid properties such as the saturation temperature T_{sat}^* , the liquid density ρ_f^* , the vapor density ρ_g^* , the liquid saturation enthalpy h_{sat}^* , and the saturation enthalpy difference Δh_{fg}^* are set) is kept at a constant value in the model, i.e. it is space- and time-independent. Since the fractional change in the pressure from the channel inlet to the channel exit is relatively small, this assumption is a fairly good approximation and is frequently made.

III.A The Channel Flow Equations

To solve the full three-dimensional mass, momentum and energy equations that describe the coolant flow in a reactor core is computationally an arduous task, and this is rarely done. Rather, advantage is taken of the geometry and the flow in the fuel assembly to justify a one-dimensional analysis based on axial flow; thus simpler, computationally less intensive, models have been derived.²⁸⁻³⁰ The aspect ratio of a typical BWR fuel assembly, equal to the ratio of the length to the width is large (~ 25).²⁸ Moreover, the cross flow within an assembly is very small when compared to the axial flow since the physical casing of the assembly restricts the cross flow. Because of this the three-dimensional mass, momentum and energy equations in the single-phase and the two-phase regions are averaged over the cross section of the flow channel to arrive at equations that depend on one spatial variable (the axial variable) and time.²⁸

After the full three-dimensional mass, momentum and energy equations have been cross-

Figure 1: Schematic description of a BWR lattice cell.

sectionally averaged, there result one-dimensional space- (axial) and time-dependent partial differential equations (PDEs).²⁸ These are comprised of the 3 equations for mass, momentum and energy continuity in the single-phase region, and 3 equations for mass, momentum and energy continuity in each of the two phases of the two-phase regions, i.e. a total of 6 equations in the two phases. In addition three constitutive relationships for the mass, momentum and energy transfer between the two phases are needed to solve the 6 equations in the two-phase region. Additional approximations are made to reduce the 6 equations to 3 equations in the two-phase region by assuming a drift velocity V_{gj} between the two phases, assuming a thermal equilibrium (i.e. both phases are at the saturation temperature at the prevailing pressure), and allowing for the concentration parameter C_o , which represents the global effect due to the radially nonuniform void distribution and the velocity profiles, to be different from 1.0. This model, called the drift flux model, is used frequently for the analysis of flows at low pressure and/or low flow rate under steady-state or near-steady-state conditions. Further approximations, where V_{gj} is assumed zero, i.e. the liquid and vapor in the two-phase region are assumed to have the same velocity, and C_o is taken as 1, i.e. no concentration profile effects are taken into consideration, leads to the homogeneous equilibrium model. The homogeneous equilibrium model is useful for high pressure, high flow rate conditions, such as those in a BWR, and can be used to study the axial flow in a BWR channel.

Hence, in this section we begin from the equations that describe the fluid flow in each channel, i.e. the one-dimensional time-dependent mass, momentum and energy conservation equations in the single-phase region and the homogeneous equilibrium model equations in the two-phase region, and develop time-dependent ODEs from these PDEs. In order to reduce

the space- and time-dependent PDEs to time-dependent ODEs we use a weighted residuals procedure in which *reasonable* quadratic spatial approximations for the space- and time-dependent single-phase enthalpy and the two-phase quality are introduced. We established that these spatial approximations are reasonable by first “validating” them for a uniformly heated channel with space- and time-independent heat flux.²⁴

Results of that validation study²⁴ can be summarized as follows. We introduced various spatial approximations for the space- and time-dependent single-phase enthalpy and two-phase quality, and developed dynamical systems comprised of a set of ODEs that describe the dynamics of well-studied related systems — uniformly heated channels that *do not* involve neutron kinetics and fuel rod heat conduction. We then compared the results — in the form of the predicted SBs in an operating parameter plane — from these simple dynamical systems with the results of the rather complicated *exact* mathematical analysis for similar heated channels.^{31,32} We showed that with quadratic spatial approximations there is a good agreement, thereby validating the approximations. Further, since the steady-state *exact* enthalpy profile is linear, approximating the space- and time-dependent enthalpy by a quadratic spatial profile is very reasonable, since at steady-state it becomes identical to the exact linear profile. Hence, here we start from the PDEs for the fluid flow that now *do* incorporate the effects of BWR neutron kinetics and fuel rod heat conduction, and using a weighted residuals procedure with quadratic spatial approximations for the single-phase enthalpy and the two-phase quality that we introduced and validated earlier,²⁴ we develop the time-dependent ODEs.

In all, five ODEs are developed. These are for the phase variables $a_1(t)$, $a_2(t)$, $s_1(t)$, $s_2(t)$

and $v_{inlet}(t)$ in each channel. The variables $a_1(t)$ and $a_2(t)$ are respectively, the coefficients of the (spatially) linear and (spatially) quadratic terms, of the spatially quadratic representation for the space- and time-dependent single-phase enthalpy. The variables $s_1(t)$ and $s_2(t)$ are respectively, the coefficients of the (spatially) linear and (spatially) quadratic terms, of the spatially quadratic representation for the space- and time-dependent two-phase quality. The variable $v_{inlet}(t)$ is the channel inlet velocity. The development presented below is for the l -th channel; however for simplicity of notation, we have omitted the channel index subscript l .

III.A.1 The Equations for $a_1(t)$ and $a_2(t)$

The ODEs for $a_1(t)$ and $a_2(t)$ are developed from the enthalpy equation for the incompressible liquid phase. This equation in dimensional form is²⁸

$$\rho_f^* \frac{\partial h^*(z^*, t^*)}{\partial t^*} + \rho_f^* v_{inlet}^*(t^*) \frac{\partial h^*(z^*, t^*)}{\partial z^*} = \frac{q''_{1\phi}^*(t^*) \xi_h^*}{A^*}, \quad (21)$$

where $h^*(z^*, t^*)$ is the single-phase enthalpy and $q''_{1\phi}^*(t^*)$ is the wall heat flux in the single-phase region given by

$$q''_{1\phi}^*(t^*) = h_{\infty, 1\phi}^* [T_{surf, 1\phi}^*(t^*) - T_{bulk, 1\phi}^*]. \quad (22)$$

In the above equation, $h_{\infty, 1\phi}^*$ is the single-phase convective heat transfer coefficient estimated by the Dittus-Boelter correlation,²⁸ $T_{surf, 1\phi}^*(t^*)$ is the surface temperature of the fuel rod in the single-phase region, and $T_{bulk, 1\phi}^* = (T_{sat}^* + T_{inlet}^*)/2$ is the bulk temperature of the fluid in the single-phase region.

Using the dimensionless variables and parameters defined in Appendix C, Eq. (21) is converted into the dimensionless form

$$\frac{\partial h(z, t)}{\partial t} + v_{inlet}(t) \frac{\partial h(z, t)}{\partial z} = N_\rho N_r N_{pch,1\phi}(t), \quad (23)$$

where $h(z, t)$ is the single-phase enthalpy, the dimensionless numbers $N_\rho \equiv \rho_g^*/\rho_f^*$ and $N_r \equiv \rho_f^*/(\rho_f^* - \rho_g^*)$, and $N_{pch,1\phi}(t)$ is the time-dependent phase change number in the single-phase region which is proportional to the wall heat flux $q''_{1\phi}(t^*)$. Hence $N_{pch,1\phi}(t)$ is related to the surface temperature of the fuel rod $T_{surf,1\phi}(t)$

$$N_{pch,1\phi}(t) = N_{cov,1\phi} (T_{surf,1\phi}(t) - T_{bulk,1\phi}), \quad (24)$$

where $N_{cov,1\phi}$ is the single-phase conversion number which depends on the single-phase heat transfer coefficient, $T_{surf,1\phi}(t)$ is the surface temperature of the fuel rod in the single-phase region (which is an intermediate variable, expressed in terms of the phase variables, operating parameters and design parameters of the dynamical system by Eq. (99) after the fuel rod heat conduction model is developed), and $T_{bulk,1\phi} = T_{sat} - N_{sub}N_{sbo}$ is the bulk temperature of the liquid. The number N_{sbo} is a design parameter and the subcooling number N_{sub} is an operating parameter of the dynamical system for the open loop. (Later in Section VII, when we include the recirculation loop model, N_{sub} will become a time-dependent intermediate variable that depends upon some of the phase variables.)

In the context of the weighted residuals procedure, we use a spatially quadratic approxi-

mation for the single-phase enthalpy

$$h(z, t) = h_{inlet} + a_1(t)z + a_2(t)z^2, \quad (25)$$

where $a_1(t)$ is the time-dependent coefficient of the spatially linear expansion function z , and $a_2(t)$ is the time-dependent coefficient of the spatially quadratic expansion function z^2 . At steady-state, $\tilde{a}_1 = N_\rho N_r \tilde{N}_{pch,1\phi} / \tilde{v}_{inlet}$ and $\tilde{a}_2 = 0$ (here and throughout this work a tilde on a quantity indicates its steady-state value). As mentioned above, this approximation, Eq. (25), is motivated by the good agreement we obtained,²⁴ for the case of a uniformly heated channel, between SBs from the exact model and those determined using the above approximation.

We use the weight functions 1 and z , and the limits of integration from the channel inlet $z = 0$ to the boiling boundary $z = \mu(t)$, to straightforwardly arrive at two linear equations in $da_1(t)/dt$ and $da_2(t)/dt$. These two equations are solved simultaneously for $da_1(t)/dt$ and $da_2(t)/dt$ to obtain at ODEs for the phase variables $a_1(t)$ and $a_2(t)$. The equations are

$$\frac{da_1(t)}{dt} = \frac{6}{\mu(t)} [N_\rho N_r N_{pch,1\phi}(t) - v_{inlet}(t)a_1(t)] - 2v_{inlet}(t)a_2(t) \quad (26)$$

$$\frac{da_2(t)}{dt} = -\frac{6}{\mu^2(t)} [N_\rho N_r N_{pch,1\phi}(t) - v_{inlet}(t)a_1(t)]. \quad (27)$$

At steady-state, Eqs. (26) and (27) give $\tilde{a}_1 = N_\rho N_r \tilde{N}_{pch,1\phi} / \tilde{v}_{inlet}$, and $\tilde{a}_2 = 0$; hence, the quadratic spatial profile for the single-phase enthalpy is identical to the linear profile at steady-state which in that case is an exact solution.

Equation (25) satisfies the inlet boundary condition (BC), $h(0, t) = h_{inlet}$, and must also satisfy the boiling boundary condition, $h(\mu(t), t) = h_{sat}$. Using this condition we obtain an expression for the boiling boundary in terms of the phase variables $a_1(t)$ and $a_2(t)$

$$\mu(t) = \frac{2N_\rho N_r N_{sub}}{a_1(t) + \sqrt{a_1^2(t) + 4a_2(t)N_\rho N_r N_{sub}}}. \quad (28)$$

This completes the development of the ODEs for $a_1(t)$ and $a_2(t)$, given by Eqs. (26) and (27) above.

III.A.2 The Equations for $s_1(t)$ and $s_2(t)$

The ODEs for the next two channel phase variables, $s_1(t)$ and $s_2(t)$, are developed from the cross-sectionally averaged two-phase mixture density equation, which for the homogeneous equilibrium model and time-dependent heat flux is²⁸

$$\frac{\partial \rho_m^*(z^*, t^*)}{\partial t^*} + \frac{\partial}{\partial z^*} [\rho_m^*(z^*, t^*) v_m^*(z^*, t^*)] = 0, \quad (29)$$

where $\rho_m^*(z^*, t^*)$ is the mixture density related to the void fraction $\alpha(z^*, t^*)$ through

$$\rho_m^*(z^*, t^*) = \alpha(z^*, t^*) \rho_g^* + [1 - \alpha(z^*, t^*)] \rho_f^*, \quad (30)$$

which in turn, for the homogeneous equilibrium model, is related to the mixture quality $x(z^*, t^*)$ as

$$\alpha(z^*, t^*) = \frac{1}{1 + \frac{1-x}{x} \frac{\rho_g^*}{\rho_f^*}}, \quad (31)$$

and $v_m^*(z^*, t^*)$ is the mixture velocity or the volumetric flux which satisfies

$$\frac{\partial v_m^*(z^*, t^*)}{\partial z^*} = \frac{q_{2\phi}''^*(t^*) \xi_h^* \Delta \rho^*}{\rho_f^* \rho_g^* A^* \Delta h_{fg}^*} . \quad (32)$$

The wall heat flux in the two-phase region $q_{2\phi}''^*(t^*)$, is related to the surface temperature of the fuel rod surface in the two-phase region $T_{surf,2\phi}^*(t^*)$, by the Jens-Lottes correlation^{28,29} (in SI units)

$$q_{2\phi}''^*(t^*) = \frac{e^{4p^* \times 10^{-6}/6.2}}{(25)^4 \times 10^{-6}} (T_{surf,2\phi}^*(t^*) - T_{sat}^*)^4 . \quad (33)$$

Using the dimensionless variables and parameters defined in Appendix C, Eq. (29) is converted into the dimensionless form

$$\frac{\partial \rho_m(z, t)}{\partial t} + v_m(z, t) \frac{\partial \rho_m(z, t)}{\partial z} = -N_{pch,2\phi}(t) \rho_m(z, t) , \quad (34)$$

where $\rho_m(z, t)$ is the mixture density related to the mixture quality $x(z, t)$ by Eqs. (30) and (31)

$$\rho_m(z, t) = \frac{N_\rho N_r}{N_\rho N_r + x(z, t)} , \quad (35)$$

and the mixture velocity, $v_m(z, t)$, is obtained by rewriting Eq. (32) in dimensionless form (using the variables defined in Appendix C), and integrating it from the boiling boundary $\mu(t)$ to z

$$v_m(z, t) = v_{inlet}(t) + N_{pch,2\phi}(t) [z - \mu(t)] . \quad (36)$$

The phase change number in the two-phase region $N_{pch,2\phi}(t)$, in Eqs. (34) and (36), is pro-

portional to the vapor generation rate; hence, it is proportional to the wall heat flux $q_{2\phi}''(t^*)$

$$N_{pch,2\phi}(t) = N_{cov,2\phi} [T_{surf,2\phi}(t) - T_{sat}]^4, \quad (37)$$

where $N_{cov,2\phi}$ is the two-phase conversion number which depends on the system pressure p^* , and $T_{surf,2\phi}(t)$ is the surface temperature of the fuel rod in the two-phase region. Analogously to $T_{surf,1\phi}$, $T_{surf,2\phi}$ is an intermediate variable expressed in terms of the phase variables, operating parameters and design parameters of the dynamical system; it will be given by Eq. (99) below, following the development of the fuel rod conduction model.

In the context of the weighted residuals procedure, we use a spatially quadratic approximation for the quality

$$x(z, t) = N_\rho N_r \{s_1(t) [z - \mu(t)] + s_2(t) [z - \mu(t)]^2\}, \quad (38)$$

where $s_1(t)$ is the time-dependent coefficient of the spatially linear expansion function $[z - \mu(t)]$, and $s_2(t)$ is the time-dependent coefficient of the spatially quadratic expansion function $[z - \mu(t)]^2$. This spatial approximation is motivated by the good results we obtained in earlier work²⁴ for uniformly heated channels. The particular forms of the expansion functions used automatically satisfy the BC at $z = \mu(t)$. Also, from Eqs. (34) and (35), the steady-state *exact* quality profile is linear $[\tilde{x}(z) = N_\rho N_r \tilde{N}_{pch,2\phi} / \tilde{v}_{inlet} (z - \tilde{\mu})]$. Hence, approximating the space- and time-dependent quality with the quadratic spatial profile is very reasonable since at steady-state this expression becomes identical to the exact linear profile, i.e. at the steady-state, $\tilde{s}_1 = \tilde{N}_{pch,2\phi} / \tilde{v}_{inlet}$ and $\tilde{s}_2 = 0$.

Substituting Eq. (38) into Eq. (34) gives

$$\rho_m(z, t) = \frac{1}{1 + s_1(t) [z - \mu(t)] + s_2(t) [z - \mu(t)]^2}, \quad (39)$$

which is substituted into Eq. (34). Then using the weight functions, 1 and $[z - \mu(t)]$, leads to two linear equations in $ds_1(t)/dt$ and $ds_2(t)/dt$. These two equations are solved simultaneously for $ds_1(t)/dt$ and $ds_2(t)/dt$ (see Appendix D for details) to arrive at the ODEs for the phase variables $s_1(t)$ and $s_2(t)$

$$\frac{ds_1(t)}{dt} = \frac{1}{f_2(t)} [f_3(t)f_1(t) + f_4(t)] \quad (40)$$

$$\frac{ds_2(t)}{dt} = \frac{1}{f_2(t)} [f_5(t)f_1(t) + f_6(t)], \quad (41)$$

where $f_n(t)$, $n = 2, \dots, 6$ are slightly complicated intermediate variables which depend on the phase variables, operating parameters and design parameters. The detailed expressions for these variables are given in Appendix D. The quantity $f_1(t)$ is the time derivative of the boiling boundary $\mu(t)$. From Eq. (28)

$$f_1(t) = \frac{d\mu(t)}{dt} = -\frac{\mu(t)}{\sqrt{a_1^2(t) + 4a_2(t)N_\rho N_r N_{sub}}} \left[\frac{da_1(t)}{dt} + \mu(t) \frac{da_2(t)}{dt} \right], \quad (42)$$

where $da_1(t)/dt$, $da_2(t)/dt$ and $\mu(t)$ are expressed in terms of the phase variables, operating parameters and design parameters by Eqs. (26), (27) and (28), respectively. At steady-state, Eqs. (34), (35) and (38) give $\bar{s}_1 = \bar{N}_{pch,2\phi}/\bar{v}_{inlet}$ and $\bar{s}_2 = 0$; hence, the spatial profile for the quality at steady-state is identical to the linear profile which is the exact steady-state

solution. This completes the development of the ODEs for $s_1(t)$ and $s_2(t)$, given by Eqs. (40) and (41) above.

III.A.3 The Equation for $v_{inlet}(t)$

The ODE for the final channel phase variable $v_{inlet}(t)$, is developed from the single-phase and two-phase momentum conservation equations. The momentum equation in the incompressible liquid-phase region is²⁸

$$-\frac{\partial P_{1\phi}^*}{\partial z^*} = \rho_f^* \left[\frac{dv_{inlet}^*(t^*)}{dt^*} + \frac{f_{1\phi}^*}{2D_h^*} v_{inlet}^{*2}(t^*) + g^* \right], \quad (43)$$

and in the two-phase region, described by the homogeneous equilibrium model, it is²⁸

$$-\frac{\partial P_{2\phi}^*}{\partial z^*} = \rho_m^*(z^*, t^*) \left[\frac{\partial v_m^*(z^*, t^*)}{\partial t^*} + v_m^*(z^*, t^*) \frac{\partial v_m^*(z^*, t^*)}{\partial z^*} + \frac{f_{2\phi}^*}{2D_h^*} v_m^{*2}(z^*, t^*) + g^* \right]. \quad (44)$$

Using the dimensionless variables and parameters defined in Appendix C, the two equations above are written in dimensionless form as

$$-\frac{\partial P_{1\phi}}{\partial z} = \frac{dv_{inlet}(t)}{dt} + N_{f,1\phi} v_{inlet}^2(t) + Fr^{-1}, \quad (45)$$

$$-\frac{\partial P_{2\phi}}{\partial z} = \rho_m(z, t) \left[\frac{\partial v_m(z, t)}{\partial t} + v_m(z, t) \frac{\partial v_m(z, t)}{\partial z} + N_{f,2\phi} v_m^2(z, t) + Fr^{-1} \right]. \quad (46)$$

The single-phase momentum equation [Eq. (45)] is integrated from the inlet $z = 0$, to the boiling boundary $z = \mu(t)$, to arrive at the single-phase pressure drop $\Delta P_{1\phi}(t)$. The

expressions for $v_m(z, t)$ and $\rho_m(z, t)$ from Eqs. (36) and (39) respectively, are substituted into the two-phase momentum equation [Eq. (46)], and the latter is integrated from the boiling boundary, $z = \mu(t)$, to the channel exit, $z = 1$, to arrive at the two-phase pressure drop $\Delta P_{2\phi}(t)$. These two pressure drops are summed along with the inlet and exit form pressure drops, and equated to the external pressure drop ΔP_{ext} , to arrive at an ODE for the inlet velocity $v_{inlet}(t)$

$$\frac{dv_{inlet}(t)}{dt} = \frac{1}{f_7(t)} [f_8(t) + f_9(t)f_1(t) + f_{10}(t)f_{11}(t)] . \quad (47)$$

The details of the development of the above equation are given in Appendix E. The quantity $f_1(t)$ is given by Eq. (42), and the quantities $f_n(t)$, $n = 7, \dots, 10$, are slightly complicated intermediate variables that depend on the phase variables, operating parameters and design parameters. The detailed expressions for these variables also are developed in Appendix E. The quantity $f_{11}(t)$ is the time derivative of the phase change number $N_{pch,2\phi}(t)$. An expression for $f_{11}(t)$ in terms of the phase variables, operating parameters and design parameters is obtained from Eq. (37). Since this expression depends on the phase variables that represent the fuel rod heat conduction, the detailed expression for $f_{11}(t)$ will be given by Eq. (107) below, after the fuel rod heat conduction model is developed. The externally imposed pressure drop ΔP_{ext} across the channel appearing in $f_8(t)$ is an operating parameter, and it is required as a BC to be the same for all the representative channels.

III.A.4 The Expression for $\alpha(t)$

The ODEs for the five channel variables $a_1(t)$, $a_2(t)$, $s_1(t)$, $s_2(t)$ and $v_{inlet}(t)$, have been developed in the subsections above. A detailed expression for the intermediate variable $\alpha(t)$ — the void fraction spatially averaged over the two-phase region — which appeared in Eq. (20) is now derived. Equation (31) is made dimensionless using the dimensionless numbers N_ρ and N_r defined in Appendix C. This results in

$$\alpha(z, t) = N_r \frac{x(z, t)}{N_r N_\rho + x(z, t)}. \quad (48)$$

The void fraction spatially averaged over the two-phase region is defined as

$$\alpha(t) = \frac{1}{1 - \mu(t)} \int_{\mu(t)}^1 \alpha(z, t) dz. \quad (49)$$

Substituting the expression for $x(z, t)$ from Eq. (38) into Eq. (48) and carrying out the integration in the above equation, we get

$$\alpha(t) = N_r \frac{1}{1 - \mu(t)} [s_1(t)M_2(t) + s_2(t)M_3(t)] , \quad (50)$$

where $M_2(t)$ and $M_3(t)$ are slightly complicated functions of $s_1(t)$, $s_2(t)$ and $\mu(t)$. The quantity $M_2(t)$ is defined by Eq. (D.24) in Appendix D, and $M_3(t)$ is defined by Eq. (E.56) in Appendix E.

III.B The Fuel Rod Heat Conduction Model

In this section we start from the one-dimensional (radial), time-dependent heat conduction equation for the fuel rod and develop ODEs that represent the heat conduction in a BWR fuel rod. In order to reduce the space- and time-dependent PDE to time-dependent ODEs, we use a variational method and introduce a *reasonable* spatial approximation for the space- and time-dependent fuel rod temperature. This spatial approximation is reasonable in the sense that it already has been “validated”.⁴

In that validation study,⁴ we started from the one-dimensional time-dependent heat conduction equation for the fuel rod — which for simplicity, was taken as a single region, i.e. both the clad and the gap between the pin and the clad were omitted — and developed time-dependent ODEs using an eigenfunction expansion method and a variational method. We then developed very simple time-dependent ODEs to represent the fluid flow through the channel and combined them with the fuel rod heat conduction ODEs to construct two BWR dynamical systems — one resulting from the eigenfunction expansion ODEs and the other resulting from the variational method ODEs. Using the first dynamical system, we determined stability boundaries in an operating parameter plane and showed that as the number of eigenfunctions retained in the expansion of the fuel rod temperature is increased (and therefore the number of ODEs in the dynamical system is increased), the SB converges, i.e. beyond a certain number of eigenfunctions (hence, ODEs) there is an imperceptible shift in the SB, indicating convergence to the “exact” SB. This however, occurred only after a large number of eigenfunctions — 64 — were retained (hence a large number of ODEs — 69 — were used), making the resulting dynamical system very large, and hence complicated.

