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Discontinuous Galerkin for Hyperbolic Systems with Stiff Relaxation

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Abstract. A Discontinuous Galerkin method is applied to hyperbolic systems that contain stiff relaxation terms. We demonstrate that when the relaxation time is unresolved, the method is accurate in the sense that it accurately represents the system's Chapman-Enskog approximation. Results are presented for the hyperbolic heat equation and coupled radiation-hydrodynamics.

1 Introduction

Many physical systems can be modeled by hyperbolic systems that contain relaxation terms, written in the form

$$\partial_t \mathbf{u} + \partial_x \mathbf{f}(\mathbf{u}) = \mathbf{s}(\mathbf{u}), \quad (1)$$

where $\mathbf{u}$, $\mathbf{f}(\mathbf{u})$, and $\mathbf{s}(\mathbf{u})$ are vectors of length p . Let τ represent the relaxation time contained in $\mathbf{s}(\mathbf{u})$. For small τ , we assume that (1) can be accurately represented by the reduced system

$$\partial_t \mathbf{v} + \partial_x \mathbf{g}(\mathbf{v}) = \tau \partial_x [\mathbf{D}(\mathbf{v}) \partial_x \mathbf{v}], \quad (2)$$

where $\mathbf{v}$ and $\mathbf{g}(\mathbf{v})$ are vectors of length q , with $q < p$, and $\mathbf{D}(\mathbf{v})$ is a $q \times q$ matrix. This reduced system is derived through a Chapman-Enskog expansion [1] and is sometimes referred to as the continuum or equilibrium-diffusion limit.

A simple example is the hyperbolic heat equation [2,3]:

$$\partial_t u + \partial_x v = 0, \quad (3a)$$

$$\partial_t v + \partial_x u = -v/\tau. \quad (3b)$$

For small τ , this system can be accurately represented by the heat equation, $\partial_t u = \tau \partial_{xx} u$. Although (3) appears simple, it presents a challenge for numerical methods. The difficulty is that for a given mesh and timestep, an accurate discretization of (1) may not asymptotically reduce to an accurate discretization of (2) as $\tau \rightarrow 0$.

Methods for stiff systems have recently been an active area of research [2,4, and many others]. Such systems can be found, for example, in combustion, multi-phase flows, and rarefied gas dynamics. In addition, the equations

that govern neutron and radiation transport are stiff, where Discontinuous Galerkin (DG) with linear elements has been used with good success [5,6]. In our experience, DG performs very well for other stiff systems and in this paper we summarize the results of two examples. More detailed results and discussion are presented in [8].

2 The Discontinuous Galerkin Method

In this section we give a brief outline of our implementation of DG for a single space dimension, with an emphasis on the source-term treatment. For more details, see [7,8]. Let the spatial mesh be made up of cells defined as the intervals $[x_{m-1/2}, x_{m+1/2}]$ with $\Delta x_m = x_{m+1/2} - x_{m-1/2}$. For each cell m , the interval $[x_{m-1/2}, x_{m+1/2}]$ is mapped to $\xi \in [-1, 1]$ and $\mathbf{u}$ is approximated as

$$\mathbf{u}|_m(\xi, t) = \sum_{j=0}^k \mathbf{U}_{m,j}(t) \phi_j(\xi).$$

The basis set $\{\phi\}$ are Lagrange-Legendre polynomials, with $\xi_0 = -1 < \xi_1 < \dots < \xi_k < \xi_{k+1} = 1$ the Gauss-Lobatto integration points. This basis satisfies

$$\phi_i(\xi_j) = \delta_{i,j}, \quad (4)$$

so that $\mathbf{U}_{m,j}$ represents the value of the solution with respect to cell m at ξ_j .

