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Abstract

In this paper, by systematically treating the integrals involved in the piezoelectric inclusion problem, explicit results were obtained for the piezoelectric Eshelby tensors for a spheroidal inclusion aligned along the axis of the anisotropy in a transversely isotropic piezoelectric material. This problem was first treated by Dunn and Wienecke (1996) using a Green's function approach, which closely follows Withers' approach (1989) for an ellipsoidal inclusion problem in a transversely isotropic elastic medium. The same problem was recently treated by Michelitsch and Levin (2000) also using a Green's function approach. In this paper, a different method was used to obtain the explicit results for the piezoelectric Eshelby tensors for a spheroidal inclusion. The method is a direct extension of a more unified approach, which has been recently developed by Mikata (2000), which is based on Deeg's results (1980) on a piezoelectric inclusion problem. The main advantage of this method is that it is more straightforward and simpler than Dunn and Wienecke (1996), or Michelitsch and Levin (2000), and the results are a little bit more explicit than their solutions. The key step of this paper is an analytical closed form evaluation of several integrals, which was made possible after a careful treatment of a certain bi-cubic equation.

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Abstract

In this paper, by systematically treating the integrals involved in the piezoelectric inclusion problem, explicit results were for the piezoelectric Eshelby tensors for a spheroidal inclusion aligned along the axis of the anisotropy in a transversely isotropic piezoelectric material. This problem was first treated by Dunn and Wienecke (1996) by using a Green's function approach, which closely follows Withers' approach (1989) for an ellipsoidal inclusion problem in a transversely isotropic elastic medium. The same problem was recently treated by Michelitsch and Levin (2000) by also using a Green's function approach. In this paper, a different method was used to obtain the explicit results for the piezoelectric Eshelby tensors for a spheroidal inclusion. The method is a direct extension of a more unified approach, which has been recently developed by Mikata (2000), which is based on Deeg's results (1980) on a piezoelectric inclusion problem. The main advantage of this method is that it is more straightforward and simpler than Dunn and Wienecke (1996), or Michelitsch and Levin (2000), and the results are a little bit more explicit than their solutions. The key step of this paper is an analytical closed form evaluation of several integrals, which were made possible after a careful treatment of a certain bi-cubic equation.

1. Introduction

The importance of piezoelectric composites has been well documented in recent years in relation to smart materials and smart structures as well as electronic packaging (Taya, 1995). For production and application of the piezoelectric composites, the characterization of piezoelectric composites becomes very important. In the characterization of the linear elastic composite materials, Eshelby tensor has played a dominant role (Mura, 1987, Mori and Tanaka, 1973). Similarly, the central issue in the characterization of piezoelectric composites is determination of the piezoelectric Eshelby tensors (Mikata, 2000).

This paper treats the explicit determination of the piezoelectric Eshelby tensors for a spheroidal inclusion aligned along the axis of the anisotropy in a transversely isotropic piezoelectric material. This problem was first treated by Dunn and Wienecke (1996), and recently by Michelitsch and Levin (2000). There have been a number of studies on piezoelectric inclusion problems as well as piezoelectric composites (Deeg, 1980, Benveniste, 1992, Wang, 1992, Dunn and Taya, 1993 a & b, Dunn, 1994, Huang and Yu, 1994, Dunn and Wienecke, 1996, Huang, 1996, Michelitsch and Levin, 2000, Mikata, 2000). However, only three (Huang and Yu, 1994, Dunn and Wienecke, 1996, Michelitsch and Levin, 2000) of the above studies have considered the piezoelectric Eshelby tensors for a spheroidal inclusion in more details. In particular, Dunn and Wienecke (1996) have obtained the piezoelectric Eshelby tensors for a spheroidal inclusion explicitly using a Green's function approach, which closely follows Withers' approach (1989) for an ellipsoidal inclusion problem in a transversely isotropic elastic medium. Michelitsch and Levin (2000) also recently obtained piezoelectric Eshelby tensors for a spheroidal inclusion explicitly by deriving Green's functions for transversely isotropic piezoelectric materials. In this paper, we also obtain the piezoelectric Eshelby tensors for a spheroidal inclusion explicitly, but using a different approach. The method is a direct extension of a more unified approach, which has been recently developed by Mikata (2000). The main advantage of this method is that it is more straightforward and simpler than Dunn and Wienecke (1996), or Michelitsch and Levin (2000), and the results are a little bit more explicit than their solutions.

The general strategy of this paper largely follows the one employed in the recent publication by the present author (Mikata, 2000), where Deeg's results (1980) on a piezoelectric inclusion problem were used. The key step of this paper is an analytical evaluation of several integrals, which were made possible after a careful treatment of a certain bi-cubic equation.

2. Governing equations of piezoelectricity

The governing equations of piezoelectricity are given by

$$\sigma_{ij,i} + f_j = 0 \tag{1}$$

$$\sigma_{ij} = C_{ijmn}\, \varepsilon_{mn} - e_{nij}\, E_n \tag{2}$$

$$\varepsilon_{mn} = \frac{1}{2}(u_{m,n} + u_{n,m}) \tag{3}$$

$$D_{i,i} = \rho \tag{4}$$

$$D_i = e_{imn}\, \varepsilon_{mn} + \kappa_{in}\, E_n \tag{5}$$

$$E_n = -\, \phi_{,n} \tag{6}$$

where σ_{ij}, ε_{mn}, u_m and f_j are stress, strain, displacement field and body force, respectively, D_i, E_i, ρ, and ϕ are electric displacement, electric field, electric charge density and electric potential, respectively, and C_{ijmn}, κ_{in} and e_{nij} are elastic moduli, permittivity and piezoelectric constants, respectively. Eqs. (1) - (3) describe the elasticity of the material, whereas Eqs. (4) - (6) describe the electrostatics of the material. The coupling between elasticity and electrostatics, i.e., piezoelectricity, is provided by the piezoelectric constants e_{nij}. It should be noted here that the electrostatic part (Eqs. (4) - (6)) is written in the rationalized MKSA system (see Jackson, 1975).

Following Barnett and Lothe (1975) and Deeg (1980), we will rewrite the above governing equations by defining the following variables.

$$U_M = \begin{cases} u_m & \text{for } M\ (= m) = 1,2,3 \\ \phi & \text{for } M = 4 \end{cases} \tag{7}$$

$$Z_{Mn} = \begin{cases} \varepsilon_{mn} & \text{for } M\ (= m) = 1,2,3 \\ -\, E_n & \text{for } M = 4 \end{cases} \tag{8}$$

$$\Sigma_{iJ} = \begin{cases} \sigma_{ij} & \text{for } J\ (= j) = 1,2,3 \\ D_i & \text{for } J = 4 \end{cases} \tag{9}$$

$$\rho_J = \begin{cases} f_j & \text{for } J\ (= j) = 1,2,3 \\ -\, \rho & \text{for } J = 4 \end{cases} \tag{10}$$

$$F_{iJMn} = \begin{cases} C_{ijmn} & \text{for } J, M = 1,2,3 \\ e_{nij} & \text{for } J = 1,2,3 \ ; M = 4 \\ e_{imn} & \text{for } J = 4 \ ; M = 1,2,3 \\ -\kappa_{in} & \text{for } J = M = 4 \end{cases} \tag{11}$$

where U_M, Z_{Mn}, Σ_{iJ}, ρ_J and F_{iJMn} are displacement - electric potential, strain - electric field, stress - electric displacement, body force - electric charge density and piezoelectric moduli, respectively. With the help of (7) - (11), the governing equations of piezoelectricity, i.e., Eqs. (1) - (6), can be compactly rewritten as (see Mikata, 2000)

