**FAST 3-D SEISMIC MODELING AND
PRESTACK DEPTH MIGRATION USING
GENERALIZED SCREEN METHODS**

FINAL REPORT

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MP Dvorscak

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Date

Mark P. Dvorscak

(630) 252-2393

E-mail: mark.dvorscak@ch.doe.gov

Office of Intellectual Property Law

DOE Chicago Operations Office

M. Nafi Toksöz

**Earth Resources Laboratory
Department of Earth, Atmospheric, and Planetary Sciences
Massachusetts Institute of Technology
Cambridge, Massachusetts 02139**

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OVERVIEW

The objective of this project has been to develop and test new methods for the modeling and migration of seismic data in complex media in 3-D, for the purpose of improved subsurface imaging. Seismic imaging of complex structures and stratigraphy in the subsurface is crucial to the petroleum industry for energy resource exploration and production, and is also an important part of the basic study of the earth's crust for understanding tectonics, deformation, and sedimentation processes. Imaging can be improved in several ways—by the development of new 3-D migration algorithms, the development of forward modeling methods for 3-D elastic media, and the development of innovative analysis methods to extract geological information from seismic images. In this project, we addressed each of the following areas.

1. We conducted a theoretical analysis of generalized screen methods with an emphasis on stability and accuracy.
2. We developed more efficient finite difference methods using variable grids and attenuation.
3. We investigated new methods of extracting stratigraphic information from images of the subsurface through attributes based on scale analysis of seismic reflection signatures.

Our contribution in each of these areas is presented below.

SCREEN METHODS—THEORETICAL DEVELOPMENTS

We completed a theoretical analysis of phase screen propagators to answer several critical questions: the existence of a singularity in the Green's function for the case of a zero vertical wavenumber, the stability and accuracy of such propagators, and the effects of backscattering for large contrast heterogeneous media. The results of this development are given in the following manuscript by Roque Szeto. In summary, the theory is based on separating the wavefield into foreshattering and backscattering parts. The resulting two-equation boundary value problem is equivalent to the Lippmann-Schwinger integral equation. This approach is robust and appropriate for earth structures with high velocity contrasts. This theory also resolves the apparent singularity problem that has persisted in generalized screen propagator formulations. With this formulation we have studied the effects of the commonly used approximations as a function of the degree of velocity contrast in the media. We find that the local Born approximation does very well for models with velocity contrasts up to about 20%. The screen propagator formulations developed by our collaborators at Los Alamos National Laboratory and the University of California at Santa Cruz are significantly more accurate than the local Born approximation. For velocity variations greater than 60% (such as might exist in sub-salt imaging applications in the Gulf of Mexico), the effects of backscattering are important and need to be included.

Acoustic Wave Propagation in Heterogeneous Media

Roque Kwok-Hung Szeto

Earth Resources Laboratory, Department of Earth, Atmospheric, and Planetary Sciences,
Massachusetts Institute of Technology, 42 Carleton Street, Cambridge, MA 02142-1324

The scalar acoustic wave equation has been widely used to study wave phenomena. For many optical and some ocean acoustical applications in which the backscattering effects can be ignored, we are justified to use the well-known paraxial/parabolic approximation. In many instances the small refractive-index-variation Born approximation or the smooth refractive-index-variation Rytov approximation have also often been used. For applications in which the propagation wavelength is large compared with the size of the heterogeneities, and if the refractive index variation function is neither small nor smooth, these approximations are of considerable concern. Alternate efficient, accurate, and robust methods are necessary and important to the better understanding of wave propagation phenomena. The work undertaken here is to develop a set of equations which do not make use of the above approximations. The development is based on splitting the wavefield into a fore scattering part and a back scattering part. It is shown that the developed system of differential equations and boundary conditions are equivalent to the commonly-known Lippmann-Schwinger integral equation for wave propagation of an impinging wavefield into a heterogeneous slab. The theory is applied to resolve the zero vertical wavenumber Green's function singularity paradox for parabolic one-way equation solutions. Numerical examples are given for layered medium to derive an understanding of the quantitative aspects of backscattering effects and the small-perturbation Born approximation solution.

Keywords: wave propagation, Lippmann-Schwinger equation, Born series, multiple scattering

1. Introduction

The effects of scattering for acoustic wave propagation in general heterogeneous medium is studied in this work. The equations arise in many fields across many scientific disciplines. Even though the governing equation is linear and that an integral equation was written down as early as the work by Breit [1] in the study of scattering of slow neutron by protons bound in a molecule, and which equation was generalized using variational principles in a systematic treatment to the quantum theory of scattering by Lippmann and Schwinger [2], no known closed form solution has been found for general arbitrary scattering media. Ultimately, numerical solution is at this time the only recourse. The difficulty and challenge lies in the immense requirement of computational resources for three-dimensional problems.

Because of the computational requirements for three dimensional problem applications, there have been many different approximations used. In optical propagation through atmospheric turbulence, paraxial approximation has been used in the Helmholtz scalar reduced wave equation associated with the time-dependent acoustic wave equation [3]. Implicit in the assumption is that there is a preferred propagation direction and that the scattering angle is small. These approximations have also been found in the study of numerical optical propagation through fiber optics [4], where a propagation is divided into two steps — a Fresnel diffraction integral propagation in vacuum and a phase only correction due to the scattering medium. The use of paraxial differential equation can also be found in underwater acoustics propagation applications [5]. Pseudo-differential operator theory was later used to incorporate higher order terms to allow more effective treatment for wider angle of scattering. These developments led to the many versions of the wide angle parabolic differential equations [6-8]. The wide-angle parabolic approximation theory has subsequently also been found in geophysics application, mainly used in migration of seismic data where diffraction effects are important and that the time of first arrival (phase of the wavefield) is of primary interest [9-12].

Other attempts in the solution is the use of the WKB method [13,16], the assumption being that the wavelength associated with the temporal frequency and the velocity of the background medium

is considerably smaller than the size of the scatterers in the medium.

Further common approximations are that either the refractive index variation associated with the heterogeneities is small resulting in the well-known Born approximation or the refractive index variations is smooth resulting in the Rytov approximation [14-15]. Attempts using the perturbations series to correct for multiple backscattering can also be found in [16-17,19]

Necessitated by the existence of many relevant applications such as in seismology in which both the backscattering effects are large and the refractive index variations are neither small nor smooth, we have undertaken an effort to derive equations to study the validity of the small backscattering assumption and the Born/Rytov approximations. Alternate robust, efficient, and accurate solution method for general heterogeneous media is necessary to study these matters to shed light to a better understanding of wave scattering phenomena.

The analysis is for a single temporal frequency. That is, we use the Helmholtz reduced wave equation. (For many applications of interest, there are obvious advantages in using the Helmholtz reduced wave equation as compared with the time-dependent acoustic wave equation.) Our approach is to start with the Breit-Lippmann-Schwinger integral equation for general acoustic medium. The wavefield is separated into two parts: the fore scattering part and the back scattering part. Two first order differential equations and boundary conditions are obtained for wave propagation of an impinging wavefield into a heterogeneous slab. When the vertical wavenumber goes to zero, the Breit-Lippmann-Schwinger integral equation has an apparent singularity [1]. By introducing a scaled sum-difference set of variables this singularity paradox can be explained. The equation for zero and small vertical wavenumber is shown to have a very simple solution. Numerical examples are given to illustrate the theory and to identify the backscattering effects and the validity of the small refractive index variation Born approximation.

The presentation is organized as follows. The mathematical problem is formulated and its equivalent Breit-Lippmann-Schwinger integral equation is given in the next section. In Section 3, the notion of multiple effective reverse direction scattering is introduced, and a new infinite series

solution is constructed, the terms of which can be associated with either fore scattering or back scattering. The results obtained in Section 3 are used to construct the two-equation boundary value problem for the fore scattering wavefield and the back scattering wavefield, and for the scaled sum-difference dependent variables. The latter set of equations is valid for all values of vertical wavenumber and is therefore more suitable for numerical implementation. This is the main result of the work and is given in Section 4. Numerical examples are shown in Section 5 to illustrate the theory. Appendix A contains a detailed derivation of the sequence of initial value problems for the terms of the new infinite series based on the theory of multiple effective reverse direction scattering in Section 3.

2. The Mathematical Problem and the Equivalent Integral Equation

We seek solution $u(\mathbf{r}, z, t)$ for a forward impinging wavefield at $z = 0$ into a heterogeneous slab, $0 < z < L$, satisfying the three dimensional acoustic wave equation,

$$\left(\partial_z^2 + \nabla_{\perp}^2 - \frac{1}{c^2(\mathbf{r}, z)} \partial_t^2 \right) u(\mathbf{r}, z, t) = 0. \quad (1)$$

The notations t denotes time, z denotes the position along the propagation path and $\mathbf{r}$ denotes a two-component position vector, with components (x, y) , defining position in a plane perpendicular to the propagation path, $\nabla_{\perp}^2$ denotes the two-dimensional Laplacian, $\partial^2/\partial x^2 + \partial^2/\partial y^2$, ∂_z and ∂_t denotes the partial derivatives, $\partial/\partial z$ and $\partial/\partial t$ respectively.

We next introduce the Fourier transform of $u(\mathbf{r}, z, t)$ in time,

$$\tilde{u}(\mathbf{r}, z; \omega) = \int dt \exp(-i\omega t) u(\mathbf{r}, z, t), \quad (2)$$

and if we use the notation k to denote the wavenumber corresponding to a constant background velocity, c_0 , and $\delta n(\mathbf{r}, z)$ to denote the refractive index variation function,

$$k = \frac{\omega}{c_0}, \quad (3)$$

$$\delta n(\mathbf{r}, z) = \frac{c_0^2}{c^2(\mathbf{r}, z)} - 1, \quad (4)$$

then upon taking the Fourier transform of Eq. (1) in time, we obtain the Helmholtz reduced wave equation,

$$(\partial_z^2 + \nabla_\perp^2 + k^2 [1 + \delta n(\mathbf{r}, z)]) \tilde{u}(\mathbf{r}, z; \omega) = 0. \quad (5)$$

For the discussion in the next three sections we shall use the following refractive index variation model that $\delta n(\mathbf{r}, z)$ is a piecewise continuous with at most jump discontinuities but otherwise arbitrary function for values of $z \in (0, L)$, and that it vanishes outside the slab, $z \leq 0$ and $z \geq L$. To avoid any back scattering at the slab entrance, $z = 0$, and at the slab exit, $z = L$, it is necessary that the derivative of $\delta n(\mathbf{r}, z)$ also vanishes at both places. It is not difficult that the refractive index variation function for most applications can be put into (or approximated by) this form. A forward impinging wavefield at $z = 0$ is a boundary condition particularly relevant to geophysical migration/inversion applications. We next introduce the $\mathbf{r}$ -Fourier-transform,

$$\phi(z; \mathbf{k}_\perp, \omega) = \int d\mathbf{r} \exp(-i\mathbf{r} \cdot \mathbf{k}_\perp) \tilde{u}(\mathbf{r}, z; \omega), \quad (6)$$

where $\mathbf{k}_\perp$ is a two-component wavenumber vector with components (k_x, k_y) . We can write the Helmholtz reduced wave equation,

$$\partial_z^2 \phi(z; \mathbf{k}_\perp, \omega) + k_z^2 \phi(z; \mathbf{k}_\perp, \omega) + k^2 \widetilde{\delta n}(z; \mathbf{k}_\perp) \star \phi(z; \mathbf{k}_\perp, \omega) = 0, \quad (7)$$

where k_z is the vertical wavenumber and it satisfies the equation,

$$k_z = (k^2 - k_\perp^2)^{1/2}, \quad (8)$$

and $\star$ indicates convolution operation — the Fourier transform of the product of two functions is equal to the convolution of their respective Fourier transform,

$$\int d\mathbf{r} \exp(-i\mathbf{k}_\perp \cdot \mathbf{r}) \delta n(\mathbf{r}, z) \tilde{u}(\mathbf{r}, z; \omega) = \frac{1}{2\pi} \int d\mathbf{k}_\perp' \widetilde{\delta n}(z; \mathbf{k}_\perp - \mathbf{k}_\perp') \phi(z; \mathbf{k}_\perp', \omega). \quad (9)$$

(There are different types and order of expansion of the square root in approximating the vertical wavenumber, $k_z = \sqrt{k^2 - k_\perp^2}$. For example, the first term in the Taylor series approximation for

very large wavenumber gives rise to the paraxial approximation (Fresnel diffraction integral in optical propagation); the different order terms of the Pade series approximation give the parabolic wide angle equations for migration in geophysics, wave propagation in ocean acoustics; and absorbing/outgoing boundary conditions used in finite difference methods for the numerical approximations of solutions of the wave equations in infinite domains [21-23]. The approach we take in this work does not need to use any of the above approximations because we perform our numerical integration in the transverse wavenumber space.)

