Regularization Analysis of SAR Superresolution

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ABSTRACT

Superresolution concepts offer the potential of resolution beyond the classical limit. This great promise has not generally been realized. In this study we investigate the potential application of superresolution concepts to synthetic aperture radar. The analytical basis for superresolution theory is discussed. In a previous report the application of the concept to synthetic aperture radar was investigated as an operator inversion problem. Generally, the operator inversion problem is *ill posed*. This work treats the problem from the standpoint of regularization. Both the operator inversion approach and the regularization approach show that the ability to superresolve SAR imagery is severely limited by system noise.

ACKNOWLEDGEMENTS

CONTENTS

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1.   Introduction

The purpose of this study is to investigate the potential of applying regularization to the SAR superresolution problem.[1,2] This work is an extension of the work reported in SAND2001-1532.[3] As with the previous study, we do not obtain an advantage using regularization.

In this study we use regularization to construct families of approximate solutions that are compatible with a given image. The principle of regularization uses additional information say, a solution set that is restricted to functions satisfying a smoothness condition, to obtain a better approximation to the image. By considering only well-behaved approximations, one might expect that it is possible to use more than N_c eigenfunctions in the inverse operation, and, in this way, achieve higher fidelity. We show, however, that regularized solutions exhibit essentially the same "ill-conditioned" behavior as the solution we obtain if we invert the data directly by using finitely many eigenfunctions (see Eq. (5) in SAND2001-1532).

2. Regularization

Our objective is to analyze the signal-to-noise ratio required to obtain spectral components beyond the bandpass of the system for the case of regularized solutions to the image restoration problem. A regularized solution is a solution to the restoration problem constrained to a class of "well-behaved" functions. For large space-bandwidth products, we show that, to recover $b\log(c)$ terms beyond the degrees of freedom in the system, the signal-to-noise ratio must grow "exponentially" in b; here b is a positive constant. As a practical limit, in terms of the band-limited noise level, ε (see Lemma 2 in SAND2001-1532), we can recover at most on the order of $|\log(\varepsilon)|\log(c)$ spectral components outside the degrees of freedom in the system. In other words, regularized solutions have essentially the same SNR requirements as the inverse (see Eq. (5) in SAND2001-1532) obtained by using only finitely many eigenfunctions.

In an attempt to control error propagation, constraints are often introduced to restrict the class of admitted solutions; this is the concept of regularization due to Tikhonov and Arsenin.[4] We define a regularized solution, in the presence of noise, to be the minimum over a finite dimensional space S of the functional,

$$\Phi(f) = \|Lf - (s + \eta)\|^2 \tag{1}$$

subject to the constraint $\Omega(f) \equiv \|Bf\|^2 \leq E^2$, where the operator B^*B is positive and B^* is the adjoint of B. Here, the operator L is defined by Eq. (10), $s \equiv L\sigma = \sum_{k=1}^{\infty} \lambda_k a_k \varphi_k$,

and η is the band-limited noise with $\eta_k \equiv (\eta, \varphi_k)$, $\langle \eta_k \rangle = 0$, $\langle \eta_k^2 \rangle = \varepsilon^2 \lambda_k$, see Eq. (36) and Eq. (46) in SAND2001-1532. We have imposed the finite dimensional assumption on

S and the positivity assumption on B^*B to ensure that $\Omega(f)$ satisfies the definition of a stabilizing functional, in particular, this guarantees that $K_E \equiv \left\{ f \in S : \|Bf\|^2 \leq E^2 \right\}$ is compact. Since it can be shown that $\Omega(f)$ is a stabilizing functional, it follows that the minimization problem has a unique solution, see Tikhonov and Arsenin.[4] The simplest choice for B is $B = I$ (I is the identity operator), so that, for object restoration, a bound is imposed on the total energy. Another possibility is to let B denote a differential operator, so the constraint gives a bound on the derivatives, and the approximate solution must satisfy a smoothness requirement. Our goal is to provide an estimate of the superresolution capabilities of regularized solutions for a wide class of stabilizing functionals $\Omega(f)$.