$$F_{iJMn} U_{M,ni} = - \rho_J \tag{12}$$

3. Piezoelectric inclusion problem

Let us consider a piezoelectric inclusion problem where a region Ω in an infinite domain $\mathbf{R}^3$ has a constant eigenstrain - eigen electric field $\mathbf{Z}^*$, which is both stress free and electric displacement free (see Fig. 1). There are no body force and no charge density for this problem. Mathematically, the problem is defined as follows:

$$\Sigma_{iJ,i} = 0 \tag{13}$$

$$\Sigma_{iJ} = F_{iJMn}[Z_{Mn} - Z^*_{Mn}(\mathbf{x})] \tag{14}$$

$$F_{iJMn}\, Z_{Mn} = F_{iJMn}\, U_{M,n} \tag{15}$$

where the eigenstrain - eigen electric field $Z^*_{Mn}(\mathbf{x})$ is given by

$$Z^*_{Mn}(\mathbf{x}) = \begin{cases} Z^*_{Mn} & \mathbf{x} \in \Omega \\ 0 & \mathbf{x} \in R^3 - \Omega \end{cases} \tag{16}$$

Substituting (14) and (15) into (13), we obtain

$$F_{iJMn}\, U_{M,ni} = F_{iJMn}\, \partial_i\, Z^*_{Mn}(\mathbf{x}) \tag{17}$$

where ∂_i denotes the partial differentiation with respect to x_i. It is seen from (17) that $F_{iJMn}\, \partial_i\, Z^*_{Mn}(\mathbf{x})$ acts as a body force - electric charge density. Deeg (1980) has obtained a fairly general result for this problem in an integral form. The case when the shape of the inclusion Ω is an ellipsoid, however, is the most interesting. In this case, the strain - electric field Z in Ω resulting from Z* can be sometimes determined explicitly by evaluating the integral analytically. Deeg (1980) did not do this explicit evaluation in his dissertation. The result obtained by Deeg (1980) for the ellipsoidal case can be recast into the following form.

$$Z_{Mn} = S_{MnAb}\, Z^*_{Ab} \qquad \text{in } \Omega \tag{18}$$

where S_{MnAb} is a piezoelectric analog of Eshelby tensor, and is given by

$$S_{MnAb} = \begin{cases} \dfrac{1}{8\pi} F_{iJAb}\,(I_{inmJ} + I_{imnJ}) & \text{when} \quad M = 1,2,3 \\ \dfrac{1}{4\pi} F_{iJAb} I_{in4J} & \text{when} \quad M = 4 \end{cases} \tag{19}$$

$$I_{inMJ} = a_1\, a_2\, a_3 \int_{|\mathbf{x}|=1} \frac{1}{\mu^3}\, x_i\, x_n\, K^{-1}_{MJ}\, dS \tag{20}$$

$$\mu = \sqrt{a_1^2\, x_1^2 + a_2^2\, x_2^2 + a_3^2\, x_3^2} \tag{21}$$

$$K_{MJ} = F_{pMJq}\, x_p\, x_q = F_{pJMq}\, x_p\, x_q \tag{22}$$

where a_i is the length of the semi-axis of the ellipsoid in the x_i-direction. In light of the fact that S_{MnAb} consists of 4 different tensors, in this paper, it shall be called piezoelectric Eshelby tensors (cf., Mikata, 2000). $|\mathbf{x}| = 1$ is the surface of the unit sphere and K^{-1}_{MJ} is the inverse of 4x4 matrix K_{MJ}, which is defined by (22). The shape of the ellipsoid will affect the piezoelectric Eshelby tensors S_{MnAb} through μ in the integrand. It should be mentioned here that the coordinate axes are chosen such that they coincide with the axes of the ellipsoid.

4. Piezoelectric Eshelby tensors

The piezoelectric Eshelby tensors are defined by Eqs. (19) through (22). The key part of the definition is the integral I_{inMJ}. Using the results of our previous paper (Mikata, 2000), we have

$$I_{inMJ} = \int_{-1}^{1} dt \int_{0}^{2\pi} G_{inMJ}\left(\frac{y_1}{a_1}, \frac{y_2}{a_2}, \frac{y_3}{a_3}\right) d\phi \tag{23}$$

where y_1, y_2, and y_3 are given by

$$y_1 = \sqrt{1 - t^2}\cos\phi, \qquad y_2 = \sqrt{1 - t^2}\sin\phi, \qquad y_3 = t \tag{24}$$

and

$$G_{inMJ}(\mathbf{x}) = x_i\, x_n\, K^{-1}_{MJ} \tag{25}$$

Let us now specifically consider a transversely isotropic piezoelectric material. The constitutive equations for the transversely isotropic piezoelectric material are given by

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{zx} \\ \sigma_{xy} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(C_{11} - C_{12}) \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ 2\varepsilon_{yz} \\ 2\varepsilon_{zx} \\ 2\varepsilon_{xy} \end{bmatrix}$$

$$- \begin{bmatrix} 0 & 0 & e_{31} \\ 0 & 0 & e_{31} \\ 0 & 0 & e_{33} \\ 0 & e_{15} & 0 \\ e_{15} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} \tag{26}$$

$$
\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & e_{15} & 0 \\ 0 & 0 & 0 & e_{15} & 0 & 0 \\ e_{31} & e_{31} & e_{33} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ 2\varepsilon_{yz} \\ 2\varepsilon_{zx} \\ 2\varepsilon_{xy} \end{bmatrix} + \begin{bmatrix} \kappa_{11} & 0 & 0 \\ 0 & \kappa_{11} & 0 \\ 0 & 0 & \kappa_{33} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} \tag{27}
$$

It should be noted here that the anisotropy axis is along the x_3-axis. By using the definition (11) of F_{iJMn}, K_{MJ} of (22) is given by (see Mikata, 2000)

$$
K_{MJ} = \begin{bmatrix} C_{11} x_1^2 + C_{44} x_3^2 + \frac{1}{2}(C_{11} - C_{12}) x_2^2 & \frac{1}{2}(C_{11} + C_{12}) x_1 x_2 & (C_{13} + C_{44}) x_3 x_1 & (e_{15} + e_{31}) x_3 x_1 \\ \frac{1}{2}(C_{11} + C_{12}) x_1 x_2 & \frac{1}{2}(C_{11} - C_{12}) x_1^2 + C_{11} x_2^2 + C_{44} x_3^2 & (C_{13} + C_{44}) x_2 x_3 & (e_{15} + e_{31}) x_2 x_3 \\ (C_{13} + C_{44}) x_3 x_1 & (C_{13} + C_{44}) x_2 x_3 & C_{44} (x_1^2 + x_2^2) + C_{33} x_3^2 & e_{15} (x_1^2 + x_2^2) + e_{33} x_3^2 \\ (e_{15} + e_{31}) x_3 x_1 & (e_{15} + e_{31}) x_2 x_3 & e_{15} (x_1^2 + x_2^2) + e_{33} x_3^2 & -\kappa_{11} (x_1^2 + x_2^2) - \kappa_{33} x_3^2 \end{bmatrix} \tag{28}
$$