A complete theory has been developed for all real values of the vertical wavenumbers k_z^2 's. We shall however first concentrate on $k_z^2 > 0$.

To proceed, we seek solution to the Helmholtz reduced wave equation satisfying the following boundary conditions. The first condition is that if we denote the part of fore scattering wavefield by $\phi^f(z; \mathbf{k}_\perp, \omega)$, then the prescribed impinging wavefield at the slab entrance can be written as the (incomplete) boundary condition at $z = 0$,

$$\phi^f(0; \mathbf{k}_\perp, \omega) = \phi_0(\mathbf{k}_\perp, \omega), \quad (10)$$

where $\phi_0(\mathbf{k}_\perp, \omega)$ is the Fourier transform of the impinging wavefield at the slab entrance. The boundary condition is incomplete because the back scattering part of the wavefield is not given — it is to be determined as a part of the solution.

The second set of boundary condition is the Sommerfeld radiation boundary condition as $|z| \rightarrow \infty$,

$$\partial_z \phi(z_a; \mathbf{k}_\perp, \omega) + ik_z \phi(z_a; \mathbf{k}_\perp, \omega) = 0, \quad z_a \rightarrow (-\infty), \quad (11)$$

$$\partial_z \phi(z_b; \mathbf{k}_\perp, \omega) - ik_z \phi(z_b; \mathbf{k}_\perp, \omega) = 0, \quad z_b \rightarrow \infty. \quad (12)$$

The solution of the Helmholtz reduced wave equation Eq. (7) satisfying the boundary conditions Eq.'s (10)-(12) is a multi-point — the prescribed fore scattering wavefield at the slab entrance, $z = 0$, and the radiation conditions as z tends to $\pm\infty$ — boundary value problem for a second order ordinary

differential equation. Treating the heterogeneous term in the Helmholtz reduced wave equation as a forcing term, the corresponding integral equation can be obtained as follows. The solution of the homogeneous differential equation satisfying the inhomogeneous boundary conditions,

$$\partial_z^2 \phi_1(z; \mathbf{k}_\perp, \omega) + k_z^2 \phi_1(z; \mathbf{k}_\perp, \omega) = 0, \quad 0 \leq z < \infty, \quad (13)$$

$$\phi_1(0; \mathbf{k}_\perp, \omega) = \phi_0(\mathbf{k}_\perp, \omega), \quad \partial_z \phi_1(L; \mathbf{k}_\perp, \omega) - ik_z \phi_1(L; \mathbf{k}_\perp, \omega) = 0. \quad (14)$$

This problem has solution,

$$\phi_1(z; \mathbf{k}_\perp, \omega) = \phi_0(\mathbf{k}_\perp, \omega) \exp(ik_z z). \quad (15)$$

The Green's function solution for the background medium satisfying the Sommerfeld radiation boundary condition as $|z| \rightarrow \infty$,

$$\partial_z^2 G(z, z'; \mathbf{k}_\perp, \omega) + k_z^2 G(z, z'; \mathbf{k}_\perp, \omega) = \delta(z - z'), \quad (16)$$

$$\partial_z G(z_a, z'; \mathbf{k}_\perp, \omega) + ik_z G(z_a, z'; \mathbf{k}_\perp, \omega) = 0, \quad z_a \rightarrow (-\infty), \quad (17)$$

$$\partial_z G(z_b, z'; \mathbf{k}_\perp, \omega) - ik_z G(z_b, z'; \mathbf{k}_\perp, \omega) = 0, \quad z_b \rightarrow \infty. \quad (18)$$

This has the solution [27],

$$G(z, z'; \mathbf{k}_\perp, \omega) = \frac{1}{2ik_z} \exp(ik_z |z - z'|). \quad (19)$$

We can now combine our solutions to write the integral equation,

$$\phi(z; \mathbf{k}_\perp, \omega) = \phi_0(\mathbf{k}_\perp, \omega) \exp(ik_z z) - \frac{k^2}{2ik_z} \int_0^L dz' \exp(ik_z |z - z'|) \widetilde{\delta n}(z'; \mathbf{k}_\perp) \star \phi(z'; \mathbf{k}_\perp, \omega). \quad (20)$$

Eq. (20) is commonly known as the Lippmann-Schwinger integral equation [2] — though strictly speaking it should have been attributed to Breit [1] who obtained the integral equation in the study of neutron scattering.

Splitting the integrals into two parts, we have

$$\begin{aligned} \phi(z; \mathbf{k}_\perp, \omega) = & \phi_0(\mathbf{k}_\perp, \omega) \exp(ik_z z) - \frac{k^2}{2ik_z} \exp(+ik_z z) \int_0^z dz' \exp(-ik_z z') \widetilde{\delta n}(z'; \mathbf{k}_\perp) \star \phi(z'; \mathbf{k}_\perp, \omega) \\ & - \frac{k^2}{2ik_z} \exp(-ik_z z) \int_z^L dz' \exp(ik_z z') \widetilde{\delta n}(z'; \mathbf{k}_\perp) \star \phi(z'; \mathbf{k}_\perp, \omega). \end{aligned} \quad (21)$$

The first term on the right hand side is the incident wavefield due to the prescribed impinging data at the slab entrance. The second term is the fore scattering wavefield due to the heterogeneities of the medium. (The commonly used Born approximation solution equation is derived when the unknown wavefield inside the integrand is substituted by the incident wavefield.) The third term is the back scattering wavefield due to heterogeneities of the medium — this is the term which is ignored in all paraxial/parabolic approaches. Part of the objective of the present work is to derive equations to study the effects on solutions with or without the back scattering term.

The Breit-Lippmann-Schwinger integral equation solution can be extended to the whole space — outside the slab homogeneous medium (vacuum) propagation applies. In the next section, we shall use the Breit-Lippmann-Schwinger integral equation to derive a sequence of initial value problems, which solutions approximate the multiple fore scattering waves and the multiple back scattering waves.

3. The Theory of Multiple Effective Reverse Direction Scattering

We make the assumption that the solution of the integral equation for general heterogeneous medium can be written as an infinite Born series,

$$\phi(z; \mathbf{k}_\perp, \omega) = \sum_{j \geq 0} \phi^{(j)}(z; \mathbf{k}_\perp, \omega), \quad (22)$$

where the first term of the series, $\phi^{(0)}(z; \mathbf{k}_\perp, \omega)$, is the incident wavefield term,

$$\phi^{(0)}(z; \mathbf{k}_\perp, \omega) = \phi_0(\mathbf{k}_\perp, \omega) \exp(ik_z z). \quad (23)$$

Higher order terms of the series are given by the equation,

$$\phi^{(j)}(z; \mathbf{k}_\perp, \omega) = -k^2 \int_0^L dz' G(z - z') \widetilde{\delta n}(z'; \mathbf{k}_\perp) * \phi^{(j-1)}(z'; \mathbf{k}_\perp, \omega), \quad j \geq 1. \quad (24)$$

We note that the small perturbation Born series in Eq. (22) for many applications of interest does not converge. To address this, we offer the following heuristic explanation. The intermediate equations

we are to develop in this section are valid approximation only if the infinite series (to be defined below) associated with a given refractive index variation function is a convergent series, otherwise their solutions should be treated as terms of an asymptotic series. However, the system of two equations we obtain in the next section which is a re-summation of the Born series can be regarded as analytic continuation in some norm of the refractive index variation and is therefore valid for all refractive index variation functions. To justify this, the system of two equations will be independently verified to satisfy the scalar Helmholtz reduced wave equation.

The integral on the right hand side of Eq. (24) can be split into two parts. We write $\phi^{(1)}(z; \mathbf{k}_\perp, \omega)$ as

$$\begin{aligned}\phi^{(1)}(z; \mathbf{k}_\perp, \omega) &= \int_0^L dz' G(z - z') \widetilde{\delta n}(z'; \mathbf{k}_\perp) \star \phi^{(0)}(z'; \mathbf{k}_\perp, \omega) \\ &= \left(\int_0^z + \int_z^L \right) dz' G(z - z') \widetilde{\delta n}(z'; \mathbf{k}_\perp) \star \phi^{(0)}(z'; \mathbf{k}_\perp, \omega).\end{aligned}\quad (25)$$

The first integral on the right hand side is the contribution at z due to the effect of fore scattering from all values of z' less than z . We therefore associate the first integral as a fore scattering integral contribution. Accordingly, we use the notation, Fv , to denote the fore scattering integral contribution,

$$Fv = -k^2 \int_0^z dz' G(z - z') \widetilde{\delta n}(z'; \mathbf{k}_\perp) \star v(z'; \mathbf{k}_\perp, \omega). \quad (26)$$

Analogously, we associate the second integral in Eq. (25) as a back scattering integral contribution and we use the notation, Bv , to denote the back scattering integral contribution,

$$Bv = -k^2 \int_z^L dz' G(z - z') \widetilde{\delta n}(z'; \mathbf{k}_\perp) \star v(z'; \mathbf{k}_\perp, \omega). \quad (27)$$

We can now rewrite the equation for $\phi^{(1)}(z; \mathbf{k}_\perp, \omega)$ as

$$\phi^{(1)}(z; \mathbf{k}_\perp, \omega) = F\phi^{(0)} + B\phi^{(0)}. \quad (28)$$

For the second term, we make use of the definitions of both $\phi^{(2)}(z; \mathbf{k}_\perp, \omega)$ and $\phi^{(1)}(z; \mathbf{k}_\perp, \omega)$ to write

$$\phi^{(2)}(z; \mathbf{k}_\perp, \omega) = -k^2 \int_0^L dz_2 G(z - z_2) \widetilde{\delta n}(z_2; \mathbf{k}_\perp) \star \phi^{(1)}(z_2; \mathbf{k}_\perp, \omega)$$

$$\begin{aligned}
&= (-1)^2 (k^2)^2 \int_0^L dz_2 G(z - z_2) \widetilde{\delta n}(z_2; \mathbf{k}_\perp) \\
&\quad * \int_0^L dz_1 G(z_2 - z_1) \widetilde{\delta n}(z_1; \mathbf{k}_\perp) * \phi^{(0)}(z_1; \mathbf{k}_\perp, \omega) \\
&= (-1)^2 (k^2)^2 \left(\int_0^z + \int_z^L \right) dz_2 G(z - z_2) \widetilde{\delta n}(z_2; \mathbf{k}_\perp) \\
&\quad * \left(\int_0^{z_2} + \int_{z_2}^L \right) dz_1 G(z_2 - z_1) \widetilde{\delta n}(z_1; \mathbf{k}_\perp) * \phi^{(0)}(z_1; \mathbf{k}_\perp, \omega). \quad (29)
\end{aligned}$$

We can split up the double integral on the right hand side into four parts,

$$\phi^{(2)}(z; \mathbf{k}_\perp, \omega) = FF\phi^{(0)} + BF\phi^{(0)} + BB\phi^{(0)} + FB\phi^{(0)}. \quad (30)$$

It is now evident that $\phi^{(3)}(z; \mathbf{k}_\perp, \omega)$ contains 8 terms, and in general $\phi^{(j)}(z; \mathbf{k}_\perp, \omega)$ contains 2^j terms of the form $F^{k_1} B^{k_2} \dots F^{k_{j-1}} B^{k_j} F^{k_{j+1}} \phi^{(0)}$, $k_i \geq 0$, and $k_1 + k_2 + \dots + k_j + k_{j+1} = j$.