Following the 'quadrature free' approach [7], DG(k) for Eq. (1) can be written as

$$\dot{\mathbf{U}} + \frac{1}{\Delta x_m} \mathbf{C}_B \mathbf{B} - \mathbf{C}_V \mathbf{V} = \mathbf{S}, \quad (5)$$

where the subscript m is henceforth dropped unless needed, and

$$\begin{aligned} \mathbf{U} &= (\mathbf{U}_0, \mathbf{U}_1, \dots, \mathbf{U}_{k+1})^T, & \mathbf{B} &= (\mathbf{F}_{m-1/2}, \mathbf{F}_{m+1/2})^T, \\ \mathbf{S} &= (\mathbf{s}_0, \mathbf{s}_1, \dots, \mathbf{s}_{k+1})^T, & \mathbf{V} &= (\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_{n_V})^T, \\ \mathbf{s}_j &= \mathbf{s}(\mathbf{U}_j), & \mathbf{f}_j &= \mathbf{f}(\mathbf{u}(\xi_j^V, t)). \end{aligned}$$

To account for the possible nonlinearity of $\mathbf{f}(\mathbf{u})$, its volume integral is approximated using n_V -quadrature points, at locations $\{\xi^V\}$, with $n_V \geq k+1$. The matrix $\mathbf{C}_B$ is $(k+1) \times 2$ and $\mathbf{C}_V$ is $(k+1) \times n_V$. Both of these matrices are independent of the particular element and can be computed once and stored.

The quantity $\mathbf{F}_{m+1/2} \equiv \mathbf{F}(\mathbf{U}_{m,k+1}, \mathbf{U}_{m+1,0})$ is any suitable flux function. At least in this study, the wave decomposition in the flux solver is based on the 'frozen' flux $\mathbf{f}(\mathbf{u})$, as opposed to including any effects of the 'equilibrium' waves defined by $\mathbf{g}(\mathbf{v})$. This issue is discussed further in [8].

The time integration is carried out using a simple predictor-corrector approach. For the predictor, (5) is discretized as

$$\frac{\mathbf{U}^{n+1/2} - \mathbf{U}^n}{\Delta t/2} + \frac{1}{\Delta x_m} \mathbf{C}_B \mathbf{B}^n - \mathbf{C}_V \mathbf{V}^n = \mathbf{S}^{n+1/2},$$

where the superscript- n denotes the time-level. The corrector-step is

$$\frac{\mathbf{U}^{n+1} - \mathbf{U}^n}{\Delta t} + \frac{1}{\Delta x_m} \mathbf{C}_B \mathbf{B}^{n+1/2} - \mathbf{C}_V \mathbf{V}^{n+1/2} = \theta \mathbf{S}^{n+1} + (1 - \theta) \mathbf{S}^n,$$

where $0 \leq \theta \leq 1$. We typically use $\theta = 1$ (fully implicit), which in general is only first-order in time. An alternative is $\theta = 1/2$ (Crank-Nicholson), which is nominally second-order when the relaxation time is resolved, but is well known to give oscillatory behavior if the relaxation time is unresolved. In practice, if the relaxation time is unresolved, a fully implicit treatment is sufficiently accurate since the remaining terms are treated with second-order accuracy [8].

Note that each step is *point implicit*, where by 'point' we mean a single $\mathbf{U}_j$. It is critical that the basis satisfy (4); otherwise, in general, the method would be implicit over all $(k+1)$ -values of $\mathbf{U}_j$ within an element.

3 Hyperbolic Heat Equation

In this section we present results for (3) on the domain $x \in [0, 1]$, with periodic boundary conditions, and the initial condition $u = v = \cos(2\pi x)$. For reference, we compare with a high-resolution (HR) finite-volume method that uses a central-difference slope reconstruction and the time-integrator presented in the previous section.

The DG results were run at a Courant number of 0.3 (stability limit is $1/3$), while the HR results used a Courant number of 0.8 (stability limit is 1). The time-integrator used $\theta = 1/2$, although little difference was observed in the values of u with $\theta = 1$. No slope-limiting was applied in either method.

Figure 1 shows the results for two values of τ and the final time set to $0.01/\tau$. The exact total amount of damping is the same for both cases. The results show that DG(1) is fairly independent of τ , whereas for $\tau = 1 \times 10^{-5}$, the HR results are significantly over-damped, even for 80-mesh cells.

