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THE SHAPE OF A GRAVITY FINGER

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TABLE OF CONTENTS

Abstract	v
1 Introduction	1
2 Governing Equations and Boundary Conditions.....	2
3 The Interface Shape Using Conformal Mapping	10
4 The Saffman-Taylor Limit	17
5 Numerical Results	19
6 Conclusion.....	22
References	24
A Useful Transformations for Principal Value Integrals.....	25

Abstract

A new gravity finger model is proposed in this report in the absence of interfacial tension but in the presence of gravities. This model considers difference in density and viscosity of the two fluids. Thus, it is able to represent both stable and unstable displacements, and the finger development along either the upper or the bottom walls of a channel. This solution recovers the *Saffman - Taylor* solution [8] if gravity is neglected. The results of the solution are very similar to the solutions proposed by Brener et al. [1] for the gravity number up to 10. However, the solution provided in this work only has one free parameter while the solution of Brener et al. has three.

1 Introduction

Displacing a more viscous fluid from a reservoir by a less viscous fluid results in unstable viscous fingers. Saffman and Taylor made a historical discovery in a narrow Hele-Shaw cell experiment in which they found that the ratio of width of finger to width of channel λ is close to $\frac{1}{2}$ [8]. They also developed an analytical expression for the finger interface in which λ serves as a free parameter. If λ equals $\frac{1}{2}$, the analytical expression fits the developed finger shape of the experiment quite well. From then on, extensive studies have been performed to understand the selection of the parameter λ and its relationship to the instability of fluid displacements [7].

When the density of the resident fluid is different from that of the displacing, gravity will complicate the analysis because it may improve or deteriorate the stability of displacement. Craig et al. [3] used five-spot and line-drive experiments to demonstrate the significant effects of gravity fingers on oil recovery. Greenkorn and colleagues [5] studied the interface of the two phase immiscible flow by using Hele-Shaw cell experiments. They proposed a series solution to describe the interface shape of the finger as follows

$$x = \frac{Gy}{\lambda} - h \sum_{n=1}^{\infty} A_n \exp\left\{-\frac{n\pi Gy}{\lambda h}\right\} \cos \frac{n\pi y}{\lambda h} \quad (1)$$

where G is the gravity dimensionless number, h is the height of the cell and λ is the finger-cell width ratio. However, they did not specify how to determine these infinite coefficients A_n in (1). They also extended their research into a glass bead porous medium and observed that gravity can damp out some of the instability of the fingers and that the width of the developed finger depends strongly on the velocity, height of media and other fluid properties [9]. Fayers and Muggeridge provided a model for calculating gravity finger between miscible fluids in tilted reservoirs [4], based on the numerical solution of a partial differential equation to obtain the finger width. More recently, Brener et al. [1, 2] proposed a three-parameter

expression for describing the stable gravity finger shape, which reads as follows

$$x = \frac{W}{\pi} [(1 - \lambda) \ln \sin(2\alpha) + 2y_0 \ln \tan(\alpha)] \quad (2)$$

$$y = W \left[y_0 - \frac{\lambda}{2} + \frac{2\lambda \cos(\epsilon\pi/2)}{\pi} \int_0^\alpha (\tan \alpha')^\epsilon d\alpha' \right]. \quad (3)$$

where λ is the relative width of finger, y_0 denotes the y coordinate of the center position of the finger departed from the center of a channel, $\epsilon = 1 - 2/\pi\beta$, $\beta = \arctan(1/\hat{N}_g\lambda)$, α takes values from 0 to $\pi/2$, W is the width of the channel and $\hat{N}_g$ is the gravity number that is defined as follows

$$\hat{N}_g = \frac{(g\rho \sin \omega)b^2}{12\mu V} \quad (4)$$

where ω represents tilt angle, ρ and μ are the density and viscosity of the resident fluid, V is the fluid velocity far ahead of interface, and b is the thickness of a Hele-Shaw cell. Like in [5, 8], they neglected the density and viscosity of displacing fluid.

In the following, we propose a new gravity finger model for two-phase fluid displacement in absent of interfacial tension. It has the advantages of the equations by Greenkorn et al. [5] and Brener et al. [1, 2] but without their drawbacks: (a) the model considers density and viscosity of both resident and displacing fluids; (b) it only involves one free parameter (as (1)); (c) it provides a closed form solution of the interface shape of a developed gravity finger (as (2) and (3)).

2 Governing Equations and Boundary Conditions

A typical schematic of a gravity finger is shown in Figure 1. The displacement occurs in a vertical cross-section of a channel of width H , where the flow direction is perpendicular to

Figure 1: Schematic of a gravity finger

that of gravity. The displaced fluid, with subscript 1, has larger density, thus, the displacing fluid (subscript 2) will evolve into a fully-developed single finger along the upper wall. Displacement at constant injection velocity V is considered. The interface of the two fluids will be denoted as AB , while the bottom wall is denoted as ED . The coordinate of the x axis is along ED . Because the injected fluid occupies less than the entire space, fluid 2 moves with velocity U for upstream, while the displaced fluid 1 moves with velocity V , far downstream. Without loss in generality, the streamfunction is zero along the bottom wall.