Using the second dynamical system with a far smaller number of ODEs (9), we showed that the variational method yielded SBs that matched well with those obtained using the large number of eigenfunctions. Based on the conflicting objectives of improving the SB accuracy and reducing the number of equations in the dynamical system, the variational method was determined to be the best choice; therefore, we use that variational method here.

III.B.1 The Heat Conduction ODEs

For each channel, four ODEs are developed from the heat conduction PDE. These ODEs are for the two coefficients of the two spatially-piecewise-quadratic representation of the fuel pellet temperature in the single-phase region of the channel, $T_{1,1\phi}(t)$ and $T_{2,1\phi}(t)$, and for the two coefficients of the analogous spatial representation in the two-phase region, $T_{1,2\phi}(t)$ and $T_{2,2\phi}(t)$. Here and everywhere below, for simplicity of notation, we have omitted the channel index subscript l on all quantities.

The heat conduction equations for the fuel rod with the assumption of angular symmetry — conduction in the radial direction only — and with a time-dependent spatially-uniform volumetric heat generation rate $q'''(t^*)$ [proportional to the neutron density, i.e.

$$q'''(t^*) = c_q^* n_0^*(t^*) + c_q^* \xi n_1^*(t^*)] \text{ are}^{29}$$

$$\rho_p^* c_p^* \frac{\partial T_{p,j\phi}^*(r^*, t^*)}{\partial t^*} = k_p^* \left[\frac{\partial^2 T_{p,j\phi}^*(r^*, t^*)}{\partial r^{*2}} + \frac{1}{r^*} \frac{\partial T_{p,j\phi}^*(r^*, t^*)}{\partial r^*} \right] + q'''(t^*), \quad 0 \leq r^* \leq r_p^*, \quad (51)$$

$$\rho_c^* c_c^* \frac{\partial T_{c,j\phi}^*(r^*, t^*)}{\partial t^*} = k_c^* \left[\frac{\partial^2 T_{c,j\phi}^*(r^*, t^*)}{\partial r^{*2}} + \frac{1}{r^*} \frac{\partial T_{c,j\phi}^*(r^*, t^*)}{\partial r^*} \right], \quad r_g^* \leq r^* \leq r_c^*, \quad (52)$$

with the BCs

$$\frac{\partial T_{p,j\phi}^*(0, t^*)}{\partial r^*} = 0, \quad (53)$$

$$-k_p^* \frac{\partial T_{p,j\phi}^*(r_p^*, t^*)}{\partial r^*} = \frac{r_g^*}{r_p^*} h_g^* [T_{p,j\phi}^*(r_p^*, t^*) - T_{c,j\phi}^*(r_g^*, t^*)], \quad (54)$$

$$-k_c^* \frac{\partial T_{c,j\phi}^*(r_g^*, t^*)}{\partial r^*} = h_g^* [T_{p,j\phi}^*(r_p^*, t^*) - T_{c,j\phi}^*(r_g^*, t^*)], \quad (55)$$

$$-k_c^* \frac{\partial T_{c,j\phi}^*(r_c^*, t^*)}{\partial r^*} = h_{\infty,j\phi}^* [T_{c,j\phi}^*(r_c^*, t^*) - T_{bulk,j\phi}^*], \quad (56)$$

where $T_{p,j\phi}^*(r^*, t^*)$ and $T_{c,j\phi}^*(r^*, t^*)$ — the fuel pellet temperature and the clad temperature respectively, in the single-phase region for $j = 1$ and in the two-phase region for $j = 2$ — are treated separately in the regions below and above the moving boiling boundary, and the solutions are not required to be continuous at the interface between these two regions. The quantities ρ_p^* , c_p^* and k_p^* are the UO_2 fuel pellet density, specific heat, and thermal conductivity, respectively; and the quantities ρ_c^* , c_c^* and k_c^* are the zircaloy clad density, specific heat, and thermal conductivity, respectively. As shown in Fig. 1, r_p^* is the radius of the fuel pellet, r_g^* is the inner radius of the clad and r_c^* is the outer radius of the clad which also is the outer radius of the fuel rod. The gap between the pellet and the clad ($r_g^* - r_p^*$) has a gap conductance h_g^* . The BC in Eq. (53) follows from the assumed radial symmetry of the fuel rod. The BCs, Eqs. (54) and (55) at $r^* = r_p^*$ and $r^* = r_g^*$ respectively, follow from the fact that the heat stored in the fuel-clad gap is negligible, i.e. all the heat emitted at the surface of the pellet is transferred to the clad, or $q_{surface}^*(r_p^*, t^*) = q_{surface}^*(r_g^*, t^*)$. Of course, Eqs. (54) and (55) could be combined to give this same relationship explicitly as

$$-r_p^* k_p^* \frac{\partial T_{p,j\phi}^*(r_p^*, t^*)}{\partial r^*} = -r_g^* k_c^* \frac{\partial T_{c,j\phi}^*(r_g^*, t^*)}{\partial r^*}, \quad (57)$$

which could be used as a BC instead of Eqs. (54) and (55). The last condition, Eq. (56), is the BC at the surface of the fuel rod where $h_{\infty,j\phi}^*$ is the convective heat transfer coefficient, and the heat flux due to thermal radiation has been neglected. As mentioned earlier, the single-phase convective heat transfer coefficient $h_{\infty,1\phi}^*$ [see Eq. (22)] is estimated by the Dittus-Boelter correlation, and the two-phase heat transfer coefficient $h_{\infty,2\phi}^*$ is estimated using the Jens-Lottes correlation (in SI units)

$$h_{\infty,2\phi}^* = 2.56e^{4p^*/6.2 \cdot 10^6} \left(\tilde{T}_{surf,2\phi}^* - T_{sat}^* \right)^3, \quad (58)$$

which is obtained by equating the right hand side of Eq. (33) at steady-state to the right hand side of Eq. (56) at steady-state for $j = 2$. The temperatures, $\tilde{T}_{surf,2\phi}^*$ and T_{sat}^* , in Eq. (33) are equal to $\tilde{T}_{c,2\phi}^*(r_c^*)$ and $T_{bulk,2\phi}^*$ in Eq. (56), respectively, at steady-state.

Equations (51) through (56) are made dimensionless by using the dimensionless quantities α_p , T_p , c_q , α_c , T_c , and the Biot numbers Bi_p , Bi_g and $Bi_{c,j\phi}$ defined in Appendix A. Further, for simplicity of notation, we omit the subscript $j\phi$ from all the terms everywhere in this section with the understanding that the analysis applies to both the single-phase ($j = 1$) and two-phase ($j = 2$) regions. The dimensionless heat conduction equations then become

$$\frac{1}{\alpha_p} \frac{\partial T_p(r, t)}{\partial t} = \frac{\partial^2 T_p(r, t)}{\partial r^2} + \frac{1}{r} \frac{\partial T_p(r, t)}{\partial r} + c_q n(t) + c_q \xi n_1(t), \quad 0 \leq r \leq r_p, \quad (59)$$

$$\frac{1}{\alpha_c} \frac{\partial T_c(r, t)}{\partial t} = \frac{\partial^2 T_c(r, t)}{\partial r^2} + \frac{1}{r} \frac{\partial T_c(r, t)}{\partial r}, \quad r_g \leq r \leq r_c, \quad (60)$$

with the BCs

$$\frac{\partial T_p(0, t)}{\partial r} = 0, \quad (61)$$

$$-\frac{\partial T_p(r_p, t)}{\partial r} = \frac{Bi_p}{r_p} [T_p(r_p, t) - T_c(r_g, t)], \quad (62)$$

$$-\frac{\partial T_c(r_g, t)}{\partial r} = \frac{Bi_g}{r_g} [T_p(r_p, t) - T_c(r_g, t)], \quad (63)$$

$$-\frac{\partial T_c(r_c, t)}{\partial r} = \frac{Bi_c}{r_c} [T_c(r_c, t) - T_{bulk}], \quad (64)$$

where the coefficient $\xi_l \equiv (1, \psi_1(\underline{r}))_{V_l} / (1, \psi_0(\underline{r}))_{V_l}$ is given in terms of the inner products over the volumes associated with the relevant planar regions of the reactor V_l , and the bulk temperatures are given by, $T_{bulk,1\phi} = T_{sat} - N_{sub}N_{sbo}$, and $T_{bulk,2\phi} = T_{sat}$.

Before applying the variational method, these equations are further rearranged. We introduce the temperature deviations from their steady-state values as new variables

$$\theta_p(r, t) = T_p(r, t) - \tilde{T}_p(r), \quad 0 \leq r \leq r_p, \quad (65)$$

$$\theta_c(r, t) = T_c(r, t) - \tilde{T}_c(r), \quad r_g \leq r \leq r_c, \quad (66)$$

where the tilde ($\sim$) on a quantity indicates the steady-state solution obtained by solving Eqs. (59) and (60) at steady-state subject to the BCs, Eqs. (61) – (64). These solutions are

$$\tilde{T}_p(r) = -c_q \tilde{n}_0 \frac{r^2}{4} + \tilde{b}_1, \quad 0 \leq r \leq r_p, \quad (67)$$

$$\tilde{T}_c(r) = \tilde{b}_2 \log(r) + \tilde{b}_3, \quad r_g \leq r \leq r_c, \quad (68)$$

where

$$\tilde{b}_1 = -\frac{\tilde{b}_2}{Bi_g} + c_q \tilde{n}_0 \frac{r_p^2}{4} + \tilde{b}_2 \log(r_g) + \tilde{b}_3, \quad (69)$$

$$\tilde{b}_2 = -c_q \tilde{n}_0 \left(\frac{Bi_g}{Bi_p} \right) \frac{r_p^2}{2}, \quad (70)$$

$$\tilde{b}_3 = -\frac{\tilde{b}_2}{Bi_c} - \tilde{b}_2 \log(r_c) + T_{bulk}. \quad (71)$$

Equations (59) through (64) then become

$$\begin{aligned} \frac{1}{\alpha_p} \frac{\partial \theta_p(r, t)}{\partial t} &= \frac{\partial^2 \theta_p(r, t)}{\partial r^2} + \frac{1}{r} \frac{\partial \theta_p(r, t)}{\partial r} + c_q (n_0(t) - \tilde{n}_0) + c_q \xi n_1(t), \\ 0 &\leq r \leq r_p, \end{aligned} \quad (72)$$

$$\frac{1}{\alpha_c} \frac{\partial \theta_c(r, t)}{\partial t} = \frac{\partial^2 \theta_c(r, t)}{\partial r^2} + \frac{1}{r} \frac{\partial \theta_c(r, t)}{\partial r}, \quad r_g \leq r \leq r_c, \quad (73)$$

$$\frac{\partial \theta_p(0, t)}{\partial r} = 0, \quad (74)$$

$$-\frac{\partial \theta_p(r_p, t)}{\partial r} = \frac{Bi_p}{r_p} [\theta_p(r_p, t) - \theta_c(r_g, t)], \quad (75)$$

$$-\frac{\partial \theta_c(r_g, t)}{\partial r} = \frac{Bi_g}{r_g} [\theta_p(r_p, t) - \theta_c(r_g, t)], \quad (76)$$

$$-\frac{\partial \theta_c(r_c, t)}{\partial r} = \frac{Bi_c}{r_c} \theta_c(r_c, t), \quad (77)$$

which are the final forms of the PDEs and BCs, from which the ODEs that represent the fuel rod heat conduction dynamics, are developed. We first use Eqs. (73), (76) and (77) to develop an approximate expression for the temperature in the clad. We then employ Eqs. (72), (74) and (75) to determine the distribution of the temperature in the fuel pellet

via a variational principle which leads to ODEs that represent the fuel rod heat conduction.

The thermal diffusivity α_c^* — a measure of how rapidly heat diffuses through a material — of the clad is large, i.e. it is about ten times larger than that of the pellet, α_p^* . Therefore, during a transient the heat is conducted very rapidly through the clad and there is no significant change in the clad temperature (spatial) profile from its initial steady-state distribution. We exploit this fact in modeling the clad heat conduction dynamics, and do not solve the transient heat conduction equation in the clad explicitly. Rather, we use the steady-state clad temperature distribution and the two BCs for the clad, i.e. Eqs (76) and (77), to arrive at an approximate expression for the clad temperature $\theta_c(r, t)$.

Based on the logarithmic spatial distribution of the steady-state clad temperature [Eq. (68)], we assume the following space- and time-dependence for $\theta_c(r, t)$

$$\theta_c(r, t) = b_2(t) \log r + b_3(t), \quad (78)$$

where the time dependent functions, $b_2(t)$ and $b_3(t)$, are determined from the BCs, Eqs. (76) and (77). Substituting Eq. (78) into Eqs. (76) and (77), and solving for $b_2(t)$ and $b_3(t)$ gives

$$b_2(t) = \frac{-Bi_g Bi_c}{Bi_g + Bi_c + Bi_g Bi_c \log(r_c/r_g)} \theta_p(r_p, t), \quad (79)$$

$$b_3(t) = \frac{Bi_g(1 + Bi_c \log r_c)}{Bi_g + Bi_c + Bi_g Bi_c \log(r_c/r_g)} \theta_p(r_p, t), \quad (80)$$

which make $\theta_c(r, t)$ time-dependent only through the surface value, $\theta_p(r_p, t)$, which depends on the pellet temperature at the surface $T_p(r_p, t)$ and the coolant bulk temperature T_{bulk} through Eqs. (65), (67), (69) and (71). Substituting Eqs. (79) and (80) into Eq. (78), and

evaluating at $r = r_g$ gives

$$\theta_c(r_g, t) = \frac{Bi_g [1 + Bi_c \log(r_c/r_g)]}{Bi_g + Bi_c + Bi_g Bi_c \log(r_c/r_g)} \theta_p(r_p, t) . \quad (81)$$

Substituting this expression into Eq. (75), we rewrite Eq. (75) as

$$-\frac{\partial \theta_p(r_p, t)}{\partial r} = \frac{Bi_p^+}{r_p} \theta_p(r_p, t) , \quad (82)$$

where Bi_p^+ is the resulting modified Biot number

$$Bi_p^+ = Bi_p \frac{Bi_c}{Bi_g + Bi_c + Bi_g Bi_c \log(r_c/r_g)} . \quad (83)$$

It may appear from Eq. (82) that the temperature distribution in the pellet is independent of the coolant temperature. However, it is clear that this is not the case when one recalls that $\theta_p(r, t)$ is only the deviation of the pellet temperature from steady-state value, and the pellet temperature itself $T_p(r_p, t)$ does depend upon T_{bulk} through Eqs. (65), (67), (69) and (71).

The problem then is reduced to solving the heat conduction equation in the fuel pellet, Eq. (72), subject to the BCs, Eqs. (74) and (82). We follow the variational approach we introduced⁴ by rewriting Eq. (72) as

$$H\theta_p = f, \quad (84)$$

where the operator

$$H = \nabla^2 \quad (85)$$

$$\begin{aligned} \theta_p(r, t) &= T_1(t) + \alpha_1(t)r + \alpha_2(t)r^2, \quad 0 < r < r_d, \\ &= T_2(t) + \beta_1(t)r + \beta_2(t)r^2, \quad r_d < r < r_p, \end{aligned} \quad (88)$$

a two-piecewise quadratic function

To determine such a function $\varphi = \theta_p$, based on our earlier analysis,⁴ we write $\theta_p(r, t)$ as

and (82).

the function $\varphi = \theta_p$, that approximates the solution to Eq. (84) subject to the BCs, Eqs (74) of the discontinuity, respectively. The function that minimizes the functional in Eq. (87) is done here. The subscripts l and r stand for the limits of the functions from the *left* and *right* empirically to be $r_d = 0.83 r_p$. Hence, this value is used in all the analyses and calculations The value of r_d that leads to a SB which best matches the reference SB was determined⁴ from $r = 0$ to $r = r_p$, and the radius r_d is the point of discontinuity between $r = 0$ and $r = r_p$. where the notation (α, β) indicates the inner product of α and β with the limits of integration

$$\begin{aligned} \mathcal{F}(\varphi) = & (\varphi, H\varphi) - 2(\varphi, f) + \left[\frac{r_p^2}{2} \left(\frac{\partial \varphi}{\partial r} \right)^2 + \frac{2B_{l+}^p}{r_p^2} \left(\frac{\partial \varphi}{\partial r} \right)^2 - \frac{2}{B_{l+}^p} \varphi^2 \right]_{r=r_p} \\ & - \left[\frac{r_p^2}{2} \left(\frac{\partial \varphi}{\partial r} \right)^2 + r_d \left[\varphi_l \frac{\partial \varphi_r}{\partial r} - \varphi_r \frac{\partial \varphi_l}{\partial r} \right] \right]_{r=r_d}, \end{aligned} \quad (87)$$

which need not satisfy the BCs, Eqs. (74) and (82),

We introduce a general functional $\mathcal{F}$ that accommodates discontinuous trial functions $\varphi(r, t)$,

$$f = \frac{1}{\theta_p} \frac{\partial \theta_p}{\partial t} (n_0(t) - n_0) + c_q^g n_1(t). \quad (86)$$

is the Laplacian in cylindrical coordinates and

where we make $\theta_p(r_d, t)$ and $\partial\theta_p(r_d, t)/\partial r$ continuous at $r = r_d$. ($\partial^2\theta_p(r_d, t)/\partial^2 r$ is a discontinuous function at that point.) Using these two continuity conditions and using the two BCs, Eq. (74) and Eq. (82), we determine $\alpha_1(t)$, $\alpha_2(t)$, $\beta_1(t)$ and $\beta_2(t)$ in Eq. (88), in terms of $T_1(t)$ and $T_2(t)$

$$\begin{aligned}
\alpha_1(t) &= 0, \\
\alpha_2(t) &= a_{2a}T_1(t) + a_{2b}T_2(t), \\
\beta_1(t) &= b_{1a}T_1(t) + b_{1b}T_2(t), \\
\beta_2(t) &= b_{2a}T_1(t) + b_{2b}T_2(t),
\end{aligned} \tag{89}$$

where a_{2a} , a_{2b} , b_{1a} , b_{1b} , b_{2a} and b_{2b} are constants, expressions for which were straightforwardly arrived at by using the symbolic manipulation package Mathematica.³³ The expressions are several pages long; hence are not explicitly included here. Substituting Eq. (89) into Eq. (88) and rearranging the terms gives

$$\begin{aligned}
\theta_p(r, t) &= T_1(t)v_{1a}(r) + T_2(t)v_{1b}(r), \quad 0 < r < r_d, \\
&= T_1(t)v_{2a}(r) + T_2(t)v_{2b}(r), \quad r_d < r < r_p,
\end{aligned} \tag{90}$$

where

$$\begin{aligned}
v_{1a}(r) &= 1 + a_{2a}r^2, \\
v_{1b}(r) &= a_{2b}r^2, \\
v_{2a}(r) &= b_{1a}r + b_{2a}r^2, \\
v_{2b}(r) &= 1 + b_{1b}r + b_{2b}r^2.
\end{aligned} \tag{91}$$

We follow the Rayleigh-Ritz procedure^{34,35} to approximately minimize the functional [Eq. (87)] and arrive at ODEs for $T_1(t)$ and $T_2(t)$. In this procedure, the trial function given by Eq. (90) is substituted into the functional $\mathcal{F}$, and $T_1(t)$ and $T_2(t)$ are determined so that the functional $\mathcal{F}$ is minimized. This is accomplished by setting the variations $\delta\mathcal{F}/\delta T_1$ and $\delta\mathcal{F}/\delta T_2$ equal to zero. The resulting two equations are solved simultaneously to arrive at ODEs for $T_1(t)$ and $T_2(t)$.

$$\frac{dT_s(t)}{dt} = l_{1,s}T_1(t) + l_{2,s}T_2(t) + l_{3,s}[c_q(n_0(t) - \tilde{n}_0) + c_q\xi n_1(t)], \quad s = 1, 2, \tag{92}$$

where $l_{1,s}$, $l_{2,s}$ and $l_{3,s}$ are slightly complicated constants which depend on the design parameters r_d , r_p , r_g , r_c , Bi_p , Bi_g and Bi_c . Expressions for them are very long; hence, they are not written out explicitly here. They were determined using the symbolic manipulation package Mathematica,³³ and they were directly included in the stability analysis and the numerical simulation codes (written in the C programming language) to carry out the stability studies reported below. The Mathematica input file, that was used to determine these expressions, is available.³⁶ Since the last term in the above equation is zero at steady-state (because

$\tilde{n}_1 = 0$), and since in general $l_{1,1}l_{2,2} \neq l_{1,2}l_{2,1}$, the steady-state values $\tilde{T}_s, s = 1, 2$, are zero. This is consistent with the fact that the phase variables $T_s(t), s = 1, 2$, are the time dependent coefficients in the spatial approximation for $\theta_p(r, t)$, which at steady-state is equal to zero [see Eq. (65)]. This completes the development of the ODEs [Eq. (92)], that represent the fuel rod heat conduction in the BWR model.

III.B.2 Expressions for Intermediate Variables

Expressions for the intermediate variables, which depend on the phase variables $T_s(t)$, are now derived. They are the average temperature in the fuel rod, $T_{avg}(t)$, the surface temperatures, $T_{surf,1\phi}$ and $T_{surf,2\phi}$, and $f_{11}(t) (\equiv dT_{surf,2\phi}(t)/dt)$ which appeared in Eqs. (20), (24), (37) and Eq. (47), respectively.

We operate on Eq. (65) with $1/(\pi r_p^2) \int_0^{r_p} 2\pi r dr$, and define $T_{avg,j\phi}(t)$, the average pellet temperature in each phase $j\phi, j = 1, 2$, of the channel, as

$$T_{avg,j\phi}(t) = \frac{2}{r_p^2} \int_0^{r_p} T_{p,j\phi}(r, t) r dr, \quad (93)$$

to obtain

$$T_{avg,j\phi}(t) = \theta_{avg,j\phi}(t) + \tilde{T}_{avg,j\phi}, \quad j = 1, 2. \quad (94)$$

The steady-state average temperature $\tilde{T}_{avg,j\phi}$, is determined by operating on Eq. (67) with $1/(\pi r_p^2) \int_0^{r_p} 2\pi r dr$, to obtain

$$\tilde{T}_{avg,j\phi} = -c_q \tilde{n}_0 \frac{r_p^2}{8} + \tilde{b}_{1,j\phi}. \quad (95)$$

The definition

$$\theta_{avg,j\phi}(t) = \frac{2}{r_p^2} \int_0^{r_p} \theta_{p,j\phi}(r,t) r dr , \quad (96)$$

relates $T_{avg,j\phi}(t)$ to the phase variables $T_{s,j\phi}(t)$, $s = 1, 2$, $j = 1, 2$, through Eq. (90). Here and everywhere below, we retain the subscript $j\phi$ on all terms in order to clearly distinguish between the overall average temperature of the fuel rod $T_{avg}(t)$ in the channel used in Eq. (20), and the average temperature of the fuel rod in each of the two axial regions [$z < \mu(t)$ and $z > \mu(t)$] of the channel $T_{avg,j\phi}(t)$, $j = 1, 2$. Substituting Eq. (90) into the above equation and carrying out the integrations gives

$$\theta_{avg,j\phi}(t) = A_{j\phi} T_{1,j\phi}(t) + C_{j\phi} T_{2,j\phi}(t) , \quad (97)$$

where $A_{j\phi}$ and $C_{j\phi}$ are slightly complicated constants which depend on the design parameters r_d , r_p , r_g , r_c , Bi_p , Bi_g and Bi_c . Expressions for them are not written out explicitly since doing so would require several pages. Instead, they were determined from the symbolic manipulation package Mathematica³³ and directly included in the stability analysis and the numerical simulation codes. The Mathematica input file used to generate them, is available.³⁶ The average temperature of the fuel rod $T_{avg}(t)$, used in Eq. (20), is then defined as the weighted average of the single- and two-phase region average temperatures

$$T_{avg}(t) = \mu(t) T_{avg,1\phi}(t) + [1 - \mu(t)] T_{avg,2\phi}(t). \quad (98)$$

This completes the discussion of the average temperature of the fuel rod.