We next introduce the notion of multiple effective reverse direction scattering. A wave is defined to have undergone an effective reverse direction scattering if two adjacent scattering operators are not the same. Recalling that the $\phi^{(0)}(z; \mathbf{k}_\perp, \omega)$ is a fore scattering wavefield, we accordingly interpret it as a "F". Thus, we conclude that $F\phi^{(0)}$ has undergone zero multiple effective reverse direction scattering because the two operators in the sequence $F"F$ are the same. On the other hand, $B\phi^{(0)}$ has undergone a single multiple reverse direction scattering, because we interpret it as a $B"F$ sequence.

The number of multiple effective reverse direction scattering for the four terms of $\phi_2(z; \mathbf{k}_\perp, \omega)$ in Eq. (30) can now easily be deduced: $FF\phi^{(0)}$ has none because the sequence $FF"F$ has the same operators throughout; $BF\phi^{(0)}$ ($BF"F$) has undergone a single multiple effective reverse direction scattering because of the subsequence BF ; $BB\phi^{(0)}$ ($BB"F$) has also undergone a single multiple effective reverse direction scattering because of the subsequence $B"F$; and $FB\phi^{(0)}$ ($FB"F$) has undergone a double multiple effective reverse direction scattering because of the two subsequences, FB and $B"F$. In general, the sequence operating on $\phi^{(0)}(z; \mathbf{k}_\perp, \omega)$, $B^{k_1} F^{k_2} \dots B^{k_q} F^{k_{q+1}} \phi^{(0)}$, $k_i > 0, i \leq q$, and $k_{q+1} \geq 0$ has undergone q multiple effective reverse direction scatterings.

We can now construct a new series based on multiple effective reverse direction scattering. We write

$$\phi(z; \mathbf{k}_\perp, \omega) = \sum_{\alpha \geq 0} \sigma^{(\alpha)}(z; \mathbf{k}_\perp, \omega). \quad (31)$$

Eq. (31) is called the Merds series for the wavefield $\phi(z; \mathbf{k}_\perp, \omega)$. The terms $\sigma^{(\alpha)}(z; \mathbf{k}_\perp, \omega)$, $\alpha \geq 0$, are themselves infinite series characterized by α , the number of multiple effective reverse direction scattering. That is, the series in Eq. (31) is a double infinite series.

Detailed derivation of the governing equations for $\sigma^{(\alpha)}(z; \mathbf{k}_\perp, \omega)$ can be found in Appendix A. We summarize the equations here,

$$\partial_z \sigma^{(0)}(z; \mathbf{k}_\perp, \omega) = ik_z \sigma^{(0)}(z; \mathbf{k}_\perp, \omega) + \frac{ik_z^2}{k_z} \widetilde{\delta n}(z; \mathbf{k}_\perp) * \sigma^{(0)}(z; \mathbf{k}_\perp, \omega), \quad (32)$$

$$\sigma^{(0)}(0; \mathbf{k}_\perp, \omega) = \phi_0(\mathbf{k}_\perp, \omega), \quad (33)$$

$$\begin{aligned} \partial_z \sigma^{(2q)}(z; \mathbf{k}_\perp, \omega) &= ik_z \sigma^{(2q)}(z; \mathbf{k}_\perp, \omega) + \frac{ik_z^2}{2k_z} \widetilde{\delta n}(z; \mathbf{k}_\perp) * [\sigma^{(2q-1)}(z; \mathbf{k}_\perp, \omega) + \sigma^{(2q)}(z; \mathbf{k}_\perp, \omega)], \\ q &> 0, \end{aligned} \quad (34)$$

$$\sigma^{(2q)}(0; \mathbf{k}_\perp, \omega) = 0, \quad (35)$$

$$\begin{aligned} \partial_z \sigma^{(2q+1)}(z; \mathbf{k}_\perp, \omega) &= -ik_z \sigma^{(2q+1)}(z; \mathbf{k}_\perp, \omega) - \frac{ik_z^2}{2k_z} \widetilde{\delta n}(z; \mathbf{k}_\perp) * [\sigma^{(2q+1)}(z; \mathbf{k}_\perp, \omega) \sigma^{(2q)}(z; \mathbf{k}_\perp, \omega)], \\ q &\geq 0, \end{aligned} \quad (36)$$

$$\sigma^{(2q+1)}(L; \mathbf{k}_\perp, \omega) = 0. \quad (37)$$

We observe that the terms of the series are solutions of initial value problems for ordinary differential equations (in the propagation direction variable z). That is, the original Helmholtz reduced wave equation boundary value problem for an impinging wavefield into a heterogeneous

slab has now been reduced to a sequence of initial-value problems, and these can be efficiently and accurately calculated successively.

The significance of the terms $\sigma^{(\alpha)}(z; \mathbf{k}_\perp, \omega)$ is as follows. All the even order terms have undergone multiple effective even number of reverse direction scattering, resulting the wavefield contribution in a forward direction. They are therefore considered as fore scattering terms. Correspondingly, all the odd order terms have undergone multiple effective odd number of reverse direction of scattering, resulting the wavefield contribution in a backward direction. These are treated as back scattering terms.

We can now make the following general comments in comparing our series solution with the Born approximation solution and the paraxial/parabolic-approximation solution. The zero-th order term, $\sigma^{(0)}(z; \mathbf{k}_\perp, \omega)$, should give better approximation than the commonly used Born approximation and the paraxial-parabolic-approximation calculations because it contains all orders of the fore scattering terms of the Born series solution and that no square-root approximation of the vertical wavenumber is necessary in our derived equations. Further, the first order term, $\sigma^{(1)}(z; \mathbf{k}_\perp, \omega)$, should provide the first correction due to the back scattering effects, which the paraxial-parabolic approximation solution does not treat.

In the next section, we shall utilize the results we have obtained to develop a two-equation two-point boundary value problem (for ordinary differential equations), and which formulation is valid for all vertical wavenumbers.

4. Reduction to a Two-Equation-Problem

We introduce $\phi^f(z; \mathbf{k}_\perp, \omega)$ to denote the part of wavefield which has undergone an even number of multiple effective reverse direction scattering (which results the wavefield in the fore scattering direction), and $\phi^b(z; \mathbf{k}_\perp, \omega)$ which has undergone an odd number of multiple effective reverse direction

scattering (which results the wavefield in the back scattering direction),

$$\phi^f(z; \mathbf{k}_\perp, \omega) = \sum_{q \geq 0} \sigma^{(2q)}(z; \mathbf{k}_\perp, \omega), \quad (38)$$

$$\phi^b(z; \mathbf{k}_\perp, \omega) = \sum_{q \geq 0} \sigma^{(2q+1)}(z; \mathbf{k}_\perp, \omega). \quad (39)$$

From the definition of the multiple effective reverse direction scattering series in Eq. (31), we see that

$$\phi(z; \mathbf{k}_\perp, \omega) = \phi^f(z; \mathbf{k}_\perp, \omega) + \phi^b(z; \mathbf{k}_\perp, \omega). \quad (40)$$

It is now a simple matter to obtain the two equation boundary value problem. For the governing equation for $\phi^f(z; \mathbf{k}_\perp, \omega)$, we use the differential equations for $\sigma^{(0)}(z; \mathbf{k}_\perp, \omega)$ in Eq. (32) and $\sigma^{(2q)}(z; \mathbf{k}_\perp, \omega)$, $q > 0$ in Eq. (34). For the governing equation for $\phi^b(z; \mathbf{k}_\perp, \omega)$, we use the differential equations for $\sigma^{(2q-1)}(z; \mathbf{k}_\perp, \omega)$, $q > 0$ in Eq. (36),

$$\partial_z \phi^f(z; \mathbf{k}_\perp, \omega) - ik_z \phi^f(z; \mathbf{k}_\perp, \omega) - \frac{ik^2}{2k_z} \widetilde{\delta n}(z; \mathbf{k}_\perp) \star \phi^f(z; \mathbf{k}_\perp, \omega) - \frac{ik^2}{2k_z} \widetilde{\delta n}(z; \mathbf{k}_\perp) \star \phi^b(z; \mathbf{k}_\perp, \omega) = 0, \quad (41)$$

$$\partial_z \phi^b(z; \mathbf{k}_\perp, \omega) + ik_z \phi^b(z; \mathbf{k}_\perp, \omega) + \frac{ik^2}{2k_z} \widetilde{\delta n}(z; \mathbf{k}_\perp) \star \phi^b(z; \mathbf{k}_\perp, \omega) + \frac{ik^2}{2k_z} \widetilde{\delta n}(z; \mathbf{k}_\perp) \star \phi^f(z; \mathbf{k}_\perp, \omega) = 0. \quad (42)$$

Using the slab entrance conditions in Eq. (33) and Eq. (35) and the slab exit condition in Eq. (37), we can write the boundary conditions for $\phi^f(z; \mathbf{k}_\perp, \omega)$ and $\phi^b(z; \mathbf{k}_\perp, \omega)$,

$$\phi^f(0; \mathbf{k}_\perp, \omega) = \phi_0(\mathbf{k}_\perp, \omega), \quad (43)$$

$$\phi^b(L; \mathbf{k}_\perp, \omega) = 0. \quad (44)$$

Because the equations are in the fore scattering waves and in the back scattering waves, we call Eq.'s (41)-(44) the two-equation boundary value problem in the fore scattering wave and back scattering wave (f-b) formulation. We can see that if $\phi^b(z; \mathbf{k}_\perp, \omega)$ is set to zero in Eq. (41), we recover the various forms of the paraxial/parabolic wave propagator depending on the square root approximation of the vertical wavenumber in Eq. (8).

The derivation of the two-equation boundary value problem was based on the re-summation of the terms of the Born series into a series consisting of terms of multiple effective reverse direction scattering, which itself is an infinite series. It appeared our theory was based on the premise that these infinite series exist. Therefore, this leads us to the question of the validity of the two-equation boundary value problem. To address this, we introduce two functions which are sum and difference of the fore scattering wavefield, $\phi^f(z; \mathbf{k}_\perp, \omega)$, and the back scattering wavefield, $\phi^b(z; \mathbf{k}_\perp, \omega)$,

$$\psi^+(z; \mathbf{k}_\perp, \omega) = \phi^f(z; \mathbf{k}_\perp, \omega) + \phi^b(z; \mathbf{k}_\perp, \omega), \quad (45)$$

$$\psi^-(z; \mathbf{k}_\perp, \omega) = \phi^f(z; \mathbf{k}_\perp, \omega) - \phi^b(z; \mathbf{k}_\perp, \omega). \quad (46)$$

We next form the sum and the difference of the two differential equations in the two-equation (f-b) formulation boundary value problem. For the sum of Eq.'s (41)-(42), we have

$$\partial_z \psi^+(z; \mathbf{k}_\perp, \omega) - ik_z \psi^-(z; \mathbf{k}_\perp, \omega) = 0. \quad (47)$$

For the difference between Eq. (41) and Eq. (42) we have

$$\partial_z \psi^-(z; \mathbf{k}_\perp, \omega) - ik_z \psi^+(z; \mathbf{k}_\perp, \omega) - \frac{ik^2}{k_z} \widetilde{\delta n}(z; \mathbf{k}_\perp) * \psi^+(z; \mathbf{k}_\perp, \omega) = 0. \quad (48)$$

We take derivative with respect to z of Eq. (47), and we substitute the derivative $\partial_z \psi^-(z; \mathbf{k}_\perp, \omega)$ from the right hand side of Eq. (48) to obtain the differential equation for the sum function, $\psi^+(z; \mathbf{k}_\perp, \omega)$,

$$\partial_z^2 \psi^+(z; \mathbf{k}_\perp, \omega) + k_z^2 \psi^+(z; \mathbf{k}_\perp, \omega) + k^2 \widetilde{\delta n}(z; \mathbf{k}_\perp) * \psi^+(z; \mathbf{k}_\perp, \omega) = 0. \quad (49)$$

This is precisely the Helmholtz reduced wave equation in Eq. (7) obtained in Section 2. Therefore, we have proved that the solution of the two-equation boundary value problem solves the impinging wavefield Helmholtz reduced wave equation and the outgoing boundary condition as $|z| \rightarrow \infty$, or that the two equation boundary value problem is another form of the Breit-Lippmann-Schwinger integral equation.