If we now define the relative finger width as $\lambda \equiv \frac{H_2}{H}$, where H_2 is the finger width at infinity upstream, the mass balances of two fluids read

$$V = \lambda U + (1 - \lambda) \frac{U}{M} \quad (5)$$

$$U = \frac{V}{\lambda + \frac{1-\lambda}{M}}. \quad (6)$$

Here, we can see that if $M \geq 1$, $U \geq V$, and vice-versa.

Assuming incompressible fluids, we can write the following governing equations:

$$\vec{u}_1 = -\frac{k}{\mu_1}(\nabla p_1 - \rho_1 \vec{g}) \quad (7)$$

$$\vec{u}_2 = -\frac{k}{\mu_2}(\nabla p_2 - \rho_2 \vec{g}) \quad (8)$$

$$\nabla \cdot \vec{u}_1 = 0 \quad (9)$$

$$\nabla \cdot \vec{u}_2 = 0 \quad (10)$$

where $\vec{u}_j$, μ_j , p_j , ρ_j and k denote fluid velocities, fluid viscosities, fluid pressures, fluid densities and porous media permeability, $j = 1, 2$ represent the two fluids, and $\vec{g}$ is the acceleration of gravity vector. A solution of this problem is $u_{2x} = U$ and $u_{2y} = 0$ inside fluid 2, namely,

$$-\frac{k}{\mu_2} \frac{\partial p_2}{\partial x} = U \quad (11)$$

$$-\frac{k}{\mu_2} \left(\frac{\partial p_2}{\partial y} + \rho_2 g \right) = 0 \quad (12)$$

Integrating (12), we obtain the pressure in fluid 2 as

$$p_2 = -\rho_2 g y + c(x) \quad (13)$$

where $c(x)$ is a constant of integration obtained by substituting (13) into (11), and integrating

$$c(x) = -\frac{\mu_2 U}{k} x + \hat{a} \quad (14)$$

where $\hat{a}$ is a constant. Combining (13) and (14), the pressure in fluid 2 is

$$p_2 = -\rho_2 g y - \frac{\mu_2 U x}{k} + \hat{a} \quad (15)$$

Inside fluid 1, we do not have a simple solution for the pressure. Because of continuity of pressure on the interface, we have at any point

$$p_1 = p_2 = -\rho_2 g y - \frac{\mu_2 U x}{k} + \hat{a} \quad (16)$$

We can also deduce certain properties upstream and downstream. Far upstream, $u_{1y} = 0$, i.e.,

$$u_{1y} = -\frac{k}{\mu_1} \left(\frac{\partial p_1}{\partial y} + \rho_1 g \right) \Rightarrow 0 \quad (17)$$

Integrating the above equation, the far upstream pressure in fluid 1 is

$$p_1 \approx -\rho_1 g y + e(x), \quad (18)$$

where $e(x)$ is a constant of integration. We note that at that limit

$$\frac{\partial p_1}{\partial x} = e'(x). \quad (19)$$

Also far upstream, the interface is parallel to the x-axis, the y component velocities are null and in the absence of capillarity, the pressures in the two fluids satisfy

$$\frac{\partial p_1}{\partial x} = \frac{\partial p_2}{\partial x} \quad (20)$$

Combining equations (18), (19), and (20), the coefficient $e(x)$ can be found

$$e'(x) = -\frac{\mu_2 U}{k} \quad (21)$$

or

$$e(x) = -\frac{\mu_2 U x}{k} + \hat{e} \quad (22)$$

where $\hat{e}$ is another integration constant. Thus, the pressure of fluid 1 far upstream is

$$p_1 = -\rho_1 g y - \frac{\mu_2 U x}{k} + \hat{e} \quad (23)$$

Far downstream, the condition of a constant fluid velocity at steady-state gives

$$-\frac{k}{\mu_1} \frac{\partial p_1}{\partial x} = V \quad (24)$$

Up to now, we have solved the pressure in displacing fluid 2 and have found certain pressure boundary conditions in the displaced fluid 1. To proceed further, we will transform the above equations in the streamfunction and potential domain to solve for the fluid interface explicitly. Define the potential in displacing fluid 1 as

$$\Phi_1 = -\frac{k}{\mu_1}(p_1 + \rho_1 gy + \hat{d}) \quad (25)$$

where $\hat{d}$ is a constant, or using the pressure as a dependent variable

$$p_1 = -\frac{\mu_1 \Phi_1}{k} - \rho_1 gy - \hat{d} \quad (26)$$

At the interface between the two fluids, the potential can be obtained from equations (16) and (26),

$$\Phi_1 = \frac{U}{M}x - \frac{(\rho_1 - \rho_2)gyk}{\mu_1} - \frac{k}{\mu_1}\hat{d} - \hat{a} \quad (27)$$

Far upstream, equations (23) and (26) give

$$\Phi_1 = -\frac{k}{\mu_1}(\hat{d} + \hat{e}) + \frac{\mu_2 U x}{\mu_1} \quad (28)$$