Next, the fuel rod surface temperatures, $T_{surf,j\phi}(t)$, $j = 1, 2$, are defined by evaluating Eq. (66) at $r = r_c$

$$T_{surf,j\phi}(t) = T_{c,j\phi}(r_c, t) = \theta_{c,j\phi}(r_c, t) + \tilde{T}_{c,j\phi}(r_c), \quad j = 1, 2, \quad (99)$$

where the steady-state clad surface temperature, $\tilde{T}_{c,j\phi}(r_c) = \tilde{T}_{surf,j\phi}$, is determined from Eq. (68)

$$\tilde{T}_{surf,j\phi} = \tilde{T}_{c,j\phi}(r_c) = \tilde{b}_{2,j\phi} \log r_c + \tilde{b}_{3,j\phi}, \quad (100)$$

and the temperature deviation $\theta_{c,j\phi}(r_c, t) = \theta_{surf,j\phi}(t)$ is determined from Eq. (78) as

$$\theta_{surf,j\phi}(t) = \theta_{c,j\phi}(r_c, t) = b_{2,j\phi}(t) \log r_c + b_{3,j\phi}(t). \quad (101)$$

The fuel rod surface temperatures — which are the same as the clad surface temperatures — were used in the definitions of the phase change numbers $N_{pch,j\phi}$ [Eqs. (24) and (37)], which subsequently were used in Eqs. (26), (27), (40) and (41). The clad surface temperature deviation, $\theta_{c,j\phi}(r_c, t)$, is thus related to the pellet surface temperature deviation $\theta_{p,j\phi}(r_p, t)$, via Eqs. (79) and (80), as

$$\theta_{surf,j\phi}(t) = \theta_{c,j\phi}(r_c, t) = \frac{Bi_g}{Bi_g + Bi_{c,j\phi} + Bi_g Bi_{c,j\phi} \log(r_c/r_g)} \theta_{p,j\phi}(r_p, t). \quad (102)$$

The pellet surface temperature deviation can be evaluated in two different ways — either directly from the expression for the pellet temperature [Eq. (90)] by evaluating it at $r = r_p$

to give

$$\theta_{p,j\phi}(r_p, t) = T_{1,j\phi}(t)v_{2a,j\phi}(r_p) + T_{2,j\phi}(t)v_{2b,j\phi}(r_p) , \quad (103)$$

or by relating it to the average quantity $\theta_{avg,j\phi}(t)$ defined in Eq. (93). In this second approach, Eq. (72) is operated on by $1/(\pi r_p^2) \int_0^{r_p} 2\pi r dr$, and the BCs, Eqs (74) and (82), are used to obtain

$$\theta_{p,j\phi}(r_p, t) = \frac{r_p^2}{2Bi_p^+} \left\{ c_q [n(t) - \tilde{n}_0] + c_q \xi n_1(t) - \frac{1}{\alpha_p} \frac{d\theta_{avg,j\phi}(t)}{dt} \right\}, \quad j = 1, 2 . \quad (104)$$

Although both definitions give the same surface temperature at steady-state, the results vary for the time-dependent problem, and hence, also for the SB calculation. We showed⁴ that more accurate results — as measured by the computed SB in an operating parameter plane when compared to a reference SB generated using the eigenfunction function expansion method — are obtained by relating the surface temperature to the average temperature [Eq. (104)] rather than using the direct definition [Eq. (103)]. Therefore, here we use Eq. (104) for calculating $\theta_{surf,j\phi}(t)$. Substituting Eq. (104) into Eq. (102) gives

$$\begin{aligned} \theta_{surf,j\phi}(t) = \theta_{c,j\phi}(r_c, t) = & \left\{ \frac{r_p^2}{2Bi_p^+} \frac{Bi_g}{[Bi_g + Bi_{c,j\phi} + Bi_g Bi_{c,j\phi} \log(r_c/r_g)]} \right\} \\ & \left\{ c_q [n(t) - \tilde{n}_0] + c_q \xi n_1(t) - \frac{1}{\alpha_p} \frac{d\theta_{avg,j\phi}(t)}{dt} \right\} , \\ & j = 1, 2 . \end{aligned} \quad (105)$$

The derivative $d\theta_{avg,j\phi}(t)/dt$ is straightforwardly determined in terms of the phase variables $T_s(t)$, $s = 1, 2$, by taking the time derivative of Eq. (97), and using Eq. (92) to substitute for

$dT_s(t)/dt$, $s = 1, 2$, on the right hand side of the resulting expression, giving

$$\begin{aligned} \frac{d\theta_{avg,j\phi}(t)}{dt} = & A_{j\phi} \left\{ l_{1,1,j\phi} T_{1,j\phi}(t) + l_{2,1,j\phi} T_{2,j\phi}(t) + \right. \\ & \left. l_{3,1,j\phi} [c_q(n_0(t) - \bar{n}_0) + c_q \xi n_1(t)] \right\} + \\ & C_{j\phi} \left\{ l_{1,2,j\phi} T_{1,j\phi}(t) + l_{2,2,j\phi} T_{2,j\phi}(t) + \right. \\ & \left. l_{3,2,j\phi} [c_q(n_0(t) - \bar{n}_0) + c_q \xi n_1(t)] \right\}. \end{aligned} \quad (106)$$

This completes the development of the equations for the fuel rod surface temperatures.

Finally, we determine $f_{11}(t)$ ($\equiv dT_{surf,2\phi}(t)/dt$) which appeared in Eq. (47). From Eq. (66) at $r = r_c$, we have $d\theta_{c,2\phi}(r_c, t)/dt = dT_{c,2\phi}(r_c, t)/dt$, and since $T_{c,2\phi}(r_c, t) = T_{surf,2\phi}(t)$, we have from Eq. (105)

$$\begin{aligned} f_{11}(t) \equiv \frac{d\theta_{c,2\phi}(r_c, t)}{dt} = & \left\{ \frac{r_p^2}{2Bi_p^+} \frac{Bi_g}{[Bi_g + Bi_{c,2\phi} + Bi_g Bi_{c,2\phi} \log(r_c/r_g)]} \right\} \\ & \left[c_q \frac{dn_0(t)}{dt} + c_q \xi \frac{dn_1(t)}{dt} - \frac{1}{\alpha_p} \frac{d^2\theta_{avg,2\phi}(t)}{dt^2} \right], \end{aligned} \quad (107)$$

where dn_0/dt and $dn_1(t)/dt$ were expressed in terms of the phase variables of the dynamical system in Eq. (17), and $d^2\theta_{avg,2\phi}(t)/dt^2$ is determined from Eq. (106)

$$\begin{aligned} \frac{d^2\theta_{avg,2\phi}(t)}{dt^2} = & A_{2\phi} \left\{ l_{1,1,2\phi} \frac{dT_{1,2\phi}(t)}{dt} + l_{2,1,2\phi} \frac{T_{2,2\phi}(t)}{dt} + \right. \\ & \left. l_{3,1,2\phi} \left[c_q \frac{n_0(t)}{dt} + c_q \xi \frac{n_1(t)}{dt} \right] \right\} + \\ & C_{2\phi} \left\{ l_{1,2,2\phi} \frac{T_{1,2\phi}(t)}{dt} + l_{2,2,2\phi} \frac{T_{2,2\phi}(t)}{dt} + \right. \\ & \left. l_{3,2,2\phi} \left[c_q \frac{n_0(t)}{dt} + c_q \xi \frac{n_1(t)}{dt} \right] \right\}, \end{aligned}$$

where dn_0/dt and $dn_1(t)/dt$ are again given by Eq. (17), and $dT_{s,2\phi}/dt, s = 1, 2$ are given by Eq. (92).

This completes the development of the model as a dynamical system comprised of time-dependent nonlinear ODEs. In the next section we summarize the dynamical system, and clearly identify the phase variables, the operating parameters, and the design parameters.

IV THE DYNAMICAL SYSTEM

The dynamical system that results from the model developed above is of the form

$$\dot{\underline{X}}(t) = \underline{F}(\underline{X}; \underline{\gamma}, \underline{\kappa}) , \quad (108)$$

where $\underline{X}(t)$ is the vector of phase variables

$$\begin{aligned} \underline{X}(t) = & (n_k(t), u_k(t), a_{1,l}(t), a_{2,l}(t), s_{1,l}(t), s_{2,l}(t), v_{inlet,l}(t), T_{s,j\phi,l}(t))^T, \\ & k = 0, 1, \quad l = 0, 1, \quad s = 1, 2, \quad j = 1, 2, \end{aligned} \quad (109)$$

$\underline{\gamma}$ is the vector of system operating parameters

$$\underline{\gamma} = (\rho_{ext}, N_{sub}, \Delta P_{ext})^T, \quad (110)$$

and $\underline{\kappa}$ is the vector of design parameters

$$\begin{aligned} \underline{\kappa} = & (Bi, Fr, L, N_{cov}, N_f, N_r, N_{sbo}, N_\rho, T_{sat}, c_1, c_2, c_q, \\ & k_{exit}, k_{inlet}, l_1, l_2, l_3, \tau_c, \tau_d, \tau_g, \tau_p, \Lambda, \alpha_p, \beta, \lambda, \omega)^T, \end{aligned} \quad (111)$$

[which will be omitted from all future references herein to the dynamical system, Eq. (108)].

The dynamical system is comprised of twenty-two phase variables, three operating parameters and many design parameters. We have not listed all the design parameters in the vector κ above. We only have shown the parameters that actually appear in the final ODEs. These parameters, which in turn depend on those listed in Appendix A, are defined in Appendix C.

The set of twenty-two ODEs include four for the neutron kinetics model, Eqs. (16) and (17) for $n_k(t)$ and $u_k(t)$, $k = 0, 1$, respectively. Ten ODEs describe the fluid flow in the two channels (actually five for each channel $l = 0, 1$). These are Eqs. (26), (27), (40) (41), and Eq. (47) for $a_1(t)$, $a_2(t)$, $s_1(t)$, $s_2(t)$ and $v_{inlet}(t)$, respectively, where the subscript l is omitted. Finally, there are eight ODEs that describe the fuel rod heat conduction (actually two for each axial region $j\phi$, $j = 1, 2$, in each channel $l = 0, 1$). These are given by Eq. (92) for $T_s(t)$, $s = 1, 2$, where the subscripts $j\phi$ and l are omitted.

The twenty-two phase variables of the dynamical system are

- $n_0(t)$: coefficient of the fundamental neutron number density mode.
- $n_1(t)$: coefficient of the first harmonic neutron number density mode.
- $u_k(t)$, $k = 0, 1$: coefficient of the eigenvector $\underline{\phi}_{1k}(\underline{r})$ in Eq. (9).

- $a_{1,l}(t), l = 0, 1$: coefficient of the spatially linear component of the quadratic representation for the single-phase enthalpy in channel l .
- $a_{2,l}(t), l = 0, 1$: coefficient of the spatially quadratic component of the quadratic representation for the single-phase enthalpy in channel l .
- $s_{1,l}(t), l = 0, 1$: coefficient of the spatially linear component of the quadratic representation for the two-phase quality in channel l .
- $s_{2,l}(t), l = 0, 1$: coefficient of the spatially quadratic component of the quadratic representation for the two-phase quality in channel l .
- $v_{inlet,l}(t), l = 0, 1$: inlet velocity in channel l .
- $T_{s,j\phi,l}(t), s = 0, 1, j = 1, 2, l = 0, 1$: s -th coefficient of the best trial function in the variational method used to solve for the fuel pellet temperature in each axial region $j\phi$ of channel l .

The three operating parameters are

- ρ_{ext} : total control rod induced external reactivity. Its modal analogs $\rho_{ext,kml}, k = 0, 1, m = 0, 1$, for each channel $l = 0, 1$ satisfy Eq. (B.16).
- N_{sub} : channel inlet subcooling number, a measure of how subcooled the liquid is at the inlet of the channel. Because the flow in the lower plenum is assumed to be well mixed, the channel inlet subcooling is the same for all the representative channels.

- ΔP_{ext} : external pressure drop across a channel. Because the flow in the two representative channels is from the same lower plenum to the same upper plenum, the external pressure drop is the same across the representative channels.

V STABILITY ANALYSES

V.A Stability Boundaries

The stability of the fixed points^a of the twenty-two dimensional dynamical system was analyzed by first determining the stability boundary (SB) and then carrying out various numerical simulations. The SB was obtained by supplementing the 22 nonlinear steady-state equations by the linear system generated by the Jacobian $J = \underline{F}_X(\underline{\tilde{X}}(\underline{\gamma}); \underline{\gamma})$, of the original system and then solving the extended system comprised of the original 22 fixed point equations plus the two equations that follow from setting the real and imaginary parts of the determinant of $J - i\Omega I$ equal to zero. These last two equations, which equivalently can be obtained by linearizing the original system and seeking solutions of the form $\exp(i\Omega t)$, provide the necessary condition for a Hopf bifurcation which leads to the loss of stability as a parameter is varied across the SB.

Actually, the steady-state forms of the original 22 equations given by Eq. (108), were first reduced to three equations with three unknowns (steady-state values of three phase variables). This was done as follows:

^aA fixed point (FP) is the steady-state solution determined for a specific set of operating parameters.

1. The four equations given by Eqs. (16) and (17) give the condition, $\tilde{\rho}_{km} = 0$, which is the total reactivity in the reactor at steady-state. From Eq. (20), this condition for $k = m = 0$ gives

$$\rho_{ext} + \sum_{l=0}^1 c_{1,00l} [\tilde{\alpha}_l - \alpha_{o,l}] + \sum_{l=0}^1 c_{2,00l} [\tilde{T}_{avg,l} - T_{avg,o,l}] = 0 , \quad (112)$$

where $\rho_{ext} = \rho_{ext,000} + \rho_{ext,001}$ is the total control rod induced reactivity equal to the sum of the rod induced reactivities over all the reactor regions l . $\rho_{ext,kml}$, for $k \neq 0$ and $m \neq 0$, are similarly obtained using Eq. (20). The steady-state value $\tilde{n}_0$, which corresponds to the coefficient of the fundamental mode [see Eqs. (4), (9) and (15) respectively], is an unknown value (> 0). The steady-state values of the other three phase variables are zero; these are $\tilde{n}_1$ which corresponds to the coefficient of the first harmonic mode and hence is zero at steady-state, and $\tilde{u}_0$ and $\tilde{u}_1$ whose zero values follow from Eq. (17) at steady-state.

2. Equations (26) and (27) give $\tilde{a}_1 = N_\rho N_r \tilde{N}_{pch,1\phi} / \tilde{v}_{inlet}$ and $\tilde{a}_2 = 0$. The single-phase phase change number $\tilde{N}_{pch,1\phi}$ depends only on $\tilde{n}_0$ via Eqs. (24) and (99). Hence, $\tilde{a}_1$ can be straightforwardly determined in terms of $\tilde{n}_0$ and $\tilde{v}_{inlet}$.

3. In Eqs. (40) and (41), the term $\tilde{f}_1$ is 0 because it is the time derivative of the boiling boundary [see Eq. (42)]. Hence, $\tilde{f}_4$ and $\tilde{f}_6$ are zero. These two conditions give $\tilde{s}_1 = \tilde{N}_{pch,2\phi} / \tilde{v}_{inlet}$ and $\tilde{s}_2 = 0$. The two-phase phase change number $\tilde{N}_{pch,2\phi}$ depends only on $\tilde{n}_0$ via Eqs. (37) and (99); hence, $\tilde{s}_1$ can easily be written in terms of $\tilde{n}_0$ and $\tilde{v}_{inlet}$.

4. Equation (47) reduces to $\tilde{f}_8 = 0$ because both time derivatives, $\tilde{f}_1$ and $\tilde{f}_{11}$, are zero at steady-state (f_1 is the time derivative of the boiling boundary [see Eq. (42)] and f_{11} is the time derivative of the fuel rod surface temperature in the two-phase region [see Eq. (107)]).

From Eq. (E.64) the steady-state condition $\tilde{f}_8 = 0$ gives

$$\begin{aligned} \Delta P_{ext} = & \Delta \tilde{P}_{inlet} - \Delta \tilde{P}_{exit} - \Delta \tilde{P}_{fric,1\phi} - \Delta \tilde{P}_{grav,1\phi} - \\ & \Delta \tilde{P}_{acc2,2\phi} - \Delta \tilde{P}_{fric,2\phi} - \Delta \tilde{P}_{grav,2\phi} , \end{aligned} \quad (113)$$

which fixes the external pressure drop in each representative channel.

5. Finally, Eq. (92) at steady-state is trivial, since $\tilde{T}_s = 0, s = 1, 2$.

Thus, the 22 equations were reduced to the 3 equations — Eq. (112), and Eq. (113) with the suppressed $l = 0, 1$, for each of the two representative channels. These three equations, in conjunction with the two additional equations that follow from setting the real and imaginary parts of the determinant of $J - i\Omega I$ equal to zero, formed a set of 5 equations and 5 unknowns. Of the 5 unknowns, 3 were the steady-state values of the phase variables $\tilde{n}_0$, $\tilde{v}_{inlet,0}$ and $\tilde{v}_{inlet,1}$. (The analytical expressions are used to determine the steady-state values of the four phase variables, $\tilde{a}_{1,0}$, $\tilde{a}_{1,1}$, $\tilde{s}_{1,0}$ and $\tilde{s}_{1,1}$ from $\tilde{n}_0$, $\tilde{v}_{inlet,0}$ and $\tilde{v}_{inlet,1}$. The remaining 15 phase variables $\tilde{n}_1$, $\tilde{u}_0$, $\tilde{u}_1$, $\tilde{a}_{2,0}$, $\tilde{a}_{2,1}$, $\tilde{s}_{2,0}$, $\tilde{s}_{2,1}$, $T_{s,j\phi,l}$, $s = 1, 2$, $j = 1, 2$, $l = 1, 2$, are always zero at steady-state.) The other two unknowns in the set of 5 equations were usually the oscillation frequency Ω and one of the operating parameters ρ_{ext} , N_{sub} or ΔP_{ext} .

The relevant three-dimensional parameter space, the $\rho_{ext} - N_{sub} - \Delta P_{ext}$ space, is formed by the three operating parameters in Eq. (110). Further, since the external reactivity is fixed on the 100% rod line ^b, the two-dimensional section of this parameter space corresponding

^bThe 100% rod line, also called the 100% flow-control line, is a line on the BWR power-flow map which passes through the normal operating point of the BWR (100% power and 100% flow rate). Further, the control rods do not move on this line; hence, operating points on the line are identified by a fixed ρ_{ext} value equal to the value at the normal operating point. BWRs operate on this line during startup and shutdown operations by flow control via the recirculation pumps.^{29, 37}

to this fixed value is the appropriate plane ($N_{sub} - \Delta P_{ext}$) in which to study the stability of steady-state BWR operating points on the 100% rod line. Thus, a point on the SB was determined by first fixing the external reactivity to the 100% rod line value, then choosing a value for N_{sub} , and solving the five equations for the three steady-state phase variables $\tilde{n}_0$, $\tilde{v}_{inlet,0}$ and $\tilde{v}_{inlet,1}$, the system parameter ΔP_{ext} and the frequency Ω of the oscillatory solution that results from the Hopf bifurcation. The entire SB was generated by varying one of the three quantities, N_{sub} or $\tilde{n}_0$ or $\tilde{v}_{inlet,0}$, in a sequence of calculations and solving for the remaining five unknowns.

The *stability boundary* code used in the calculations is written in C, and it uses the International Mathematics and Statistics Library (IMSL) C subroutine *imsl_d_zeros_sys_eqn* to solve the system of five nonlinear equations. This IMSL subroutine solves a general system of N nonlinear equations $f_n(x_1, \dots, x_n) = 0, n = 1, \dots, N$ using a modified Powell hybrid algorithm.³⁸

V.B Stability Boundaries for the 100% Rod Line

The results of the stability analysis, carried out as summarized above for a fixed $\rho_{ext} = 0$ equal to the 100% rod line value, are shown in Fig. 2 where the SB is presented in the inlet-subcooling-number/external-pressure-drop operating parameter plane. Also shown are the points on the 100% rod line — on which BWRs are operated — transformed from the BWR power-flow map to this parameter space. Since the $\tilde{n}_0$, $\tilde{v}_{inlet,0}$ and ρ_{ext} values are known for the points on the 100% rod line, the other two unknown operating parameters, N_{sub} and ΔP_{ext} , were determined using the *steady-state* code (see Section VI below); hence, this way

the 100% rod line was transformed from the power-flow map to the $N_{sub} - \Delta P_{ext}$ plane. Each point in this plane determines a unique steady-state (fixed point) value. All points on the right side of the SB, of course, correspond to stable steady-states and all points on the left side correspond to unstable steady-states. However, since this plane is for a fixed $\rho_{ext} = 0$, equal to the 100% rod line value, only the points on this plane corresponding to the 100% rod line of the power-flow map are relevant and of practical interest. Thus, only those points are studied here. The solid curve is the SB based on the BWR model described above which includes the full space-time modal kinetics. The dashed curve (indistinguishable from the solid SB curve in Fig. 2) is the SB based on the reduced model in which the modal kinetics [Eqs. (16) and (17) with $k = 0, 1$] is replaced by point kinetics [Eqs. (16) and (17) with $k = 0$]; this curve is indistinguishable from the solid curve on the scale of Fig. 2 (actually, the points differ only in their third digits). It is not surprising that the two SBs are almost identical, since the higher modes drop out of the steady-state modal kinetics equations leaving them the same as those for point kinetics; thus the fixed points on the 100% rod line are identical, and the small differences (in the third digits) between those on the SBs are due the differences that appear in the Jacobians of the two systems (i.e. their linearizations about the steady-states). The fact that all the points on the 100% rod line in Fig. 2 are on the stable side of the SB, of course, shows that all these points give parameter values at which the corresponding BWR steady-states will be stable. Reference will be made to the operating point indicated by the asterisk in Fig. 2 in Section VI on numerical simulations. As we shall discuss there, even though the SBs that result from the two models — the one using the modal kinetics equations and the one using the point kinetics equations — are essentially the same, our numerical simulations show that the two models can lead to very different nonlinear time

evolutions for the same system starting from the same initial condition (IC), which in turn have very significant implications about the necessity to include the modal kinetics equations in the model.

Figure 2: Parameter plane. Stability boundary (SB) in the inlet-subcooling-number/external-pressure-drop operating parameter plane (the section of the 3-D parameter space corresponding to the external reactivity fixed at the 100% rod line value). The SB divides the operating parameter plane into stable and unstable regions. The two SBs are indistinguishable in this figure.

Because operational guidance and regulation of BWRs with respect to their stability is focused around the so-called power-flow map, we also have transformed the calculated SBs from the parameter plane in Fig. 2 to a two dimensional projection of the phase space — the channel-zero fundamental-mode-neutron-density(power)/inlet-velocity(flow) plane, where the 100% rod line is a straight line. Thus, the SBs and the 100% rod line also are shown in

Fig. 3 in the power-flow plane which is more familiar to regulatory agencies and operating engineers. Although a steady-state is uniquely determined by fixing the three operating parameters $\underline{\gamma} = (\rho_{ext}, N_{sub}, \Delta P_{ext})$ and solving Eq. (108) at steady-state $[F(\tilde{X}(\underline{\gamma}); \underline{\gamma}) = 0]$, it also can be uniquely determined by fixing the steady-state values of any two phase variables and any one operating parameter. In the power-flow plane for example, each point determines a unique steady-state when the steady-states of the two phase variables, $\tilde{n}_0$ and $\tilde{v}_{inlet,0}$, are fixed and any one operating parameter is fixed. If the operating parameter ρ_{ext} is fixed equal to the 100% rod line value, only the points corresponding to the 100% rod line (shown in Fig. 3) in this plane are relevant and of practical interest, and since the SB plotted in this figure corresponds to fixed ρ_{ext} equal to the 100% rod line value, only the points on the 100% rod line can be related to the SB in this plane.