An observation of practical significance in this analysis can be stated regarding the decomposition of the wavefield into a fore scattering part and a back scattering part. The multiple scattering

effects due to the heterogeneities of the medium on the wavefield when evaluated at the entrance of the slab (back scattering portion of the wavefield) and at the exit of the slab (fore scattering portion of the wavefield) can be quantified and compared with measurement data for relevant applications in the applied sciences, *e.g.* optical wave propagation, underwater ocean acoustics, and exploration geophysics.

The theory developed for real vertical wavenumber goes through without any modification to imaginary vertical wavenumber, where oscillatory waves now become evanescent waves. The Sommerfeld outgoing radiation wave condition at infinity, $\exp(ik_z z)$, for real vertical wavenumber, *i.e.* k_z real, now becomes exponential decaying wave at infinity when we substitute k_z by ik'_z , where k'_z is real and is positive, in the outgoing radiation condition. That is, we now speak of projection condition at infinity that solution decays to zero at infinity.

For non-monochromatic wave propagation which is more relevant in underwater ocean acoustic applications and in geophysical applications, we often encounter the study of the part of the spectrum in which the vertical wavenumber, k_z , defined in Eq. (8) goes to zero. Here, we see that when the vertical wavenumber, k_z , goes to zero, there appears to be a singularity in the Breit-Lippmann-Schwinger integral equation in Eq. (20) as well as the two-equation (f-b) formulation boundary value problem in Eq.'s (41)-(44), giving rise to the difficulty in computing solutions for very small vertical wavenumber. We also note that the difficulties in using the one-way (neglecting backscattering effects) wave equation for very small vertical wavenumber calculation are well documented in the literature [12,14-15,18]. Of course, there is no physical and mathematical justification for the existence of the vertical wavenumber singularity in the solution of the two-way wave equation. This not so well-understood zero vertical wavenumber singularity paradox can be explained as follows.

We use the notation $k_\perp^c$ to denote the set of critical transverse wavenumber vectors at which the vertical wavenumber k_z is equal to zero, *i.e.*

$$|k_\perp^c|^2 = k^2. \quad (50)$$

When the vertical wavenumber k_z is equal to zero, we see that from the Breit-Lippmann-Schwinger integral equation in Eq. (20) $\phi(z; \mathbf{k}_\perp^c, \omega)$ must vanish in such a way that

$$\phi^f(z; \mathbf{k}_\perp^c, \omega) = -\phi^b(z; \mathbf{k}_\perp^c, \omega). \quad (51)$$

That is, as $\mathbf{k}_\perp$ tends to $\mathbf{k}_\perp^c$, $\phi(z; \mathbf{k}_\perp, \omega)$ goes to zero as a linear function of $k_z \left(\equiv [k^2 - k_\perp^2]^{1/2} \right)$.

Therefore, to obtain a formulation suitable for all values of the vertical wavenumber, we go to the scaled sum-difference transformation,

$$\phi^+(z; \mathbf{k}_\perp, \omega) = \frac{1}{k_z} [\phi^f(z; \mathbf{k}_\perp, \omega) + \phi^b(z; \mathbf{k}_\perp, \omega)], \quad (52)$$

$$\phi^-(z; \mathbf{k}_\perp, \omega) = \phi^f(z; \mathbf{k}_\perp, \omega) - \phi^b(z; \mathbf{k}_\perp, \omega). \quad (53)$$

The propagation equations and boundary conditions for the scaled sum-difference variables become

$$\partial_z \phi^+(z; \mathbf{k}_\perp, \omega) = i\phi^-(z; \mathbf{k}_\perp, \omega), \quad (54)$$

$$\partial_z \phi^-(z; \mathbf{k}_\perp, \omega) = ik_z^2 \phi^+(z; \mathbf{k}_\perp, \omega) + ik_z^2 \widetilde{\delta n}(z; \mathbf{k}_\perp) * \phi^+(z; \mathbf{k}_\perp, \omega), \quad (55)$$

$$k_z \phi^+(0; \mathbf{k}_\perp, \omega) + \phi^-(0; \mathbf{k}_\perp, \omega) = 2\phi_0(\mathbf{k}_\perp, \omega), \quad (56)$$

$$k_z \phi^+(1; \mathbf{k}_\perp, \omega) - \phi^-(1; \mathbf{k}_\perp, \omega) = 0. \quad (57)$$

The two-equation boundary value problem in scaled sum-difference (s-d) formulation as given by Eq.'s (54)-(57) are the main results in this work. For numerical implementation, Eq.'s (54)-(57) are more appropriate than the two-equation (f-b) formulation equations.

The condition in Eq. (51) further illustrates the important effects of the back scattering wave-field, $\phi^b(z; \mathbf{k}_\perp, \omega)$, in the solution of wave propagation in heterogeneous medium. It is without this term that all paraxial/parabolic approaches encounter computational difficulties at and near zero vertical wavenumber.

It is now straightforward to obtain the zero vertical wavenumber equations and boundary conditions by simply setting the value of k_z equals to zero in Eq.'s (54)-(57),

$$\partial_z \phi^+(z; \mathbf{k}_\perp^c, \omega) = i\phi^-(z; \mathbf{k}_\perp^c, \omega), \quad (58)$$

$$\partial_z \phi^-(z; \mathbf{k}_\perp^c, \omega) = +ik_z^2 \widetilde{\delta n}(z; \mathbf{k}_\perp^c) * \phi^+(z; \mathbf{k}_\perp^c, \omega), \quad (59)$$

$$\phi^-(0; \mathbf{k}_\perp^c, \omega) = 2\phi_0(\mathbf{k}_\perp^c, \omega), \quad (60)$$

$$\phi^-(1; \mathbf{k}_\perp^c, \omega) = 0. \quad (61)$$

From the scaled sum-difference variables we can use Eq.'s (51) and (53) to obtain the result that for zero vertical wavenumber,

$$\phi^f(z; \mathbf{k}_\perp^c, \omega) = \frac{1}{2}\phi^-(z; \mathbf{k}_\perp^c, \omega). \quad (62)$$

We shall call $\phi^+(z; \mathbf{k}_\perp^c, \omega)$ the limiting function at zero k_z vertical wavenumber. For very small k_z values,

$$\phi^+(z; \mathbf{k}_\perp, \omega) \approx k_z \phi^+(z; \mathbf{k}_\perp^c, \omega). \quad (63)$$

5. Numerical Examples

Our objective in this section is to apply the theory we have developed in the previous sections to evaluate the effects of backscattering and the Born approximation. We compare numerical solutions of the two-equation boundary value problem with the Born approximation solution, and with the various orders of the Merds series solution. To this end, we restrict our examples to layered media only — even though the derivation of the two-equation boundary value problem in the previous sections is for general arbitrary refractive index variation function with at most jump discontinuities.

For layered media, the two-equation boundary value problem reduces to the standard two point boundary value problem for ordinary differential equations — the corresponding theory for numerical solutions is well-established [23]. The results reported here use the second order centered-Euler finite difference method to approximate the differential equations, resulting in a block tridiagonal matrix equations which can be solved efficiently. For the numerical solution to maintain second order accuracy, the layer boundaries must lie on a computational grid point [23]. (Arbitrary stepsize in

z can be used to improve efficiency and higher order solutions can be readily obtained using either Richardson extrapolation or deferred correction [23].)

Numerical solutions using finite difference methods (and to some extent using the discrete wavenumber method [24]) for the time-dependent wave equation have been the *de facto* standard in the validation of approximation methods relating to the Helmholtz reduced wave equations (as discussed in the Introduction in Section 1). No attempt will be made in this work to compare our numerical solutions with solutions obtained using these (*de facto*) computational methods because we have shown that from our developed theory in Section 4 that the solutions of the two-equation boundary value problem is not an approximation but they are exact solutions of the Helmholtz reduced wave equation.

The parameters for the calculations are as follows. We use a background wavelength, denoted by λ , of 100m — this corresponds to the use of the Ricker wavelet function with a centered-frequency, denoted by f_c , of 20 Hz, and the use of a background velocity, denoted by c_0 , of 2000 m/s. We use a slab thickness, L , of 500 m (corresponding to 5λ) in the examples.

The three refractive index variation functions used in the calculations are shown in Fig. 1. The solid line corresponds to weak perturbation with the largest velocity perturbation from background velocity of 2000m/s at about 15%, the dotted line corresponds to mild perturbation with the maximum velocity perturbation at about 25%, and the dashed line corresponds to strong perturbation with the maximum velocity perturbation at about 38%.

For the results presented here, we use the normalized boundary condition at $z = 0$,

$$\phi(0, \mathbf{k}_\perp, \omega) = 1. \quad (64)$$

The incident wavefield, $\phi(0, \mathbf{k}_\perp, \omega) \exp(i\mathbf{k}_z z)$, is subtracted from the numerical solutions in all the figures presented here. For the wavefields shown here, we plot the fore scattering part, the back scattering part, and its sum (fore scattering part + back scattering part).

The figures are arranged into the following groups: (1) effects of refractive index variation function on the wavefields on the two-equation boundary value problem; (2) effects of number of terms of the Merds series solutions and the Born series solutions for the layered models in Fig. 1; and (3) comparison of Born series solutions, Merds series solutions, and the two-equation boundary value problem solution for the weakest perturbation and the strongest perturbation shown in Fig. 1. The computations are done for the part of the wavefield which corresponds to the set of transverse wavenumber vectors, $k_{\perp}$, with magnitude equal to $\frac{1}{4}k \equiv \pi/(2\lambda) = 0.005\pi$. This implies that the vertical wavenumber k_z has the value $[(2\pi f_c)/c_0] \left[1 - \left(\frac{1}{4}\right)^2\right]^{1/2} = \sqrt{15}/200$.

For the first group, the results are shown in Fig.'s 2-4. They are solutions of the two-equation boundary value problem: Fig. 2 plots the real part and the imaginary part of the fore scattering wavefield, Fig. 3 plots the real part and the imaginary part of the back scattering wavefield, and Fig. 4 plots the real part and the imaginary part of the total wavefield. There are three curves in each of the subfigures, the solid lines correspond to 15% velocity perturbation the dotted lines correspond to 25% velocity perturbation and the dashed lines correspond to 38% velocity perturbation. Comparing Fig. 2 with Fig. 3 we see that the amplitude of the fore scattering wavefields is approximately an order of magnitude higher than the amplitude of the back scattering wavefields. We also see the the magnitude of the refractive index variation functions has more pronounced effects on the magnitude of the back scattering wavefields than on the magnitude of the fore scattering wavefields. These results are indicative that the two way wave equation approach discussed in this work in general is more appropriate and robust for acoustic wave propagation in general heterogeneous media.

We next consider the second group of figures for the partial sums of the Merds series solution. The first two terms of the Merds series solution are the fore scattering wavefield and the back scattering wavefield, $\sigma^{(0)}(z; k_{\perp}, \omega)$ and $\sigma^{(1)}(z; k_{\perp}, \omega)$ respectively. These and their sum are shown in Fig.'s 5-7. The corresponding figures for the sum of the first four terms of the Merds series solution are shown in Fig.'s 8-10. These results compare very well with solutions of the two-equation

boundary value problem shown in Fig.'s 2-4 for both the 15% velocity perturbation and the 25% velocity perturbations. The comparison for the 38% velocity perturbation results is acceptable. The corresponding results for the Born series solutions are shown in Fig.'s 11-12 for the first term and the second term of the Born series respectively. Here, we see that the results for both the 25% velocity perturbation and the 38% velocity perturbation diverge. Perhaps, this is the reason that it is very seldom that calculations for practical applications using the Born series solution go beyond the first few terms of the Born series — the Born series has a much smaller radius of convergence when compared with the Merds series. Even though a very few terms are required when the Born series converges for very small velocity perturbation, the examples here appear to indicate that the approximation diverges after the first term for even quite mild (25%) velocity perturbation.