4 Radiation Hydrodynamics

The non-relativistic Euler equations of gas dynamics, coupled with a gray P_1 -model of radiation transport, can be written in non-dimensional form as

Fig. 1. Results for the hyperbolic heat equation. DG(1) and HR used the same time integrator.

[9,10]

$$\partial_t \rho + \partial_x(\rho v) = 0, \quad (6a)$$

$$\partial_t(\rho v) + \partial_x(\rho v^2 + p) = -\mathcal{P} S_F, \quad (6b)$$

$$\partial_t(\rho E) + \partial_x[(\rho E + p)v] = -\mathcal{P} C S_E, \quad (6c)$$

$$\partial_t E_r + C \partial_x F_r = C S_E, \quad (6d)$$

$$\partial_t F_r + \frac{1}{3} C \partial_x E_r = C S_F, \quad (6e)$$

where ρ is the material density, v the velocity, E the total material specific energy, p the material pressure, E_r the radiation energy density, and F_r the radiation flux. The coupling terms are given by

$$S_E = \sigma_t(T^4 - E_r) + \sigma_t \frac{v}{c} (F_r - \frac{4}{3} \frac{v}{c} E_r), \quad (6f)$$

$$S_F = -\sigma_t(F_r - \frac{4}{3} \frac{v}{c} E_r) + \sigma_t \frac{v}{c} (T^4 - E_r), \quad (6g)$$

where σ_t is the flow-length scale over the photon mean-free-path and T is the temperature. There are two non-dimensional constants in this system:

$$\mathcal{C} = c/a_\infty, \quad \mathcal{P} = a_R T_\infty^4 / \rho_\infty a_\infty^2,$$

where c is the lightspeed, a_R the radiation constant, and ' ∞ ' denotes reference conditions, with a_∞ the reference soundspeed.

For large σ_t , (6) can be approximated by

$$\partial_t \rho + \partial_x(\rho v) = 0, \quad (7a)$$

$$\partial_t(\rho v) + \partial_x(\rho v^2 + p^*) = 0, \quad (7b)$$

$$\partial_t(\rho E^*) + \partial_x[(\rho E^* + p^*)v] = \partial_x \left[\frac{pC}{3\sigma_t} \partial_x T^4 \right], \quad (7c)$$

where $p^* = p + \mathcal{P}T^4/3$ and $E^* = E + \mathcal{P}T^4/\rho$. This limit is often referred to as the *equilibrium-diffusion limit* [9,10].

The results in Fig. 2 show the effects of σ_t on a Riemann problem. The slope limiter used for these results is described in [8]. For $\sigma_t = 0$, radiation and hydrodynamics decouple, and the exact hydrodynamic solution is two shocks moving to the left separated by a contact discontinuity. As $\sigma_t \rightarrow \infty$, the exact solution approaches a single shock moving to the left at Mach 10. Starting errors are apparent at $x = 0.4$ and $x = 0.7$ in Fig. 2c. We suspect these starting errors would be smaller if the Riemann solver included source-term effects. Nevertheless, these results are of reasonable quality for problems where standard finite-volume methods would be impractical.

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Fig. 2. DG(1) results for a Riemann problem in radiation hydrodynamics, $t = 0.05$, Courant number 0.3, $\theta = 1$, $\gamma = 5/3$, $\mathcal{P} = 44.930910$, $\mathcal{C} = 100$. The initial condition is $(\rho, u, p) = (1.0, 0.0, 0.6)$ for $x < 0.75$ and $(\rho, u, p) = (3.122689, -6.797632, 2.874136)$ for $x > 0.75$, with E_r and F_r set to their equilibrium values. Symbols are the cell averages of the 200-cell solution. In each plot, the solid line is the following: a) exact solution, b) 2000-cell solution (mesh converged to plotting scale), c) exact solution for $\sigma_t \rightarrow \infty$.