Before we present an orthonormal expansion for the regularized solution, we specify the space S, and the number of components, M, beyond the degrees of freedom. Let us consider a case that appears to be very favorable to the recovery of information outside of the bandpass, namely, we assume that the true solution, $\sigma \equiv \sum_{k=1}^{\infty} a_k \varphi_k$, approximately resides in a small number of components, $M = O(\log(c))$, beyond the degrees of freedom in the system. Specifically, we assume $\left\| \sigma - \sum_{k=1}^{N+M} a_k \varphi_k \right\|^2 \leq \varepsilon^2$, where $N = N_c \equiv [2wx]$, $c = \pi wx$, and $M = O(\log(c))$. We define the space S to be equal to the span of the functions $\varphi_k, k = 1,...,N+M$, and we refer to the projection, $\sigma_{N+M} = \sum_{k=1}^{N+M} a_k \varphi_k$, of σ onto S as the object. The projection, h, of $s + \eta$ onto LS

(image   of S under L)   is   given   by $h = \sum_{k=1}^{N+M} (\lambda_k a_k + \eta_k)\varphi_k$,   and

$L^{-1}h = \sum_{k=1}^{N+M} (\lambda_k a_k + \eta_k)\varphi_k / \lambda_k$. Since $(s + \eta) - h$ is orthogonal to LS we may define the

regularized solution as the minimum over S, of the functional,

$$\Phi(f) = \|Lf - h\|^2, \tag{2}$$

where f satisfies the condition $\Omega(f) \equiv \|Bf\|^2 \leq E^2$. It can be shown, since B^*B is

positive, that for sufficiently large E, the set K_E contains $L^{-1}h$, and we set E_0 equal to

the smallest E such that $L^{-1}h \in K_E$.

To estimate the discrepancy between the object and the regularized solution, we

need an approximation for the eigenvalues. It is possible to give a simple asymptotic

expression   for λ_{N+M} if   we   assume   that M takes   the   form

$N + M = \left[(2/\pi)(c + b\log(2\sqrt{c})) \right]$, that is, $M = \left(\frac{2}{\pi}c - N + \frac{2}{\pi}b\log(2\sqrt{c}) \right) + \theta_c$, for some fixed

$b \geq 0$ and θ_c where $0 \leq \theta_c < 1$; it follows that $M \sim \dfrac{2}{\pi}b\log(2\sqrt{c})$, as $c \to \infty$. In this

case, we have

$$\lambda_{N+M} \sim \left(1 + e^{\pi b} \right)^{-1}, \tag{3}$$

as $c \to \infty$, Slepian.[5] We seek an estimate of the discrepancy when not all the energy of

the object, σ_{N+M}, resides in the first N components, that is $\left\| \sum_{k=N+1}^{N+M} a_k \varphi_k \right\|^2 \gg \varepsilon^2$, and,

in particular, we assume $|a_{N+M}| \gg \varepsilon$, so this term cannot be neglected.

Since Φ is convex and K_E is compact, if $E^2 \leq E_0^2$, the minimum must lie on the surface of the set K_E; that is, the inequality may be replaced by the equality $\|Bf\|^2 = E^2$. It follows that the minimization problem may be solved by the method of Lagrange multipliers, namely, the problem of finding the unconditional minimum of the functional

$$\Phi_\alpha(f) = \|Lf - h\|^2 + \alpha\|Bf\|^2 \tag{4}$$

where the parameter α is to be determined. The f_α minimizing the expression in Eq. (4) is given by the solution to the Euler equation associated with the functional Φ_α (Bertero and Boccacci[6]),

$$\left(L^*L + \alpha B^*B\right)f_\alpha = L^*h \tag{5}$$

where L^* denotes the adjoint of L ($L^* = L$ since L is self-adjoint). For $\alpha \geq 0$, the operator $L^*L + \alpha B^*B$ is self-adjoint and positive on S, so Eq. (5) has a unique solution for every non-negative α. If there exists an α, say α', for which $\|Bf_{\alpha'}\|^2 = E^2$ then $f_{\alpha'}$ also solves the *constrained* minimum problem.

To obtain a series expansion for f_α, we need to also assume that the operators L and B commute (the range of B is contained in S and $LB = BL$). This implies that the prolate spheroidal wave functions are also eigenfunctions for B; that is, $B\varphi_k = \mu_k\varphi_k$, $k = 1,...,N + M$, and since B^*B is positive, $|\mu_k| \geq \mu$ for some $\mu > 0$.