We have gathered the results and plotted in the next six figures to compare the Born series solution (dotted lines), the Merds series solution (dashed-dotted lines for fore scattering wavefield and dashed lines for the sum of fore scattering and back scattering wavefields), and the two-equation boundary value problem (solid lines) for the 15% velocity perturbation and the 38% velocity perturbation refractive index variation functions.

In Fig.'s 13-15 we show comparisons for the 15% velocity perturbation using one term, two terms, and five terms of the Born Series and the Merds series with solutions of the two-equation boundary value problem. (These figures plot not only the fore scattering wavefield of the Merds series but also the total scattering wavefield, therefore strictly speaking we are also comparing the first $(2n)$ terms, (for $n = 1, 2, 5$) of the Merds series. We observe that even though the Born series does not diverge, the results are relatively less accurate than the Merds series. This should not be a surprise as the analysis in the Appendix shows that first few terms of the Born series is the first few terms of $\sigma^{(0)}(z; \mathbf{k}_\perp, \omega)$.

Fig.'s 16-18 show the corresponding comparisons for the 38% velocity perturbation. Not surprisingly, the results of the Born series calculations diverge. However, not withstanding this relatively

moderate velocity perturbation, the results of the Merds series calculations show no indication of divergence. Indeed, it appears that the five term Merds series calculations in Fig. 18 better approximates the two equation boundary value problem calculations than the two term Merds series calculations in Fig. 17.

These results are indicative that the Merds series solution provide better approximations than the commonly used Born series solution at very little additional computational complexities.

6. Summary

To summarize, a theory has been developed for the Helmholtz reduced wave equation solution of a prescribed impinging wavefield into a heterogeneous slab. The theory is based on separating the wavefield into a fore scattering part and a back scattering part. It is shown that the derived two-equation boundary value problem is equivalent to the commonly-known Lippmann-Schwinger integral equation. The derived equations are therefore most appropriate for problems in which backscattering effects are important. The two-equation boundary value problem applies to all piecewise continuous refractive index variation functions with at most jump discontinuities (laterally varying layered media), which accounts for almost all problems of practical interest in the applied sciences. Also, the scaled sum-difference formulation is valid for all values of the vertical wavenumber.

For applications in which the Merds series solution (constructed based on the theory of multiple effective reverse direction scattering) converges, the sequence of initial value problems derived in Section 3 can also (and should) be used because not only the terms of the series can be efficiently and accurately computed using existing theory for numerical initial value problems in ordinary differential equations [25-26] but they also provide accurate calculations of each of the physically meaningful multiple scattering terms, $\sigma^{(\alpha)}(z; \mathbf{k}_\perp, \omega)$. Further, the simple numerical examples reported in this

work appear to indicate that the Merds series solution is more accurate and has a larger radius of convergence than the Born series solution.

The numerical solution of the two-equation boundary value problem for general heterogeneous media poses additional challenges because the standard finite difference method which was used to generate the results for layered media in this work is perhaps not appropriate, the reason being that the transverse wavenumber wavefields, $\phi(z; \mathbf{k}_\perp, \omega)$, are coupled because of the transverse wavenumber convolution operator in the differential equations. The discussion of accurate and efficient solution methods for general heterogeneous media is beyond the scope of this presentation and will appear elsewhere.

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Appendix A. The Initial Value Problems for The Merds Series

In this appendix, detailed derivation of the governing equations for the terms, $\sigma^{(\alpha)}(z; \mathbf{k}_\perp, \omega)$, of the Merds series in Eq. (31) of Section 3 are given.

We begin with the zero-th order term, $\sigma^{(0)}(z; \mathbf{k}_\perp, \omega)$. From the definition of the Merds series in Eq. (31) in Section 3, $\sigma^{(0)}(z; \mathbf{k}_\perp, \omega)$ has not undergone any multiple effective reverse direction scattering. Accordingly, it is made up of terms, $F^k \phi^{(0)}$, $k \geq 0$. (The commonly used Born approximation solution is equal to the first term of the zero-th order term of this new series.) We write the operator equation for $\sigma^{(0)}(z; \mathbf{k}_\perp, \omega)$,

$$\sigma^{(0)}(z; \mathbf{k}_\perp, \omega) = \phi^{(0)}(z; \mathbf{k}_\perp, \omega) + F\phi^{(0)} + F^2\phi^{(0)} + F^3\phi^{(0)} + \dots$$

$$= \phi^{(0)}(z; \mathbf{k}_\perp, \omega) + F \left[\phi^{(0)}(z_1; \mathbf{k}_\perp, \omega) + F\phi^{(0)} + F^2\phi^{(0)} + \dots \right]. \quad (65)$$

The expression inside the square bracket on the second line is precisely $\sigma^{(0)}(z_1; \mathbf{k}_\perp, \omega)$,

$$\sigma^{(0)}(z; \mathbf{k}_\perp, \omega) = \phi^{(0)}(z; \mathbf{k}_\perp, \omega) + F\sigma^{(0)}. \quad (66)$$

Using the definition of the fore scattering operator in Eq. (26) in Section 3, we can express the operator equation as

$$\sigma^{(0)}(z; \mathbf{k}_\perp, \omega) = \phi^{(0)}(z; \mathbf{k}_\perp, \omega) - k^2 \int_0^z dz_1 G(z - z_1) \widetilde{\delta n}(z_1; \mathbf{k}_\perp) * \sigma^{(0)}(z_1; \mathbf{k}_\perp, \omega). \quad (67)$$

We next take derivative with respect to z , and we use Leibnitz's rule for differentiation under the integral sign and the Green's function solution in Eq. (19) in Section 2 to obtain the differential equation for $\sigma^{(0)}(z; \mathbf{k}_\perp, \omega)$,

$$\partial_z \sigma^{(0)}(z; \mathbf{k}_\perp, \omega) = ik_z \sigma^{(0)}(z; \mathbf{k}_\perp, \omega) + \frac{ik^2}{k_z} \widetilde{\delta n}(z; \mathbf{k}_\perp) * \sigma^{(0)}(z; \mathbf{k}_\perp, \omega). \quad (68)$$

This is to be solved subject to the initial condition which can be obtained by evaluating the integral equation, Eq. (67), at the slab entrance, $z = 0$,

$$\sigma^{(0)}(0; \mathbf{k}_\perp, \omega) = \phi_0(\mathbf{k}_\perp, \omega). \quad (69)$$

We next consider the first order term $\sigma^{(1)}(z; \mathbf{k}_\perp, \omega)$. It is also itself an infinite series of the terms in the Born series which have undergone a multiple effective single reverse direction scattering. Evidently, it is made up of terms of the form $B^{k_1} F^{k_2} \phi^{(0)}$, $k_1 > 0, k_2 \geq 0$,

$$\begin{aligned} \sigma^{(1)}(z; \mathbf{k}_\perp, \omega) = & B\phi^{(0)} + (BF + BB)\phi^{(0)} + (BFF + BBF + BBB)\phi^{(0)} \\ & + (BFFF + BBFF + BBBF + BBBB)\phi^{(0)} + \dots \end{aligned} \quad (70)$$

Factoring out the back scattering operator B , and rearranging the terms which can now be seen to be made up of F^{k_1} , and $B^{k_2} F^{k_3}$, $k_1, k_2 > 0, k_3 \geq 0$, the former being terms of $\sigma^{(0)}(z; \mathbf{k}_\perp, \omega)$ and the

latter being terms of $\sigma^{(1)}(z; \mathbf{k}_\perp, \omega)$,

$$\begin{aligned}
\sigma^{(1)}(z; \mathbf{k}_\perp, \omega) &= B \left[\phi^{(0)} + (F + B) \phi^{(0)} + (FF + BF + BB) \phi^{(0)} \right. \\
&\quad \left. + (FFF + BFF + BBF + BBB) \phi^{(0)} + \dots \right] \\
&= B \left[\phi^{(0)} + \{F + FF + FFF + \dots\} \phi^{(0)} \right. \\
&\quad \left. + \{B + (BF + BB) + (BFF + BBF + BBB) + \dots\} \phi^{(0)} \right] \\
&= B \left[\sigma^{(0)} + \sigma^{(1)} \right]. \tag{71}
\end{aligned}$$

We can use the definition of the back scattering operator, B , as given by Eq. (27) in Section 3 to write out the integral equation for $\sigma^{(1)}(z; \mathbf{k}_\perp, \omega)$,

$$\sigma^{(1)}(z; \mathbf{k}_\perp, \omega) = -k^2 \int_z^L dz_1 G(z - z_1) \widetilde{\delta n}(z_1; \mathbf{k}_\perp) \star \left[\sigma^{(0)}(z_1; \mathbf{k}_\perp, \omega) + \sigma^{(1)}(z_1; \mathbf{k}_\perp, \omega) \right]. \tag{72}$$

To obtain the corresponding differential equation, we again take derivative of both sides with respect to z , and we use Leibnitz's rule for differentiation under the integral sign,

$$\partial_z \sigma^{(1)}(z; \mathbf{k}_\perp, \omega) = -ik_z \sigma^{(1)}(z; \mathbf{k}_\perp, \omega) - \frac{ik^2}{k_z} \widetilde{\delta n}(z; \mathbf{k}_\perp) \star \left[\sigma^{(0)}(z; \mathbf{k}_\perp, \omega) + \sigma^{(1)}(z; \mathbf{k}_\perp, \omega) \right]. \tag{73}$$

This is to be solved subject to the boundary condition which can be obtained by evaluating the integral equation, Eq. (72), at the slab exit, $z = L$,

$$\sigma^{(1)}(L; \mathbf{k}_\perp, \omega) = 0. \tag{74}$$

This is an initial value problem, integrating backwards from the slab exit back to the slab entrance. The solution provides the first order correction to the wavefield due to the effects of back scattering for propagation in heterogeneous medium.

In general, higher order terms of the new series can be very simply obtained as follows. For the even order terms we write

$$\sigma^{(2q)}(z; \mathbf{k}_\perp, \omega) = \sum_{j \geq 2q} s_j^{(2q)}(z; \mathbf{k}_\perp, \omega)$$

$$= \sum_{j \geq 2q} \left(\sum_{\substack{k_i \geq 1, i \leq 2q \\ k_{2q+1} \geq 0 \\ k_1 + \dots = j}} F^{k_1} B^{k_2} F^{k_3} \dots B^{k_{2q}} F^{k_{2q+1}} \phi^{(0)} \right). \quad (75)$$

For the first term of this new series, $s_{2q}^{(2q)}(z; k_{\perp}, \omega)$, we can factor out the first fore scattering operator to obtain the simple relation,

$$s_{2q}^{(2q)}(z; k_{\perp}, \omega) = F s_{2q-1}^{(2q-1)}(z; k_{\perp}, \omega). \quad (76)$$

All other higher order terms, $s_j^{(2q)}(z; k_{\perp}, \omega)$, $j \geq 2q+1$, can be written as two partial sums which we can simplify to obtain a simple expression relating the function $s_j^{(2q)}(z; k_{\perp}, \omega)$ to $s_{j-1}^{(2q)}(z; k_{\perp}, \omega)$ and $s_{j-1}^{(2q-1)}(z; k_{\perp}, \omega)$,

$$\begin{aligned} s_j^{(2q)}(z; k_{\perp}, \omega) &= \sum_{\substack{\mu_i \geq 1, i \leq 2q \\ \mu_{2q+1} \geq 0 \\ \mu_1 + \mu_2 + \dots = j}} F^{\mu_1} B^{\mu_2} \dots B^{\mu_{2q}} F^{\mu_{2q+1}} \phi^{(0)} \\ &= F \sum_{\substack{\mu_i \geq 1, 1 < i \leq 2q \\ \mu_{2q+1} \geq 0 \\ \mu_2 + \dots = j-1}} B^{\mu_2} \dots B^{\mu_{2q}} F^{\mu_{2q+1}} \phi^{(0)} + F \sum_{\substack{\mu_1 - 1 > 1 \\ \mu_i \geq 1, 1 < i \leq 2q \\ \mu_{2q+1} \geq 0 \\ \mu_1 - 1 + \dots = j-1}} F^{\mu_1 - 1} B^{\mu_2} \dots B^{\mu_{2q}} F^{\mu_{2q+1}} \phi^{(0)} \\ &= F \sum_{\substack{\mu_i \geq 1, 1 < i \leq 2q \\ \mu_{2q+1} \geq 0 \\ \mu_2 + \dots = j-1}} B^{\mu_2} \dots B^{\mu_{2q}} F^{\mu_{2q+1}} \phi^{(0)} + F \sum_{\substack{\mu'_1 \geq 1 \\ \mu_i \geq 1, 1 < i \leq 2q \\ \mu_{2q+1} \geq 0 \\ \mu'_1 + \dots = j-1}} F^{\mu'_1} B^{\mu_2} \dots B^{\mu_{2q}} F^{\mu_{2q+1}} \phi^{(0)} \\ &= F s_{j-1}^{(2q-1)}(z; k_{\perp}, \omega) + F s_{j-1}^{(2q)}(z; k_{\perp}, \omega). \end{aligned} \quad (77)$$