The stability of the operating points on the 100% rod line can easily be inferred from either Fig. 2 or Fig. 3 by comparing their distances from the SB. Clearly, the points move closer to the unstable region as the flow rate is reduced from 100%. Hence, they become less stable, with the least stable points being in the low-flow/high-power region. Although all the points on the 100% rod line in Fig. 3 are in the stable region (in correct correspondence with Fig. 2), some — especially those nearby points 6 and 7 — are very close to the SB. Thus the nature of the Hopf bifurcation that occurs as a parameter is varied across the SB becomes very important. If it is supercritical, *stable* limit cycle oscillations exist for parameter values nearby the SB in the region where the steady-state is *unstable* (the region to the left of the SB in Fig. 2). Whereas if it is subcritical, *unstable* limit cycle oscillations exist for parameter values nearby the SB in the region where the steady-state is *stable* (the region to the right of

Figure 3: Power-flow plane. Transformed stability boundary (TSB) for the 100% rod line in the power-flow plane (a 2-D projection of the 22-D phase space). The entire boundary is included in order to separate the plane into the stable and unstable steady-states, even though not all points on it are achievable. For example, only the portion of the unstable region to the right of the natural circulation curve is relevant since only this portion is accessible during normal BWR operation.

the SB in Fig. 2). In this latter case, an IC nearby a *stable* steady-state on the 100% rod line (in Fig. 3), which can be generated by a small but finite perturbation of that steady-state, can lead to *growing*, and even *completely divergent* oscillations. Thus, numerical simulations were also carried out, which are reported in Section VI.B, to deduce whether the bifurcation is supercritical or subcritical.

VI NUMERICAL SIMULATIONS

VI.A Numerical Simulations Approach

Several sets of numerical simulations were performed to study the nonlinear effects not captured by the local bifurcation and stability analyses. In particular, they were performed to study the growing and decaying oscillations, to determine the nature of the Hopf bifurcation, to study the decay ratios, and to examine the frequencies of the oscillations. A set of simulations also was carried out to identify important differences between BWR behavior predicted by the present model and that predicted when the simple point kinetics equations, rather than space-time modal kinetics equations, are used in it.

The numerical simulations were carried out using two codes written in C — the *steady-state* code and the *numerical simulation* code. The first one — which also was used for the stability analyses reported in Section V — determined the steady-state values of the phase variables, for fixed values of the three operating parameters $(\rho_{ext}, N_{sub}, \Delta P_{ext})$, by solving the three equations for the three unknowns, $\tilde{n}_0$, $\tilde{v}_{inlet,0}$ and $\tilde{v}_{inlet,1}$. (See Section V for details.)

The analytical expressions were then used to determine the steady-state values of the four phase variables, $\tilde{a}_{1,0}$, $\tilde{a}_{1,1}$, $\tilde{s}_{1,0}$ and $\tilde{s}_{1,1}$. The remaining 15 phase variables $\tilde{n}_1$, $\tilde{u}_0$, $\tilde{u}_1$, $\tilde{a}_{2,0}$, $\tilde{a}_{2,1}$, $\tilde{s}_{2,0}$, $\tilde{s}_{2,1}$, $T_{s,j\phi,l}$, $s = 1, 2$, $j = 1, 2$, $l = 1, 2$, are always zero at steady-state.

Alternatively, an option in the code enabled it to determine the steady-state, by fixing any one of the three operating parameters and any two of the three phase variables, $\tilde{n}_0$, $\tilde{v}_{inlet,0}$ or $\tilde{v}_{inlet,1}$. The code then solved the three equations to determine the remaining two operating parameters and the the third phase variable. These two operating parameters combined with the third operating parameter of course, determine the unique steady-state. This option was essential to determine the steady-state for points on the power-flow map when a combination of phase variables and operating parameters were fixed (such as on the 100% rod line where $\tilde{n}_0$, $\tilde{v}_{inlet,0}$, and ρ_{ext} are known).

The IMSL C subroutine *imsl_d_zeros_sys_eqn* was used to solve the system of the three nonlinear equations in the *steady-state* code. The second code, the *numerical simulation* code solved the 22 time-dependent nonlinear ODEs. The initial conditions were set “close” to the steady-state values obtained from the first code. Since it is well known that the neutron kinetics equations make the dynamical system *stiff*^{39,40} with some of the eigenvalues of the Jacobian matrix $J = \underline{F}_{\underline{X}}(\underline{\tilde{X}}(\underline{\gamma}); \underline{\gamma})$ becoming large and negative, numerical differential equation solvers such as the Runge-Kutta method are inefficient, and in some instances fail completely, and special methods are often required. To solve the stiff set of ODEs here, we used the IMSL C solver *imsl_d_ode_adams_gear* which is based on Gear’s methods for stiff ODEs.^{41,42}

VI.B The Nature of the Hopf Bifurcation

For ICs only slightly away from the fixed points (FP), the results of the numerical simulations always agreed with those of the stability analyses, i.e. growing oscillations resulted for points on the unstable side of the SB in the parameter plane and decaying oscillations (back to the FP) resulted for points on the stable side. To specifically study the nature of the Hopf bifurcation as the SB in Fig. 2 is crossed, numerical simulations were carried out starting with ICs nearby the FP indicated by point 6 on the 100% rod line. It is in the stable region of Fig. 2 and is determined by the system parameters $(\rho_{ext}, N_{sub}, \Delta P_{ext}) = (0.0, 1.12, 8.2)$. The resulting trajectories for two nearby ICs are plotted in Fig. 4. We observe that for IC-1, which is a small perturbation of four of the FP components ($n_0 : 0.5800 \rightarrow 0.5886$; $a_{1,0} : 0.26608 \rightarrow 0.25610$; $s_{1,0} : 4.9306 \rightarrow 4.7367$; $v_{inlet,0} : 0.39400 \rightarrow 0.39383$), the oscillations damp with the (solid) trajectory returning toward the FP when the simulation was terminated. This is expected since the FP is stable. However, what is interesting is that when starting from a nearby IC, labeled IC-2 (identical to IC-1 except $n_0 = 0.5916$), the oscillations grow with the dashed trajectory diverging away from the FP when the simulation was terminated. This suggests that the bifurcation on the SB in Fig. 2 (and Fig. 3) is a *subcritical* Hopf bifurcation that can lead in practice to growing oscillations even when a steady-state, such as point 6 on the 100% rod line, is stable. This is consistent with the fact that the growing oscillations, that result from the simulations we performed for parameter values $(\rho_{ext}, N_{sub}, \Delta P_{ext}) = (0.0, 1.12, 7.5)$ at point * in Fig. 2, did not saturate at a stable limit cycle which would indicate that the Hopf bifurcation is *supercritical*. Hence, when the system is operated at the allegedly “safe” parameter values corresponding to point 6, for which the FP is stable,

an IC that could result due to a finite perturbation from that FP would lead to growing oscillations, not oscillations that would die away and evolve to the FP. Thus, although the steady-state corresponding to the point 6 in Figs. 2 and 3 is stable, it may not necessarily be a safe point at which to operate the BWR and should be avoided during normal startup and shutdown operations, since a small but finite perturbation could lead to growing nuclear-coupled density-wave oscillations as indeed the simulations summarized in Fig. 4 show. This, of course, typically is the case when a subcritical Hopf bifurcation occurs. These results are consistent with the results reported by Tsuji et al.¹⁶ who also showed in the stability analysis of their model that a subcritical Hopf bifurcation occurred as the SB was crossed.

Figure 4: Subcritical Hopf bifurcation. Phase portrait showing the evolution of the phase space trajectories in the projection of the phase space onto the $v_{inlet,0} - n_0 - n_1$ subspace for parameter values at point 6 in Fig. 2 and for two nearby ICs. They indicate a *subcritical* Hopf bifurcation at the SB and suggest the existence of an unstable limit cycle in the phase space nearby the stable FP.

VI.C Decay Ratios and Oscillation Frequencies

The decay ratio (DR) is a traditional stability index also used to characterize BWR stability. It is the ratio of two consecutive maxima during the decay of an oscillation¹² and is a measure of how fast ($DR \approx 0$) or how slowly ($DR \approx 1$) oscillations damp. The DRs and frequencies of oscillations were studied at various points on the power-flow map. Specifically, they were studied on the 100% rod line as the flow rate is reduced, by calculating them from the numerical simulations carried out at various points on that rod line. The DRs calculated are tabulated in the insert in Fig. 3 and also are plotted in Fig. 5 against the points on the 100% rod line. It is clear from Fig. 5 that, moving down the 100% rod line from point 1 to 6, the DRs increase and they approach unity in the low-flow/high-power region of the power-flow map. This is consistent with our stability analyses and shows that the low-flow/high-power region is the least stable region. In fact, from points 4 to 7, the NRC requirement¹² of $DR < 0.75 - 0.8$ is not rigorously satisfied and those points should be avoided during normal startup and shutdown operations of a BWR.

The frequencies of oscillations were calculated from the numerical simulations at various points on the 100% rod line. They are shown in Fig. 6 plotted against the position of points 1 through 7 on the rod line. Clearly, as the operating point is moved down the 100% rod line, the frequency decreases more or less linearly from $0.91Hz$ at point 1 (the 100% power operating point) to $0.45Hz$ at point 6. The frequencies at points 5 - 6 which are in the low-flow/high-power region of the power-flow map, agree well with the frequency of approximately $0.5Hz$ observed during BWR stability tests at the Forsmark-1 BWR¹² and during the incident at the LaSalle-2 BWR.¹⁵

Figure 5: Decay ratios (DRs) for points on the 100% rod line. As the operating point is moved down the 100% rod line, from points 1 to 6, the DR increases indicating decreasing stability. The least stable points, points 5 to 7, are in the low-flow/high-power region of the power-flow map.

Figure 6: Frequencies of oscillation for points on the 100% rod line. As the operating point is moved down the 100% rod line, the frequency decreases from 0.91 Hz at point 1 (the normal operating point at 100% power) to 0.45 Hz at point 6 (a point in the low-flow/high-power region of the power-flow map). The frequencies for points 5 - 6 which are in the low-flow/high-power region, agree well with the 0.5 Hz observed during BWR stability tests and instability incidents.

VI.D The Effect of the First Harmonic Mode

To determine the effect of the first harmonic mode on stability, we studied the phase flows predicted by the model with the space-time modal kinetics equations and compared them with those predicted when the modal kinetics equations are replaced by simple point kinetics equations [$k = 0$ only in Eq. (17)]. The two trajectories — one calculated using modal kinetics, the other using point kinetics — shown in Fig. 7 start from the same IC-2 of Fig. 4. We observe that while the general model with the modal kinetics equations leads to growing oscillations, the model with the point kinetics equations leads to damped oscillations. Thus, these out-of-phase growing oscillations result from the effects of the first harmonic mode. Stated mathematically, the basin of attraction of the FP shrinks because of the first harmonic mode. From either perspective, the low-flow/high-power region on the BWR power-flow map, which is the least stable region, in fact, is even less stable because of the first harmonic mode, and the results based on the use of point kinetics fail to reflect this decreased stability. This nonconservative feature of the model based on the widely used point kinetics equations indicates that the use of the point kinetics approximation may be inappropriate in many cases for the analysis of density-wave instabilities in BWRs.

VI.E Sinusoidal Stabilization

Some exploratory numerical simulations were carried out to study the possibility of stabilizing the growing oscillations shown in Figs. 4 and 7, by sinusoidally forcing the system operating parameters $N_{sub,l}$, $\Delta P_{ext,l}$ and $\rho_{ext,kml}$. Both in-phase forcing (in which the sinu-

Figure 7: A comparison between space-time kinetics and point kinetics. Evolution of the phase space trajectories in the projection of the phase space onto the $v_{inlet,0} - n_0 - n_1$ space for parameter values at point 6 in Fig. 2 and IC-2 in Fig. 4. The two trajectories, one with the effect of the first harmonic and the other without it, show the destabilizing effect of the first harmonic mode which is inaccessible when space-time kinetics is replaced by point kinetics; hence, they demonstrate the nonconservative nature of the point kinetics equations in the study of BWR stability.

soidal variation of the parameter was in-phase in the two halves $l = 0, 1$ of the reactor) and out-of-phase forcing (in which it was out-of-phase) were studied.

For in-phase forcing, with $N_{sub,l}(t) = \tilde{N}_{sub,l} \{1 + 0.01 \sin [0.045 (2\pi t^*)]\}$ — where the amplitude is 1% $\tilde{N}_{sub,l}$, the frequency is 1/10th the natural system frequency of 0.45 *Hz*, and t^* is in seconds — the oscillations emanating from IC-2, that grew for the autonomous (unforced) system, now actually damp. Thus, this in-phase forcing of $N_{sub,l}$ has a stabilizing effect on the system. The resulting damped oscillations along with the original growing oscillations from Figs. 4 and 7 are plotted as phase portraits in Fig. 8. When the forcing frequency was reduced to 1/20th the natural frequency, the oscillations damped even faster. Conversely, when it was increased to the natural system frequency, it resulted in a substantially faster growth of oscillation amplitude and hence, decreased the time for the oscillations to grow to the reference cutoff value we set at 114% $\tilde{n}_0$. A further increase in the frequency above 0.45 *Hz* only resulted in slower growth of oscillation amplitude and hence, increased the time for the oscillations to reach the reference cutoff value. All further frequency increases up to the maximum value studied, 20 times the natural system frequency, also led to growing oscillations. The frequency was then fixed at 0.045 *Hz*, and the forcing amplitude was varied. As the amplitude was increased from 1% $\tilde{N}_{sub,l}$ the oscillations damped faster, and as it was decreased to zero they became growing oscillations as expected. Similar behavior was observed as a function of both frequency and amplitude when $\Delta P_{ext,l}(t)$ was sinusoidally forced. However, for in-phase forcing of $\rho_{ext,kml}(t)$ no decaying oscillations were ever observed over the broad range of frequencies and amplitudes studied.

For out-of-phase forcing, with $N_{sub,l}(t) = \tilde{N}_{sub,l} \{1 + (1 - 2l)0.01 \sin [0.045 (2\pi t^*)]\}$, $l =$

Figure 8: Stabilization of growing oscillations. Evolution of the of the phase space trajectories in the projection of the phase space onto the $v_{inlet,0} - n_0 - n_1$ subspace for parameter values at point 6 in Fig. 2 and IC-2 in Fig. 4. The figure shows the stabilizing effect of the in-phase sinusoidal forcing of $N_{sub,l}(t)$.

0, 1, unlike in the in-phase case, the oscillations here were still divergent; furthermore, they were divergent for the entire range of frequencies studied. However, at the fixed frequency of 0.045 *Hz* when the forcing amplitude was increased to 2% $\tilde{N}_{sub,l}$ (and beyond) the oscillations decayed. Thus for this fixed frequency, the lower bound of the amplitude above which the forcing of $N_{sub,l}(t)$ is stabilizing, is higher for out-of-phase forcing than for in-phase forcing. However, for out-of-phase forcing of $\Delta P_{ext,l}(t)$, exactly the opposite effects were observed. At the fixed frequency of 0.045 *Hz*, damped oscillations were observed for out-of-phase forcing amplitudes as small as 0.5% $\Delta \tilde{P}_{ext,l}$ while growing oscillations were observed for all in-phase forcing amplitudes below 1% $\Delta \tilde{P}_{ext,l}$. Thus, for this fixed frequency, the lower bound of the amplitude above which forcing of $\Delta P_{ext,l}(t)$ is stabilizing, is lower for out-of-phase forcing than for in-phase forcing. As in the in-phase case, no convergent oscillations were observed for out-of-phase forcing of $\rho_{ext,kml}(t)$ over the broad range of frequencies and amplitudes studied.

These exploratory simulations show that the stability of operating points can be increased (or decreased) by simple sinusoidal forcing of certain system parameters. Although this does not indicate that such forcing is a viable way of controlling BWR instabilities, it does demonstrate that system components external to the reactor core do affect BWR stability. This follows from the fact that some constants, which appear as parameters in the nuclear-coupled thermal-hydraulic BWR core model developed in Sections II and III above, become time-dependent and depend on system phase variables when additional components of the system are included in the model. For example, the inlet subcooling $N_{sub,l}$, which now is a parameter in the model, will depend upon the phase variables through the channel exit quality when

the recirculation loop is included. Motivated by concern about potentially significant affects of this important system component, in the next section we extend the model developed and analyzed above, to include a simple representation of a BWR recirculation loop.

VII A SIMPLE RECIRCULATION LOOP MODEL

In the model developed in Sections II and III, which was used in Sections V and VI for stability analyses and numerical simulations, the the core was isolated from the rest of the reactor by specifying pressure BCs across the core and by specifying the core inlet subcooling. These constant (or time-dependent) pressure drop BCs and core inlet subcooling BC are appropriate to represent certain transients in the core. However, the dynamics of a BWR core are, in general, also influenced by the dynamics that occur outside the core. In fact, it is particularly important to include the effects of the recirculation loop in the development of a BWR model to be used in stability analyses. Hence, in this section we report the effects of the inclusion of a BWR recirculation loop on stability.

We first track the flow of the water-steam working fluid from the core exit back into the core inlet. The two-phase mixture that exits the channels rises in the upper plenum and passes through the steam separators and dryers. There the steam separates from the water and dries, with the separated water flowing down and collecting in the reservoir between the core shroud and the vessel wall. The dried steam leaves the vessel, goes to the turbines where it passes through a high pressure turbine and two low pressure turbines, driving them to generate electricity, and finally flows into the condenser. Depending on the power

load, a control mechanism also allows for a portion of the steam to bypass the turbines and throttle directly into the condenser. Another control mechanism allows steam from the turbines, called the extraction steam, to pass through the feedwater heaters and heat the feed water. The extraction steam eventually also flows into the condenser after passing through a series of feedwater heaters. The steam in the condenser is condensed and pumped as condensate water through the demineralizers, feedwater heaters and to the feed pumps. The feed pumps in turn pump the heated water, called the feedwater, back into the reactor vessel. The feedwater flows into the reservoir, where it mixes with the separated water that has flowed down from the steam separators and dryers. Some of the water, equal to about a third of the total flow to the core, is pumped by a set of two recirculation pumps into ten jet pump assemblies (in General Electric BWRs⁴³). Because of the suction action of the jet pumps, this water sucks with it additional water from the reservoir, equal to about two-thirds of the total flow to the core. The entire driving flow and suction flow mixture flows through the jet pumps into the lower plenum of the reactor vessel and then flows up into the core.

The detailed dynamics of the entire recirculation loop could be studied by solving the equations of continuity, momentum and energy within each of the components of the recirculation loop — the upper plenum, the steam separators, the dryers, the turbines, the condenser, the condensate pumps, the feedwater heaters, the feed pumps, the recirculation pumps, the jet pumps, and the lower plenum. However, it is not our intention here to model the entire system. Rather, we develop a very simple recirculation loop model that directly relates the delayed effects of phase variables, at the channel exits, upon the channel inlet

subcooling. Hence the simple model captures the most important effect — the effect, on the state of the water entering the channels at any time, of the state of water exiting the channel at an earlier time. Using this simple model, we study how the recirculation loop affects the overall stability by determining conditions under which it has a stabilizing effect and conditions under which it has a destabilizing effect. In particular, we study the effect of time delays associated with the steam separators and those associated with the jet pumps, as parameters, and carry out stability analyses and several sets of numerical simulations to study their effects on the stability of operating points.

VII.A The Simple Model

A simple recirculation loop model is developed by appropriately modifying the channel inlet subcooling number N_{sub} , along the lines of the very simple loop model developed by Rizwanuddin and Dorning.¹⁴ The inlet subcooling is made a function of the delayed effects of the quality, mixture density and mixture velocity at the channel exit, where the time delays are associated with the effects of the steam separators and the jet pumps. Also included are the delayed effects of controlling the feedwater flow rate (via the feed pumps) and the feedwater subcooling (via the feedwater heaters), where the time delays are associated with the effects of the jet pumps.

Assuming the saturated water exiting the steam separators ($\dot{m}_{sep}$) and the feedwater ($\dot{m}_{feed}$) in each region of the reservoir are well mixed, the reservoir enthalpy $h_{resv}(t_1)$ at any

instant t_1 in each region is

$$h_{resv}(t_1) = \frac{\dot{m}_{sep}(t_1)h_{sat} + \dot{m}_{feed}(t_1)h_{feed}(t_1)}{\dot{m}_{sep}(t_1) + \dot{m}_{feed}(t_1)}, \quad (114)$$

where h_{sat} is the saturation enthalpy and $h_{feed}(t_1)$ is the feedwater enthalpy. Here and everywhere below, the channel index subscript l , which also represents a reservoir region, has been omitted. Assuming a time delay τ_1 associated with the steam separators, the exit flow rate from them to the reservoir $\dot{m}_{sep}(t_1)$, is related to the channel exit quantities, the quality x_{exit} , the mixture density $\rho_{m,exit}$, and the mixture velocity $v_{m,exit}$, through the delay τ_1 as

$$\dot{m}_{sep}(t_1) = [1 - x_{exit}(t_1 - \tau_1)] \rho_{m,exit}(t_1 - \tau_1) v_{m,exit}(t_1 - \tau_1) NA, \quad (115)$$

where x_{exit} , $\rho_{m,exit}$ and $v_{m,exit}$ are determined from Eqs. (38), (35) and (36) respectively, at the exit of the channel, $z = 1$. Here the channel cross sectional flow area is A , and the number of channels in each region l of the reactor are N . The flow rate exiting from each region to the turbines as pure steam, of course, is $x_{exit}(t_1)\rho_{m,exit}(t_1)v_{m,exit}(t_1)NA$, which at steady-state also is equal to the feedwater flow rate $\tilde{m}_{feed}$, i.e.

$$\begin{aligned} \tilde{m}_{feed} &= \tilde{x}_{exit} \tilde{\rho}_{m,exit} \tilde{v}_{m,exit} NA \\ &= \tilde{x}_{exit} \tilde{v}_{inlet} NA, \end{aligned} \quad (116)$$

where we have used the steady-state continuity condition, $\tilde{\rho}_{m,exit} \tilde{v}_{m,exit} = \tilde{v}_{inlet}$. Further, taking the time-dependence of the feedwater flow rate as a function $\chi_1(t_1)$, which is determined by the control action on the feed pumps, the feedwater flow rate $\dot{m}_{feed}(t_1)$ is related

to its steady-state value $\tilde{m}_{feed}$, as

$$\begin{aligned}\dot{m}_{feed}(t_1) &= \tilde{m}_{feed}\chi_1(t_1) \\ &= \tilde{x}_{exit}\tilde{v}_{inlet}NA\chi_1(t_1) .\end{aligned}\tag{117}$$

Next, we relate the channel inlet enthalpy h_{inlet} to the reservoir enthalpy h_{resv} through a delayed time τ_2 , associated with the reactor jet pumps

$$h_{inlet}(t_1) = h_{resv}(t_1 - \tau_2) .\tag{118}$$

Taking the time-dependence of the feedwater subcooling as a function $\chi_2(t_1)$, which is determined by the control action on feedwater heaters and relates $N_{sub,feed}(t_1)$ to its steady-state value $\tilde{N}_{sub,feed}$, as

$$N_{sub,feed}(t_1) = \tilde{N}_{sub,feed}\chi_2(t_1) ,\tag{119}$$

we obtain the channel inlet subcooling $N_{sub}(t)$ by substituting Eqs. (115) and (117) into Eq. (114), substituting the resulting expression for h_{resv} into Eq. (118), and using the definitions of the channel inlet subcooling number N_{sub} and the feedwater subcooling number $N_{sub,feed}$ given in Appendix C. The result is

$$N_{sub}(t) = \frac{\tilde{x}_{exit}\tilde{v}_{inlet}\chi_1(t - \tau_2)\tilde{N}_{sub,feed}\chi_2(t - \tau_2)}{[1 - x_{exit}(t - \tau_{12})]\rho_{m,exit}(t - \tau_{12})v_{m,exit}(t - \tau_{12}) + \tilde{x}_{exit}\tilde{v}_{inlet}\chi_1(t - \tau_2)} ,\tag{120}$$

where $\tau_{12} = \tau_1 + \tau_2$. This equation relates the delayed effects of the channel exit quality x_{exit} , mixture density $\rho_{m,exit}$ and mixture velocity $v_{m,exit}$ to the channel inlet subcooling

$N_{sub}(t)$. It also includes the (delayed) time-dependence of the feedwater flow rate through $\chi_1(t)$, and the (delayed) time-dependence of the feed water subcooling through $\chi_2(t)$. The time delays τ_1 and τ_2 , generally depend on the complex dynamics of the steam separators and jet pumps. However, since it is not our intention here to develop a detailed model for these reactor components, we have represented them as simple time delays in the model. By carrying out the stability analyses and several sets of numerical simulations, we study the time delays as design parameters to determine conditions under which they have a stabilizing effect and the conditions under which they have a destabilizing effect. The time-dependent functions $\chi_1(t)$ and $\chi_2(t)$ can be identified with control mechanisms for the feedwater pumps and the feedwater heaters. Thus, when $\chi_1 = 1$ and $\chi_2 = 1$, the feedwater flow rate and subcooling are fixed at their steady-state values and the channel inlet subcooling varies only according to the delayed effects of the exit variables — the exit quality, the exit mixture density and the exit mixture velocity.