We seek a solution to the Euler equation in terms of the functions φ_k; substituting

$$L^*h = \sum_{k=1}^{N+M}\lambda_k(\lambda_k a_k + \eta_k)\varphi_k \text{ and } f_\alpha = \sum_{k=1}^{N+M}b_k\varphi_k \text{ into Eq. (5) yields,}$$

$$\sum_{k=1}^{N+M}\left(\lambda_k^2 + \alpha|\mu_k|^2\right)b_k\varphi_k = \sum_{k=1}^{N+M}\lambda_k(\lambda_k a_k + \eta_k)\varphi_k . \tag{5a}$$

It follows that $b_k = \dfrac{\lambda_k (\lambda_k a_k + \eta_k)}{\lambda_k^2 + \alpha |\mu_k|^2}$ and the regularized solution is given by

$$f_\alpha = \sum_{k=1}^{N+M} \frac{\lambda_k (\lambda_k a_k + \eta_k)}{\lambda_k^2 + \alpha |\mu_k|^2} \varphi_k, \qquad (5b)$$

where α is the unique non-negative root of the equation

$$\|Bf_\alpha\|^2 \equiv \sum_{k=1}^{N+M} \frac{|\mu_k|^2 \lambda_k^2 (\lambda_k a_k + \eta_k)^2}{\left(\lambda_k^2 + \alpha |\mu_k|^2\right)^2} = \mathrm{E}^2. \qquad (5c)$$

Equation (5c) has a non-negative root since the function, $F(\alpha)$, defined by

$F(\alpha) = \sum_{k=1}^{N+M} |\mu_k|^2 \lambda_k^2 (\lambda_k a_k + \eta_k)^2 \Big/ \left(\lambda_k^2 + \alpha |\mu_k|^2\right)^2$ is   monotonically   decreasing

($\mathbf{F'}(\alpha) < 0$) with $F(0) = \mathrm{E}_0^2 \geq \mathrm{E}^2$ and $F(\alpha) \to 0$ as $\alpha \to \infty$. Here, we use the fact that

$E_0^2 \equiv \left\|B(L^{-1}h)\right\|^2 = \sum_{k=1}^{N+M} |\mu_k|^2 (\lambda_k a_k + \eta_k)^2 / \lambda_k^2$ since $L^{-1}h \in K_{E_0}$ and $L^{-1}h \notin K_E$ for

$E < E_0$ (recall that E_0 is the smallest E such that $L^{-1}h \in K_E$).

We seek an estimate of the average discrepancy between the regularized solution f_α and

the object, $\sigma_{N+M} = \sum_{k=1}^{N+M} a_k \varphi_k$, namely

$$\left\langle \|f_\alpha - \sigma_{N+M}\|^2 \right\rangle = \sum_{k=1}^{N+M} \left\langle \left(\frac{\lambda_k (\lambda_k a_k + \eta_k)}{\lambda_k^2 + \alpha |\mu|^2} - a_k \right)^2 \right\rangle, \qquad (6)$$

for different values of E. The kth term in the sum may be written

$$\left\langle \left(\frac{\lambda_k(\lambda_k a_k + \eta_k)}{\lambda_k^2 + \alpha|\mu_k|^2} - a_k \right)^2 \right\rangle =$$

$$\left\langle \left(\frac{\lambda_k^2 a_k}{\lambda_k^2 + \alpha|\mu_k|^2} - a_k \right)^2 \right\rangle - \left\langle \frac{2\lambda_k \eta_k a_k}{\lambda_k^2 + \alpha|\mu_k|^2} \right\rangle + \left\langle \frac{\lambda_k^2 \eta_k^2}{\left(\lambda_k^2 + \alpha|\mu_k|^2\right)^2} \right\rangle \qquad (7)$$