The inverse matrix K_{MJ}^{-1} is calculated as (see Huang and Yu, 1994, Mikata, 2000)

$$K_{MJ}^{-1} = \frac{1}{D}\begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix} \quad (29)$$

where

$$D(x_1, x_2, x_3) = -PQ$$

$$P(x_1, x_2, x_3) = (C_{11} - C_{12})\, z + 2\, C_{44}\, x_3^2 \quad (30)$$

$$Q(x_1, x_2, x_3) = q_1 z^3 + q_2 z^2 x_3^2 + q_3 z x_3^4 + q_4 x_3^6$$

$$z = x_1^2 + x_2^2$$

and

$$b_{11} = (r_{111} x_1^2 + r_{112} x_2^2)\, z^2 + (r_{113} x_1^2 + r_{114} x_2^2)\, x_3^2 z + (r_{115} x_1^2 + r_{116} x_2^2)\, x_3^4 + r_{117} x_3^6$$

$$b_{12} = x_1 x_2 [r_{121} z^2 + r_{122} z x_3^2 + r_{123} x_3^4]$$

$$b_{13} = x_1 x_3 P [r_{131} z + r_{132} x_3^2]$$

$$b_{14} = x_1 x_3 P [r_{141} z + r_{142} x_3^2]$$

$$b_{21} = b_{12}$$

$$b_{22} = (r_{221} x_1^2 + r_{222} x_2^2)\, z^2 + (r_{223} x_1^2 + r_{224} x_2^2)\, x_3^2 z + (r_{225} x_1^2 + r_{226} x_2^2)\, x_3^4 + r_{227} x_3^6 \quad (31)$$

$$b_{23} = x_2 x_3 P [r_{231} z + r_{232} x_3^2]$$

$$b_{24} = x_2 x_3 P [r_{241} z + r_{242} x_3^2]$$

$$b_{31} = b_{13}$$

$$b_{32} = b_{23}$$

$$b_{33} = P [r_{331} z^2 + r_{332} z x_3^2 + r_{333} x_3^4]$$

$$b_{34} = P\,[r_{341}\, z^2 + r_{342}\, z\, x_3^2 + r_{343}\, x_3^4]$$

$$b_{41} = b_{14}$$

$$b_{42} = b_{24}$$

$$b_{43} = b_{34}$$

$$b_{44} = P\,[r_{441}\, z^2 + r_{442}\, z\, x_3^2 + r_{443}\, x_3^4]$$

q_i (i = 1 - 4) in (30) and r_{ijk} in (31) are functions of piezoelectric material parameters, and are given in Appendix A. Since K^{-1}_{MJ} in (25) is a symmetric matrix, we have

$$I_{inMJ} = I_{inJM} \tag{32}$$

The piezoelectric Eshelby tensors S_{MnAb} defined by (19) have already been obtained in terms of I_{inMJ} for an arbitrary ellipsoid in a transversely isotropic material whose axes coincide with the axes of the anisotropy (see Mikata, 2000).

5. Spheroidal inclusion along the x_3-axis

The spheroidal inclusion along the x_3-axis can be represented by

$$a_1 = a, \qquad \frac{a_1}{a_2} = 1 \quad , \qquad \frac{a_1}{a_3} = \beta \tag{33}$$

Substituting (33) into (23), we obtain

$$I_{inMJ} = \int_{|y|=1} G_{inMJ}\left(\frac{y_1}{a}, \frac{y_2}{a}, \frac{\beta y_3}{a}\right) dS = \int_{|y|=1} G_{inMJ}\left(y_1, y_2, \beta y_3\right) dS \tag{34}$$

The second equality in (34) follows from the fact that G_{inMJ} (**x**) is a homogeneous function of order zero. Substituting (25) into (34), we obtain the non-zero components of I_{inMJ} as

$$I_{11MJ} = \int_{|y|=1} y_1^2 K_{MJ}^{-1}(y_1, y_2, \beta y_3)\, dS$$

$$I_{22MJ} = \int_{|y|=1} y_2^2 K_{MJ}^{-1}(y_1, y_2, \beta y_3)\, dS$$

$$I_{33MJ} = \int_{|y|=1} \beta^2 y_3^2 K_{MJ}^{-1}(y_1, y_2, \beta y_3)\, dS$$

$$I_{1212} = \int_{|y|=1} y_1 y_2 K_{12}^{-1}(y_1, y_2, \beta y_3)\, dS \tag{35}$$

$$I_{1313} = \int_{|y|=1} \beta\, y_1 y_3 K_{13}^{-1}(y_1, y_2, \beta y_3)\, dS$$

$$I_{1314} = \int_{|y|=1} \beta\, y_1 y_3 K_{14}^{-1}(y_1, y_2, \beta y_3)\, dS$$

$$I_{2323} = \int_{|y|=1} \beta\, y_2 y_3\, K_{23}^{-1}(y_1, y_2, \beta y_3)\, dS$$

$$I_{2324} = \int_{|y|=1} \beta\, y_2 y_3\, K_{24}^{-1}(y_1, y_2, \beta y_3)\, dS$$

Let us parametrize the unit sphere as follows.

$$y_1 = \sin\theta \cos\phi, \qquad y_2 = \sin\theta \sin\phi y_3 = \cos\theta$$
$$0 \le \theta \le \pi, \qquad 0 \le \phi \le 2\pi \tag{36}$$

The area element is given by

$$dS = \sin\theta\, d\theta\, d\phi \tag{37}$$

It can be easily seen from (30) and (36) that $D(y_1, y_2, \beta y_3)$ does not depend on ϕ. Substituting (36) and (37) into (35), and performing the integration with respect to ϕ, and further changing the variables from θ to t by

$$t = \cos\theta, \qquad dt = -\sin\theta\, d\theta, \qquad \sin^2\theta = 1 - t^2 \tag{38}$$

we finally obtain

$$I_{1111} = I_{2222} = \frac{\pi}{2} \int_0^1 \frac{1 - t^2}{D} [(3r_{111} + r_{112})(1 - t^2)^3 + \beta^2 (3r_{113} + r_{114})(1 - t^2)^2 t^2$$
$$+ \beta^4 (3r_{115} + r_{116})(1 - t^2)\, t^4 + 4\beta^6 r_{117}\, t^6]dt$$

$$I_{1122} = I_{2211} = \frac{\pi}{2} \int_0^1 \frac{1 - t^2}{D} [(3r_{221} + r_{222})(1 - t^2)^3 + \beta^2 (3r_{223} + r_{224})(1 - t^2)^2 t^2$$
$$+ \beta^4 (3r_{225} + r_{226})(1 - t^2)\, t^4 + 4\beta^6 r_{227}\, t^6]dt$$

$$I_{1133} = I_{2233} = -2\pi \int_0^1 \frac{1 - t^2}{Q} [r_{331} (1 - t^2)^2 + \beta^2 r_{332} (1 - t^2)\, t^2 + \beta^4 r_{333}\, t^4]dt$$

$$I_{1144} = I_{2244} = -2\pi \int_0^1 \frac{1-t^2}{Q} [r_{441}(1-t^2)^2 + \beta^2 r_{442}(1-t^2)t^2 + \beta^4 r_{443} t^4]\, dt$$

$$I_{1134} = I_{2234} = -2\pi \int_0^1 \frac{1-t^2}{Q} [r_{341}(1-t^2)^2 + \beta^2 r_{342}(1-t^2)t^2 + \beta^4 r_{343} t^4]\, dt$$

$$I_{3311} = I_{3322} = 2\pi\beta^2 \int_0^1 \frac{t^2}{D} [(r_{111} + r_{112})(1-t^2)^3 + \beta^2 (r_{113} + r_{114})(1-t^2)^2 t^2$$