We can now use the results in Eq.'s (76)-(77) to obtain an operator equation for $\sigma^{(2q)}(z; k_{\perp}, \omega)$, $q > 0$,

$$\begin{aligned} \sigma^{(2q)}(z; k_{\perp}, \omega) &= \sum_{j \geq 2q} s_j^{(2q)}(z; k_{\perp}, \omega) \\ &= s_{2q}^{(2q)}(z; k_{\perp}, \omega) + \sum_{j \geq 2q+1} s_j^{(2q)}(z; k_{\perp}, \omega) \\ &= F \left(s_{2q-1}^{(2q-1)}(z; k_{\perp}, \omega) + \sum_{j \geq 2q+1} [s_{j-1}^{(2q-1)}(z; k_{\perp}, \omega) + s_{j-1}^{(2q)}(z; k_{\perp}, \omega)] \right) \\ &= F \left(s_{2q-1}^{(2q-1)}(z; k_{\perp}, \omega) + \sum_{j \geq 2q} [s_j^{(2q-1)}(z; k_{\perp}, \omega) + s_j^{(2q)}(z; k_{\perp}, \omega)] \right) \\ &= F (\sigma^{(2q-1)} + \sigma^{(2q)}). \end{aligned} \quad (78)$$

Following the same steps as was used in the derivation for $\sigma^{(0)}(z; \mathbf{k}_\perp, \omega)$, we obtain the corresponding integral equation,

$$\sigma^{(2q)}(z; \mathbf{k}_\perp, \omega) = -k^2 \int_0^z dz_1 G(z - z_1) \widetilde{\delta n}(z_1; \mathbf{k}_\perp) * [\sigma^{(2q-1)}(z_1; \mathbf{k}_\perp, \omega) + \sigma^{(2q)}(z_1; \mathbf{k}_\perp, \omega)], \quad (79)$$

and the equivalent initial value problem,

$$\partial_z \sigma^{(2q)}(z; \mathbf{k}_\perp, \omega) = ik_z \sigma^{(2q)}(z; \mathbf{k}_\perp, \omega) + \frac{ik^2}{2k_z} \widetilde{\delta n}(z; \mathbf{k}_\perp) * [\sigma^{(2q-1)}(z; \mathbf{k}_\perp, \omega) + \sigma^{(2q)}(z; \mathbf{k}_\perp, \omega)]. \quad (80)$$

$$\sigma^{(2q)}(0; \mathbf{k}_\perp, \omega) = 0. \quad (81)$$

We can proceed in analogous fashion to obtain the higher odd order terms,

$$\partial_z \sigma^{(2q+1)}(z; \mathbf{k}_\perp, \omega) = -ik_z \sigma^{(2q+1)}(z; \mathbf{k}_\perp, \omega) - \frac{ik^2}{2k_z} \widetilde{\delta n}(z; \mathbf{k}_\perp) * [\sigma^{(2q+1)}(z; \mathbf{k}_\perp, \omega) \sigma^{(2q)}(z; \mathbf{k}_\perp, \omega)], \quad (82)$$

$q \geq 0.$

$$\sigma^{(2q+1)}(L; \mathbf{k}_\perp, \omega) = 0. \quad (83)$$

In summary, we have derived a sequence of initial value problems for the terms of the Merds series: Eq.'s (68)-(69) for the zero-th order term $\sigma^{(0)}(z; \mathbf{k}_\perp, \omega)$; Eq.'s (80)-(81) for higher even order terms, $\sigma^{(2q)}(z; \mathbf{k}_\perp, \omega)$, $q > 0$, and Eq.'s (82)-(83) for higher odd order terms, $\sigma^{(2q-1)}(z; \mathbf{k}_\perp, \omega)$, $q > 0$. We see that to obtain solution for $\sigma^{(q)}(z; \mathbf{k}_\perp, \omega)$, $q > 0$, we need only information of $\sigma^{(q-1)}(z; \mathbf{k}_\perp, \omega)$.

Figure Captions

Figure 1. The layered models (15%, 25%, and 38% velocity perturbation) — There are three models used in the calculations. They all have seven layers. The solid line corresponds to 15% velocity perturbation, the dotted line corresponds to 25% velocity perturbation, and the dashed-dotted line corresponds to 38% velocity perturbation.

Figure 2. The fore scattering wavefield solution of the two equation boundary value problem for the layered models in Fig. 1 — The fore scattering wavefield, $\phi^f(z; k_{\perp}, \omega)$, for the layered models displayed in Fig. 1 are shown: solid line (15% velocity perturbation), dotted line (25% velocity perturbation), and dashed-dotted line (38% velocity perturbation). We see that there is not a very large variation in the amplitude of these wavefields for the three models.

Figure 3. The back scattering wavefield solution of the two equation boundary value problem for the layered models shown in Fig. 1 — The back scattering wavefield, $\phi^b(z; k_{\perp}, \omega)$, for the layered models in Fig. 1 are shown: solid line (15% velocity perturbation), dotted line (25% velocity perturbation), and dashed-dotted line (38% velocity perturbation). The amplitude is smaller than the fore scattering wavefield in Fig. 2. However, the larger the velocity perturbation, the stronger are the effects on the back scattering wavefield.

Figure 4. The total wavefield solution of the two equation boundary value problem for the layered models in Fig. 1 — The total scattering wavefield, $[\phi^f(z; k_{\perp}, \omega) + \phi^b(z; k_{\perp}, \omega)]$, for the layered models in Fig. 1 are shown: solid line (15% velocity perturbation), dotted line (25% velocity perturbation), and dashed-dotted line (38% velocity perturbation). We see that the fore scattering part is the dominant contribution to the total wavefield.

Figure 5. The first term of the Merds series, $\sigma^{(0)}(z; \mathbf{k}_\perp, \omega)$ for the layered models in Fig. 1 — The first term of the Merds series, $\sigma^{(0)}(z; \mathbf{k}_\perp, \omega)$, is a fore scattering contribution. There is reasonable agreement with the two equation boundary value problem results in Fig. 2.

Figure 6. The second term of the Merds series, $\sigma^{(1)}(z; \mathbf{k}_\perp, \omega)$ for the layered models in Fig. 1 — The second term of the Merds series, $\sigma^{(1)}(z; \mathbf{k}_\perp, \omega)$, is a back scattering contribution. For moderate velocity perturbation, the results are quite different in form when compared with the two equation boundary value problem results shown in Fig. 3.

Figure 7. $[\sigma^{(0)}(z; \mathbf{k}_\perp, \omega) + \sigma^{(1)}(z; \mathbf{k}_\perp, \omega)]$ for the layered models in Fig. 1 — The results here should be compared with the total wavefield solution of the two equation boundary value problem shown in Fig. 4. Here, we see that deviation becomes more pronounced at increasing propagation distance for the 38 % velocity perturbation.

Figure 8. $[\sigma^{(0)}(z; \mathbf{k}_\perp, \omega) + \sigma^{(2)}(z; \mathbf{k}_\perp, \omega)]$ for the layered models in Fig. 1 — We plot the sum of the first two terms of $\phi^f(z; \mathbf{k}_\perp, \omega)$ for the layered models. There appears to be an overcorrection for the 38% velocity perturbation when compared with the result shown in Fig. 2.

Figure 9. $[\sigma^{(1)}(z; \mathbf{k}_\perp, \omega) + \sigma^{(3)}(z; \mathbf{k}_\perp, \omega)]$ for the layered models in Fig. 1 — We plot the sum of the first two terms of $\phi^b(z; \mathbf{k}_\perp, \omega)$ for the layered models in Fig. 1. There appears to be an overcorrection for the solution corresponding to the 38% velocity perturbation when compared with the result shown in Fig. 3 for the backscattering part.

Figure 10. $[\sigma^{(0)}(z; \mathbf{k}_\perp, \omega) + \dots + \sigma^{(3)}(z; \mathbf{k}_\perp, \omega)]$ for the layered models in Fig. 1 — We plot the sum of the first four terms of the Merds series for the layered models in Fig. 1. The results should be compared with Fig. 4. The fore scattering part dominates the back scattering part. The results

for both the 15% velocity perturbation and the 25% velocity perturbation appear to give good agreement.

Figure 11. The first term of the Born series for the layered models in Fig. 1 — We plot the first term of the Born series for the layered models in Fig. 1. Here, we have to use a different axis scaling. When compared with the results in Fig. 2, the amplitude for the 38% velocity perturbation is over a factor of two of the results shown in Fig. 2.

Figure 12. The sum of the first two terms of the Born series for the layered models in Fig. 1 — We plot the sum of the first two terms of the Born series for the layered models in Fig. 1. The results for both the 25% velocity perturbation and the 38% velocity perturbation appear to be diverging.

Figure 13. One term comparison for the 15% velocity perturbation — There are four curves in the subfigures: two equation boundary value problem (solid); first term of the Born series (dotted); $\sigma^{(0)}(z; k_{\perp}, \omega)$ of the Merds series (dashed-dotted); and $\sigma^{(0)}(z; k_{\perp}, \omega) + \sigma^{(1)}(z; k_{\perp}, \omega)$ of the Merds series (dashed). The Born series result is slightly "out-of-phase" with the results of both the Merds series result and the two equation boundary value problem. There is very good agreement between the Merds series results and the two equation boundary value problem results.

Figure 14. Two term comparison for the 15% velocity perturbation — There are four curves in the subfigures: two equation boundary value problem (solid); first two terms of the Born series (dotted); the sum of the first two foreshattering terms, $[\sigma^{(0)}(z; k_{\perp}, \omega) + \sigma^{(2)}(z; k_{\perp}, \omega)]$ of the Merds series (dashed-dotted); and the sum of the first two foreshattering terms and the first two backscattering terms $[\sigma^{(0)}(z; k_{\perp}, \omega) + \dots + \sigma^{(3)}(z; k_{\perp}, \omega)]$ of the Merds series (dashed).

Figure 15. Five term comparison for the 15% velocity perturbation — Five term comparison for the 38% velocity perturbation — There are four curves in the subfigures: two equation boundary

value problem (solid); the sum of the first five terms of the Born series (dotted); the sum of the first five forescattering terms of the Merds series, $[\sigma^{(0)}(z; \mathbf{k}_\perp, \omega) + \sigma^{(2)}(z; \mathbf{k}_\perp, \omega) + \dots + \sigma^{(8)}(z; \mathbf{k}_\perp, \omega)]$ (dashed-dotted); and the sum of the first five forescattering terms and the first five backscattering terms of the Merds series, $[\sigma^{(0)}(z; \mathbf{k}_\perp, \omega) + \dots + \sigma^{(9)}(z; \mathbf{k}_\perp, \omega)]$ (dashed). There is noticeable improvement in the Born series solution.

Figure 16. One term comparison for the 38% velocity perturbation — There are four curves in the subfigures: two equation boundary value problem (solid); first term of the Born series (dotted); $\sigma^{(0)}(z; \mathbf{k}_\perp, \omega)$ of the Merds series (dashed-dotted); and $[\sigma^{(0)}(z; \mathbf{k}_\perp, \omega) + \sigma^{(1)}(z; \mathbf{k}_\perp, \omega)]$ of the Merds series (dashed).