We note, to guarantee that the first two terms on the right are small for arbitrary a_k and $|\mu_k| \geq \mu$, we must have that α is small; that is, if we expand about $\alpha = 0$ we see that the first two terms tend to zero, in general, only if α approaches zero. In particular, for $k = N + M$, since this term cannot be neglected, we must have that $\alpha << \lambda_{N+M}^2$. From Eq. (5c), it follows that E must approach E_0. For E sufficiently close to E_0 we may expand the right-hand-side of expression (6) about $\alpha = 0$ to obtain

$$\left\langle \| f_\alpha - \sigma_{N+M} \|^2 \right\rangle \sim \sum_{k=1}^{N+M} \frac{\langle \eta_k^2 \rangle}{\lambda_k} = \varepsilon^2 \sum_{k=1}^{N+M} \frac{1}{\lambda_k}, \qquad (8)$$

as $E \rightarrow E_0$ (or as $\alpha \rightarrow 0$). The expected discrepancy is bounded below, by using expression (3),

$$\left\langle \| f_\alpha - \sigma_{N+M} \|^2 \right\rangle \geq \frac{\varepsilon^2}{2} \sum_{k=1}^{N+M} \frac{1}{\lambda_k} \geq \frac{\varepsilon^2}{2} \frac{1}{\lambda_{N+M}} \sim \frac{\varepsilon^2}{2}\left(1 + e^{\pi b}\right)^{-1} \geq \varepsilon^2 \frac{e^{\pi b}}{4}, \qquad (9)$$

provided E is sufficiently close to E_0, and c is large.

We have arrived at our desired result; namely, to recover order $b\ln(c)$ components of spectral information outside the bandpass of the system, the signal-to-noise ratio, $\dfrac{E_\eta}{\varepsilon}$, (see expression (47) in SAND2001-1532) must grow exponentially in b, that is,

$$e^{-\pi b/2} \geq \left(1 + e^{\pi b}\right)^{-1/2} \sim \sqrt{\lambda_{N+M}} \gg \frac{\varepsilon}{E_\eta}. \tag{10}$$

It follows that the regularized solution suffers from the same "ill-conditioning" or SNR requirements as the restriction, to the space S, of the inverse operator defined by Eq. (5) in SAND2001-1532; namely, $\sqrt{\lambda_{N+M}} \gg \dfrac{\varepsilon}{E_\eta}$, compare expressions (47) in SAND2001-1532 and (10).

Additionally, we may write the number of components M in terms of the level of the bandlimited noise, ε; that is, if $b = \dfrac{p}{\pi}\left|\log(\varepsilon)\right|$, $p > 0$,

$$\left\langle \left\| f_\alpha - \sigma_{N+M} \right\|^2 \right\rangle \geq \varepsilon^2 \frac{e^{\pi b}}{4} \geq \frac{\varepsilon^{2-p}}{4}, \tag{11}$$

It follows that $0 < p < 2$; for if $p \geq 2$, the average discrepancy is bounded away from zero. Conversely, using expression (8), it can be shown that if $b = \dfrac{p}{\pi}\left|\log(\varepsilon)\right|$, $0 < p < 2$, the average discrepancy tends to zero as $\varepsilon \downarrow 0$. In other words, as $\varepsilon \downarrow 0$ and $c \to \infty$, we can recover at most

$$M \sim \frac{2}{\pi} b \log\left(2\sqrt{c}\right) = \frac{2p}{\pi^2}\left|\log(\varepsilon)\right|\log\left(2\sqrt{c}\right) \sim \frac{p}{\pi^2}\left|\log(\varepsilon)\right|\log(c), \tag{12}$$

spectral components beyond the degrees of freedom in the system. Roughly, as a practical limit, we can recover at most order $\left|\log(\varepsilon)\right|\log(c)$ components outside the bandpass.

One might expect that, by imposing a smoothness criterion on the solution set, the resulting solutions might provide a better approximation to the object; however, as we have seen, regularized solutions exhibit essentially the same ill-conditioning as the

inverse of the restricted operator. The point is that, for the case considered here, regularization does not solve the superresolution problem. Other types of regularization procedures involving say convolution operators suffer from similar limitations, see Tikhonov and Arsenin[4] or Bertero and Boccacci.[6]

3. Summary

Although regularization offers the potential of resolution beyond the bandpass of the system, this potential has not been realized. In fact, as we have seen, regularization has the same SNR requirements as the operator inversion approach. An increase in the number of spectral components beyond N_c, in either case, the operator inversion or regularization approach, is severely limited by noise. As a practical limit we can recover at most on the order of $\left|\log(\varepsilon)\right|\log(c)$ spectral components beyond N_c, where ε is the band-limited noise level. Roughly, the constraints imposed on the solution set, by the regularization technique, do not produce a "better" behaved class of solutions, instead we recover the original set of solutions. In this sense, regularization does not provide additional information for the image restoration problem, and does not significantly improve image restoration.

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