Using the conventional definition $M \equiv \frac{\mu_1}{\mu_2}$, the condition on Φ_1 upstream can also be expressed as

$$\Phi_1 = \frac{U}{M}x - \frac{k}{\mu_1}(\hat{d} + \hat{e}) \quad (29)$$

and in differential form

$$\frac{\partial \Phi_1}{\partial x} = \frac{U}{M}, \quad (x \rightarrow -\infty) \quad (30)$$

Far downstream, the differential condition from (24) and (26) gives

$$\frac{\partial \Phi_1}{\partial x} = V, \quad (x \rightarrow +\infty) \quad (31)$$

To make further progress and under steady-state conditions, we define the traveling coordinate, $X = x - Ut$, and re-write $Y = y$. Then equations (27), (30) and (31) become

$$\Phi_1 = \frac{U}{M}X - \frac{\Delta \rho g k}{\mu_1}Y - \frac{\Delta \rho g k}{\mu_1}H \quad (\text{on } AB) \quad (32)$$

$$\frac{\partial \Phi_1}{\partial X} = \frac{U}{M} \quad (X \rightarrow -\infty) \quad (33)$$

$$\frac{\partial \Phi_1}{\partial X} = V \quad (X \rightarrow +\infty) \quad (34)$$

and $\Phi_1(X = 0, Y = H) = 0$.

To obtain the interface function, we need the information on the streamfunction. Along the interface AB , the normal velocity can be written as follows

$$u_{1n} = \frac{\partial \Phi_1}{\partial n} = U \cos \theta \quad (35)$$

Because of the *Cauchy-Riemann* condition, $\frac{\partial \Phi_1}{\partial n} = -\frac{\partial \Psi_1}{\partial s}$, we also have

$$-\frac{\partial \Psi_1}{\partial s} = U \cos \theta \quad (36)$$

Now knowing the relationship $\cos \theta ds = -dY$, $\Psi_1|_{Y=H} = VH$, and $\Psi_1|_{Y=0} = 0$, we integrate Ψ_1 in equation (36) to give streamfunction on AB

$$\Psi_1 = UY + H(V - U) \quad (37)$$

Thus, our objective is to find two harmonic functions Φ_1 and Ψ_1 subject to the following boundary conditions

$$\Psi_1 = VH; \quad \text{on } Y = H \quad (38)$$

$$\Psi_1 = (V - U)H + UY; \quad \text{on } AB \quad (39)$$

$$\Phi_1 = \frac{UX}{M} - \frac{\Delta \rho g k}{\mu_1} Y + \frac{\Delta \rho g k H}{\mu_1}; \quad \text{on } AB \quad (40)$$

$$\Psi_1 = 0; \quad \text{at } Y = 0 \quad (41)$$

$$\frac{\partial \Phi_1}{\partial X} = \frac{U}{M}; \quad X \rightarrow -\infty \quad (42)$$

$$\frac{\partial \Phi_1}{\partial X} = V; \quad X \rightarrow +\infty \quad (43)$$

Without loss the generality, we take $H = 1$, and transform equations (38) - (43) using the transformation

$$\tilde{W}_1 = \Phi_1 + i \Psi_1 - \frac{U}{M} Z \quad (44)$$

Then, we normalize the transformed equations by $V - \frac{U}{M}$, use Φ and Ψ as the normalized potential and streamfunction respectively, replace X and Y by x and y , and define the gravity segregation number

$$\begin{aligned}
N_g &= \frac{\Delta \rho g k}{\mu_1 (V - \frac{U}{M})} \\
&= \frac{\Delta \rho g k}{\mu_1 (V - \frac{V}{\lambda M + (1-\lambda)})} \\
&= \frac{\Delta \rho g k [\lambda M + (1-\lambda)]}{\mu_1 \lambda V (M-1)}
\end{aligned} \tag{45}$$

Equations (38) to (43) are further simplified to

$$\Psi = 1; \quad \text{on } y = 1 \tag{46}$$

$$\Psi = 1 - \frac{1-y}{\lambda}; \quad \text{on } AB \tag{47}$$

$$\Phi = N_g (1-y); \quad \text{on } AB \tag{48}$$

$$\Psi = 0; \quad \text{on } y = 0 \tag{49}$$

$$\frac{\partial \Phi}{\partial X} = 0; \quad X \rightarrow -\infty \tag{50}$$

$$\frac{\partial \Phi}{\partial X} = 1; \quad X \rightarrow +\infty \tag{51}$$

3 The Interface Shape Using Comformal Mapping

To seek the solution of interface, we consider application of conformal mapping. Figure 2 shows the $\Phi - \Psi$ domain where Φ and Ψ need to be solved. With the definition of a new pseudo-streamfunction