With the simple recirculation loop model included, the dynamical system that results has the same phase variables as the original one presented in Section IV without the recirculation loop model. The only difference is that the channel inlet subcooling, which earlier was an operating parameter, has become a time-dependent function which depends on some of the phase variables. The operating parameters, which uniquely determine a steady-state, are now the control rod induced external reactivity ρ_{ext} , the feedwater subcooling $\tilde{N}_{sub,feed}$, and the channel external pressure drop ΔP_{ext} . That is, $\tilde{N}_{sub,feed}$ has replaced N_{sub} as an operating parameter. The steady-state of this new dynamical system is identical to that of the dynamical system without the recirculation loop, uniquely determined by $(\rho_{ext}, \tilde{N}_{sub}, \Delta P_{ext})$,

provided $\tilde{N}_{sub,feed}$ and the channel inlet subcooling $\tilde{N}_{sub}$ satisfy Eq. (120) at steady-state, i.e.

$$\tilde{N}_{sub} = \tilde{x}_{exit} \tilde{N}_{sub,feed} . \quad (121)$$

However, in the time-dependent case the two models differ significantly. Hence, we carried out stability analyses and several sets of numerical simulations to study the effects of the simple recirculation loop model. The results of these stability analyses are presented in the next subsection and the results of the numerical simulations are reported below in Subsection VII.C.

The *stability boundary* code (see Section V) and the *numerical simulation* code (see Section VI) developed earlier were appropriately modified to incorporate the time dependence of $N_{sub}(t)$. However, there was no need to modify the *steady-state* code (see Section VI) since the time dependence of $N_{sub}(t)$ does not affect the steady-state.

VII.B Stability Analyses

We determined the SBs at various values of the time delays τ_1^* and τ_2^* for a fixed $\rho_{ext} = 0$, equal to the 100% rod line value. Since the inlet subcooling number no longer is an operating parameter but a time-dependent variable [see Eq. (120)], the appropriate plane to study the SBs is the steady-state inlet-subcooling-number/external-pressure-drop dependent-variable/operating-parameter plane, since this plane is the closest analogue to the $N_{sub} - \Delta P_{ext}$ parameter plane that was relevant to the model that did not include the recirculation loop. The effect of the recirculation loop can then easily be studied by comparing the new

SBs with those determined previously in Section V. We also transformed the calculated SBs from the $\tilde{N}_{sub} - \Delta P_{ext}$ plane to a two dimensional projection of the phase space, the channel-zero fundamental-mode-neutron-density(power)/inlet-velocity(flow) plane, as we did in Section V.

In Eq. (120) when χ_1 and χ_2 are 1, i.e. the feedwater flow rate and subcooling are fixed at their steady-state values, only the sum of the time delays τ_1^* and τ_2^* appears in that equation. Hence for the stability analyses, only a single parameter $\tau_{12}^* = \tau_1^* + \tau_2^*$ was varied. The total time delay τ_{12}^* was varied from 1 s to 10 s in increments of 1 s, and the corresponding SBs are plotted in Figs. 9 - 12. In the first two figures, Figs. 9 and 10, the SBs are plotted for $\tau_{12}^* = 1, 2, 3, 4$ and 5 s. In the next two figures, Figs. 11 and 12, the SBs are plotted for $\tau_{12}^* = 6, 7, 8, 9$ and 10 s. Also plotted in these four figures is the original SB from Fig. 2 without the recirculation loop effect. As observed in Fig. 9, the effect of the loop at $\tau_{12}^* = 1$ s is very significant. The SB shifts towards the right, crossing the 100% rod line at point 4 indicating that points 5, 6 and 7 clearly are unstable. At $\tau_{12}^* = 2$ s however, the SB shifts to the left of point 6 and points 4, 5 and 6 are stable again, while point 7 remains unstable. In addition to the SB, a small island of instability also appears for this τ_{12}^* around $(\tilde{N}_{sub}, \Delta P_{ext}) = (1.0, 20.0)$ which is shown in Fig. 10. However, it has no appreciable effect on the stability of points on the 100% rod line since it is far away from that rod line. At $\tau_{12}^* = 3$ s the SB again shifts to the right and the island of instability disappears. The island appears again at $\tau_{12}^* = 4$ s; it is much larger than it was for $\tau_{12}^* = 2$ s (see Fig. 10) and it even encompasses some of the 100% rod line. The SB for $\tau_{12}^* = 4$ s, however, shifts back to the left (see Fig. 9), but not as far left as for $\tau_{12}^* = 2$ s; thus, point 6 is still unstable. At

$\tau_{12}^* = 5$ s, the SB shifts right and the island of instability becomes smaller.

Figure 9: Simple SBs in the $\tilde{N}_{sub} - \Delta P_{ext}$ plane for $\tau_{12}^* = 1, 2, 3, 4$ and 5 s. The SBs actually cross the 100% rod line making points in the low-flow/high-power region, such as point 7, clearly unstable. The SBs however, show a stabilizing effect (compared to the SB without the loop) for certain points when τ_{12}^* is close to integer multiples of the natural period of oscillation ($= 2.2$ s). This can be seen for point 6 from the SB for $\tau_{12}^* = 2$ s. For τ_{12}^* values close to half-integer multiples ($= 1, 3$ and 5 s) of the natural period exactly the opposite effect can be observed. The SBs in this case move further to the right in the plane indicating that even more points on the rod line are unstable. Island SBs for $\tau_{12}^* = 2, 4$ and 5 s are shown in Fig. 10.

The stability results for $\tau_{12}^* = 6 - 10$ s are plotted in Figs. 11 and 12. For $\tau_{12}^* = 6$ s the island of instability disappears once again. However, the SB for $\tau_{12}^* = 6$ s is far to the right and almost is identical to that for $\tau_{12}^* = 1$ s. The SBs for $\tau_{12}^* = 7, 8, 9$ and 10 s generally cross the 100% rod line between points 4 and 5 and the islands of instability encompass more of the rod line indicating that an even larger fraction of the points on the 100% rod line are

Figure 10: Islands of instability in the $\tilde{N}_{sub} - \Delta P_{ext}$ plane for $\tau_{12}^* = 2, 4$ and $5s$. These islands along with the SBs of Fig. 9 generally increase the unstable region in the plane and consequently even more points on the 100% rod line are unstable. These islands of instability do not exist for $\tau_{12}^* = 1, 3s$. The simple SBs are shown in Fig. 9.

unstable.

Figure 11: Simple SBs in the $\tilde{N}_{sub} - \Delta P_{ext}$ plane for $\tau_{12}^* = 6, 7, 8, 9$ and 10 s. The SBs show the destabilizing effect of the recirculation loop for points on the 100% rod line. They actually cross the 100% rod line showing that points in the low-flow/high-power region are unstable. Island SBs for $\tau_{12}^* = 7, 8, 9$ and 10 s are shown in Fig. 12.

The SBs were transformed from the $\tilde{N}_{sub} - \Delta P_{ext}$ plane to the $\tilde{n}_0 - \tilde{v}_{inlet,0}$ power-flow plane and are plotted in Fig. 13 for $\tau_{12}^* = 2, 5$ and 10 s. Also plotted is the transformed SB from Fig. 3 without the loop effect. As can be observed from the figure, unlike the TSB without the loop effect, the TSB for $\tau_{12}^* = 2$ s actually crosses the 100% rod line showing that points such as point 7 are unstable. By increasing τ_{12}^* to 5 s and then to 10 s the corresponding TSBs cross the rod line at points further to the right indicating that even more points such as points 4, 5 and 6 are unstable. These results, of course, are the same as those discussed

above based on Figs. 9 - 12.

Figure 12: Islands of instability in the $\tilde{N}_{sub} - \Delta P_{ext}$ plane for $\tau_{12}^* = 7, 8, 9$ and $10 s$. These islands along with the SBs of Fig. 11 generally increase the unstable region in the plane and consequently even more points on the 100% rod line are unstable.

It is important to gain some physical insight into the behavior of the “simple” SBs (see Figs. 9 and 11) and the “islands of instability” (see Figs. 10 and 12). Although for increasing τ_{12}^* , the shifting of the simple SBs may seem unsystematic at first, a closer look suggests that there indeed is a pattern to the shifting which can be explained by both mathematical and physical arguments. For τ_{12}^* values equal to 2 and 4 s for example, it was shown in Fig. 9 that the SBs are in the left part of the plane when compared with the SBs plotted for $\tau_{12}^* = 1, 3$ and 5 s. Hence, the effect of the loop for $\tau_{12}^* = 2$ and 4 s is not as destabilizing as the effect for $\tau_{12}^* = 1, 3$ and 5 s. In fact, when compared to the SB without the loop, the effect of the

Figure 13: TSBs in the $\tilde{n}_0 - \tilde{v}_{inlet,0}$ phase variable plane for $\tau_{12}^* = 2, 5$ and 10 s. The “simple” TSBs and “island” TSBs show the destabilizing effect of the recirculation loop for points on the 100% rod line.

loop for $\tau_{12}^* = 2\text{ s}$ is stabilizing for certain points, such as point 6. The stabilizing effect at this value of τ_{12}^* can be explained by relating it to the natural period of oscillation at point 6 which is about 2.22 s for the original system without the recirculation loop. (From Fig. 6, the natural frequency for point 6 on the 100% rod line is 0.45 Hz which makes the period 2.22 s without the loop effect. The natural period with loop effect is not very different from this value). Numerical simulations carried out showed damped oscillations for parameter values at point 6 with $\tau_{12}^* = 4.44\text{ s}$, thereby indicating that the effect of the loop also is stabilizing for a value of τ_{12}^* equal to twice the natural period. Thus it can be concluded that for certain points on the rod line, the loop is stabilizing for the first few integer multiples of the natural period and it is destabilizing at half-integer multiple values (point 6 is unstable when $\tau_{12}^* = 1, 3$ and 5 s). Physically, the stabilizing effect may be argued as follows. Suppose the inlet subcooling increases and it leads to a lower quality of steam at the exit. The lower exit quality will result in more saturated water in the reservoir which will increase the overall reservoir enthalpy. Since the time-delay associated with the loop is in-phase with the natural period of oscillation, this effect will appear in the inlet subcooling after the in-phase delay, and as a result, it will decrease the inlet subcooling (thereby stabilizing the system). Although such simple explanations may hold true for the first few integer multiples of the natural period, the dynamics generally gets very complex at higher integer values and such simple explanations are not sufficient to completely describe the behavior of the system. This indeed can be seen from Fig. 11 for τ_{12}^* around 7 and 9 s, which are about 3 and 4 times the natural period respectively, where the stabilizing effect is not observed.

Although the above simple arguments seem to explain the effect of τ_{12}^* on the simple SBs,

it is not obvious that there is a simple explanation for the appearance, disappearance, and variation in size of the islands of instability. More study seems to be needed to understand the behavior of the islands as a function of τ_{12}^* and to relate it to the natural periods for points on the 100% rod line.

The most important effect of the recirculation loop, however, is very obvious from the simple SBs and the islands of instability. The simple SBs actually cross the 100% rod line making points in the low-flow/high-power region (such as point 7) clearly unstable for all values of τ_{12}^* . This was not the case for the SB without the loop effect. Although that SB did show a decreased stability of points on the 100% rod line in the low-flow/high-power region, it did not cross the rod line. The effect of the simple SBs, combined with the effect of the island SBs, is even more destabilizing since it makes more points on the 100% rod line unstable. Further, the realistic time delays associated with the steam separators and the downcomer region are of the order of several seconds or more. Therefore the results for $\tau_{12}^* = 10\text{ s}$ are more relevant than those for 1 or 2 s. At $\tau_{12}^* = 10\text{ s}$, the effect of the loop clearly is destabilizing for even more points on the 100% rod line (points 4, 5 and 6, and not just point 7 as was the case at $\tau_{12}^* = 2\text{ s}$). Hence these results, which are based on the salient effects of the recirculation loop, show that the loop generally has a destabilizing effect on the 100% rod line, making points in the low-flow/high-power region unstable.

VII.C Numerical Simulations

Several sets of numerical simulations were carried out to supplement the results of the stability analyses that were reported and discussed in the previous subsection. The first of

these was done, with $\tau_{12}^* = 4\text{ s}$, to establish the nature of the Hopf bifurcations that occur as the SBs are crossed when a parameter value is varied. This was done both for the simple SBs, which are analogous to the one that was found in Section V to exist for the system that did not include the recirculation loop model, and for the SBs that are the perimeters of the islands of instability — which did not exist for the system that did not include the recirculation loop model. As will be shown below, both these Hopf bifurcations are subcritical. Next, a set of simulations was carried out with $\tau_{12}^* = 10\text{ s}$ to determine the natures of the growing and damped oscillations that occur for parameter values in the vicinity of the intersection points of the simple SB and the island SB (even though these intersection points are not on the 100% rod line). Finally, additional sets of numerical simulations were carried out to study the effect of the time delays τ_1^* and τ_2^* on the time evolutions of phase variables starting from IC-1 and IC-2 of Fig. 4 for parameter values corresponding to point 6 in Fig. 2.

The calculations just summarized are now discussed in detail. The simple SB for $\tau_{12}^* = 4\text{ s}$ in Fig. 9 and the island SB for $\tau_{12}^* = 4\text{ s}$ in Fig. 10 are plotted together in Fig. 14. Only the most relevant region, the region where the SBs intersect the 100% rod line, is shown in this figure by zooming in around the points on the 100% rod line in the low-flow/high-power region. The nature of the Hopf bifurcation was studied for three points — points I, II, and III — indicated by asterisks in the figure, where the 100% rod line intersects the SBs. Specifically, point I with $(N_{sub}, \Delta P_{ext}) = (1.0, 8.847)$ is the intersection point for the simple SB, point II with $(N_{sub}, \Delta P_{ext}) = (0.885, 10.20)$ is the intersection point for the left branch on the island SB and point III with $(N_{sub}, \Delta P_{ext}) = (0.666, 14.782)$ is the intersection point for the right branch on the island SB.

Figure 14: Simple and island SBs for $\tau_{12}^* = 4\text{ s}$ in the $\tilde{N}_{sub} - \Delta P_{ext}$ plane. Also plotted is the 100% rod line to show the three intersection points of the SBs with the rod line — points I, II, and III.

Several simulations were done³⁶ for parameter values on both stable and unstable side of each of the points, point I, II and III. These simulations show that the Hopf bifurcations, that occur as the simple SB and each branch of the island SB is crossed, all are subcritical. We show in Fig. 15, in the projection of the phase space into a 3-D subspace, the resulting trajectories for the two nearby ICs at a FP corresponding to parameter values $(N_{sub}, \Delta P_{ext}) = (0.885, 10.1545)$, which is in the stable region slightly to the left of point II. The solid trajectory is for the damped oscillations and it returns toward the FP. (The FP is indicated by the asterisk in the figure.) The dashed trajectory is for the growing oscillations and it diverges away from the FP. This suggests that the bifurcation on the simple SB in

Fig. 14 is a *subcritical* Hopf bifurcation. We recall that in Section VI.B, we observed a similar behavior for the system without the recirculation loop and concluded that the nature of the Hopf bifurcation also was subcritical for that system.

Figure 15: Subcritical Hopf bifurcation. Phase portrait showing the evolution of two nearby ICs in the projection of the phase space onto the $v_{inlet,0} - n_0 - n_1$ subspace for parameter values $(N_{sub}, \Delta P_{ext}) = (0.885, 10.1545)$, which is in the stable region slightly to the left of point II in Fig. 14. They indicate a *subcritical* Hopf bifurcation at the SB and suggest the existence of an unstable limit cycle in the phase space nearby the stable FP indicated by an asterisk in the figure.

Next, a set of simulations was carried out with $\tau_{12}^* = 10\text{ s}$ to determine the natures of the growing and damped oscillations that occur for parameter values in the vicinity of the intersection points of the simple SB and the island SB (even though these intersection points are not on the 100% rod line). The simple SB for $\tau_{12}^* = 10\text{ s}$ in Fig. 11 and the inland SB for $\tau_{12}^* = 10\text{ s}$ in Fig. 12 are plotted together in Fig. 16. The region where the SBs intersect each other at $(N_{sub}, \Delta P_{ext}) = (1.688, 23.6)$, is emphasized in this figure which

is zoomed in around that point. There also is another point $(N_{sub}, \Delta P_{ext}) = (0.698, 11.9)$ in the plane where the two SBs intersect, but we will study only the intersection point at $(N_{sub}, \Delta P_{ext}) = (1.688, 23.6)$. The two curves that intersect at this point divide the neighborhood into 4 regions: region 0, region 1a, region 1b and region 2. Region 0 is the stable region where all eigenvalues of the Jacobian matrix of the dynamical system have real parts < 0 . Region 1a is the unstable region where a complex conjugate pair of eigenvalues has a positive real part. Specifically, this pair is the one which, in fact, has a zero real part on the simple SB. Region 1b is the unstable region where a complex conjugate pair of eigenvalues also has a positive real part. However, this pair is different from the one in region 1a. It is the pair that has a zero real part on the island SB. Region 2 is the unstable region where two pairs — the one with a zero real part on the simple SB and the one with a zero real part on the island SB — have positive real parts. Since the unstable eigenspaces of the linearized operator are different in these 4 regions, it is of some interest to examine the growing and damped oscillations that occur for parameter values in the vicinity of this intersection point.

Numerical simulations were carried out³⁶ for one set of parameter values in each of the 4 regions. The 4 sets of parameter values, which are close to the intersection point, are indicated by the 4 asterisks in Fig. 16. We shall identify them as points 0, 1a, 1b and 2 corresponding to their respective region numbers. Point 0 is at $(N_{sub}, \Delta P_{ext}) = (1.72, 24.0)$, point 1a is at $(N_{sub}, \Delta P_{ext}) = (1.7, 23.2)$, point 1b is at $(N_{sub}, \Delta P_{ext}) = (1.68, 24.0)$ and point 2 is at $(N_{sub}, \Delta P_{ext}) = (1.66, 23.2)$. The results of the numerical simulations at point 0 in the form of decaying oscillations — consistent with the stability analysis — with persistent

Figure 16: The $(N_{sub}, \Delta P_{ext})$ space in the vicinity of the intersection point $(N_{sub}, \Delta P_{ext}) = (1.688, 23.6)$, of the simple and island SBs for $\tau_{12}^* = 10 s$. The intersecting SBs divide the neighborhood into 4 regions. Region 0 is the stable region. In region 1a, a complex conjugate pair of eigenvalues of the Jacobian matrix of the dynamical system has a positive real part. In region 1b a different complex conjugate pair of eigenvalues has a positive real part. In region 2, two complex conjugate pairs of eigenvalues have positive real parts.

beats are shown in Fig. 17. The results of the numerical simulations carried out at the other three points³⁶ are in the form of growing oscillations — again consistent with the stability analysis — with persistent beats. Numerical simulations also were run for parameter values $(N_{sub}, \Delta P_{ext}) = (0.7, 16.4)$ very close to the 100% rod line and slightly to the right of the right branch of the island SB for $\tau_{12}^* = 10 s$ (see Fig. 12). The results are shown in Fig. 18. Two observations can be made by comparing Figs. 17 and 18. First, the effect of the recirculation loop, which is felt only after 10 s have elapsed, is very clear in each figure in the form

of a sudden jump in $n_0(t)$. A less abrupt change occurs at 20 s as the effect propagates, but thereafter abrupt changes are no longer obvious. Second, persistent beats that can be seen in the decaying oscillations in Fig. 17, are not directly due to the discontinuity in $n_0(t)$; rather they are the result of the two interacting frequencies corresponding to the two complex conjugate pairs of eigenvalues associated with the two SBs. For parameter values in the vicinity of the intersection point of the simple and island SBs, these beats persist throughout the decay or growth and continue long after the effect of the discontinuity in $n_0(t)$ has damped. Conversely, for parameter values that are not near the intersection point, yet are very close to the island SB, the phenomenon that appears in the form of initial beats, associated with the discontinuity in $n_0(t)$, dies away (see Fig. 18).

Additional sets of numerical simulations were carried out to study the effect of the time delays τ_1^* and τ_2^* on the time evolutions of phase variables starting from IC-1 and IC-2 of Fig. 4 for parameter values corresponding to point 6 in Fig. 2. We chose these two sets of ICs and parameter values corresponding to point 6 to run the numerical simulations here because these cases were studied in some detail in Section VI for the original system without the recirculation loop model. Hence, the effect of the recirculation loop can be studied by comparing these results with those reported previously in Section VI. The range of τ^* 's was determined for which the previously growing oscillations (without the recirculation loop), emanating from IC-2, became decaying oscillations; and the range for which the previously growing oscillations grew even faster also was established. The range of τ^* 's for which the previously decaying oscillations (in the absence of the recirculation loop), for IC-1, decayed faster also was established, as was the range for which they became growing oscillations.

Figure 17: Time evolution of $n_0(t)$ for parameter values $(N_{sub}, \Delta P_{ext}) = (1.72, 24.0)$ in region 0 of Fig. 16. The IC is a small perturbation of the FP ($a_{1,0} : 0.2312 \rightarrow 0.2382$; $s_{1,0} : 4.284 \rightarrow 4.254$; $v_{inlet,0} : 0.829 \rightarrow 0.839$). The interaction of the two oscillation frequencies, associated with the two pairs of complex conjugate eigenvalues of the Jacobian with real parts close to zero, causes the regular beats in the decaying oscillations. The jump in $n_0(t)$, which can be clearly seen at 10 s, is due to the effect of the loop.

Figure 18: Time evolution of $n_0(t)$ for parameter values $(N_{sub}, \Delta P_{ext}) = (0.70, 16.4)$ slightly to the right of the right branch of the island SB for $\tau_{12}^* = 10$ s in Fig. 12. The IC is a small perturbation of the FP ($a_{1,0} : 0.2143 \rightarrow 0.2103$; $s_{1,0} : 3.972 \rightarrow 3.902$; $v_{inlet,0} : 0.6420 \rightarrow 0.6390$).

As reported in Subsection VII.B, for the system that includes the recirculation loop model with fixed feedwater flow rate and feedwater subcooling, only the sum of the time delays τ_1^* and τ_2^* is relevant. Hence, here again only the single parameter $\tau_{12}^* = \tau_1^* + \tau_2^*$ was varied. At $\tau_{12}^* = 0.2$ s, the time evolution of the phase variables emanating from IC-2, which previously exhibited growing oscillations (see Fig. 4), now exhibit decaying oscillations. However, as τ_{12}^* was increased from 0.2 s to 0.4 s, the oscillations damped more slowly, and at $\tau_{12}^* = 0.6$ s, the oscillations grew. It was observed that the oscillations continued to grow for values of τ_{12}^* upto 1.97 s, above which the oscillations once again damped. This behavior is consistent with our stability analyses which led to a SB for $\tau_{12}^* = 1$ s, shown in Fig. 9, that is far to the right in the $\tilde{N}_{sub} - \Delta P_{ext}$ plane showing that points such as point 6, are very unstable. However, for $\tau_{12}^* = 2$ s the SB shifts to the left of point 6 indicating that it is stable once again. At $\tau_{12}^* = 2.4$ s, the oscillations damped fastest and at $\tau_{12}^* = 2.6$ s, the oscillations grew. The oscillations continued to grow for values of τ_{12}^* beyond 2.6 s for $\tau_{12}^* = 3, 4, 5, 6, 7, 8, 9$ and 10 s. This behavior, of course, is consistent with the SBs shown in Figs. 9, 10, 11 and 12, plotted for $\tau_{12}^* = 1$ s to 10 s in increments of 1 s, where point 6 is in the unstable region for all the values of $\tau_{12}^* > 2$ s that were studied.