Figure 17. Two term comparison for the 38% velocity perturbation — There are four curves in the subfigures: two equation boundary value problem (solid); first two terms of the Born series (dotted); the sum of the first two forescattering terms of the Merds series, $[\sigma^{(0)}(z; \mathbf{k}_\perp, \omega) + \sigma^{(2)}(z; \mathbf{k}_\perp, \omega)]$ (dashed-dotted); and the sum of the first two forescattering terms and the first two backscattering terms of the Merds series, $[\sigma^{(0)}(z; \mathbf{k}_\perp, \omega) + \dots + \sigma^{(3)}(z; \mathbf{k}_\perp, \omega)]$ (dashed). The Born series result appears to have diverged. The Merds series results are acceptable for this level of velocity perturbation.

Figure 18. Five term comparison for the 38% velocity perturbation — There are four curves in the subfigures: two equation boundary value problem (solid); the sum of the first five terms of the Born series (dotted); the sum of the first five forescattering terms of the Merds series, $[\sigma^{(0)}(z; \mathbf{k}_\perp, \omega) + \sigma^{(2)}(z; \mathbf{k}_\perp, \omega) + \dots + \sigma^{(8)}(z; \mathbf{k}_\perp, \omega)]$ (dashed-dotted); and the sum of the first five forescattering terms and the first five backscattering terms of the Merds series, $[\sigma^{(0)}(z; \mathbf{k}_\perp, \omega) + \dots + \sigma^{(9)}(z; \mathbf{k}_\perp, \omega)]$ (dashed). Here, the Merds series appear to give better results than the results shown in the two term comparison in Fig. 17.

Figure 1

Figure 2

Figure 3

Figure 4

Figure 5

Figure 6

Figure 7

Figure 8

Figure 9

Figure 10

Figure 11

Figure 12

Figure 13

Figure 14

Figure 15

Figure 16

Figure 17

Figure 18

VARIABLE GRID FINITE DIFFERENCE METHOD

We developed a two-dimensional viscoelastic finite difference modeling method for highly complex surface topography and subsurface structures. This method allows us to include strong lateral velocity heterogeneity, complex topographic relief, and large attenuation in our models of the subsurface. We use a velocity-stress staggered grid formulation and employ an $O(2,4)$ accurate scheme. The implementation includes an irregular free surface condition for topographic relief and a variable grid technique in the complex parts of the model. Several methods of a free surface condition were benchmarked, and an accurate, simple, and stable condition was developed that performs stress calculations to ensure that the shear and normal stresses at the surface are zero while particle velocities are set to zero above the surface. The variable grid method, using a finer grid in heterogeneous regions, results in a more efficient modeling method for complex structures. Stability of the approach is improved by using a weighting method for updating the coarse grid node values from the fine grid nodes. The method has been tested on a variety of models, and results have been successfully compared to seismic data acquired for a near surface application. Koichi Hayashi's Master's thesis titled "Variable Grid Finite-Difference Modeling Including Surface Topography" that follows explains the method in more detail.

Variable Grid Finite-difference Modeling Including Surface Topography

By

Koichi Hayashi

B.S., Earth Sciences (1990)
Chiba University

Submitted to Department of Earth, Atmospheric, and Planetary Sciences
in the partial fulfillment of the requirements
for the degree of Master of Sciences

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

August 6, 1999

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Signature of Author

Department of Earth, Atmospheric, and Planetary Sciences
August 6, 1999

Certified by

M. Nafi Toksöz
Professor of Geophysics
Thesis advisor

Accepted by

Ronald G. Prinn
Chairman

Department of Earth, Atmospheric, and Planetary Sciences

Variable Grid Finite-difference Modeling Including Surface Topography

By

Koichi Hayashi

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Abstract

We have developed a two-dimensional viscoelastic finite-difference modeling method for highly complex surface topography and subsurface structures. Realistic modeling of seismic wave propagation in the near surface region is complicated by many factors, such as strong heterogeneity, topographic relief and large attenuation. In order to account for these complications, we use a velocity-stress staggered grid and employ an $O(2,4)$ accurate viscoelastic finite-difference scheme. The implementation includes an irregular free surface condition for topographic relief and a variable grid technique in the shallow parts of the model.

Several methods of free surface condition are bench marked, and an accurate and simple condition is proposed. In the proposed free surface condition, stresses are calculated so that the shear and normal stresses perpendicular to the boundary are zero. The calculation of particle velocities does not involve any specific calculations, and the particle velocities are set to zero above the free surface. A stable variable grid method is introduced, where we use a three times finer grid in the near surface or low velocity region compared to the rest of the model. In order to reduce instability, we apply averaging or weighting to the replacement of the coarse grid components within the fine grid. The method allows us to avoid any limitation of the shape of the grid size boundary. Numerical tests indicate that approximately ten grid-points per shortest wavelength with the variable grid method results in accurate calculations. The method requires a stair-shaped discretization of a free surface. We investigated the stair-shaped structures, and found that the cause of the dispersion from irregular free surface is mainly a numerical error due to the large grid sizes rather than the Rayleigh waves scattering due to the stair-shaped boundary.

The finite-difference modeling is applied to the investigation of near surface wave propagation. Several numerical simulations are performed to show the characters of wave propagation in the near surface region. The simulations show that the low velocity thin layers just below the surface and anelastic attenuation have significant effect on surface seismic record. The 2-D modeling of near surface structure beneath a 2-D refraction survey line is carried out. The comparison of the observed data with theoretical waveforms is performed. The characters in the observed data can be explained by a subsurface model constructed by P-wave traveltime tomography.

Thesis Supervisor: M. Nafi Toksöz
Title: Professor of Geophysics

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Chapter 1

Introduction

Surface seismic methods have been widely used for civil engineering investigations and environmental problems. However, there are increasing demand for investigations of more complex geological features. The shallow seismic method will play an important role in such investigations, but requires accurate and reliable data, interpolation and modeling (Steeple, 1998). In order to improve the shallow seismic methods, accurate and efficient numerical modeling methods are needed. The finite-difference method is one such technique that can be used for complex structures, because it can account for factors that complicate seismic wave propagation, such as large velocity contrasts, strong heterogeneity, topographic relief, and attenuation. The modeling of irregular free surface topography and the inclusion of attenuation are two factors that are particularly important in modeling.

The real earth is not perfectly elastic, and propagating waves attenuate with time due to various energy-loss mechanisms. The effect of attenuation on wave propagation is generally very large in near surface, and therefore should be considered in the studies of near surface wave propagation. Anelastic attenuation can be described using a viscoelastic model (Lay and Wallace, 1995). Numerical modeling of linear viscoelastic seismic responses in the time domain has recently become practical through algorithms based on the superposition of relaxation mechanisms. The standard linear solid (SLS) is a simple viscoelastic model consisting of a spring in parallel with a dashpot in a series (Pipkin, 1986). Day and Minster (1984) described a method to connect several SLSs in parallel to yield an excellent approximation to a constant Q in a predefined frequency band. Blanch (1995) showed that several SLSs connected in parallel can be tuned through a single parameter to yield a constant Q approximation. Robertsson et al. (1994) presented a viscoelastic finite-difference method using memory variables to eliminate the convolution terms of viscoelastic equations. These algorithms enable us to calculate attenuation

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efficiently.

Generally, the solid earth's surface is not flat. Since sources and receivers are usually placed on the ground surface, topography may have a significant influence on recorded data. For finite-difference modeling of irregular free surfaces, several computational methods have been presented. These methods can be classified into two main groups. The first is the method in which the finite-difference grid is deformed to match exactly the free surface relief (Hestholm and Ruud, 1994). It is effective for relatively smooth topography, but has limitations for steep topography. The second method employs a rectangular grid and generalizes the free surface condition (Robertsson, 1996). Unlike the first method, the second method has no limitations on the shape of topographic relief. However, its drawback is that it requires very fine gridding, requiring at least 15 grid-points per wavelength (Robertsson, 1996).

An efficient solution to the above dilemma is to use a finer sampling of grid in the vicinity of the irregular free surface compared to the deeper parts of the model. The method is called a multi-grid or variable grid method. Several seismological studies describing the variable grid approach have been reported. McLaughlin and Day (1994) employed a variable grid scheme to seismic wave simulations using a 3-D elastic velocity-displacement finite-difference method. Falk et al. (1995) used a varying grid spacing technique to model tube waves. De Lilla (1997) proposed a variable grid finite-difference method that can handle any integer number for a grid size ratio. Robertsson and Holliger (1997) applied the variable grid method to rough topography. This approach enables us to handle rough topography efficiently.

In this thesis, I will show that the generalized free surface condition, combined with the variable grid approach, enables us to apply a two-dimensional, viscoelastic, finite-difference modeling to steep and complex structures. In Chapter 2, I derive velocity-stress elastic wave equations and extend them to the viscoelastic case. Then, I show its finite-

difference approximation and fundamental benchmark tests to demonstrate the accuracy and reliability of the finite-difference code. In Chapter 3, I present an accurate and stable free surface condition that can handle irregular free surface, perform benchmark tests, and present criteria for accuracy are presented. In chapter 4, I present a stable variable grid method, perform several numerical tests, and combine the method with the irregular free surface condition. In Chapter 5, I apply finite-difference modeling to the investigation of near surface wave propagation, with numerical simulations, and carry out calculations to match the observed waveform data obtained by a 2-D seismic field experiment.

Chapter 2

Viscoelastic Finite-difference Method

Seismologists began using finite-difference methods to solve wave propagation problems some 30 years ago. Most of the early works regarding the application of finite-difference methods to seismic wave propagation were based on the second-order elastic wave equation in which the displacement of media was directly solved. Alterman and Kornfeld (1968), Alterman and Karal (1968) developed the algorithm for homogeneous media. Boore (1972) and Kelly et al. (1976) developed the algorithm for heterogeneous media. Virieux (1986) proposed a alternative method in which the second-order wave equation was reformulated to first-order hyperbolic equations using the velocity-stresses staggered grid scheme for the P-SV problem. Levander (1988) extended the staggered grid scheme to a fourth-order finite-difference method. Carcione (1993), Robertsson et al. (1994), and Xu and McMechan (1998) presented viscoelastic finite-difference method based on the fourth-order staggered grid scheme. Jih et al. (1988), Tessmer et al. (1992), Hestholm and Ruud (1994), Graves (1996), and Robertsson (1996) developed the algorithm for irregular topography.

In this chapter, I derive velocity-stress elastic wave equations in the first section. In the second section, I extend it to the viscoelastic case and show its finite-difference implementation. Finally, I show some fundamental benchmark tests to demonstrate the accuracy and the reliability of a finite-difference code that I developed.

2.1 Velocity-stress Elastic Wave Equations

Although my finite-difference modeling is based on the viscoelastic wave equation, the method has been extended from elastic velocity-stress finite-difference modeling. The

derivation of the velocity-stress elastic wave equations gives the fundamental idea and mathematical foundation of the wave equation. The relationship between forces and deformation in infinitesimal strain theory is largely empirically based and given by a constitutive law called Hook's law. The deformation is a function of material properties of the body such as density, rigidity (resistance to shear), and incompressibility (resistance to change in volume). The material properties are known as elastic moduli. When stress varies with time, strain varies similarly, and the balance between stress and strain results in seismic waves. These waves travel at velocities that depend on the elastic moduli and are governed by equations of motion.