$$\chi = \Psi - y \tag{52}$$

equations (47) and (48) can be rewritten as

$$\chi = (1 - \Phi)(1 - \lambda) \quad (\text{on } AB) \quad (53)$$

Then our task changes to finding the solution of the Laplace equation for χ

$$\nabla^2 \chi = 0 \quad (54)$$

with boundary conditions

$$\chi = 0; \quad \text{on } \Psi = 1 \quad (55)$$

$$\chi = 0; \quad \text{on } \Psi = 0 \quad (56)$$

$$\chi = 0; \quad \text{on } \Phi \rightarrow +\infty \quad (57)$$

$$\chi = (1 - \Psi)(1 - \lambda); \quad \text{on } AB \quad (58)$$

To solve this problem, we use a *Schwarz-Christoffel* transformation [6] for mapping the strip domain in the $\Phi - \Psi$ plane into the $\xi - \eta$ upper half plane, as illustrated in Figure 3. Here, we let $d \equiv \frac{1}{\lambda N_g}$ and $\alpha \equiv \arctan(\frac{1}{d})$. Then, the transform angles are (see Figure 3)

$$\begin{aligned} \alpha_1 &= 0 \\ \alpha_2 &= \alpha \\ \alpha_3 &= \pi - \alpha \end{aligned} \quad (59)$$

The *Schwarz - Christoffel* transform can be obtained as follow

$$\frac{\partial Z}{\partial \zeta} = a(\zeta + 1)^{\frac{\alpha}{\pi}-1} (\zeta - 1)^{-\frac{\alpha}{\pi}}, \quad (60)$$

Figure 2: Potential - streamfunction plane

or in integral form

$$Z = \Phi + i\Psi = a \int_0^\zeta \frac{1}{(\zeta - 1)^{\frac{\alpha}{\pi}} (\zeta + 1)^{1 - \frac{\alpha}{\pi}}} d\zeta + b \quad (61)$$

where a and b are constants of integration to be determined by the appropriate choice of the corresponding transformed points between the Z and ζ planes. Before we determine these constants, we first derive a relationship for the boundary values in the segment A_2A_3 between the two planes. From (47) and (48), we know that $\Phi = d(1 - \Psi)$, thus, the transformation (61) becomes

$$d(1 - \Psi) + i\Psi = a \int_0^\zeta \frac{1}{(\zeta - 1)^{\frac{\alpha}{\pi}} (\zeta + 1)^{1 - \frac{\alpha}{\pi}}} d\zeta + b \quad (62)$$

Figure 3: Schwarz-Christoffel transformation

and provides $\Psi(\xi)$ on $A'_2A'_3$. Hence, the relation $\chi = (1 - \lambda)(1 - \Psi)$ in Z maps onto $\chi = F(\xi)$ in ζ . Also, equation (52) gives the ordinate in Z by Ψ and χ , thus,

$$y = \Psi + \chi \quad (63)$$

If we find ω , the harmonic counterpart of χ , the relation between the boundary position A_2A_3 in Z and the harmonic functions on $A'_2A'_3$ in ζ can be obtained, namely

$$z = x + iy = \Phi + i\Psi + \omega + i\chi + c \quad (64)$$

The harmonic functions χ and ω are obtained by

$$\begin{aligned} \omega(\xi, \eta) &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(t - \xi)F(t)}{(t - \xi)^2 + \eta^2} dt \\ &= \frac{1}{\pi} \int_{-1}^1 \frac{(t - \xi)F(t)}{(t - \xi)^2 + \eta^2} dt \end{aligned} \quad (65)$$

$$\chi(\xi, \eta) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(t - \xi)\omega(t, 0)}{(t - \xi)^2 + \eta^2} dt$$

where all integrals are understood to be principal value integrals. Substituting equations (65), (47) and (48) into (64), we establish the interface shape as

$$\begin{aligned} x &= \frac{d(1 - y)}{\lambda} + \omega(\xi, 0) + c \\ &= \frac{d(1 - y)}{\lambda} + \frac{1}{\pi} \int_{-1}^1 \frac{(t - \xi)F(t)}{(t - \xi)^2} dt + c \\ &= \frac{d(1 - y)}{\lambda} + \frac{1}{\pi} \int_{-1}^1 \frac{F(t)}{(t - \xi)} dt + c \end{aligned} \quad (66)$$

where

$$F(\xi) = (1 - \lambda)(1 - \Psi) \quad (67)$$

ξ is related to y through the following transformation (formula (61))

$$\begin{aligned} Z &= \Phi + i\Psi \\ &= \frac{d(1 - \lambda)}{\lambda} + i\left(1 - \frac{1 - y}{\lambda}\right) \\ &= a \int_0^\zeta \frac{1}{(\zeta - 1)^{\frac{\alpha}{\pi}}(\zeta + 1)^{1 - \frac{\alpha}{\pi}}} d\zeta + b \end{aligned} \quad (68)$$

and Ψ is related to ξ through (62). Therefore, (66), (68) and (62) provide the equations for solving the interface shape. To determine the coefficients a and b in (62) and (68), we fix the corresponding points between Z and ζ as $Z = i \Leftrightarrow \zeta = -1$ and $Z = d \Leftrightarrow \zeta = 1$. After some algebraic manipulations, the coefficients are given by