Numerical simulations also were carried out at these values of τ_{12}^* , for parameter values corresponding to point 6 in Fig. 2 and IC-1 of Fig. 4, for which the oscillations without the recirculation loop were damped. It was observed that the time evolution of $n_0(t)$ at various τ_{12}^* were qualitatively the same as those described above. The only difference was in the amplitudes of the oscillations; they were slightly less for both decaying and growing oscillations, i.e. the decaying oscillations decayed slightly faster and the growing oscillations

grew slightly slower. This is not unexpected since previously without the recirculation loop the oscillations decayed for IC-1 and grew for IC-2. The phase portraits showing the evolution of the phase space trajectories in the projection of the phase space onto the $v_{inlet,0} - n_0 - n_1$ subspace for $\tau_{12}^* = 2.52 s$ are shown in Fig. 19. The solid trajectory shows the damped oscillations for IC-1 without the loop and the dashed trajectory shows the growing oscillations that result from the effect of the loop.

Figure 19: Destabilizing effect of the simple recirculation loop. Phase portrait showing the phase space trajectories in the projection of the phase space onto the $v_{inlet,0} - n_0 - n_1$ subspace for parameter values at point 6 in Fig. 2 and IC-1 in Fig. 4 with the simple recirculation loop ($\tau_{12}^* = 2.52 s$) and without it. The oscillations that decayed in the absence of the loop (solid trajectory) become growing oscillations (dashed trajectory) because of the destabilizing effect of the loop with $\tau_{12}^* = 2.52 s$.

Finally, numerical simulations, analogous to those described in Subsection VI.E, were carried out for the system with the simple recirculation loop model to explore the effect of simple sinusoidal forcings in ΔP_{ext} , χ_1 and χ_2 . We did not study a wide range of forcing

frequencies and amplitudes here, since we already did a somewhat detailed study in Subsection *VI.E*. Rather, based on the results summarized there, we studied only the in-phase forcing with a fixed forcing frequency of 1/10th the natural system frequency of 0.45Hz , and incremented the forcing amplitude to find the amplitudes that stabilize the growing oscillations. Since the sinusoidal forcings here have a time delay τ_2^* associated with the jet pumps, we also briefly examined the influence of τ_2^* on the effects of the sinusoidal forcings.

The first of the simulations was done by fixing $\tau_1^* = 1\text{ s}$ and $\tau_2^* = 1.6\text{ s}$ in Eq. (120), and sinusoidally forcing ΔP_{ext} , which can be related to the control of the recirculation pumps. The forcing amplitude was increased from 1% $\Delta \tilde{P}_{ext}$ in increments of 1%. Unlike the growing oscillations for the original model which were stabilized by small forcing amplitudes (see Subsection *VI.E*), similar oscillations were stabilized here only when large values of the forcing amplitude ($\geq 7\% \Delta P_{ext}$) were used. Shown in Fig. 20 are the time evolutions of the phase variable $n_0(t)$ with and without the sinusoidal forcing. The solid curve shows the decaying oscillations that result from $\Delta P_{ext}(t) = \Delta \tilde{P}_{ext} \{1 + 0.08 \sin [0.045 \ 2\pi(t^* - \tau_2^*)]\}$, and the dashed curve shows the growing oscillations that occur when there is no forcing. Additional numerical simulations were carried out by fixing $\tau_{12}^* = \tau_1^* + \tau_2^*$, and varying τ_1^* and τ_2^* to determine the effect of the time delay τ_2^* when the system was sinusoidally forced. The sinusoidal forcing stabilized the growing oscillations only in the limited range of $1\text{ s} \leq \tau_2^* \leq 2\text{ s}$. For all other values of τ_2^* studied, the oscillations remained growing oscillations.

Simulations also were used to investigate the possibility of stabilizing growing oscillations by forcing the feedwater flow rate. This was done by replacing $\chi_1 = 1$, by $\chi_1(t) =$

Figure 20: Time evolution of $n_0(t)$ (solid curve) for $(\tau_1^*, \tau_2^*) = (1.0, 1.6) s$ for parameter values at point 6 in Fig. 2 and IC-2 in Fig. 4 resulting from sinusoidal forcing in $\Delta P_{ext}(t)$ with frequency $0.045 Hz$ and amplitude 8%. Also shown is the time evolution (dashed curve) without the sinusoidal forcing. The figure shows the stabilizing effect of the sinusoidal forcing in $\Delta P_{ext}(t)$.

$\{1 + A_1 \sin[0.045 \cdot 2\pi(t^* - \tau_2^*)]\}$, where τ_1^* was fixed at $1 s$ and τ_2^* was fixed at $1.6 s$. The forcing frequency was fixed at $0.045 Hz$ and the forcing amplitude A_1 was increased from 1% in increments of 1%. The result was that the oscillations damped only at $A_1 \geq 10\%$. Then, the amplitude was fixed at 10%, τ_{12}^* was fixed at $2.6 s$, and τ_2^* was varied to determine its influence on the stabilizing effects of the sinusoidal forcing. The sinusoidal forcing stabilized the growing oscillations for the entire range of τ_2^* studied, i.e. $0 - 2.6 s$.

Sinusoidal forcing of the feedwater subcooling, controlled by the feedwater heaters in

the actual BWR system, was included in the model via the representation of $\chi_2(t) = \{1 + A_2 \sin[0.045 \, 2\pi(t^* - \tau_2^*)]\}$, where τ_1^* was fixed at $1 \, s$ and τ_2^* was fixed at $1.6 \, s$. The forcing frequency was again fixed at $0.045 \, Hz$ and the forcing amplitude A_2 was varied from 1% in increments of 1%. The growing oscillations became damped oscillations only when the forcing amplitude was made $\geq 7\%$. Finally, A_2 was fixed at 8%, τ_{12}^* was fixed at $2.6 \, s$, and τ_2^* was varied to determine how it modifies the effects of sinusoidal stabilization. As in the case in which the feedwater flow rate was varied by forcing $\chi_1(t)$, the sinusoidal forcing of the feedwater subcooling via $\chi_2(t)$ stabilized the growing oscillations for the entire range of τ_2^* studied, i.e. $0 - 2.6 \, s$.

VIII DISCUSSION

In Subsection *VI.E*, it was shown that the sinusoidal forcing of the external pressure drop across the channels and of the channel inlet subcooling — both operating parameters in the original BWR core model developed in Section III — could have a significant effect on the dynamics of the system, converting growing oscillations to damped oscillations even with small forcing amplitude values. Although these two quantities were indeed parameters in the original core model, they clearly are not fixed operational parameters in a real BWR system. Rather, they are time-dependent quantities that couple the recirculation loop dynamics to the core dynamics. Thus, the fact that their sinusoidal variation had significant effects on stability, even when small forcing amplitudes were used, motivated the extension of the original nuclear-coupled thermal-hydraulic core model to include equations representing the

dynamics of the recirculation loop. This model led to important changes in the stability predicted by the model — substantial movement of the “simple” stability boundary from its location for the original model, and the appearance of totally new “islands of instability” which did not exist for the original core model, or for the other core models studies previously.^{4,14,16,17,21} Thus, the extension of the original model to include a simple representation of the recirculation loop was very important, as indeed was heralded by the significant effects of the sinusoidal forcing of $\Delta P_{ext}(t)$ and $N_{sub}(t)$ reported in Subsection VI.E.

When the extended model with the simple recirculation loop was forced sinusoidally, there were no significant effects on stability for forcing functions of low amplitude, as reported at the end of Subsection VII.C. This suggests that further extensions of the model to include additional components of the whole BWR system *may* not be as important as the extension to include the recirculation loop, and that the extended model developed and analyzed in Section VII may reflect the essential dynamics and stability features of the overall system. Nevertheless, the fact that a totally new phenomenon — islands of instability — appeared when the model was extended suggest that future research along two lines would be useful. These are: (1) replacing the very simple model of the recirculation loop developed, included and analyzed as summarized in Section VII, by a more detailed model; and (2) significantly further extending the BWR model developed and analyzed here to include ODEs that represent the balance of the plant at a level and detail that is comparable to that which now is in the nuclear-coupled thermal-hydraulic core model.

Although the simple model described in Section VII for the recirculation loop captures the main features of the loop, it does not represent the loop dynamics at the same level

of detail as the original model (Section III) represents the core dynamics. For example, the downward flow between the core shroud and the vessel wall is not represented by flow equations analogous to those that were developed for the upward flow in the core coolant channels. A model including such equations should be developed. That model also should include representations of the two-phase flow through the steam separators and dryers, the water flow back down to the reservoir, the flow through the jet pumps, and a simple representation of the recirculation pumps⁴⁴ that drive the jet pumps. It is essential that the representation of these loop components, which were shown in Section VII above to have important effects, be kept simple, i.e. in the form of nonlinear ODEs, but that they also include a level of detail that is comparable to that of the original core model. The development and careful analysis of such a model would not be a trivial extension of the research reported here. However, it would be very valuable in that the effects on the “simple” SBs and the “islands of instability” discovered in Section VII here could be studied further by using a more detailed model.

The fact that the studies of the effects of sinusoidally forcing the extended model, reported at the end of Section VII, showed that only fairly large amplitude forcing (≥ 7 or 8%) had significant effects on the system dynamics and stability, suggest (not surprisingly) that the dynamics of the balance of the plant are not as important with respect to stability as the dynamics of the core and the recirculation loop. However, under some circumstances large variations in thermal-hydraulic variables can occur in components in the balance of the plant (a particularly relevant example is the large variation that occurred in the feedwater temperature due to a valving error during the LaSalle-2 incident¹⁵). Thus, developing a

model for the balance of the plant and carrying out stability analyses using it, would be a worthwhile extension of the research reported here. This, of course, would not be a trivial extension. However, if the spirit of the model described here (based on a modest number of nonlinear ODEs) was retained, it would be manageable. This, in fact, emphasizes the value of the approach used here — develop a model that is simple, in that it is representable in the form of a reasonable number of ODEs, but also is sufficiently detailed that it includes the essential features of all the system components, and is consistent in that it reflects appropriately comparable levels of detail for all the main system components. Such an overall BWR model would be tractable and could be used for detailed stability analyses and numerical simulations analogous to those reported here. This, of course, is very much in contrast to the development of a system representation based on very detailed models of all the main components which would include for example, the detailed PDEs of the fluid flow in the core channels, the detailed PDEs of the space-time neutron kinetics, the detailed PDEs of the fuel pellet and clad heat conduction, the detailed recirculating flow in the reservoir and jet pumps, etc. Huge system models of this type would not only be extremely costly to develop and extremely computationally intensive to run, but they also would not permit analytical studies and extensive parametric studies and numerical simulations of the type reported here that yield so much valuable information about BWR stability.

IX SUMMARY AND CONCLUSIONS

IX.A Summary of the Model

We have developed a model for BWR stability analyses that consists only of ODEs, and while still simple, it captures the salient engineering and physics features of a BWR and is a considerable improvement over models reported previously.^{4,9} We developed the new model by first deriving four modal kinetics equations to represent the space-time neutron kinetics. These equations included the effects of the fundamental mode and the most relevant harmonic mode, the first azimuthal harmonic, which plays a very important role in the out-of-phase instability. The coupling to the thermal-hydraulics in the channels was included through the void and Doppler feedback reactivities.

Then we modeled the single- and two-phase fluid flow in the fuel centered boiling channels by using a weighted residuals procedure, and introducing quadratic spatial approximations for the space- and time-dependent single-phase enthalpy and two-phase quality, to reduce the single- and two-phase continuity, momentum and energy equations to five time-dependent nonlinear ODEs. Next, we modeled the heat conduction in the fuel rod by reducing the one-dimensional (radial) heat conduction equation to four ODEs using a variational method. Hence, a total of nine equations were developed to represent the thermal-hydraulics in each flow channel. In order to study the out-of-phase oscillations, we introduced two representative channels, each for a planar half of the reactor, with the spatial interaction between them included through the space-time neutron kinetics.

A nonlinear dynamical system was constructed. It was comprised of twenty-two time-

dependent nonlinear ODEs — four from the neutron kinetics and nine from each of the two representative channels. Using this dynamical system we studied the stability of a BWR by carrying out detailed stability analyses and performing numerical simulations to gain additional insight.

Finally, we included a simple recirculation loop model by relating the time-delayed effects of the phase variables at the exit of the channel to the channel inlet subcooling. The effects of the simple recirculation loop model on the overall BWR stability were studied by carrying out additional stability analyses and additional numerical simulations.

IX.B Summary of the Results, and Conclusions

We studied the stability of points on the 100% rod line by determining the SB, for a fixed control rod induced external reactivity equal to that on the 100% rod line value. We presented the SB in the relevant $N_{sub} - \Delta P_{ext}$ operating parameter plane; and then transformed that SB to the power-flow plane familiar to BWR operating engineers and regulatory agencies. We showed that while the normal operating point on the 100% rod line is very stable, the stability of points on the rod line decreases as the flow rate is reduced, and that the operating points are least stable in the low-flow/high-power region of the power-flow map.

The SB also was determined for a reduced model, in which the space-time modal kinetics equations were replaced by the point kinetics equations. We observed that this SB was almost identical to that based on the model with the full space-time modal kinetics equations, indicating that the first harmonic does not have a significant influence on the SB calculations.

Numerical simulations were done to show the growing and decaying oscillations for parameter values close to (but on opposite sides of) the SB in the $N_{sub} - \Delta P_{ext}$ operating parameter space. Additional numerical simulations indicated that the Hopf bifurcation that occurs as the SB is crossed is *subcritical*; hence, a point on the 100% rod line in the low-flow/high-power region of the power-flow map, although stable, may nevertheless be a point at which a BWR should not be operated. These results are consistent with observed BWR stability behavior and they reinforce BWR power-flow map operating guidelines.

Decay ratios (DRs) and frequencies of oscillations were determined for various points on the 100% rod line. It was shown that the NRC requirement of $DR < 0.75 - 0.8$ is not rigorously satisfied by operating points in the low-flow/high-power region of the power-flow map, and these points should be avoided during normal startup and shutdown operations of a BWR. The frequency of oscillation was shown to decrease on the 100% rod line as the flow rate is reduced and the frequency of $0.5Hz$ observed in the low-flow/high-power region is consistent with that observed during actual BWR instability incidents and experiments.

Although we showed that the first harmonic had no perceptible influence on the SB, we also showed that growing oscillations due to this mode could result at operating conditions in the low-flow/high-power region of the power-flow map, and that the use of point kinetics — which failed to predict these growing oscillations — could lead to dramatically nonconservative results. Hence, because of the important effect of the first harmonic mode, this region of the power-flow map is even more problematic than previously thought. Therefore, the use of point kinetics in BWR stability analyses and in BWR simulations may be very inappropriate, and perhaps even misleading.

Finally, the effect of a simple recirculation loop model that we developed was studied by carrying out additional stability analyses and additional numerical simulations. It was shown that the loop had a stabilizing effect on certain points on the 100% rod line for time delays equal to integer multiples of the natural period of oscillation, whereas it has a destabilizing effect for half-integer multiples. However, for more practical time delays of the order of eight to ten seconds, it was determined that the overall effect generally is destabilizing with the SBs actually crossing the 100% rod line and making many points in the low-flow/high-power region unstable. Our discovery of islands of instability — additional unstable regions in the parameter plane and the power-flow map — which prior to this research were unknown, and which do not exist in the absence of the recirculation loop, has important additional implications concerning BWR stability. For the more realistic, longer time delays, these islands actually include portions of the 100% rod line; hence, operating points in these regions also are unstable.

APPENDICES

A TYPICAL BWR PARAMETERS

The following typical BWR design and operating parameters were used.^{9, 17, 28, 30, 37, 40, 45}

$$A^* = 1.442 \times 10^{-4} \text{ m}^2$$

$$L^* = 3.81 \text{ m}$$

$$D^* = 1.424 \text{ cm}$$

$$R^* = 2.32 \text{ m}$$

$$T_{inlet,o}^* = 551 \text{ K}$$

$$T_o^* = 561 \text{ K}$$

$$T_{sat}^* = 561 \text{ K}$$

$$c_{1,0} = -0.101$$

$$c_{1,1} = -0.100$$

$$c_{2,0}^* = -4.3 \times 10^{-5} \text{ K}^{-1}$$

$$c_{2,1}^* = -4.3 \times 10^{-5} \text{ K}^{-1}$$

$$c_c^* = 330.0 \text{ J kg}^{-1} \text{ K}^{-1}$$

$$c_f^* = 5.307 \times 10^3 \text{ J kg}^{-1} \text{ K}^{-1}$$

$$c_p^* = 325.0 \text{ J kg}^{-1} \text{ K}^{-1}$$

$$c_q^* = 4.6901 \times 10^{-6} \text{ W}$$

$$f_{1\phi} = 0.01467$$

$$f_{2\phi} = 0.016357$$

$$g^* = 9.81 \text{ m s}^{-2}$$

$$h_g^* = 5678.2 \text{ W m}^{-2} \text{ K}^{-1}$$

$$k_c^* = 17.0 \text{ W m}^{-1} \text{ K}^{-1}$$

$$k_{exit} = 1.35$$

$$k_f^* = 5.74 \times 10^{-1} \text{ W m}^{-1} \text{ K}^{-1}$$

$$k_{inlet} = 19$$

$$k_p^* = 2.7 \text{ W m}^{-1} \text{ K}^{-1}$$

$$n_o^* = 3.6245 \times 10^7 \text{ cm}^{-3}$$

$$p^* = 7.2 \times 10^6 \text{ N m}^{-2}$$

$$p_c^* = 16.2 \times 10^{-3} \text{ m}$$

$$r_c^* = 6.135 \times 10^{-3} \text{ m}$$

$$r_g^* = 5.322 \times 10^{-3} \text{ m}$$

$$r_p^* = 5.2 \times 10^{-3} \text{ m}$$

$$v^* \equiv \frac{1}{\nu \Sigma_f^* \Lambda^*} = 1.48987 \times 10^4 \text{ m s}^{-1}$$

$$v_o^* = 2.67 \text{ m s}^{-1}$$

$$\Delta h_{fg}^* = 1494.2 \times 10^3 \text{ J kg}^{-1}$$

$$\Lambda^* = 4.0 \times 10^{-5} \text{ s}$$

$$\Sigma_a^* = 0.016527585 \text{ cm}^{-1}$$

$$\alpha_c^* = 7.925 \times 10^{-6} \text{ m}^2 \text{ s}^{-1}$$

$$\alpha_p^* = 0.797 \times 10^{-6} \text{ m}^2 \text{ s}^{-1}$$

$$\beta = 0.0056$$

$$\rho_c^* = 6.5 \times 10^3 \text{ kg m}^{-3}$$

$$\rho_f^* = 736.49 \text{ kg m}^{-3}$$

$$\rho_g^* = 37.71 \text{ kg m}^{-3}$$

$$\rho_p^* = 10.422 \times 10^3 \text{ kg m}^{-3}$$

$$\lambda^* = 0.08 \text{ s}^{-1}$$

$$\mu_f^* = 9.693 \times 10^{-5} \text{ N m}^{-2} \text{ s}$$

$$\nu \Sigma_f^* = 0.01678 \text{ cm}^{-1}$$

$$\omega_{00}^* = 0 \text{ s}^{-1}$$

$$\omega_{10}^* = -140.08 \text{ s}^{-1}$$

$$\omega_{01}^* = -0.0571756 \text{ s}^{-1}$$

$$\omega_{11}^* = -490.78 \text{ s}^{-1}$$

$$\xi_h^* = 3.855 \times 10^{-2} \text{ m}$$

B MODAL ANALOGS OF POINT KINETICS PARAMETERS

Numerical values for the modal analogs of generation time Λ_{ik} defined by Eq. (18), and the reactivity feedback coefficients, $c_{1,kml}$ and $c_{2,kml}$, in Eq. (20), are obtained from typical BWR operating data (see Appendix A) by following the procedure outlined below. We first develop the procedure for Λ_{ik} and then for $c_{1,kml}$ and $c_{2,kml}$.

The theoretical value for the fundamental mode quantity Λ_{00} , from Eq. (18)

$$\Lambda_{00}|_{theoretical} = \frac{(\phi_{00}^\dagger(\underline{r}), \phi_{00}(\underline{r}))}{(\psi_0(\underline{r}), \nu \nu \Sigma_f(\underline{r}) \psi_0(\underline{r}))} , \quad (B.1)$$

is scaled by introducing an appropriate scale factor c_Λ , to make it proportional to the typical BWR generation time Λ , given in Appendix A

$$\Lambda_{00} = c_\Lambda \Lambda_{00}|_{theoretical} , \quad (B.2)$$

where

$$c_\Lambda = \Lambda \frac{(\psi_0(\underline{r}), \nu \nu \Sigma_f(\underline{r}) \psi_0(\underline{r}))}{(\psi_0(\underline{r}), \psi_0(\underline{r}))} , \quad (B.3)$$

and the factor multiplying Λ is the reciprocal of the simple λ -mode expression usually used in the definition of Λ . This procedure results in Λ_{00} having the appropriate value for a typical BWR. Equation (18), relates the theoretical expressions for the higher mode quantities Λ_{ik} to the theoretical fundamental mode analog Λ_{00} . Based on that relation, we write an expression

for the higher mode quantities Λ_{ik} by relating them to the empirically scaled fundamental mode value Λ_{00}

$$\Lambda_{ik} = \Lambda_{00} \frac{\frac{(\phi_{ik}^\dagger(\underline{r}), \phi_{ik}(\underline{r}))}{(\phi_{00}^\dagger(\underline{r}), \phi_{00}(\underline{r}))}}{\frac{(\psi_k(\underline{r}), \nu \nu \Sigma_f(\underline{r}) \psi_k(\underline{r}))}{(\psi_0(\underline{r}), \nu \nu \Sigma_f(\underline{r}) \psi_0(\underline{r}))}}, \quad i = 0, 1, \quad k = 0, 1, \quad (\text{B.4})$$

where the quantity on the right hand side multiplying Λ_{00} is the geometric factor. Actually, for the one-region reactor from which the development here began, Eq. (B.4) when combined with Eqs. (B.2) and (B.3) straightforwardly reduces to a very simple expression

$$\Lambda_{ik} = \Lambda \left[1 + \frac{\lambda \beta \nu \nu \Sigma_f}{(\omega_{ik} + \lambda)^2} \right] \quad i = 0, 1, \quad k = 0, 1. \quad (\text{B.5})$$

Using this expression, the modal generation times Λ_{ik} are obtained from the typical BWR operating data listed in Appendix A [the eigenvalues $\omega_{ik}, i = 0, 1, k = 0, 1$, used in the expression above were determined from Eq. (8) and are given in Appendix A.]

Next, expressions for the feedback coefficients, $c_{1,km}$ and $c_{2,km}$, in Eq. (20) are obtained from Eq. (19)

$$\rho_{km}(t) = \frac{(\phi_{ik}^\dagger(\underline{r}), \delta \underline{A}(\underline{r}, t) \phi_{jm}(\underline{r}))}{(\psi_k(\underline{r}), \nu \nu \Sigma_f(\underline{r}) \psi_m(\underline{r}))}, \quad k = 0, 1, \quad m = 0, 1. \quad (\text{B.6})$$

Because of the zeros in the last column of $\delta \underline{A}(\underline{r}, t)$ [see Eq. (11)], the expressions given by the equation above become independent of the index j when the inner products are taken. They also are made independent of the index i by neglecting the term $(\lambda \beta \nu \nu \delta \Sigma_f(\underline{r}, t))/(\omega_{ik} + \lambda)$ which appears in the numerators and which is indeed extremely small when compared to the term $\nu \nu \delta \Sigma_f(\underline{r}, t)$, to which it is added. Finally, we also neglect the term $\beta \nu \nu \delta \Sigma_f(\underline{r}, t)$ which

also appears in the numerators and which also is very small when compared to $\nu\nu\delta\Sigma_f(\underline{r}, t)$.