$$+ \beta^4 (r_{115} + r_{116})(1-t^2) t^4 + 2\beta^6 r_{117} t^6] dt \qquad (39)$$

$$I_{3333} = -4\pi\beta^2 \int_0^1 \frac{t^2}{Q} [r_{331}(1-t^2)^2 + \beta^2 r_{332}(1-t^2)t^2 + \beta^4 r_{333} t^4]\, dt$$

$$I_{3344} = -4\pi\beta^2 \int_0^1 \frac{t^2}{Q} [r_{441}(1-t^2)^2 + \beta^2 r_{442}(1-t^2)t^2 + \beta^4 r_{443} t^4]\, dt$$

$$I_{3334} = -4\pi\beta^2 \int_0^1 \frac{t^2}{Q} [r_{341}(1-t^2)^2 + \beta^2 r_{342}(1-t^2)t^2 + \beta^4 r_{343} t^4]\, dt$$

$$I_{1212} = \frac{\pi}{2} \int_0^1 \frac{(1-t^2)^2}{D} [r_{121}(1-t^2)^2 + \beta^2 r_{122}(1-t^2)t^2 + \beta^4 r_{123} t^4]\, dt$$

$$I_{1313} = I_{2323} = -2\pi\beta^2 \int_0^1 \frac{t^2(1-t^2)}{Q} [r_{131}(1-t^2) + \beta^2 r_{132} t^2]\, dt$$

$$I_{1314} = I_{2324} = -2\pi\beta^2 \int_0^1 \frac{t^2(1-t^2)}{Q} [r_{141}(1-t^2) + \beta^2 r_{142} t^2]\, dt$$

where

$$D = - PQ$$

$$P = (C_{11} - C_{12})(1 - t^2) + 2C_{44}\,\beta^2 t^2 \qquad (40)$$

$$Q = q_1 (1 - t^2)^3 + \beta^2 q_2 (1 - t^2)^2 t^2 + \beta^4 q_3 (1 - t^2)\, t^4 + \beta^6 q_4\, t^6$$

In (39), the equalities $I_{1111} = I_{2222}$, $I_{1122} = I_{2211}$, $I_{3311} = I_{3322}$, $I_{1313} = I_{2323}$, $I_{1314} = I_{2324}$, are obtained from the relations among r_{ijk} given in Appendix A. It should be noted that I_{inMJ} in (39) coincide with $\overline{G}_{MJin}$ in (20) of Huang and Yu (1994) with the following notational correspondence

$$\overline{G}_{MJin} = I_{inMJ} \qquad (41)$$

$$\rho = \beta$$

except that there are a few misprints in their paper regarding the coefficients of the polynomials in the integrands which are given in Appendix A of their paper.

By using the equation (46) of our previous paper (Mikata, 2000) and (39) above, the piezoelectric Eshelby tensors S_{MnAb} for a spheroidal inclusion along the x-axis can be obtained in terms of I_{inMJ} as

$$S_{1111} = S_{2222} = \frac{1}{4\pi}[C_{11} I_{1111} + C_{12} I_{1212} + C_{13} I_{1313} + e_{31} I_{1314}]$$

$$S_{1122} = S_{2211} = \frac{1}{4\pi}[C_{12} I_{1111} + C_{11} I_{1212} + C_{13} I_{1313} + e_{31} I_{1314}]$$

$$S_{1133} = S_{2233} = \frac{1}{4\pi}[C_{13} (I_{1111} + I_{1212}) + C_{33} I_{1313} + e_{33} I_{1314}]$$

$$S_{1143} = S_{2243} = \frac{1}{4\pi}[e_{31} (I_{1111} + I_{1212}) + e_{33} I_{1313} - \kappa_{33} I_{1314}]$$

$$S_{1212} = S_{1221} = S_{2112} = S_{2121} = \frac{1}{8\pi}(C_{11} - C_{12})[I_{1122} + I_{1212}]$$

$$S_{1313} = S_{1331} = S_{3113} = S_{3131} = S_{2323} = S_{2332} = S_{3223} = S_{3232}$$

$$= \frac{1}{8\pi}[C_{44} (I_{1133} + I_{3311} + 2I_{1313}) + e_{15} (I_{1134} + I_{1314})]$$

$$S_{1341} = S_{3141} = S_{2342} = S_{3242}$$

$$= \frac{1}{8\pi} [e_{15} (I_{1133} + I_{3311} + 2I_{1313}) - \kappa_{11} (I_{1134} + I_{1314})]$$

(42)

$$S_{3311} = S_{3322} = \frac{1}{4\pi} [C_{11} I_{1313} + C_{12} I_{2323} + C_{13} I_{3333} + e_{31} I_{3334}]$$

$$S_{3333} = \frac{1}{4\pi} [C_{13} (I_{1313} + I_{2323}) + C_{33} I_{3333} + e_{33} I_{3334}]$$

$$S_{3343} = \frac{1}{4\pi} [e_{31} (I_{1313} + I_{2323}) + e_{33} I_{3333} - \kappa_{33} I_{3334}]$$

$$S_{4113} = S_{4131} = S_{4223} = S_{4232} = \frac{1}{4\pi} [C_{44} (I_{1134} + I_{1314}) + e_{15} I_{1144}]$$

$$S_{4141} = S_{4242} = \frac{1}{4\pi} [e_{15} (I_{1134} + I_{1314}) - \kappa_{11} I_{1144}]$$

$$S_{4311} = S_{4322} = \frac{1}{4\pi} [C_{11} I_{1314} + C_{12} I_{2324} + C_{13} I_{3334} + e_{31} I_{3344}]$$

$$S_{4333} = \frac{1}{4\pi} [C_{13} (I_{1314} + I_{2324}) + C_{33} I_{3334} + e_{33} I_{3344}]$$

$$S_{4343} = \frac{1}{4\pi} [e_{31} (I_{1314} + I_{2324}) + e_{33} I_{3334} - \kappa_{33} I_{3344}]$$

$S_{MnAb} = 0$, otherwise

It will be shown in the following that we can proceed further, and in fact we can evaluate the integrals in (39) analytically in an exact closed form. To this end, let us rewrite (39) as

$$I_{1111} = I_{2222}$$

$$= - \frac{\pi}{2} I_1(2\beta^2 C_{44}, C_{11}\text{-}C_{12}; \beta^6 q_4, \beta^4 q_3, \beta^2 q_2, q_1; s_{111}, s_{112}, s_{113}, s_{114})$$

$$I_{1122} = I_{2211}$$

$$= - \frac{\pi}{2} I_1(2\beta^2 C_{44}, C_{11}\text{-}C_{12}; \beta^6 q_4, \beta^4 q_3, \beta^2 q_2, q_1; s_{221}, s_{222}, s_{223}, s_{224})$$

$$I_{1133} = I_{2233} = - 2\pi\, I_4(\beta^6 q_4, \beta^4 q_3, \beta^2 q_2, q_1; \beta^4 r_{333}, \beta^2 r_{332}, r_{331})$$

$$I_{1144} = I_{2244} = - 2\pi\, I_4(\beta^6 q_4, \beta^4 q_3, \beta^2 q_2, q_1; \beta^4 r_{443}, \beta^2 r_{442}, r_{441})$$

$$I_{1134} = I_{2234} = -2\pi\, I_4(\beta^6 q_4, \beta^4 q_3, \beta^2 q_2, q_1; \beta^4 r_{343}, \beta^2 r_{342}, r_{341})$$