$$a = \frac{d - i}{I} \quad (69)$$

$$b = \frac{I_1 d + i I_2}{I} \quad (70)$$

where I_1 , I_2 and I are the following integrals

$$I_1 = \int_{\frac{1}{2}}^1 \frac{d\tau}{\tau^{\frac{\alpha}{\pi}}(1 - \tau)^{1 - \frac{\alpha}{\pi}}} \quad (71)$$

$$I_2 = \int_0^{\frac{1}{2}} \frac{d\tau}{\tau^{\frac{\alpha}{\pi}}(1 - \tau)^{1 - \frac{\alpha}{\pi}}} \quad (72)$$

$$I = I_1 + I_2 \quad (73)$$

Upon substitution of equations (67), (69) and (70) into (62), a solution for $F(\xi)$ is obtained

$$F(\xi) = \frac{(1-\lambda)}{I(1)} I(\xi); \quad \xi \in (-1, 1) \quad (74)$$

where the $I(\xi)$ is

$$I(\xi) = \int_{\frac{1-\xi}{2}}^1 \frac{d\tau}{\tau^{\frac{\alpha}{\pi}} (1-\tau)^{1-\frac{\alpha}{\pi}}} \quad (75)$$

The principal value of (66) can be simplified as shown in the Appendix A. The final result is

$$x = \frac{d(1-y)}{\lambda} + \frac{1}{\pi} \left[\int_{-1}^1 \frac{F(t) - F(\xi)}{(t-\xi)} dt + F(\xi) \log \frac{1-\xi}{1+\xi} \right] + c \quad \xi \in (-1, 1) \quad (76)$$

Note that there is an integration constant c in the above equation. To determine c , we let $\lim_{\xi \rightarrow -1} x = 0$ at the upper channel boundary $y = 1$ in (66). The following value is obtained

$$c = -\frac{1}{\pi} \int_{-1}^1 \frac{F(t)}{t+1} dt \quad (77)$$

Thus, using expression (74), x can be written as

$$x = d \frac{I(\xi)}{I(1)} + \frac{1}{\pi} \frac{(1-\lambda)}{I(1)} \left[\int_{-1}^1 \frac{I(t) - I(\xi)}{(t-\xi)} dt + I(\xi) \log \frac{1-\xi}{1+\xi} - \int_{-1}^1 \frac{I(t)}{t+1} dt \right] \quad \xi \in (-1, 1) \quad (78)$$

The characters of two integrations in the above x expression are analyzed in the Appendix

A. Upon substitution of (69) and (70) into (68), y is

$$y = 1 - \lambda \frac{I(\xi)}{I(1)} \quad \xi \in (-1, 1) \quad (79)$$

equations (78) and (79) represent the final results of the interface functions. When ξ takes values between -1 to 1, (78) and (79) provide the interface positions between two the fluids.

4 The Saffman-Taylor Limit ($N_g \rightarrow 0$)

To verify that equations (78) and (79) recover the *Saffman - Taylor's* solution [8] when gravity is absent, we consider $N_g = 0$, and $\alpha = \frac{\pi}{2}$. Then, $I(\xi)$ can be obtained as

$$\begin{aligned} I(\xi) &= \int_{\frac{1-\xi}{2}}^1 \frac{dt}{\sqrt{t}\sqrt{1-t}} \\ &= \frac{\pi}{2} + \arcsin\xi \end{aligned}$$

From this, equations (78) and (79) are simplified to

$$x = \frac{1-\lambda}{\pi^2} \left[\int_{-1}^1 \frac{\arcsint - \arcsin\xi}{t-\xi} dt + \left(\frac{\pi}{2} + \arcsin\xi \right) \log \frac{1-\xi}{1+\xi} - \int_{-1}^1 \frac{\frac{\pi}{2} + \arcsint}{t+1} dt \right] \quad (80)$$

$$y = 1 - \frac{\lambda}{\pi} \left(\frac{\pi}{2} + \arcsin\xi \right) \quad (81)$$

The principal values in equation (80) are evaluated as follows

$$\begin{aligned} I_{x1} &= \int_{-1}^1 \frac{\arcsint - \arcsin\xi}{t-\xi} dt \\ &= \lim_{\epsilon \rightarrow 0} \left(\int_{-1}^{\xi-\epsilon} \frac{\arcsint - \arcsin\xi}{t-\xi} dt + \int_{\xi+\epsilon}^1 \frac{\arcsint - \arcsin\xi}{t-\xi} dt \right) \end{aligned}$$

which is

$$I_{x1} = \frac{\pi}{2} \log(1 - \xi^2) + \arcsin \xi \log \frac{1 + \xi}{1 - \xi} - \lim_{\epsilon \rightarrow 0} \left[\int_{-1}^{\xi - \epsilon} \frac{\log(\xi - t)}{\sqrt{1 - t^2}} dt + \int_{\xi + \epsilon}^1 \frac{\log(t - \xi)}{\sqrt{1 - t^2}} dt \right] \quad (82)$$

$$\begin{aligned} I_{x2} &= - \int_{-1}^1 \frac{\frac{\pi}{2} + \arcsin t}{t + 1} dt \\ &= - \lim_{\epsilon \rightarrow 0} \int_{-1 + \epsilon}^1 \frac{\frac{\pi}{2} + \arcsin t}{t + 1} dt \\ &= -\pi \log 2 + \lim_{\epsilon \rightarrow 0} \int_{-1 + \epsilon}^1 \log(t + 1) \frac{1}{\sqrt{1 - t^2}} dt \end{aligned} \quad (83)$$