The resulting expression for $\rho_{km}(t)$ is

$$\rho_{km}(t) = \frac{(\psi_k(\underline{r}), v\nabla \cdot \delta D(\underline{r}, t) \nabla \psi_m(\underline{r}))}{(\psi_k(\underline{r}), \nu\nu\Sigma_f(\underline{r})\psi_k(\underline{r}))} + \frac{(\psi_k(\underline{r}), [-v\delta\Sigma_a(\underline{r}, t) + \nu\nu\delta\Sigma_f(\underline{r}, t)] \psi_m(\underline{r}))}{(\psi_k(\underline{r}), \nu\nu\Sigma_f(\underline{r})\psi_k(\underline{r}))},$$

$$k = 0, 1, \quad m = 0, 1. \quad (\text{B.7})$$

We express the perturbations $\delta D(\underline{r}, t)$, $\delta\Sigma_a(\underline{r}, t)$ and $\delta\Sigma_f(\underline{r}, t)$ in terms of the void feedback, the Doppler feedback, and the effect of the control rod induced external reactivity by assuming that the void feedback affects only $D(\underline{r}, t)$, and the Doppler feedback and control rod induced external reactivity affect both $\Sigma_a(\underline{r}, t)$ and $\Sigma_f(\underline{r}, t)$ in the following manner

$$\delta D(r, z, \theta, t) = c_{\alpha, l} [\alpha_l(t) - \alpha_{o, l}] \quad (\text{B.8})$$

$$\delta\Sigma_a(r, z, \theta, t) + \nu\nu\delta\Sigma_f(r, \theta, z, t) = c_{T, l} [T_{avg, l}(t) - T_{avg, o, l}] + \rho_{ext, l}, \quad (\text{B.9})$$

where the index l corresponds to a representative flow channel. Since we are considering only two representative channels, one for each planar half of the reactor, this index also corresponds to each half of the reactor; $l = 0 : 0 < \theta < \pi$ and $l = 1 : \pi < \theta < 2\pi$. The coefficients $c_{\alpha, l}$ and $c_{T, l}$ are respectively, the void and Doppler feedback coefficients in each planar half $l = 0, 1$. The quantities $\alpha_l(t)$ and $T_{avg, l}(t)$ are respectively, the axially averaged void fractions, and axially as well as radially averaged fuel rod temperatures in each of the two channels. The $\alpha_{o, l}$ and $T_{avg, o, l}$ are their corresponding reference values at the steady-state of a reference BWR operating at 100% full power. The detailed expression for $\alpha_l(t)$ in terms

of the phase variables of the dynamical system is given by Eq. (50), following the derivation of the equations that describe the fluid flow in the channels. The detailed expression for $T_{avg,l}(t)$ is given by Eq. (98), at the end of the development of the fuel rod heat conduction model.

Substituting Eqs. (B.8) and (B.9) into Eq. (B.7), and carrying out the integrations, we arrive at Eq. (20), where the feedback coefficients, $c_{1,kml}$ and $c_{2,kml}$, are defined as

$$c_{1,kml} = c_{1,00l} \frac{\frac{(\psi_k(\mathbf{r}), v \nabla \cdot \nabla \psi_m(\mathbf{r}))_{V_l}}{(\psi_0(\mathbf{r}), v \nabla \cdot \nabla \psi_0(\mathbf{r}))_{V_l}}}{\frac{(\psi_k(\mathbf{r}), v \nu \Sigma_f(\mathbf{r}) \psi_k(\mathbf{r}))}{(\psi_0(\mathbf{r}), v \nu \Sigma_f(\mathbf{r}) \psi_0(\mathbf{r}))}}, \quad (\text{B.10})$$

$$c_{2,kml} = c_{2,00l} \frac{\frac{(\psi_k(\mathbf{r}), v \psi_m(\mathbf{r}))_{V_l}}{(\psi_0(\mathbf{r}), v \psi_0(\mathbf{r}))_{V_l}}}{\frac{(\psi_k(\mathbf{r}), v \nu \Sigma_f(\mathbf{r}) \psi_k(\mathbf{r}))}{(\psi_0(\mathbf{r}), v \nu \Sigma_f(\mathbf{r}) \psi_0(\mathbf{r}))}}, \quad (\text{B.11})$$

$$k = 0, 1, \quad m = 0, 1, \quad l = 0, 1.$$

The inner products in the numerators of the above expressions with subscripts V_l are over the volumes associated with the relevant planar halves of the reactor $l = 0, 1$. (The inner products without any subscript are over the entire volume of the reactor). The quantities on the right hand sides of the above two equations multiplying the fundamental mode values are the geometric factors which relate the higher mode quantities to their fundamental mode analogs. The fundamental mode quantities are defined as

$$c_{1,00l} = c_{\alpha,l} \frac{(\psi_0(\mathbf{r}), v \nabla \cdot \nabla \psi_0(\mathbf{r}))_{V_l}}{(\psi_0(\mathbf{r}), v \nu \Sigma_f(\mathbf{r}) \psi_0(\mathbf{r}))} \quad (\text{B.12})$$

$$c_{2,00l} = c_{T,l} \frac{(\psi_0(\mathbf{r}), v \psi_0(\mathbf{r}))_{V_l}}{(\psi_0(\mathbf{r}), v \nu \Sigma_f(\mathbf{r}) \psi_0(\mathbf{r}))}, \quad (\text{B.13})$$

$$l = 0, 1,$$

which are made proportional to the typical BWR values $c_{1,l}$ and $c_{2,l}$ given in Appendix A, by scaling $c_{\alpha,l}$ and $c_{T,l}$

$$c_{\alpha,l} = c_{1,l} \frac{(\psi_0(\underline{r}), v\nu\Sigma_f(\underline{r})\psi_0(\underline{r}))}{(\psi_0(\underline{r}), v\nabla \cdot \nabla\psi_0(\underline{r}))} \quad (\text{B.14})$$

$$c_{T,l} = c_{2,l} \frac{(\psi_0(\underline{r}), v\nu\Sigma_f(\underline{r})\psi_0(\underline{r}))}{(\psi_0(\underline{r}), \psi_0(\underline{r}))} \quad (\text{B.15})$$

$$l = 0, 1 .$$

Finally, the modal analogs $\rho_{ext,kml}$, $k = 0, 1$, $m = 0, 1$, in Eq. (20), of the control rod induced external reactivity ρ_{ext} for each reactor half $l = 0, 1$, are determined from the steady-state condition $\tilde{\rho}_{km} = 0$ (here and everywhere, a tilde on a quantity indicates its steady-state value), where the condition $\tilde{\rho}_{km} = 0$ makes the modal analogs of the total reactivity in the reactor equal to zero at steady-state [see Eq. (20)]. The expression for $\rho_{ext,kml}$ is

$$\rho_{ext,kml} = -c_{1,kml} [\tilde{\alpha}_l - \alpha_{o,l}] - c_{2,kml} [\tilde{T}_{avg,l} - T_{avg,o,l}] , \quad (\text{B.16})$$

$$k = 0, 1, \quad m = 0, 1, \quad l = 0, 1.$$

Hence, whenever we refer to *fixing* the control rod induced external reactivity ρ_{ext} , as an operating parameter, we actually mean fixing its modal analogs $\rho_{ext,kml}$, $k = 0, 1$, $m = 0, 1$, in each reactor half $l = 0, 1$, after determining them from the equation above.

C DIMENSIONLESS VARIABLES AND PARAMETERS

The dimensionless variables and parameters used in the BWR model are listed below. The asterisks indicate the original dimensional quantities. The quantities that have the subscript $j\phi$ are defined for both phases of the fluid; $j = 1$ for the single-phase region of the flow channel, and $j = 2$ for the two-phase region.

$$A^* = p_c^{*2} - \pi r_c^{*2}$$

$$N_\rho = \frac{\rho_g^*}{\rho_f^*}$$

$$Bi_{c,j\phi} = \frac{h_{\infty,j\phi}^* r_c^*}{k_c^*}$$

$$R = \frac{R^*}{L^*}$$

$$Bi_g = \frac{h_g^* r_g^*}{k_c^*}$$

$$T_{i,j\phi}(t) = \frac{T_{i,j\phi}^*(t^*)}{T_o^*}, \quad i = 1, 2, avg, bulk, surf$$

$$Bi_p = \frac{h_g^* r_g^*}{k_p^*}$$

$$T_{sat} = \frac{T_{sat}^*}{T_o^*}$$

$$Bi_{p,j\phi}^+ = Bi_p \frac{Bi_{c,j\phi}}{Bi_g + Bi_{c,j\phi} + Bi_g Bi_{c,j\phi} \log(r_c/r_g)}$$

$$c_{2,l} = \frac{c_{2,l}^*}{T_o^*}, \quad l = 0, 1$$

$$D(r, t) = \frac{D^*(r^*, t^*)}{L^*}$$

$$c_q = \frac{c_q^* n_o^* L^{*2}}{k_p^* T_o^*}$$

$$D_h^* = \frac{4A^*}{\xi_h^*}$$

$$h(z, t) = \frac{h^*(z^*, t^*)}{\Delta h_{fg}^*}$$

$$Fr = \frac{v_o^{*2}}{g^* L^*}$$

$$h_i = \frac{(h_i^* - c_f^* T_i^*)}{\Delta h_{fg}^*}, \quad i = feed, inlet, resv, sat$$

$$N_{cov,1\phi} = \frac{h_{\infty,1\phi}^* L^* T_o^* \Delta \rho^* \xi_h^*}{A^* \Delta h_{fg}^* v_o^* \rho_g^* \rho_f^*}$$

$$n_k(t) = \frac{n_k^*(t^*)}{n_o^*}, \quad k = 0, 1$$

$$N_{cov,2\phi} = \frac{2.56e^4 p^{*6} / 6.2 \cdot 10^6 L^* T_o^{*4} \Delta \rho^* \xi_h^*}{A^* \Delta h_{fg}^* v_o^* \rho_g^* \rho_f^*}$$

$$r_i = \frac{r_i^*}{L^*}, \quad i = 1, c, d, g, p$$

$$N_{f,j\phi} = \frac{f_{j\phi} L^*}{2D_h^*}$$

$$t = \frac{t^* v_o^*}{L^*}$$

$$N_r = \frac{\rho_f^*}{\Delta \rho^*}$$

$$u_k(t) = \frac{u_k^*(t^*)}{n_o^*}, \quad k = 0, 1$$

$$N_{sbo} = \frac{\Delta h_{fg}^* \rho_g^*}{2c_f^* T_o^* \Delta \rho^*}$$

$$v = \frac{v^*}{v_o^*}$$

$$N_{sub} = \frac{(h_{sat}^* - h_{inlet}^*) \Delta \rho^*}{\Delta h_{fg}^* \rho_g^*}$$

$$v_{inlet}(t) = \frac{v_{inlet}^*(t^*)}{v_o^*}$$

$$N_{sub,feed} = \frac{(h_{sat}^* - h_{feed}^*) \Delta \rho^*}{\Delta h_{fg}^* \rho_g^*}$$

$$z = \frac{z^*}{L^*}$$

$$\Delta P_{ext} = \frac{\Delta P_{ext}^*}{\rho_f^* v_o^{*2}}$$

$$\Lambda = \frac{\Lambda^* v_o^*}{L^*}$$

$$\Sigma_a = \Sigma_a^* L^*$$

$$\Sigma_f = \Sigma_f^* L^*$$

$$\alpha_c = \frac{\alpha_c^* [\equiv k_c^* / (\rho_c^* c_c^*)]}{L^* v_o^*}$$

$$\alpha_p = \frac{\alpha_p^* [\equiv k_p^* / (\rho_p^* c_p^*)]}{L^* v_o^*}$$

$$\lambda = \frac{\lambda^* L^*}{v_o^*}$$

$$\mu(t) = \frac{\mu^*(t^*)}{L^*}$$

$$\rho_m(z, t) = \frac{\rho_m^*(z^*, t^*)}{\rho_f^*}$$

$$\tau_i = \frac{\tau_i^* v_o^*}{L^*}, \quad i = 1\phi, 2\phi, 1, 2$$

$$\omega_{ik} = \frac{\omega_{ik}^* L^*}{v_o^*}$$

$$\xi_h^* = 2\pi r_c^*$$

D THE MIXTURE DENSITY EQUATION

Substituting the expression for $\rho_m(z, t)$ from Eq. (39) into the mixture density equation [Eq. (34)], using the weight functions 1 and $[z - \mu(t)]$, and integrating from the boiling boundary $z = \mu(t)$ to the channel exit $z = 1$, we arrive at two linear equations in $ds_1(t)/dt$ and $ds_2(t)/dt$

$$\begin{aligned} I_1(t) \frac{ds_1(t)}{dt} + I_3(t) \frac{ds_2(t)}{dt} - [2I_1(t)s_2(t) + I_2(t)s_1(t)] \frac{d\mu(t)}{dt} + \\ [v_{inlet}(t)I_2(t) + N_{pch,2\phi}(t)I_1(t)]s_1(t) + [2v_{inlet}(t)I_1(t) + 2N_{pch,2\phi}(t)I_3(t)]s_2(t) - \\ N_{pch,2\phi}(t)M_1(t) = 0, \quad (D.17) \end{aligned}$$

$$\begin{aligned} I_3(t) \frac{ds_1(t)}{dt} + I_4(t) \frac{ds_2(t)}{dt} - [2I_3(t)s_2(t) + I_1(t)s_1(t)] \frac{d\mu(t)}{dt} + \\ [v_{inlet}(t)I_1(t) + N_{pch,2\phi}(t)I_3(t)]s_1(t) + [2v_{inlet}(t)I_3(t) + 2N_{pch,2\phi}(t)I_4(t)]s_2(t) - \\ N_{pch,2\phi}(t)M_2(t) = 0. \quad (D.18) \end{aligned}$$

Here the $I_i(t)$, $i = 1, \dots, 6$, are defined as

$$I_1(t) = \int_{\mu(t)}^1 \frac{z - \mu(t)}{\{1 + s_1(t)[z - \mu(t)] + s_2(t)[z - \mu(t)]^2\}^2} dz, \quad (\text{D.19})$$

$$I_2(t) = \int_{\mu(t)}^1 \frac{1}{\{1 + s_1(t)[z - \mu(t)] + s_2(t)[z - \mu(t)]^2\}^2} dz, \quad (\text{D.20})$$

$$I_3(t) = \int_{\mu(t)}^1 \frac{[z - \mu(t)]^2}{\{1 + s_1(t)[z - \mu(t)] + s_2(t)[z - \mu(t)]^2\}^2} dz, \quad (\text{D.21})$$

$$I_4(t) = \int_{\mu(t)}^1 \frac{[z - \mu(t)]^3}{\{1 + s_1(t)[z - \mu(t)] + s_2(t)[z - \mu(t)]^2\}^2} dz, \quad (\text{D.22})$$

and the $M_i(t)$, $i = 1, 2$, are defined as

$$M_1(t) = \int_{\mu(t)}^1 \rho_m(z, t) dz, \quad (\text{D.23})$$

$$M_2(t) = \int_{\mu(t)}^1 \rho_m(z, t)[z - \mu(t)] dz, \quad (\text{D.24})$$

where the boiling boundary $\mu(t)$ is given by Eq. (28) and the mixture density $\rho_m(z, t)$ is given by Eq. (39).

The integrations in the $I(t)$'s and $M(t)$'s were carried out by using the symbolic manipulation package Mathematica.³³ We factored the denominators of Eqs. (D.17) – (D.24) as

$$1 + s_1(t)[z - \mu(t)] + s_2(t)[z - \mu(t)]^2 = s_2(t)[\delta(z, t) - \delta_1(t)][\delta(z, t) - 1/\delta_3(t)], \quad (\text{D.25})$$

where

$$\delta(z, t) = z - \mu(t), \quad (\text{D.26})$$

$$\delta_1(t) = -\frac{2}{s_1(t) + \sqrt{s_1^2(t) - 4s_2(t)}}, \quad (\text{D.27})$$

$$\delta_3(t) = \frac{-s_1(t) + \sqrt{s_1^2(t) - 4s_2(t)}}{2}, \quad (\text{D.28})$$

The second factor in Eq. (D.25), actually $[\delta(z, t) - \delta_2(t)]$, is rewritten by defining $\delta_2(t) = 1/\delta_3(t)$ for convenience to further simplify the $I(t)$'s and $M(t)$'s. As we shall see below, at steady-state when $\tilde{\delta}_2 \rightarrow \infty$ and $\tilde{\delta}_3 = 0$, and it is easier to simplify the $I(t)$'s and $M(t)$'s using the latter condition. Also for notational convenience and for clarity of the final equations, we define

$$\delta_e(t) = 1 - \mu(t). \quad (\text{D.29})$$

The simplified expressions for the $I(t)$'s and $M(t)$'s are

$$I_1(t) = \frac{1}{s_1^2(t) - 4s_2(t)} \left\{ \frac{\delta_1(t)}{\delta_1(t) - \delta_e(t)} + \frac{1}{1 - \delta_e(t)\delta_3(t)} - 2 + \frac{1 + \delta_1(t)\delta_3(t)}{1 - \delta_1(t)\delta_3(t)} \log \left[\frac{1 - \frac{\delta_e(t)}{\delta_1(t)}}{1 - \delta_e(t)\delta_3(t)} \right] \right\}, \quad (\text{D.30})$$

$$I_2(t) = \frac{1}{s_1^2(t) - 4s_2(t)} \left[\frac{1}{\delta_1(t) - \delta_e(t)} - \frac{1 + \delta_1(t)\delta_3(t)}{\delta_1(t)} \right] + \frac{\delta_3(t)}{s_1^2(t) - 4s_2(t)} \left\{ \frac{1}{1 - \delta_e(t)\delta_3(t)} + \frac{2}{1 - \delta_1(t)\delta_3(t)} \log \left[\frac{1 - \frac{\delta_e(t)}{\delta_1(t)}}{1 - \delta_e(t)\delta_3(t)} \right] \right\}, \quad (\text{D.31})$$

$$I_3(t) = \frac{1}{s_1^2(t) - 4s_2(t)} \left\{ \frac{\delta_1^2(t)}{\delta_1(t) - \delta_e(t)} + \frac{1}{\delta_3(t) [1 - \delta_e(t)\delta_3(t)]} - \frac{1 + \delta_1(t)\delta_3(t)}{\delta_3(t)} + \frac{2\delta_1(t)}{1 - \delta_1(t)\delta_3(t)} \log \left[\frac{1 - \frac{\delta_e(t)}{\delta_1(t)}}{1 - \delta_e(t)\delta_3(t)} \right] \right\}, \quad (\text{D.32})$$

$$I_4(t) = \frac{1}{s_1^2(t) - 4s_2(t)} \left\{ \frac{\delta_1^3(t)}{\delta_1(t) - \delta_e(t)} + \frac{1}{\delta_3^2(t) [1 - \delta_e(t)\delta_3(t)]} + \frac{\delta_1^2(t) [3 - \delta_1(t)\delta_3(t)]}{1 - \delta_1(t)\delta_3(t)} \log \left[1 - \frac{\delta_e(t)}{\delta_1(t)} \right] - \frac{[1 - \delta_1(t)\delta_3(t) + \delta_1^2(t)\delta_3^2(t) - \delta_1^3(t)\delta_3^3(t)]}{\delta_3^2(t) [1 - \delta_1(t)\delta_3(t)]} + \frac{[1 - 3\delta_1(t)\delta_3(t)] \log [1 - \delta_e(t)\delta_3(t)]}{\delta_3^2(t) [1 - \delta_1(t)\delta_3(t)]} \right\}, \quad (\text{D.33})$$

$$M_1(t) = \frac{1}{\sqrt{s_1^2(t) - 4s_2(t)}} \log \left[\frac{1 - \frac{\delta_e(t)}{\delta_1(t)}}{1 - \delta_e(t)\delta_3(t)} \right], \quad (\text{D.34})$$

$$M_2(t) = \frac{1}{\sqrt{s_1^2(t) - 4s_2(t)}} \left\{ \delta_e(t) + \delta_1(t) \log \left[1 - \frac{\delta_e(t)}{\delta_1(t)} \right] \right\} - \frac{1}{\sqrt{s_1^2(t) - 4s_2(t)}} \left\{ \delta_e(t) + \frac{1}{\delta_3(t)} \log [1 - \delta_e(t)\delta_3(t)] \right\}, \quad (\text{D.35})$$

which are functions of $s_1(t)$, $s_2(t)$ and $\mu(t)$ through Eqs. (D.27), (D.28) and (D.29). The boiling boundary, $\mu(t)$ in turn depends on the phase variables $a_1(t)$ and $a_2(t)$ via Eq. (28). Hence, the $I(t)$'s and $M(t)$'s are functions of the phase variables $s_1(t)$, $s_2(t)$, $a_1(t)$ and $a_2(t)$ only.

At steady-state $\tilde{s}_2 = 0$, and from Eq. (D.28) $\tilde{\delta}_3 = 0$. When $\delta_3(t) \rightarrow 0$, certain terms in $I_3(t)$, $I_4(t)$ and $M_2(t)$ must be evaluated very carefully, since both numerators and denomi-

nators of those terms go to zero. To simplify, we expand

$$\frac{1}{1 - \delta_e(t)\delta_3(t)} = 1 + \delta_e(t)\delta_3(t) + \delta_e^2(t)\delta_3^2(t) + O[\delta_3^3(t)] , \quad (\text{D.36})$$

$$\frac{1}{1 - \delta_1(t)\delta_3(t)} = 1 + \delta_1(t)\delta_3(t) + \delta_1^2(t)\delta_3^2(t) + O[\delta_3^3(t)] , \quad (\text{D.37})$$

$$\log[1 - \delta_e(t)\delta_3(t)] = -\delta_e(t)\delta_3(t) - \frac{\delta_e^2(t)\delta_3^2(t)}{2} + O[\delta_3^3(t)] , \quad (\text{D.38})$$

and rewrite Eqs. (D.32), (D.33) and (D.35) as

$$I_3(t) = \frac{1}{s_1^2(t) - 4s_2(t)} \left\{ \frac{\delta_1^2(t)}{\delta_1(t) - \delta_e(t)} + \frac{2\delta_1(t)}{1 - \delta_1(t)\delta_3(t)} \log \left[\frac{1 - \frac{\delta_e(t)}{\delta_1(t)}}{1 - \delta_e(t)\delta_3(t)} \right] \right\} + \frac{1}{s_1^2(t) - 4s_2(t)} \{ \delta_e(t) - \delta_1(t) + \delta_e^2(t)\delta_3(t) + \delta_e^3(t)\delta_3^2(t) + O[\delta_3^3(t)] \} , \quad (\text{D.39})$$

$$I_4(t) = \frac{1}{s_1^2(t) - 4s_2(t)} \left\{ \frac{\delta_1^3(t)}{\delta_1(t) - \delta_e(t)} + \frac{\delta_1^2(t)[3 - \delta_1(t)\delta_3(t)]}{1 - \delta_1(t)\delta_3(t)} \log \left[1 - \frac{\delta_e(t)}{\delta_1(t)} \right] \right\} + \frac{1}{s_1^2(t) - 4s_2(t)} \left\{ \left[\frac{\delta_e^2(t)}{2} - \delta_1^2(t) + 2\delta_e(t)\delta_1(t) \right] + \left[\frac{2\delta_e^3(t)}{3} + \delta_1^3(t) - \delta_e^2(t)\delta_1(t) + 2\delta_e(t)\delta_1^2(t) \right] \delta_3(t) + \left[\frac{3\delta_e^4(t)}{4} - \delta_1^4(t) + \frac{2\delta_e^2(t)\delta_1(t)}{3} + \delta_e^2(t)\delta_1^2(t) + 2\delta_e(t)\delta_1^3(t) \right] \delta_3^2(t) + O[\delta_3^3(t)] \right\} , \quad (\text{D.40})$$

$$M_2(t) = \frac{1}{\sqrt{s_1^2(t) - 4s_2(t)}} \left\{ \delta_e(t) + \delta_1(t) \log \left[1 - \frac{\delta_e(t)}{\delta_1(t)} \right] \right\} + \frac{1}{\sqrt{s_1^2(t) - 4s_2(t)}} \left\{ \frac{\delta_e^2(t)\delta_3(t)}{2} + \frac{\delta_e^3(t)\delta_3^2(t)}{3} + O[\delta_3^3(t)] \right\} , \quad (\text{D.41})$$

which can easily be evaluated as $\delta_3(t) \rightarrow 0$.