$$I_{3311} = I_{3322}$$

$$= -2\pi\beta^2\, I_2(2\beta^2 C_{44}, C_{11}\text{-}C_{12}; \beta^6 q_4, \beta^4 q_3, \beta^2 q_2, q_1; s_{331}, s_{332}, s_{333}, s_{334})$$

$$I_{3333} = -4\pi\beta^2\, I_5(\beta^6 q_4, \beta^4 q_3, \beta^2 q_2, q_1; \beta^4 r_{333}, \beta^2 r_{332}, r_{331}) \qquad (43)$$

$$I_{3344} = -4\pi\beta^2\, I_5(\beta^6 q_4, \beta^4 q_3, \beta^2 q_2, q_1; \beta^4 r_{443}, \beta^2 r_{442}, r_{441})$$

$$I_{3334} = -4\pi\beta^2\, I_5(\beta^6 q_4, \beta^4 q_3, \beta^2 q_2, q_1; \beta^4 r_{343}, \beta^2 r_{342}, r_{341})$$

$$I_{1212} = -\frac{\pi}{2}\, I_3(2\beta^2 C_{44}, C_{11}\text{-}C_{12}; \beta^6 q_4, \beta^4 q_3, \beta^2 q_2, q_1; \beta^4 r_{123}, \beta^2 r_{122}, r_{121})$$

$$I_{1313} = I_{2323} = -2\pi\beta^2\, I_6(\beta^6 q_4, \beta^4 q_3, \beta^2 q_2, q_1; \beta^2 r_{132}, r_{131})$$

$$I_{1314} = I_{2324} = -2\pi\beta^2\, I_6(\beta^6 q_4, \beta^4 q_3, \beta^2 q_2, q_1; \beta^2 r_{142}, r_{141})$$

where

$$s_{111} = 4\beta^6 r_{117}, \qquad s_{112} = \beta^4(3r_{115} + r_{116}), \qquad s_{113} = \beta^2(3r_{113} + r_{114})$$

$$s_{114} = 3r_{111} + r_{112}$$

$$s_{221} = 4\beta^6 r_{227} \qquad s_{222} = \beta^4(3r_{225} + r_{226}) \qquad s_{223} = \beta^2(3r_{223} + r_{224})$$

$$s_{224} = 3r_{221} + r_{222} \qquad (44)$$

$$s_{331} = 2\beta^6 r_{117} \qquad s_{332} = \beta^4(r_{115} + r_{116}) \qquad s_{333} = \beta^2(r_{113} + r_{114})$$

$$s_{334} = r_{111} + r_{112}$$

and

$$I_1\,(e,f; a,b,c,d; A,B,C,D)$$

$$= \int_0^1 \frac{(1-t^2)[At^6 + Bt^4(1-t^2) + Ct^2(1-t^2)^2 + D(1-t^2)^3]}{[et^2 + f(1-t^2)][at^6 + bt^4(1-t^2) + ct^2(1-t^2)^2 + d(1-t^2)^3]}\,dt$$

I_2 (e,f; a,b,c,d; A,B,C,D)

$$= \int_0^1 \frac{t^2[At^6 + Bt^4(1 - t^2) + Ct^2(1 - t^2)^2 + D(1 - t^2)^3]}{[et^2 + f(1 - t^2)][at^6 + bt^4(1 - t^2) + ct^2(1 - t^2)^2 + d(1 - t^2)^3]} dt$$

I_3 (e,f; a,b,c,d; A,B,C)

$$= \int_0^1 \frac{(1 - t^2)^2[At^4 + Bt^2(1 - t^2) + C(1 - t^2)^2]}{[et^2 + f(1 - t^2)][at^6 + bt^4(1 - t^2) + ct^2(1 - t^2)^2 + d(1 - t^2)^3]} dt$$

(45)

$$I_4 (a,b,c,d; A,B,C) = \int_0^1 \frac{(1 - t^2)[At^4 + Bt^2(1 - t^2) + C(1 - t^2)^2]}{at^6 + bt^4(1 - t^2) + ct^2(1 - t^2)^2 + d(1 - t^2)^3} dt$$

$$I_5 (a,b,c,d; A,B,C) = \int_0^1 \frac{t^2[At^4 + Bt^2(1 - t^2) + C(1 - t^2)^2]}{at^6 + bt^4(1 - t^2) + ct^2(1 - t^2)^2 + d(1 - t^2)^3} dt$$

$$I_6 (a,b,c,d; A,B) = \int_0^1 \frac{t^2(1 - t^2)[At^2 + B(1 - t^2)]}{at^6 + bt^4(1 - t^2) + ct^2(1 - t^2)^2 + d(1 - t^2)^3} dt$$

For the real piezoelectric material parameters, the above integrals are expected to be finite. In fact, this condition will impose additional constraints on the piezoelectric material parameters, which was discussed in our previous paper (Mikata, 2000), where the integrals treated were different from the above integrals. However, exactly the same constraints will be obtained from the consideration of the above integrals.

The analytical evaluations of the above integrals $I_1 \sim I_6$ are given in the following. First, let us set

$$\varepsilon = \sqrt{\frac{f}{e}} \tag{46}$$

When the condition discussed above is satisfied (see Mikata, 2000), we have the following results. Here we have assumed that all of the poles of each integrand are a simple pole. If they are not, then we would have different expressions, which are not listed in the following.

(a) When $q^2 + \frac{4}{27} p^3 \leq 0$

$$I_1 = \frac{1}{ae} [E_1 J_1(\varepsilon^2) + F_1 J_1(\alpha^2) + G_1 J_1(\beta^2) + H_1 J_1(\gamma^2)]$$

$$I_2 = \frac{1}{ae} [A + E_2 J_1(\varepsilon^2) + F_2 J_1(\alpha^2) + G_2 J_1(\beta^2) + H_2 J_1(\gamma^2)]$$

$$I_3 = \frac{1}{ae} [E_3 J_1(\varepsilon^2) + F_3 J_1(\alpha^2) + G_3 J_1(\beta^2) + H_3 J_1(\gamma^2)]$$

(47)

$$I_4 = \frac{1}{a} [F_4 J_1(\alpha^2) + G_4 J_1(\beta^2) + H_4 J_1(\gamma^2)]$$

$$I_5 = \frac{1}{a} [A + F_5 J_1(\alpha^2) + G_5 J_1(\beta^2) + H_5 J_1(\gamma^2)]$$

$$I_6 = \frac{1}{a} [F_6 J_1(\alpha^2) + G_6 J_1(\beta^2) + H_6 J_1(\gamma^2)]$$

(b) When $q^2 + \frac{4}{27} p^3 > 0$

$$I_1 = \frac{1}{ae} [K_1 J_1(\varepsilon^2) + L_1 J_1(\delta^2) + J_2(g,h;\ M_1,N_1)]$$

$$I_2 = \frac{1}{ae} [A + K_2 J_1(\varepsilon^2) + L_2 J_1(\delta^2) + J_2(g,h;\ M_2,N_2)]$$

$$I_3 = \frac{1}{ae} [K_3 J_1(\varepsilon^2) + L_3 J_1(\delta^2) + J_2(g,h;\ M_3,N_3)]$$