Substituting I_{x1} and I_{x2} into equation (80), we are able to simplify the expression for x as follows

$$x = \frac{1 - \lambda}{\pi} \log \frac{1}{2}(1 - \xi) + \frac{1 - \lambda}{\pi^2} \lim_{\epsilon \rightarrow 0} \left[- \int_{-1}^{\xi - \epsilon} \frac{\log(\xi - t)}{\sqrt{1 - t^2}} dt - \int_{\xi + \epsilon}^1 \frac{\log(t - \xi)}{\sqrt{1 - t^2}} dt + \int_{-1 + \epsilon}^1 \frac{\log(t + 1)}{\sqrt{1 - t^2}} dt \right] \quad (84)$$

After a considerable amount of algebraic manipulation, the sum of the last three integrals turns out to be zero in the limit $\epsilon \rightarrow 0$. Thus, the final result for x is

$$x = \frac{1 - \lambda}{\pi} \log \frac{1 - \xi}{2} \quad (85)$$

Combining the x and y expressions, we obtain the interface shape when gravity is negligible as follows

$$x = \frac{1 - \lambda}{\pi} \log \frac{1}{2} [1 + \cos \frac{\pi}{\lambda} (1 - y)] \quad (86)$$

which is the *Saffman - Taylor* solution.

5 Numerical Results

Numerical results for equations (79) (78) are shown in Figures 4 and 5. The unstable displacement interface corresponding to of $\lambda = 0.1$, $N_g = 0, 0.1, 5, 10$ ($\rho_1 > \rho_2$, $\mu_1 > \mu_2$, $N_g > 0$) is illustrated in Figure 4(a). The shapes of the fingers are almost identical to zero gravity curve when $N_g < 0.1$. As the effect of gravity increases, the interfaces near the center of the channel moves faster than those of small gravity number. This indicates that gravity tends to stabilize the finger tongue. When N_g is very large, the lower portion of the finger moves ahead of the upper part (not show). Although this result recovers the experimental observations in [5], we suspect that it may represent a mathematical rather than a physical reality, because there is an undetermined parameter λ in equations (79) and (78). Figure 4(b) shows interface shapes for $\lambda = 0.1, 0.3, 0.5, 0.7$ and value of gravity number equals $N_g = 0.3$. It indicates that the larger the λ , the flatter the head of a finger calculated from the solution at the condition of same gravity number. Due to the consideration of different density and viscosity between displacing and resident fluids in (79) and (78), one of the advantages of this work is that it reflects both unstable and stable displacement interfaces. Figure 4(c) shows the stable finger shapes ($\rho_1 > \rho_2$, $\mu_1 < \mu_2$, $N_g < 0$) along the upper wall for $\lambda = 0.1$ and value of gravity number $N_g = 0, 0.1, 5, 10$. For this group of fingers, the finger heads become sharper as the magnitude of the gravity number increases.

The solutions can also represents the fingers developed along the bottom wall of the channel. This group of fingers is plotted in Figure 5(a) with $\lambda = 0.1$ and $N_g = 0, 0.1, 5, 10$

Figure 4: Upper wall finger interfaces: (a) unstable fingers, $\lambda = 0.1$, $N_g = 0, 0.1, 5, 10$; (b) unstable fingers, $\lambda = 0.1, 0.3, 0.5, 0.7$, $N_g = 0.3$; (c) stable fingers, $\lambda = 0.1$, $N_g = 0, -0.1, -5, -10$.

Figure 5: Bottom wall finger interfaces: (a) stable fingers, $\lambda = 0.1$, $N_g = 0, 0.1, 5, 10$; (b) stable fingers, $\lambda = 0.1, 0.3, 0.5, 0.7$, $N_g = 0.3$; (c) unstable fingers, $\lambda = 0.1$, $N_g = 0, -0.1, -5, -10$.

($\rho_1 < \rho_2$ and $\mu_1 < \mu_2$), and Figure 5(b) with $\lambda = 0.1, 0.3, 0.5, 0.7$ for the same $N_g = 0.3$. These represent stable gravity fingers. When the magnitude of N_g increases, the finger heads tend to be sharper. The results are similar to the solution of conditionally stable gravity tongues reported by Fayers and Muggeridge [4], although there are different underlying fundamentals between the two works. Figure 5(c) shows the unstable fingers ($\rho_1 < \rho_2$, $\mu_1 > \mu_2$) developed along the bottom wall with $N_g = 0, -0.1, -5, -10$ and $\lambda = 0.1$. The finger interfaces are exactly same with the corresponding upper wall fingers in Figure 4(a).