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Model Stiff System: Generalized Hyperbolic Heat Equation

$$\partial_t u + \partial_x v = 0$$

$$\partial_t v + \partial_x u = -\frac{1}{\tau}(v - ru)$$

where $0 \leq |r| \leq 1$, $\tau > 0$. For small τ , obtain **diffusion approximation**:

$$\partial_t u + r \partial_x u = \tau(1 - r^2) \partial_{xx} u.$$

- **Uniform accuracy:** For fixed Δt , Δx , as $\tau \rightarrow 0$, discretization is consistent with diffusion approximation.
- For example, avoid truncation error terms $\mathcal{O}(\Delta x^2/\tau)$.
- Classical extensions of Godunov's method are *not* uniformly accurate.
- Can show semi-discrete, DG(1) is uniformly second-order for this system (Lowrie & Morel).

Results for Hyperbolic Heat Equation

$$u(x, 0) = v(x, 0) = \cos(2\pi x)$$

General Stiff System

$$\partial_t \mathbf{u} + \partial_x \mathbf{f}(\mathbf{u}) = \frac{1}{\tau} \mathbf{s}(\mathbf{u})$$

where $\mathbf{u}, \mathbf{f}(\mathbf{u}), \mathbf{s}(\mathbf{u}) \in \mathbb{R}^p$. Diffusion approximation:

$$\partial_t \mathbf{v} + \partial_x \mathbf{g}(\mathbf{v}) = \tau \partial_x [\mathbf{D}(\mathbf{v}) \partial_x \mathbf{v}],$$

where $\mathbf{v}, \mathbf{g}(\mathbf{v}) \in \mathbb{R}^q$, $q < p$; $\mathbf{D}(\mathbf{v})$ is a $q \times q$ matrix.

- Eigenstructure of $\partial_{\mathbf{u}} \mathbf{f}$ and $\partial_{\mathbf{v}} \mathbf{g}$ may be quite different.
- See Whitham (1974), Liu (Comm. Math. Phys., 1987), Chen, Levermore, Liu (–, 1994), Natalini (Comm. Pure App. Math., 1996), and references therein.

Past Numerical Work

- Pember (SIAM-JAM/JSC, 1993): Combustion; high-resolution methods; Euler limit.
- S. Jin (1991-): transport work; recent diffusion work; fractional-step; Lax-Friedrichs based in limit.
- Bereux & Sainsaulieu (Numerische Mathematik, 1997): Two-phase flow; unsplit; L-F based; Euler limit.
- Larsen, Morel, Miller (JCP, 1986) showed that the “Step scheme” (Godunov) does *not* give diffusion limit for steady, linear transport. Can show this extends to any reconstruction that is independent of cross-sections (Lowrie & Morel). Larsen & Morel (JCP, 1989) showed DG(1) is uniformly accurate.
- Morel, Wareing, & Smith (JCP, 1996) showed DG(1) for time-dependent, rad-transport with material energy equation is uniformly accurate.

DG(k) Method

Semi-discrete, single cell:

$$\dot{\mathbf{U}} + \frac{1}{\Delta x_m} \mathbf{C}_B \mathbf{B} - \mathbf{C}_V \mathbf{V} = \mathbf{S},$$

$$\mathbf{U} = (\mathbf{U}_0, \mathbf{U}_1, \dots, \mathbf{U}_{k+1})^{\mathbb{T}}, \quad \mathbf{B} = (\mathbf{F}_{m-1/2}, \mathbf{F}_{m+1/2})^{\mathbb{T}},$$

$$\mathbf{S} = (s_0, s_1, \dots, s_{k+1})^{\mathbb{T}}, \quad \mathbf{V} = (\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_{n_V})^{\mathbb{T}}$$

Lagrange-type basis $\rightarrow$ for implicit $\mathbf{S}$, the method is *point implicit*.