(48)

$$I_4 = \frac{1}{a} [L_4 J_1(\delta^2) + J_2(g,h;\ M_4,N_4)]$$

$$I_5 = \frac{1}{a} [A + L_5 J_1(\delta^2) + J_2(g,h;\ M_5,N_5)]$$

$$I_6 = \frac{1}{a} [L_6 J_1(\delta^2) + J_2(g,h;\ M_6,N_6)]$$

with

$$g = -2(\xi^2 - \eta^2)$$

(49)

$$h = (\xi^2 + \eta^2)^2$$

where α, β, γ, δ, ξ, η, p and q are defined in Appendix B, and the coefficients E_i (i=1~3), F_i (i=1~6), G_i (i=1~6), H_i (i=1~6), K_i (i=1~3), L_i (i=1~6), M_i (i=1~6), N_i (i=1~6) are given in Appendix C. Finally the functions J_1 and J_2 are defined as follows.

$$J_1(k) = \int_0^1 \frac{1 - t^2}{(1 - k)t^2 + k}\, dt$$

$$= \begin{cases} \dfrac{1}{k - 1} - \dfrac{1}{2(k - 1)^{\frac{3}{2}} \sqrt{k}} \log \left| \dfrac{\sqrt{k} + \sqrt{k - 1}}{\sqrt{k} - \sqrt{k - 1}} \right| & \text{when } k > 1 \\ -\dfrac{1}{1 - k} + \dfrac{1}{(1 - k)^{\frac{3}{2}} \sqrt{k}} \tan^{-1} \sqrt{\dfrac{1 - k}{k}} & \text{when } 0 < k < 1 \\ \dfrac{2}{3} & \text{when } k = 1 \end{cases} \tag{50}$$

$$J_2(g,h; M,N) = \int_0^1 \frac{(1 - t^2)[Mt^2 + N(1 - t^2)]}{t^4 + gt^2(1 - t^2) + h(1 - t^2)^2}\, dt$$

$$= \frac{N - M}{1 - g + h} + \frac{1}{(1 - g + h)^2} \left[\frac{R}{2} \log \frac{(1 + \rho)^2 + \zeta^2}{(1 - \rho)^2 + \zeta^2} - \frac{S}{\zeta} \left(\tan^{-1} \frac{1 + \rho}{\zeta} + \tan^{-1} \frac{1 - \rho}{\zeta} \right) \right] \tag{51}$$

where

$$u = -\frac{g - 2h}{2(1 - g + h)}, \qquad v = \frac{\sqrt{4h - g^2}}{2(1 - g + h)} = \frac{2\xi\eta}{1 - g + h}$$

$$r = \sqrt{u^2 + v^2}, \qquad \cos\theta = \frac{u}{\sqrt{u^2 + v^2}}, \qquad \sin\theta = \frac{v}{\sqrt{u^2 + v^2}},$$

$$\rho = \sqrt{r} \cos\frac{\theta}{2}, \qquad \zeta = \sqrt{r} \sin\frac{\theta}{2}, \tag{52}$$

and

$$T = (h - 1)M - (g - 2)N \,, \qquad U = hM - (g - 1)N$$

$$R = \frac{1}{4\rho}[T + \frac{U}{\rho^2 + \zeta^2}] \,, \qquad S = \frac{1}{4}[T - \frac{U}{\rho^2 + \zeta^2}] \tag{53}$$

By using the results of (43), (47) and (48) into (42), we obtain the piezoelectric Eshelby tensors for the spheroidal inclusion along the x_3-axis. It should be emphasized that these are exact closed form expressions for the piezoelectric Eshelby tensors.

6. Conclusion

In this paper, by systematically treating the integrals involved in the piezoelectric inclusion problem, explicit results have been obtained for the piezoelectric Eshelby tensors for a spheroidal inclusion aligned along the axis of the anisotropy in a transversely isotropic piezoelectric material. The method employed is a direct extension of a fairly unified approach, which has been recently developed by Mikata (2000), where Deeg's results (1980) on a piezoelectric inclusion problem were used. The key step of this paper is an analytical evaluation of several integrals, which were made possible after a careful treatment of a certain bi-cubic equation, whose details are given in Appendix B.

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Appendix A $q_1 \sim q_4$ and r_{ijk}

$$q_1 = C_{11}\,(C_{44}\,\kappa_{11} + e_{15}^2)$$

$$q_2 = -\,C_{13}^2\,\kappa_{11} + C_{11}\,(C_{33}\,\kappa_{11} + C_{44}\,\kappa_{33} + 2e_{15}\,e_{33}) - 2C_{13}\,(C_{44}\,\kappa_{11} + e_{15}^2) - e_{31}\,(2C_{13}\,e_{15} - C_{44}\,e_{31})$$

$$q_3 = C_{33}\,C_{44}\,\kappa_{11} - (C_{13}^2 - C_{11}\,C_{33} + 2C_{13}\,C_{44})\,\kappa_{33} + C_{33}\,(e_{15} + e_{31})^2 - e_{33}\,(2C_{13}\,e_{15} - C_{11}\,e_{33}) - 2(C_{13} + C_{44})\,e_{31}\,e_{33}$$

$$q_4 = C_{44}\,(C_{33}\,\kappa_{33} + e_{33}^2)$$

$$r_{111} = -\,\frac{C_{11} - C_{12}}{C_{11}}\,q_1$$

$$r_{112} = -\,2\,q_1$$

$$r_{113} = -\,(C_{11} - C_{12})\,(C_{33}\,\kappa_{11} + C_{44}\,\kappa_{33} + 2e_{15}\,e_{33}) - 2C_{44}\,(C_{44}\,\kappa_{11} + e_{15}^2)$$

$$r_{114} = -\,2\,q_2$$

$$r_{115} = -\,(C_{11} - C_{12})\,(C_{33}\,\kappa_{33} + e_{33}^2) - 2C_{44}\,(C_{33}\,\kappa_{11} + C_{44}\,\kappa_{33} + 2e_{15}\,e_{33})$$

$$r_{116} = -\,2\,q_3$$

$$r_{117} = -\,2\,q_4$$

$$r_{121} = \frac{C_{11} + C_{12}}{C_{11}}\,q_1$$

$$r_{122} = (C_{11} + C_{12})\,(C_{33}\,\kappa_{11} + C_{44}\,\kappa_{33} + 2e_{15}\,e_{33}) - 2\kappa_{11}\,(C_{13} + C_{44})^2 - 2\,(e_{15} + e_{31})\,[2C_{13}\,e_{15} + C_{44}\,(e_{15} - e_{31})]$$

$$r_{123} = (C_{11} + C_{12})\,(C_{33}\,\kappa_{33} + e_{33}^2) - 2\kappa_{33}\,(C_{13} + C_{44})^2 + 2C_{33}\,(e_{15} + e_{31})^2 - 4e_{33}\,(C_{13} + C_{44})\,(e_{15} + e_{31})$$