The equations (79) and (78) are also comparable with the Brener et al. solution (2) and (3). As we pointed out before, their solution involves three parameters, but without considering the density and viscosity of displacing fluid. Thus, (2) and (3) do not obtain the stable displacement. With taking $W = 1$, $\epsilon = 0$ and $y_o = 0.45$, we can compare (79) and (78) with equations (2) and (3). The results are shown in Figure 6. We could see that when for $\lambda = 0.1$ the two solutions results provide very similar interface shapes except there is difference in starting positions for the two solutions.

6 Conclusions

We have provided a new exact solution for the description of the interface shape of gravity segregated fingers. The solution considers the finite density and viscosity of the displacing fluid. Thus, it is able to represent both stable and unstable displacements, and finger development along either the upper wall or bottom wall of a channel. The solution has closed forms so that there is no need to determine the coefficients as the solution of Greenkorn et al. [5]. This solution is able to recover the *Saffman - Taylor* solution [8] if gravity is neglected. Computational results of equations (79) and (78) are very similar to the solutions (2) and (3) of Brener et al. [1] for the gravity number up to 10. However, the solution proposed in this work only has one free parameter while the solution of Brener et al. has three. The (79) and (78) also provide interface shapes similar to conditionally stable fingers calculated by

Figure 6: Comparison between solution of Brener et al. and equations (79) and (78), for $\lambda = 0.1$: (a) $N_g = 0.0001$; (b) $N_g = 0.1$; (c) $N_g = 5$; (d) $N_g = 10$.

the finger model of Fayer and Muggeridge [4], however, they are different by the underlying fundamentals.

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A Useful Transformations for Principal Value Integrals

In equation (66), $\xi \in (-1, 1)$, thus, when t integrates from -1 to 1, the integration has a singularity for each ξ . To find the principal value of the integral we use the following manipulation

$$\begin{aligned}
 I_p &= \int_{-1}^1 \frac{F(t)dt}{t - \xi} \\
 &= \lim_{\epsilon \rightarrow 0} \left[\int_{-1}^{\xi - \epsilon} \frac{F(t)dt}{t - \xi} + \int_{\xi + \epsilon}^1 \frac{F(t)dt}{t - \xi} \right] \\
 &= \lim_{\epsilon \rightarrow 0} \left[\int_{-1}^{\xi - \epsilon} \frac{F(t) - F(\xi)}{t - \xi} dt + F(\xi) \log(\xi - t) \Big|_{-1}^{\xi - \epsilon} \right] \\
 &\quad + \lim_{\epsilon \rightarrow 0} \left[\int_{\xi + \epsilon}^1 \frac{F(t) - F(\xi)}{t - \xi} dt + F(\xi) \log(t - \xi) \Big|_{\xi + \epsilon}^1 \right] \\
 &= \int_{-1}^1 \frac{F(t) - F(\xi)}{t - \xi} dt + F(\xi) \log \frac{1 - \xi}{1 + \xi}
 \end{aligned}$$

This result was used to simplify (66) to (76). Using (74), the above integration can be expressed by $I(t)$.

To evaluate x when $\xi \rightarrow -1$, it is necessary to estimate the value of the logarithmic term in (78). We proceed as follows. Let

$$\frac{1 - \xi}{2} = 1 - \epsilon \quad (87)$$

Then, $I(\xi)$ can be expressed as

$$I(\epsilon) = \int_{1-\epsilon}^1 \frac{d\tau}{\tau^{\frac{a}{\pi}} (1 - \tau)^{1 - \frac{\pi}{\pi}}} \quad (88)$$

Using the change of variables $\tau = 1 - w$, and taking the limit $\epsilon \rightarrow 0$ the above becomes

$$\begin{aligned}
I(\epsilon) &= - \int_{\epsilon}^1 \frac{dw}{(1-w)^{\frac{\alpha}{\pi}} w^{1-\frac{\alpha}{\pi}}} \\
&\approx \int_0^{\epsilon} \frac{dw}{w^{1-\frac{\alpha}{\pi}}} \\
&\approx \frac{\pi}{\alpha} \epsilon^{\frac{\alpha}{\pi}}
\end{aligned}$$

Thus, we have

$$I(\xi) \approx \frac{\pi}{2} \left(\frac{1+\xi}{2} \right)^{\frac{\alpha}{\pi}} \quad (\xi \rightarrow -1) \quad (89)$$

This gives an approximation for the second integral term in equation (78), namely

$$\begin{aligned}
I(\xi) \log \frac{1-\xi}{1+\xi} &\approx \frac{\pi}{2} \left(\frac{1+\xi}{2} \right)^{\frac{\alpha}{\pi}} [\log(1-\xi) - \log(1+\xi)] \\
&\approx -\frac{\pi}{2} \left(\frac{1+\xi}{2} \right)^{\frac{\alpha}{\pi}} \log(1+\xi) \quad (\xi \rightarrow -1)
\end{aligned}$$

Since $\lim_{x \rightarrow 0} x^{\frac{\alpha}{\pi}} \log x = \lim_{x \rightarrow 0} \frac{\alpha}{\pi} x^{\frac{\alpha}{\pi}} = 0$, we have the following final estimate of this term

$$\lim_{\xi \rightarrow -1} I(\xi) \log \frac{1-\xi}{1+\xi} \rightarrow 0 \quad (90)$$

This shows that the logarithmic term in (78) is bounded and approaches zero as $\xi \rightarrow 0$.