Equations (D.17) and (D.18) are two linear equations in $ds_1(t)/dt$ and $ds_2(t)/dt$ which

are solved simultaneously for these two derivatives to arrive at Eqs. (40) and (41). The variable $f_1(t)$ in those equations corresponds to $d\mu(t)/dt$. Detailed expressions for the other variables $f_n(t), n = 2, \dots, 6$ are

$$f_2(t) = I_3^2(t) - I_1(t)I_4(t) \quad (D.42)$$

$$f_3(t) = s_1(t) [I_1(t)I_3(t) - I_2(t)I_4(t)] + 2s_2(t) [I_3^2(t) - I_1(t)I_4(t)] , \quad (D.43)$$

$$f_4(t) = N_{pch,2\phi}(t) [M_2(t)I_3(t) - M_1(t)I_4(t)] - v_{inlet}(t)s_1(t) [I_1(t)I_3(t) - I_2(t)I_4(t)] - \quad (D.44)$$

$$[2v_{inlet}(t)s_2(t) + N_{pch,2\phi}(t)s_1(t)] [I_3^2(t) - I_1(t)I_4(t)] ,$$

$$f_5(t) = s_1(t) [I_2(t)I_3(t) - I_1^2(t)] \quad (D.45)$$

$$f_6(t) = N_{pch,2\phi}(t) [M_1(t)I_3(t) - M_2(t)I_1(t)] - v_{inlet}(t)s_1(t) [I_2(t)I_3(t) - I_1^2(t)] - \quad (D.46)$$

$$2s_2(t)N_{pch,2\phi}(t) [I_3^2(t) - I_1(t)I_4(t)] ,$$

where $I_1(t)$, $I_2(t)$, $I_3(t)$, $I_4(t)$, $M_1(t)$, $M_2(t)$ and $N_{pch,2\phi}(t)$, which depend on the phase variables, operating parameters and design parameters of the dynamical system, are given by Eqs. (D.30), (D.31), (D.39), (D.40), (D.34), (D.41) and (37), respectively.

E THE MOMENTUM EQUATIONS

The single-phase momentum equation, Eq. (45), is integrated from the inlet $z = 0$ to the boiling boundary $z = \mu(t)$ to arrive at the single-phase pressure drop $\Delta P_{1\phi}(t)$

$$\Delta P_{1\phi}(t) = \Delta P_{acc,1\phi}(t) + \Delta P_{fric,1\phi}(t) + \Delta P_{grav,1\phi}(t) , \quad (\text{E.47})$$

where the acceleration, frictional, and gravitational pressure drops are given by

$$\Delta P_{acc,1\phi}(t) = \int_0^{\mu(t)} \frac{dv_{inlet}(t)}{dt} dz = \mu(t) \frac{dv_{inlet}(t)}{dt} , \quad (\text{E.48})$$

$$\Delta P_{fric,1\phi}(t) = \int_0^{\mu(t)} N_{f,1\phi} v_{inlet}^2(t) dz = N_{f,1\phi} v_{inlet}^2(t) \mu(t) , \quad (\text{E.49})$$

and

$$\Delta P_{grav,1\phi}(t) = \int_0^{\mu(t)} Fr^{-1} dz = Fr^{-1} \mu(t). \quad (\text{E.50})$$

Substituting the expressions for $v_m(z, t)$ and $\rho_m(z, t)$ from Eqs. (36) and (39) respectively, into the two-phase momentum equation [Eq. (46)], and integrating it from the boiling boundary $z = \mu(t)$, to the channel exit $z = 1$, the two-phase pressure drop $\Delta P_{2\phi}(t)$ is obtained as

$$\Delta P_{2\phi}(t) = \Delta P_{acc1,2\phi}(t) + \Delta P_{acc2,2\phi}(t) + \Delta P_{fric,2\phi}(t) + \Delta P_{grav,2\phi}(t) , \quad (\text{E.51})$$

where the two parts of the acceleration pressure drop are

$$\begin{aligned}\Delta P_{acc1,2\phi}(t) &= \int_{\mu(t)}^1 \rho_m(z, t) \frac{\partial v_m(z, t)}{\partial t} dz \\ &= \left[\frac{dv_{inlet}(t)}{dt} - N_{pch,2\phi}(t) \frac{d\mu(t)}{dt} \right] M_1(t) + \frac{dN_{pch,2\phi}(t)}{dt} M_2(t),\end{aligned}\tag{E.52}$$

and

$$\begin{aligned}\Delta P_{acc2,2\phi}(t) &= \int_{\mu(t)}^1 \rho_m(z, t) v_m(z, t) \frac{\partial v_m(z, t)}{\partial z} dz \\ &= v_{inlet}(t) N_{pch,2\phi}(t) M_1(t) + N_{pch,2\phi}^2(t) M_2(t).\end{aligned}\tag{E.53}$$

The frictional pressure drop is

$$\begin{aligned}\Delta P_{fric,2\phi}(t) &= \int_{\mu(t)}^1 N_{f,2\phi} \rho_m(z, t) v_m^2(z, t) dz \\ &= N_{f,2\phi} [v_{inlet}^2(t) M_1(t) + 2v_{inlet}(t) N_{pch,2\phi}(t) M_2(t) + N_{pch,2\phi}^2(t) M_3(t)],\end{aligned}\tag{E.54}$$

and the gravitational pressure drop is

$$\Delta P_{grav,2\phi}(t) = \int_{\mu(t)}^1 Fr^{-1} \rho_m(z, t) dz = Fr^{-1} M_1(t).\tag{E.55}$$

In the equations above, $M_1(t)$ and $M_2(t)$ are defined by Eqs. (D.23) and (D.24) respectively in Appendix D, and $M_3(t)$ is an intermediate variable defined as

$$M_3(t) = \int_{\mu(t)}^1 \rho_m(z, t) [z - \mu(t)]^2 dz.\tag{E.56}$$

After carrying out the integration,³⁶ the resulting expression for $M_3(t)$ is

$$M_3(t) = \frac{1}{\sqrt{s_1^2(t) - 4s_2(t)}} \left\{ \frac{\delta_e^2(t)}{2} + \delta_e(t)\delta_1(t) + \delta_1^2(t) \log \left[1 - \frac{\delta_e(t)}{\delta_1(t)} \right] \right\} - \frac{1}{\sqrt{s_1^2(t) - 4s_2(t)}} \left\{ \frac{\delta_e^2(t)}{2} + \frac{\delta_e(t)}{\delta_3(t)} + \frac{1}{\delta_3^2(t)} \log [1 - \delta_e(t)\delta_3(t)] \right\}. \quad (\text{E.57})$$

At steady-state $\tilde{s}_2 = 0$, and from Eq. (D.28), $\tilde{\delta}_3 = 0$. When $\delta_3(t) \rightarrow 0$, the expression for $M_3(t)$ — similar to the expression for $M_2(t)$ [see Eq. (D.41)] — also must be evaluated very carefully. Using Eq. (D.38), we get

$$M_3(t) = \frac{1}{\sqrt{s_1^2(t) - 4s_2(t)}} \left\{ \frac{\delta_e^2(t)}{2} + \delta_e(t)\delta_1(t) + \delta_1^2(t) \log \left[1 - \frac{\delta_e(t)}{\delta_1(t)} \right] \right\} + \frac{1}{\sqrt{s_1^2(t) - 4s_2(t)}} \left\{ \frac{\delta_e^3(t)\delta_3(t)}{3} + \frac{\delta_e^4(t)\delta_3^3(t)}{4} + O[\delta_3^3(t)] \right\}. \quad (\text{E.58})$$

which can easily be evaluated as $\delta_3(t) \rightarrow 0$.

The inlet and exit pressure drops are defined as

$$\Delta P_{inlet}(t) = k_{inlet} v_{inlet}^2(t) \quad (\text{E.59})$$

and

$$\Delta P_{exit}(t) = k_{exit} \rho_m(1, t) v_m^2(1, t), \quad (\text{E.60})$$

where $\rho_m(1, t)$ and $v_m(1, t)$ are the mixture density and the mixture velocity respectively, at the exit of the channel ($z = 1$). These quantities are obtained from Eqs. (39) and (36)

respectively, to give

$$\Delta P_{exit}(t) = k_{exit} \frac{1}{1 + s_1(t)[1 - \mu(t)] + s_2(t)[1 - \mu(t)]^2} \{v_{inlet}(t) + N_{pch,2\phi}(t)[1 - \mu(t)]\}^2 \quad (\text{E.61})$$

The inlet and exit pressure loss coefficients, k_{inlet} and k_{exit} respectively, are adjusted to lump the spacer pressure losses in the flow channel at the inlet and exit of the channel.

The single-phase pressure drop $\Delta P_{1\phi}(t)$ and the two-phase pressure drop $\Delta P_{2\phi}(t)$ are summed along with the inlet and exit form pressure drops, $\Delta P_{inlet}(t)$ and $\Delta P_{exit}(t)$, and then equated to the external pressure drop ΔP_{ext} , to obtain

$$\Delta P_{ext} = \Delta P_{1\phi}(t) + \Delta P_{2\phi}(t) + \Delta P_{inlet}(t) + \Delta P_{exit}(t) , \quad (\text{E.62})$$

which is rewritten to arrive at the ODE for $v_{inlet}(t)$, i.e. Eq. (47). The quantities $f_n(t)$, $n = 7, \dots, 10$, in that equation are

$$f_7(t) = M_1(t) + \mu(t), \quad (\text{E.63})$$

$$f_8(t) = \Delta P_{ext} - \Delta P_{inlet}(t) - \Delta P_{exit}(t) - \Delta P_{fric,1\phi}(t) - \Delta P_{grav,1\phi}(t) - \quad (\text{E.64})$$

$$\Delta P_{acc2,2\phi}(t) - \Delta P_{fric,2\phi}(t) - \Delta P_{grav,2\phi}(t),$$

$$f_9(t) = N_{pch,2\phi}(t)M_1(t), \quad (\text{E.65})$$

$$f_{10}(t) = -M_2(t), \quad (\text{E.66})$$

where $M_1(t)$, $\Delta P_{inlet}(t)$, $\Delta P_{exit}(t)$, $\Delta P_{fric,1\phi}(t)$, $\Delta P_{grav,1\phi}(t)$, $\Delta P_{acc2,2\phi}(t)$, $\Delta P_{fric,2\phi}(t)$, $\Delta P_{grav,2\phi}(t)$, $N_{pch,2\phi}(t)$ and $M_2(t)$, which depend on the phase variables, operating parameters and design

parameters of the dynamical system, are given by Eqs. (D.34), (E.59), (E.60), (E.49), (E.50), (E.53), (E.54), (E.55), (37) and (D.41), respectively.

F NOMENCLATURE

Abbreviations

BC	Boundary condition
BWR	Boiling water reactor
DR	Decay Ratio
DWO	Density-wave oscillation
FP	Fixed point (steady-state)
IC	Initial condition
ODE	Ordinary differential equation
PDE	Partial differential equation
SB	Stability boundary

Symbols

A	cross sectional flow area = $p_c^2 - \pi r_c^2$
Bi	Biot number
$C(\underline{r}, t)$	averaged delayed neutron precursor group concentration
$D(\underline{r}, t)$	neutron diffusion coefficient
D_h	heated diameter = $4A/\xi_h$
Fr	Froude number
L	flow channel length

N	number of terms retained in an expansion
N_{cov}	conversion number (for phase change number)
N_f	friction number
N_{pch}	phase change number
$N(\underline{r}, t)$	neutron number density
N_{sub}	subcooling number
P	pressure
R	reactor core radius
T	temperature
$a_1(t)$	phase variable related to single-phase enthalpy [Eq. (25)]
$a_2(t)$	phase variable related to single-phase enthalpy [Eq. (25)]
c	precursor concentration
c_1	void reactivity coefficient
c_2	Doppler reactivity coefficient
c_c	clad specific heat
c_f	liquid (coolant) specific heat
c_p	fuel pellet specific heat
c_q	$c_q^* = q'''^*/n_0^*$
f	friction factor
g	gravitational constant = $9.81m/s^2$
h_∞	clad surface heat transfer coefficient
h_g	pellet-clad gap conductance

h_{inlet}	coolant inlet enthalpy
h_{sat}	coolant saturation enthalpy
k_c	clad thermal conductivity
k_{exit}	exit pressure loss coefficient (adjusted to include the effect of spacers)
k_f	liquid thermal conductivity
k_{inlet}	inlet pressure loss coefficient (adjusted to include the effect of spacers)
k_p	fuel pellet thermal conductivity
n	neutron number density
p	system pressure
p_c	BWR lattice cell pitch
q''	wall heat flux
q'''	volumetric heat generation rate
r_1	fuel rod radius = r_c
r_c	clad outer radius
r_d	point of discontinuity $0 < r_d < r_p$
r_g	clad inner radius
r_p	fuel pellet radius
$s_1(t)$	phase variable related to two-phase quality [Eq. (38)]
$s_2(t)$	phase variable related to two-phase quality [Eq. (38)]
t	time
v	neutron velocity
v	coolant velocity

x	quality
z	channel axial spatial coordinate
ΔP_{ext}	external pressure drop
Δh_{fg}	vapor-liquid enthalpy difference $\equiv h_g - h_f$
Δn	neutron density difference from steady-state value $= n - \tilde{n}$
$\Delta \rho$	liquid-vapor density difference $= \rho_f - \rho_g$
Λ	neutron generation time
Σ_a	macroscopic absorption cross section
Σ_f	macroscopic fission cross section
Ω	oscillation frequency
α	void fraction
α_c	clad thermal diffusivity $= k_c/(\rho_c c_c)$
α_p	pellet thermal diffusivity $= k_p/(\rho_p c_p)$
β	delayed neutron fraction
$\delta(z, t)$	$z - \mu(t)$
$\delta_1(t)$	$-\frac{2}{s_1(t) + \sqrt{s_1^2(t) - 4s_2(t)}}$
$\delta_2(t)$	$\frac{1}{\delta_3(t)}$
$\delta_3(t)$	$\frac{-s_1(t) + \sqrt{s_1^2(t) - 4s_2(t)}}{2}$
$\delta_e(t)$	$1 - \mu(t)$
θ	temperature difference from steady-state value $= T - \tilde{T}$
λ	precursor decay constant
χ	external control function

μ	boiling boundary
μ_f	liquid viscosity
ν	average number of neutrons produced per fission
ξ_l	volume factor = $(1, \psi_1(\underline{r}))_{V_l} / (1, \psi_0(\underline{r}))_{V_l}$, $l = 0, 1$
ξ_h	heated perimeter = $2\pi r_c$
ρ	reactivity
ρ_c	clad density
ρ_{ext}	control rod induced external reactivity
ρ_f	liquid density
ρ_g	vapor density
ρ_p	pellet density
$\tau_{1\phi}$	single-phase residence time
$\tau_{2\phi}$	two-phase residence time
τ_1	time delay associated with steam separators
τ_2	time delay associated with jet pumps
ω_{ik}	modal eigenvalues

Subscripts

1ϕ	single-phase
2ϕ	two-phase
acc	acceleration
avg	average
$exit$	channel exit

<i>ext</i>	external
<i>f</i>	liquid
<i>feed</i>	feedwater
<i>fric</i>	frictional
<i>g</i>	vapor
<i>g</i>	fuel-clad gap
<i>grav</i>	gravitational
<i>inlet</i>	channel inlet
<i>l</i>	left
<i>l</i>	channel number, reactor half, reservoir half
<i>m</i>	mixture
<i>o</i>	reference
<i>p</i>	pellet
<i>r</i>	right
<i>resv</i>	reservoir
<i>surf</i>	surface
<i>sat</i>	saturation

Superscripts

~	steady-state value (fixed point component)
*	dimensional quantity

References

- [1] OECD-NEA, "Proceedings of the International Workshop on Boiling Water Reactor Stability," Tech. Rep. CSNI 178, OECD-NEA (1990).
- [2] J. MARCH-LEUBA and J. M. REY, "Coupled Thermohydraulic-Neutronic Instabilities in Boiling Water Nuclear Reactors: A Review of the State of the Art," *Nucl. Eng. Des.*, **145**, 97 (1993).
- [3] Y. M. FARAWILA, D. W. PRUITT, and P. E. SMITH, "Analysis of the Laguna Verde Instability Event," *Proc. Am. Nucl. Soc., National Heat Transfer Conference*, **9**, 198 (1996).
- [4] A. A. KARVE, RIZWAN-UDDIN, and J. J. DORNING, "Stability Analysis of BWR Nuclear-Coupled Thermal-Hydraulics Using a Simple Model," *Nucl. Eng. Des.*, **177**, 155 (1997).
- [5] L. A. BELBLIDIA, L. E. WEAVER, and R. W. CARLSON, "Nodal Analysis of Density-Wave Oscillations in Boiling Water Nuclear Reactors," *Ann. Nucl. Energy*, **10**(10), 505 (1983).
- [6] B. J. GITNICK, R. R. GAY, R. S. BORLAND, and A. F. ANSARI, "FIBWR: A Steady-State Core Flow Distribution Code for Boiling Water Reactors," Tech. Rep. EPRI NP-1924-CCM, EPRI (1981).
- [7] D. S. ROWE, "COBRA," Tech. Rep. BNL-1695, Brookhaven National Laboratory (1973).

- [8] S. J. PENG, M. Z. PODOWSKI, R. T. LAHEY, and M. BECKER, "NUFREQ-NP: A Computer Code for the Stability Analysis of Boiling Water Nuclear Reactors," *Nucl. Sci. Eng.*, **88**, 404 (1984).
- [9] J. MARCH-LEUBA, D. G. CACUCI, and R. B. PEREZ, "Nonlinear Dynamics and Stability of Boiling Water Reactors: Part I – Qualitative Analysis," *Nucl. Sci. Eng.*, **93**, 111 (1986).
- [10] J. MARCH-LEUBA, D. G. CACUCI, and R. B. PEREZ, "Nonlinear Dynamics and Stability of Boiling Water Reactors: Part II – Quantitative Analysis," *Nucl. Sci. Eng.*, **93**, 124 (1986).
- [11] M. E. WARD and J. C. LEE, "Singular Perturbation Analysis of Limit Cycle Behavior in Nuclear-Coupled Density-Wave Oscillations," *Nucl. Sci. Eng.*, **97**, 190 (1987).
- [12] B.G. BERGDAHL, F. REISCH, R. OGUMA, J. LORENZEN, and F. AKERHIELM, "BWR Stability Investigation at Forsmark-1," *Ann. Nucl. Energy*, **16**(10), 509 (1989).
- [13] K. VALTONEN, "BWR Stability Analysis," Tech. rep., Finnish Center for Radiation and Nuclear Safety, Helsinki (1989).
- [14] RIZWAN-UDDIN and J. J. DORNING, "Density-Wave Instabilities in Advanced Boiling Water Reactors," *ANS Proc. National Heat Transfer Conference*, **6**, 161 (1992).
- [15] W. WULFF, H. S. CHENG, A. N. MALLEN, and U. S. ROHATGI, "BWR Stability Analysis With the BNL Engineering Plant Analyzer," Tech. Rep. BNL-NUREG-52312, Brookhaven National Laboratory (1992).

- [16] M. TSUJI, K. NISHIO, and M. NARITA, "Stability Analysis of BWRs Using Bifurcation Theory," *J. Nuc. Sci. and Tech.*, **30**(11), 1107 (1993).
- [17] J. MARCH-LEUBA and E. D. BLAKEMAN, "A Mechanism for Out-of-Phase Power Instabilities in Boiling Water Reactors," *Nucl. Sci. Eng.*, **107**, 173 (1991).
- [18] P. OTADUY and J. MARCH-LEUBA, "LAPUR User's Guide," Tech. rep., NUREG/CR-5421 (1989).
- [19] J. L. MUNOZ-COBO, R. B. PEREZ, D. GINESTAR, A. ESCRIVA, and G. VERDU, "Non Linear Analysis of Out of Phase Oscillations in Boiling Water Reactors," *Annals of Nucl. Energy*, **23**(16), 1301 (1996).
- [20] J. A. TURSO, J. MARCH-LEUBA, and R. M. EDWARDS, "A Modal-Based Reduced-Order Model of BWR Out-Of-Phase Instabilities," *Annals of Nucl. Energy*, **24**(12), 921 (1997).
- [21] D. D. B. VAN BRAGT and T. H. J. J. VAN DER HAGEN, "A Six-Dimensional Model for Coupled Neutronic-Thermohydraulic Out-of-Phase Oscillations in Boiling Water Reactors," *Proc. National Heat Transfer Conf.*, **HTC-Vol. 9**, 179 (1996).
- [22] J. MARCH-LEUBA, "Density-Wave Instabilities in Boiling Water Reactors," Tech. Rep. NUREG/CR-6003 ORNL/TM-12130, Oak Ridge National Laboratory (1992).
- [23] A. A. KARVE, RIZWAN-UDDIN, and J. J. DORNING, "Stability Boundaries of BWRs in Operating Parameter Space and in the Power-Flow Plane," *Trans. Am. Nucl. Soc.*, **75**, 392 (1996).

- [24] A. A. KARVE, RIZWAN-UDDIN, and J. J. DORNING, "On Spatial Approximations for Liquid Enthalpy and Two-Phase Quality During Density-Wave Oscillations," *Trans. Am. Nucl. Soc.*, **71**, 533 (1994).
- [25] J. MARSDEN and M. McCracken, *Hopf Bifurcation and Its Applications*, Springer-Verlag, New York (1976).
- [26] J. GUCKENHEIMER and P. HOLMES, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, Springer-Verlag, New York (1983).
- [27] Z. AKCASU, G. S. LELLOUCHE, and L. M. SHOTKIN, *Mathematical Methods in Nuclear Reactor Dynamics*, Academic Press, New York (1971).
- [28] N. E. TODREAS and M. S. KAZIMI, *Nuclear Systems I – Thermal Hydraulics Fundamentals*, Hemisphere, New York (1990).
- [29] R. T. LAHEY and F.J. MOODY, *The Thermal-Hydraulics of a Boiling Water Nuclear Reactor*, American Nuclear Society, LaGrange, Illinois (1977).
- [30] G. C. PARK, M. PODOWSKI, M. BECKER, and R. T. LAHEY, "The Development of NUFREQ-N: An Analytical Model for the Stability Analysis of Nuclear Coupled Density-Wave Oscillations in Boiling Water Nuclear Reactors," Tech. Rep. NUREG/CR-3375, Rensselaer Polytechnic Institute (1983).
- [31] JEAN-LUC ACHARD, D. A. DREW, and R. T. LAHEY, "The Analysis of Nonlinear Density-Wave Oscillations in Boiling Channels," *J. Fluid Mech.*, **155**, 213 (1985).

- [32] RIZWAN-UDDIN and J. J. DORNING, "Some Nonlinear Dynamics of a Heated Channel," *Nucl. Eng. Des.*, **93**, 1 (1986).
- [33] S. WOLFRAM, *The Mathematica Book, Third Edition*, Wolfram Media, Inc. and Cambridge University Press, New York (1996).
- [34] L. V. KANTOROVICH and V. I. KRYLOV, *Approximate Methods of Higher Analysis*, Interscience, New York (1964).
- [35] I. STACKGOLD, *Boundary Value Problems of Mathematical Physics*, vol. 2, Springer-Verlag, New York (1968).
- [36] A. A. KARVE, "Nuclear-Coupled Thermal-Hydraulic Stability Analysis of Boiling Water Reactors," Ph.D. thesis, University of Virginia, Charlottesville, Virginia (1998).
- [37] P. J. OTADUY, "Modeling of the Dynamic Behavior of Large BWR Cores," Ph.D. thesis, University of Florida, Gainesville, Florida (1979).
- [38] M. J. POWELL, "A Fast Algorithm for Nonlinearly Constrained Optimization Calculations," *Numerical Analysis Proceedings, Dundee 1977, Lecture Notes in Mathematics*, **630**, 144 (1978).
- [39] R. A. RYDIN, *Nuclear Reactor Theory*, University Publications, P.O.Box 47, Blacksburg, VA 24060 USA (1977).
- [40] J. J. DUDERSTADT and L. J. HAMILTON, *Nuclear Reactor Analysis*, John Wiley, New York (1976).

- [41] C. W. GEAR, *Numerical Initial Value Problems in Ordinary Differential Equations*, Prentice-Hall, Englewood Cliffs (1971).
- [42] L. F. SHAMPINE and C. W. GEAR, "A User's View of Solving Stiff Ordinary Differential Equations," *SIAM Review*, **21**, 1 (1979).
- [43] GENERAL ELECTRIC, "BWR/6 General Description of a Boiling Water Reactor," Tech. rep., Nuclear Energy Group, General Electric Company, San Jose, CA 95125 (1980).
- [44] RIZWAN-UDDIN and J. J. DORNING, "The Nonlinear Dynamics of Two-Phase Flow in a Simple Heated Loop," *Chem. Eng. Comm.*, **87**, 1 (1990).
- [45] J. H. RUST, *Nuclear Power Engineering*, Haralson Publishing Company, Buchanan, Georgia (1979).