$$r_{131} = (C_{13} + C_{44})\,\kappa_{11} + e_{15}\,(e_{15} + e_{31})$$

$$r_{132} = (C_{13} + C_{44})\,\kappa_{33} + e_{33}\,(e_{15} + e_{31})$$

$$r_{141} = C_{13}\,e_{15} - C_{44}\,e_{31}$$

$$r_{142} = -\,C_{33}\,(e_{15} + e_{31}) + e_{33}\,(C_{13} + C_{44})$$

$$r_{221} = -2q_1 = r_{112}$$

$$r_{222} = -\frac{C_{11} - C_{12}}{C_{11}} q_1 = r_{111}$$

$$r_{223} = -2q_2 = r_{114}$$

$$r_{224} = r_{113}$$

$$r_{225} = -2q_3 = r_{116}$$

$$r_{226} = r_{115}$$

$$r_{227} = -2q_4 = r_{117}$$

$$r_{231} = r_{131}$$

$$r_{232} = r_{132}$$

$$r_{241} = r_{141}$$

$$r_{242} = r_{142}$$

$$r_{331} = -C_{11}\kappa_{11}$$

$$r_{332} = -C_{44}\kappa_{11} - C_{11}\kappa_{33} - (e_{15} + e_{31})^2$$

$$r_{333} = -C_{44}\kappa_{33}$$

$$r_{341} = -C_{11}e_{15}$$

$$r_{342} = C_{13}(e_{15} + e_{31}) + C_{44}e_{31} - C_{11}e_{33}$$

$$r_{343} = -C_{44}e_{33}$$

$$r_{441} = C_{11}C_{44}$$

$$r_{442} = C_{11}C_{33} - C_{13}^2 - 2C_{13}C_{44}$$

$$r_{443} = C_{33}C_{44}$$

Appendix B Roots of the bi-cubic equation

The key to the evaluation of the integrals $I_1 \sim I_6$ is the following bi-cubic equation.

$$a z^6 + b z^4 + c z^2 + d = 0, \quad \text{or}$$

$$z^6 + \frac{b}{a} z^4 + \frac{c}{a} z^2 + \frac{d}{a} = 0 \qquad \text{(B-1)}$$

Let us set

$$p = \frac{c}{a} - \frac{1}{3}\left(\frac{b}{a}\right)^2$$

$$q = \frac{2}{27}\left(\frac{b}{a}\right)^3 - \frac{bc}{3a^2} + \frac{d}{a} \qquad \text{(B-2)}$$

Then the roots of (B-1) are given as follows.

(a) when $q^2 + \frac{4}{27} p^3 \leq 0$

$$z_1, z_2 = \pm \alpha i, \qquad z_3, z_4 = \pm \beta i, \qquad z_5, z_6 = \pm \gamma i \qquad \text{(B-3)}$$

where

$$\alpha = \sqrt{\frac{b}{3a} - 2\sqrt[3]{r}\cos\frac{\theta}{3}}$$

$$\beta = \sqrt{\frac{b}{3a} - 2\sqrt[3]{r}\cos\left(\frac{\theta}{3} + \frac{2\pi}{3}\right)}$$

$$\gamma = \sqrt{\frac{b}{3a} - 2\sqrt[3]{r}\cos\left(\frac{\theta}{3} - \frac{2\pi}{3}\right)} \qquad \text{(B-4)}$$

$$r = \sqrt{-\frac{p^3}{27}}, \quad \cos\theta = -\frac{q}{2r}, \qquad \sin\theta = \frac{1}{2r}\sqrt{-\left(q^2 + \frac{4}{27}p^3\right)}$$

(b) when $q^2 + \frac{4}{27} p^3 > 0$

$$z_1, z_2 = \pm \delta i, \qquad z_3, z_4 = \xi \pm \eta i, \qquad z_5, z_6 = -(\xi \pm \eta i) \qquad \text{(B-5)}$$

where

$$\delta = \sqrt{\frac{b}{3a} - \sqrt[3]{s} - \sqrt[3]{t}}$$

$$\xi = \sqrt{r}\cos\frac{\theta}{2}, \qquad \eta = \sqrt{r}\sin\frac{\theta}{2}$$

$$s = \frac{1}{2}\left(-q + \sqrt{q^2 + \frac{4}{27}p^3}\right)$$

$$t = \frac{1}{2}\left(-q - \sqrt{q^2 + \frac{4}{27}p^3}\right) \tag{B-6}$$

$$r^2 = \sqrt[3]{s^2} + \sqrt[3]{t^2} - \sqrt[3]{st} + \frac{b}{3a}\left(\sqrt[3]{s} + \sqrt[3]{t}\right) + \frac{1}{9}\left(\frac{b}{a}\right)^2$$

$$\cos\theta = -\frac{1}{r}\left[\frac{1}{2}\left(\sqrt[3]{s} + \sqrt[3]{t}\right) + \frac{b}{3a}\right]$$

$$\sin\theta = \frac{\sqrt{3}}{2r}\left(\sqrt[3]{s} - \sqrt[3]{t}\right)$$

When the coefficients a, b, c and d satisfy the conditions discussed in Section 5 (see Mikata, 2000), α, β, γ, and δ are all real and positive. ξ and η are always real and positive.

Appendix C Expansion coefficients E_i (i=1~3), F_i (i=1~6), G_i (i=1~6), H_i (i=1~6), K_i (i=1~3), L_i (i=1~6), M_i (i=1~6), N_i (i=1~6)

$$E_1 = \frac{-A\varepsilon^6 + B\varepsilon^4 - C\varepsilon^2 + D}{(\alpha^2-\varepsilon^2)(\beta^2-\varepsilon^2)(\gamma^2-\varepsilon^2)}$$

$$F_1 = \frac{-A\alpha^6 + B\alpha^4 - C\alpha^2 + D}{(\varepsilon^2-\alpha^2)(\beta^2-\alpha^2)(\gamma^2-\alpha^2)}$$

$$G_1 = \frac{-A\beta^6 + B\beta^4 - C\beta^2 + D}{(\varepsilon^2-\beta^2)(\alpha^2-\beta^2)(\gamma^2-\beta^2)}$$

$$H_1 = \frac{-A\gamma^6 + B\gamma^4 - C\gamma^2 + D}{(\varepsilon^2-\gamma^2)(\alpha^2-\gamma^2)(\beta^2-\gamma^2)}$$

$$K_1 = \frac{-A\varepsilon^6 + B\varepsilon^4 - C\varepsilon^2 + D}{(\delta^2-\varepsilon^2)(\varepsilon^4 - g\varepsilon^2 + h)}$$

$$L_1 = \frac{-A\delta^6 + B\delta^4 - C\delta^2 + D}{(\varepsilon^2-\delta^2)(\delta^4 - g\delta^2 + h)}$$

$$M_1 = \frac{1}{Y}[A\{h^2 + (g^2-h)\delta^2\varepsilon^2 - gh(\delta^2+\varepsilon^2)\} + B\{-g\delta^2\varepsilon^2 + h(\delta^2+\varepsilon^2)\}$$

$$-hC + D(g - \delta^2 - \varepsilon^2 + \delta^2\varepsilon^2)]$$

$$N_1 = \frac{1}{Y}[A\{gh\delta^2\varepsilon^2 - h^2(\delta^2+\varepsilon^2)\} + B(h^2 - h\delta^2\varepsilon^2) + C\{h(\delta^2+\varepsilon^2) - gh\}$$

$$+ D\{\delta^2\varepsilon^2 - g(\delta^2+\varepsilon^2) + g^2 - h\}]$$

$$E_2 = \frac{\varepsilon^2(A\varepsilon^6 - B\varepsilon^4 + C\varepsilon^2 - D)}{(\alpha^2-\varepsilon^2)(\beta^2-\varepsilon^2)(\gamma^2-\varepsilon^2)}$$

$$F_2 = \frac{\alpha^2(A\alpha^6 - B\alpha^4 + C\alpha^2 - D)}{(\varepsilon^2-\alpha^2)(\beta^2-\alpha^2)(\gamma^2-\alpha^2)}$$

$$G_2 = \frac{\beta^2(A\beta^6 - B\beta^4 + C\beta^2 - D)}{(\varepsilon^2-\beta^2)(\alpha^2-\beta^2)(\gamma^2-\beta^2)}$$