Same additional manipulations are needed for estimate of two integrals in (78) in order to calculate interface. If we expand $I(t)$ around ξ using a series expansion, then the first integral of (78) becomes

$$\begin{aligned}
\int_{-1}^1 \frac{I(t) - I(\xi)}{t - \xi} dt &= \int_{-1}^1 \frac{I(\xi) + I'(\xi)(t - \xi) + \frac{1}{2!}I''(\xi)(t - \xi)^2 + \dots - I(\xi)}{t - \xi} dt \\
&= \int_{-1}^1 [I'(\xi) + \frac{1}{2!}I''(\xi)(t - \xi) + \dots] dt
\end{aligned} \tag{91}$$

where I is given in equation (75). The integral above is dependent on the character of the derivatives of $I(\xi)$ in the domain $[-1, 1]$. The first derivative of $I(\xi)$ is

$$I'(\xi) = \frac{1}{(1 - \xi)^{\frac{\alpha}{\pi}}(1 + \xi)^{1 - \frac{\alpha}{\pi}}} \tag{92}$$

It is obvious that singular points, for $I'(\xi)$ and other higher derivatives of $I(\xi)$, occur only at $\xi = \pm 1$. Thus, equation (91) is not singular when $\xi \in (-1, 1)$ (In this work, we define $x = 0$ when $\xi = -1$. Also, we do not calculate the integral at $\xi = 1$ since it corresponds to the interface at negative infinity of x).

The second integration in equation (78) should be considered carefully, because the integration is singular when $t = -1$, i.e.

$$\begin{aligned}
I_s &\equiv \int_{-1}^1 \frac{I(t)}{t + 1} dt \\
&= \lim_{\epsilon \rightarrow 0} \int_{-1+\epsilon}^1 \frac{I(t)}{t + 1} dt \\
&\stackrel{\theta=t+1}{=} \lim_{\epsilon \rightarrow 0} \int_{\epsilon}^2 \frac{I(\theta)}{\theta} d\theta
\end{aligned}$$

The integrand of the above expression is

$$\lim_{\epsilon \rightarrow 0} \frac{I(\theta)}{\theta} = \lim_{\theta \rightarrow 0} \frac{I(\epsilon) + I'(\epsilon)(\theta - \epsilon) + \frac{1}{2!}I''(\epsilon)(\theta - \epsilon)^2 + \dots}{\theta}; \quad \epsilon \in (0, \theta)$$

Since $\theta \rightarrow 0$, when $\varepsilon \rightarrow 0$, and $I(0) = 0$, then

$$\begin{aligned}\lim_{\varepsilon \rightarrow 0} \frac{I(\theta)}{\theta} &\approx \lim_{\varepsilon \rightarrow 0} I'(\varepsilon) \\ &\approx \frac{1}{(2-\varepsilon)^{\frac{\alpha}{\pi}} \varepsilon^{1-\frac{\alpha}{\pi}}} \rightarrow \infty\end{aligned}\tag{93}$$

If we define $f(t) \equiv \frac{I(t)}{t+1}$ and $g(t) \equiv \frac{1}{(t+1)^{1-\frac{\alpha}{\pi}}}$, we have

$$\begin{aligned}\lim_{t \rightarrow -1} \frac{f(t)}{g(t)} &= \lim_{t \rightarrow -1} \frac{\frac{1}{(1-t)^{\frac{\alpha}{\pi}} (1+t)^{1-\frac{\alpha}{\pi}}}}{(1+t)^{1-\frac{\alpha}{\pi}}} \\ &= 2^{\frac{\alpha}{\pi}}\end{aligned}$$

Because $\int_{-1}^1 g(t)dt$ is convergent, I_s is also convergent. With the use of the estimate in (89), I_s can be calculated as

$$\begin{aligned}I_s &= \int_{-1}^1 \frac{I(t) - \frac{\pi}{2} \left(\frac{1+t}{2}\right)^{\frac{\alpha}{\pi}}}{t+1} dt + \int_{-1}^1 \frac{\frac{\pi}{2} \left(\frac{1+t}{2}\right)^{\frac{\alpha}{\pi}}}{t+1} dt \\ &= I_{s1} + I_{s2}\end{aligned}\tag{94}$$

where

$$I_{s1} = \begin{cases} \frac{I(t) - \frac{\pi}{2} \left(\frac{1+t}{2}\right)^{\frac{\alpha}{\pi}}}{t+1}; & t \neq -1 \\ 0; & t = -1 \end{cases}$$

and

$$I_{s2} = \frac{\pi^2}{\alpha^2}$$

Finally, I_s can be obtained as follows

$$I_s = \begin{cases} \int_{-1}^1 \frac{I(t) - \frac{\pi}{2}(\frac{1+t}{2})^{\frac{\alpha}{\pi}}}{t+1} dt + \frac{\pi^2}{\alpha^2}; & t \neq -1 \\ \frac{\pi^2}{\alpha^2}; & t = -1 \end{cases}$$