Results for Various Stiff Systems

- Hyperbolic heat equation
- Broadwell model of gas kinetics
- Euler equations with temperature relaxation (Pember)
- Radiation Hydrodynamics (Euler with P_1 -radiation)

Nondimensional P_1 RHD Equations

$$\text{Hydro-Mass: } \partial_t \rho + \partial_x(\rho v) = 0$$

$$\text{Hydro-Momentum: } \partial_t(\rho v) + \partial_x(\rho v^2 + p) = -\mathbb{P} S_F$$

$$\text{Hydro-Energy: } \partial_t(\rho E) + \partial_x(\rho E + p)v = -\mathbb{P} \mathbb{C} S_E$$

$$\text{Rad-Energy: } \partial_t E_r + \mathbb{C} \partial_x F_r = \mathbb{C} S_E$$

$$\text{Rad-Momentum: } \partial_t F_r + \frac{1}{3} \mathbb{C} \partial_x E_r = \mathbb{C} S_F$$

where $\mathbb{P}$ and $\mathbb{C}$ are non-dimensional constants and

$$S_E = \sigma_t(T^4 - E_r) + \sigma_t \frac{v}{\mathbb{C}}(F_r - \frac{4}{3} \frac{v}{\mathbb{C}} E_r)$$

$$S_F = -\sigma_t(F_r - \frac{4}{3} \frac{v}{\mathbb{C}} E_r) + \sigma_t \frac{v}{\mathbb{C}}(T^4 - E_r)$$

See Lowrie, Morel, Hittinger (Astrophysical Journal 1999).

Equilibrium Diffusion System

- When $\sigma_t \Delta x$ is large, the numerical method should reduce to an approximation of the Equilibrium Diffusion system:

$$\partial_t \begin{pmatrix} \rho \\ \rho v \\ \rho E^* \end{pmatrix} + \partial_x \begin{pmatrix} \rho v \\ \rho v^2 + p^* \\ \rho v H^* \end{pmatrix} = \partial_x \begin{pmatrix} 0 \\ 0 \\ \frac{\mathbb{P}C}{3\sigma_t} \partial_x T^4 \end{pmatrix}$$

- The left-hand side is in the form of the Euler equations, with a modified equation of state:

$$p^* = p + \frac{1}{3} \mathbb{P} T^4, \quad E^* = E + \mathbb{P} T^4 / \rho, \quad H^* = E^* + p^* / \rho.$$

Resolved Precursor, DG(1) vs. HR-Diffusion

Mach 10 stationary shock, $\sigma_t = 200$

Unresolved Precursor, DG(1) vs. DG(0)

Mach 10 stationary shock, $\sigma_t = 5 \times 10^6$

RHD Riemann Problem, $\sigma_t = 0$

DG(1) 200 cells, $t = 0.05$, solid line exact solution, initial discontinuity at $x = 0.75$.

RHD Riemann Problem, $\sigma_t = 100$

DG(1) 200 cells, $t = 0.05$, solid line mesh converged, initial discontinuity at $x = 0.75$.

RHD Riemann Problem, $\sigma_t = 10^8$

DG(1) 200 cells, $t = 0.05$, solid line exact $\sigma_t \rightarrow \infty$ solution, initial discontinuity at $x = 0.75$.

Broadwell Model

$$\partial_t \rho + \partial_x m = 0$$

$$\partial_t m + \partial_x z = 0$$

$$\partial_t z + \partial_x m = \frac{1}{2\tau}(\rho^2 + m^2 - 2\rho z)$$

- As $\tau \rightarrow 0$, $z \rightarrow (\rho^2 + m^2)/2\rho$, and the first two equations go to a system resembling the isothermal Euler equations.
- See Broadwell (Phs. Fluids, 1964), Jin (JCP, 1995), Arora & Roe (Thesis, 1996, B & C 1998).

Broadwell Riemann Problem, $\tau = 1$

DG(1) 200 cells, $t = 0.5$, solid line mesh converged, initial discontinuity at $x = 0$

Broadwell Riemann Problem, $\tau = 10^{-2}$

DG(1) 100 cells, $t = 0.5$, solid line mesh converged, initial discontinuity at $x = 0$

Broadwell Riemann Problem, $\tau = 10^{-8}$

DG(1) 200 cells, $t = 0.5$, solid line mesh converged, initial discontinuity at $x = 0$