$$H_2 = \frac{\gamma^2(A\gamma^6 - B\gamma^4 + C\gamma^2 - D)}{(\varepsilon^2 - \gamma^2)(\alpha^2 - \gamma^2)(\beta^2 - \gamma^2)}$$

$$K_2 = \frac{\varepsilon^2(A\varepsilon^6 - B\varepsilon^4 + C\varepsilon^2 - D)}{(\delta^2 - \varepsilon^2)(\varepsilon^4 - g\varepsilon^2 + h)}$$

$$L_2 = \frac{\delta^2(A\delta^6 - B\delta^4 + C\delta^2 - D)}{(\varepsilon^2 - \delta^2)(\delta^4 - g\delta^2 + h)}$$

$$M_2 = \frac{1}{Y}[A\{(2gh - g^3)\delta^2\varepsilon^2 + (g^2h - h^2)(\delta^2 + \varepsilon^2) - gh^2\}$$

$$+ B\{(g^2 - h)\delta^2\varepsilon^2 - gh(\delta^2 + \varepsilon^2) + h^2\} + C\{h(\delta^2 + \varepsilon^2) - g\delta^2\varepsilon^2\}$$

$$+ D(\delta^2\varepsilon^2 - h)]$$

$$N_2 = \frac{1}{Y}[A\{(h^2 - g^2h)\delta^2\varepsilon^2 + gh^2(\delta^2 + \varepsilon^2) - h^3\} + B\{gh\delta^2\varepsilon^2 - h^2(\delta^2 + \varepsilon^2)\}$$

$$+ C(h^2 - h\delta^2\varepsilon^2) + D\{h(\delta^2 + \varepsilon^2) - gh\}]$$

$$E_3 = \frac{A\varepsilon^4 - B\varepsilon^2 + C}{(\alpha^2 - \varepsilon^2)(\beta^2 - \varepsilon^2)(\gamma^2 - \varepsilon^2)}$$

$$F_3 = \frac{A\alpha^4 - B\alpha^2 + C}{(\varepsilon^2 - \alpha^2)(\beta^2 - \alpha^2)(\gamma^2 - \alpha^2)}$$

$$G_3 = \frac{A\beta^4 - B\beta^2 + C}{(\varepsilon^2 - \beta^2)(\alpha^2 - \beta^2)(\gamma^2 - \beta^2)}$$

$$H_3 = \frac{A\gamma^4 - B\gamma^2 + C}{(\varepsilon^2 - \gamma^2)(\alpha^2 - \gamma^2)(\beta^2 - \gamma^2)}$$

$$K_3 = \frac{A\varepsilon^4 - B\varepsilon^2 + C}{(\delta^2 - \varepsilon^2)(\varepsilon^4 - g\varepsilon^2 + h)}$$

$$L_3 = \frac{A\delta^4 - B\delta^2 + C}{(\varepsilon^2 - \delta^2)(\delta^4 - g\delta^2 + h)}$$

$$M_3 = \frac{1}{Y}[A\{h(\delta^2 + \varepsilon^2) - g\delta^2\varepsilon^2\} + B(\delta^2\varepsilon^2 - h) + C\{g - (\delta^2 + \varepsilon^2)\}]$$

$$N_3 = \frac{1}{Y}[A(h^2 - h\delta^2\varepsilon^2) + B\{h(\delta^2 + \varepsilon^2) - gh\}$$

$$+ C\{\delta^2\varepsilon^2 - g(\delta^2 + \varepsilon^2) + g^2 - h\}]$$

$$F_4 = \frac{A\alpha^4 - B\alpha^2 + C}{(\beta^2 - \alpha^2)(\gamma^2 - \alpha^2)}$$

$$G_4 = \frac{A\beta^4 - B\beta^2 + C}{(\alpha^2 - \beta^2)(\gamma^2 - \beta^2)}$$

$$H_4 = \frac{A\gamma^4 - B\gamma^2 + C}{(\alpha^2 - \gamma^2)(\beta^2 - \gamma^2)}$$

$$L_4 = \frac{A\delta^4 - B\delta^2 + C}{\delta^4 - g\delta^2 + h}$$

$$M_4 = \frac{A(h - g\delta^2) + B\delta^2 - C}{\delta^4 - g\delta^2 + h}$$

$$N_4 = \frac{-h\delta^2 A + hB + C(\delta^2 - g)}{\delta^4 - g\delta^2 + h}$$

$$F_5 = -\frac{\alpha^2(A\alpha^4 - B\alpha^2 + C)}{(\beta^2 - \alpha^2)(\gamma^2 - \alpha^2)}$$

$$G_5 = -\frac{\beta^2(A\beta^4 - B\beta^2 + C)}{(\alpha^2 - \beta^2)(\gamma^2 - \beta^2)}$$

$$H_5 = -\frac{\gamma^2(A\gamma^4 - B\gamma^2 + C)}{(\alpha^2 - \gamma^2)(\beta^2 - \gamma^2)}$$

$$L_5 = -\frac{\delta^2(A\delta^4 - B\delta^2 + C)}{\delta^4 - g\delta^2 + h}$$

$$M_5 = \frac{A\{(g^2 - h)\delta^2 - gh\} + B(h - g\,\delta^2) + \delta^2 C}{\delta^4 - g\delta^2 + h}$$

$$N_5 = \frac{A(gh\delta^2 - h^2) - h\delta^2 B + hC}{\delta^4 - g\delta^2 + h}$$

$$F_6 = \frac{A\alpha^4 - B\alpha^2}{(\beta^2 - \alpha^2)(\gamma^2 - \alpha^2)}$$

$$G_6 = \frac{A\beta^4 - B\beta^2}{(\alpha^2 - \beta^2)(\gamma^2 - \beta^2)}$$

$$H_6 = \frac{A\gamma^4 - B\gamma^2}{(\alpha^2 - \gamma^2)(\beta^2 - \gamma^2)}$$

$$L_6 = \frac{A\delta^4 - B\delta^2}{\delta^4 - g\delta^2 + h}$$

$$M_6 = \frac{A(h - g\delta^2) + B\delta^2}{\delta^4 - g\delta^2 + h}$$

$$N_6 = \frac{h(- A\delta^2 + B)}{\delta^4 - g\delta^2 + h}$$

where

$$Y = (\delta^4 - g\delta^2 + h)(\varepsilon^4 - g\varepsilon^2 + h)$$

Figure captions and figures

Fig. 1 Eigenstrain - eigen electric field **Z*** in a region Ω in an infinite piezoelectric medium

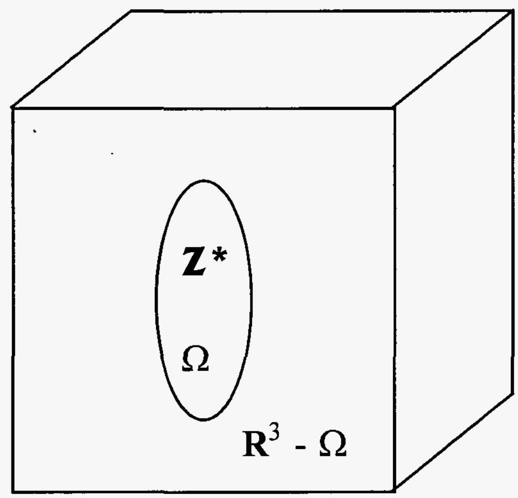

Fig. 1