In the following description of the elastic wave equations, I will employ Cartesian coordinate (x_1, x_2, x_3) . At first, I show a general three-dimensional relationships between nine strain components and three displacement components (u_1, u_2, u_3) . Normal strain can be defined by

$$\epsilon_{11} = \frac{\partial u_1}{\partial x_1} \quad \epsilon_{22} = \frac{\partial u_2}{\partial x_2} \quad \epsilon_{33} = \frac{\partial u_3}{\partial x_3}, \quad (2.1.1)$$

here, x_1, x_2, x_3 are the coordinate axes. The first subscript indicates the orientation of the line segment, and second indicates the direction of length change. Shear strains are defined by

$$\begin{aligned} \epsilon_{12} &= \frac{1}{2} \left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) & \epsilon_{21} &= \frac{1}{2} \left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2} \right) \\ \epsilon_{13} &= \frac{1}{2} \left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} \right) & \epsilon_{31} &= \frac{1}{2} \left(\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3} \right) \\ \epsilon_{23} &= \frac{1}{2} \left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} \right) & \epsilon_{32} &= \frac{1}{2} \left(\frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3} \right) \end{aligned} \quad (2.1.2)$$

These nine terms constitute the infinitesimal strain tensor, a symmetric tensor with six independent quantities that can be ordered as

$$\varepsilon_{ij} = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{1}{2} \left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2} \right) & \frac{1}{2} \left(\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3} \right) \\ \frac{1}{2} \left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2} \right) & \frac{\partial u_2}{\partial x_2} & \frac{1}{2} \left(\frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3} \right) \\ \frac{1}{2} \left(\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3} \right) & \frac{1}{2} \left(\frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3} \right) & \frac{\partial u_3}{\partial x_3} \end{pmatrix}. \quad (2.1.3)$$

We can represent all nine strain terms of (2.1.3) with compact indicial notation,

$$\varepsilon_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right). \quad (2.1.4)$$

The trace of the strain tensor is called the cubic dilatation, θ ,

$$\theta = \varepsilon_{ii} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = \nabla \cdot \mathbf{u}. \quad (2.1.5)$$

Second, I define a stress tensor. Here, we subdivide the area of fictitious plane into area elements with surface area, ΔA . A small force, ΔF , acts on each elements. The stress components acting on the plane (x_1 face) that has a normal in the x_1 direction are defined by

$$\begin{aligned} \sigma_{11} &= \lim_{\Delta A_1 \rightarrow 0} \frac{\Delta F_1}{\Delta A_1} \\ \sigma_{12} &= \lim_{\Delta A_1 \rightarrow 0} \frac{\Delta F_2}{\Delta A_1}, \\ \sigma_{13} &= \lim_{\Delta A_1 \rightarrow 0} \frac{\Delta F_3}{\Delta A_1} \end{aligned} \quad (2.1.6)$$

The first index of σ_{ij} in (2.1.6) corresponds to the direction of the normal to the plane being acted on by the force, and the second index indicates the direction of the force. For two other planes, we define six additional stress components,

$$\begin{aligned} \sigma_{21}, \sigma_{22}, \sigma_{23} &\text{ acting on the } x_2 \text{ face,} \\ \sigma_{31}, \sigma_{32}, \sigma_{33} &\text{ acting on the } x_3 \text{ face.} \end{aligned}$$

All of these are implicitly functions of space and time. A three dimensional stress tensor is defined as

$$\sigma_{ij} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}. \quad (2.1.7)$$

The diagonal terms are called normal stresses, and the off-diagonal terms are called shear stresses.

Third, I consider a force balance on a cubic element in a continuum that is under going internal motions. Applying Newton's second law to the medium gives

$$\rho \frac{\partial^2 u_i}{\partial t^2} = f_i + \frac{\partial \sigma_{ij}}{\partial x_j}. \quad (2.1.8)$$

where f_i represents body forces. This set of three equation is called the equation of motion for a continuum. For example, u_1 (x_1 component of displacement) can be written as

$$\rho \frac{\partial^2 u_1}{\partial t^2} = \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3}, \quad (2.1.9)$$

with no body forces.

Fourth, I show the relationship between stress and displacement. There are provided by constitutive laws that relate stress to strain. The most general form of constitutive law for linear elasticity is Hooke's law

$$\sigma_{ij} = C_{ijkl} \cdot \epsilon_{kl}. \quad (2.1.10)$$

The constants of proportionality, C_{ijkl} are known as elastic moduli and define the material properties of the medium. In its general form, C_{ijkl} is a third-order tensor with 81 terms relating the nine elements of the strain tensor to nine elements of the stress tensor by linear sum. In the case of an isotropic elastic substance, the elastic moduli can reduce to two independent moduli called Lamé constants, λ and μ . These are related to C_{ijkl} by

$$C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}). \quad (2.1.11)$$

where the Kronecker delta function is used. Inserting this into (2.1.10) gives,

$$\sigma_{ij} = (\lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})) \epsilon_{kl}, \quad (2.1.12)$$

which reduces to

$$\sigma_{ij} = \lambda \epsilon_{kk} \delta_{ij} + 2\mu \epsilon_{ij}. \quad (2.1.13)$$

Using the equation (2.1.5), (2.1.13) can be written as,

$$\sigma_{ij} = \lambda \theta \delta_{ij} + 2\mu \epsilon_{ij}. \quad (2.1.14)$$

Thus, the stress-strain relationship is written as,

$$\sigma_{ij} = \begin{pmatrix} \lambda\theta + 2\mu\epsilon_{11} & 2\mu\epsilon_{12} & 2\mu\epsilon_{13} \\ 2\mu\epsilon_{21} & \lambda\theta + 2\mu\epsilon_{22} & 2\mu\epsilon_{23} \\ 2\mu\epsilon_{31} & 2\mu\epsilon_{32} & \lambda\theta + 2\mu\epsilon_{33} \end{pmatrix}. \quad (2.1.15)$$

Substituting the equations (2.1.3) and (2.1.5) into (2.1.15), the stress-strain relationship can be expressed as the stress-displacement relationship as follows (only $i=1$ terms are shown),

$$\begin{aligned} \sigma_{11} &= \lambda\theta + 2\mu\epsilon_{11} = \lambda \left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} \right) + 2\mu \frac{\partial u_1}{\partial x_1} \\ \sigma_{12} &= 2\mu\epsilon_{12} = \mu \left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) \\ \sigma_{13} &= 2\mu\epsilon_{13} = \mu \left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} \right) \end{aligned} \quad (2.1.16)$$

Now, I derive velocity-stress relationship in a two-dimensional case. In the following description, I use the x, z coordinate system in which x and z means x_1 axis and x_3 axis respectively. In a two-dimensional elastic case, the equation (2.1.8) reduces to following two equations.

$$\begin{aligned} \rho \frac{\partial^2 u_x}{\partial t^2} &= \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{zx}}{\partial z} \\ \rho \frac{\partial^2 u_z}{\partial t^2} &= \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{zz}}{\partial z} \end{aligned} \quad (2.1.17)$$

And, substituting the equations (2.1.3) and (2.1.5) into (2.1.15) yields the following three equations.

$$\begin{aligned}
\sigma_{xx} &= \lambda\theta + 2\mu\varepsilon_{xx} = \lambda\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z}\right) + 2\mu\frac{\partial u_x}{\partial x} = (\lambda + 2\mu)\frac{\partial u_x}{\partial x} + \lambda\frac{\partial u_z}{\partial z} \\
\sigma_{zz} &= \lambda\theta + 2\mu\varepsilon_{zz} = \lambda\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z}\right) + 2\mu\frac{\partial u_z}{\partial z} = (\lambda + 2\mu)\frac{\partial u_z}{\partial z} + \lambda\frac{\partial u_x}{\partial x} \quad (2.1.18) \\
\sigma_{xz} &= 2\mu\varepsilon_{xz} = \mu\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right)
\end{aligned}$$

The equations (2.1.17) and (2.1.18) can be transformed into the following first order hyperbolic system,

$$\begin{aligned}
\frac{\partial v_x}{\partial t} &= b\left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xz}}{\partial z}\right) \\
\frac{\partial v_z}{\partial t} &= b\left(\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{zz}}{\partial z}\right) \\
\frac{\partial \sigma_{xx}}{\partial t} &= (\lambda + 2\mu)\frac{\partial v_x}{\partial x} + \lambda\frac{\partial v_z}{\partial z} \quad (2.1.19) \\
\frac{\partial \sigma_{zz}}{\partial t} &= (\lambda + 2\mu)\frac{\partial v_z}{\partial z} + \lambda\frac{\partial v_x}{\partial x} \\
\frac{\partial \sigma_{xz}}{\partial t} &= \mu\left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right)
\end{aligned}$$

where, $b = 1/\rho$, ρ is a density, v_x and v_z are particle velocities, σ_{xx} and σ_{zz} are normal stresses, σ_{xz} is a shear stress, and λ and μ are Lamé's constants. This system is a fundamental equations for a velocity-stress staggered grid finite difference method that is widely used in exploration seismology and that my finite-difference method is based on.

2.2 Viscoelastic Formulation Using Standard Linear Solid

In the previous section, I was concerned with an elastic case. The real Earth is, however, not perfectly elastic and propagating waves attenuate with time due to various energy-loss mechanisms. The objective of this study is the application of finite-difference modeling to near surface seismic methods. The effect of attenuation on wave propagation is generally very large in a near surface region. The attenuation should be considered in a study about near surface wave propagation. For this reason, the following discussion of the free surface condition and the variable grid method will be concerned with the viscoelastic case. The anelastic behavior can be described by a viscoelastic model. Blanch et al. (1995) presented efficient viscoelastic modeling based on the Standard Linear Solids (SLS) in which a spring and dashpot in series, in parallel with a spring. In this method, the stress and strain relaxation times can be calculated by the least square method. Robertsson et al. (1994) presented a finite-difference scheme based on the second-order accurate time, fourth-order accurate space, $O(2,4)$, velocity-stress staggered grid (Levander, 1988) for viscoelastic modeling. We employ this method because the additional computer memory requirement to that of elastic case is small compared with elastic case.

2.2.1 Viscoelastic Modeling Using τ

In this section, I describe a theoretical anelastic model based on viscoelastic theory, a phenomenological way to describe combined elastic and viscous behavior of materials. The basic hypothesis is that the current value of the stress tensor depends on the history of the strain tensor. The viscoelastic hypothesis can be described as

$$\sigma_{ij} = G_{ijkl} * \dot{\epsilon}_{kl} = \dot{G}_{ijkl} * \epsilon_{kl}, \quad (2.2.1)$$

(Christensen, 1982) where $*$ denotes time convolution, dot denotes derivative in time. The

convolution transform each strain history, $\varepsilon_{ij}(t)$, into a corresponding stress history, $\sigma_{ij}(t)$. G is a fourth-order tensor-valued function called the relaxation function. The relaxation function G determines the behavior of a material. For one-dimensional or in the special case of a simple shear in an isotropic homogeneous material, the equation (2.2.1) reduces to

$$\sigma = G * \dot{\varepsilon} = \dot{G} * \varepsilon. \quad (2.2.2)$$

The Standard Linear Solid (SLS) has been shown to be a general mechanical viscoelastic model. An array of SLS has the stress relaxation function,

$$G(t) = M_R \left(1 - \sum_{l=1}^L \left(1 - \frac{\tau_{sl}}{\tau_{ol}} \right) e^{-\frac{t}{\tau_{ol}}} \right) \theta(t) \quad (2.2.3)$$

(Blanch et al., 1993) where, $\theta(t)$ is the Heaviside function, M_R is the relaxed stress modulus corresponding to $G(t)$, and τ_{ol} and τ_{sl} are the stress and strain relaxation times for the l th SLS. M_R is related to the elastic modulus M_U (Liu et al., 1976).

$$M_U = M_R / \left(1 - \sum_{l=1}^L \left(\frac{\tau_{sl} - \tau_{ol}}{\tau_{sl}} \right) \right). \quad (2.2.4)$$

$M_U = \mu$ for transverse waves, and $M_U = \lambda + 2\mu$ for longitudinal waves (Aki and Richards, 1980). The complex stress modulus $M_c(\omega)$ is defined as the Fourier transform of the stress relaxation function. The quality factor Q is defined as

$$Q(\omega) = \frac{\text{Re}(M_c(\omega))}{\text{Im}(M_c(\omega))}, \quad (2.2.5)$$

this equation defines Q as the number of wavelengths a pulse may propagate before its amplitude drops by a factor of $e^{-\pi}$. Thus, Q is a function of frequency. For an array of Standard Linear Solids, the equation (2.2.3) and (2.2.5) yield,

$$Q(\omega) = \frac{1 - L + \sum_{l=1}^L \frac{1 + \omega^2 \tau_{sl} \tau_{ol}}{1 + \omega^2 \tau_{ol}^2}}{\sum_{l=1}^L \frac{\omega (\tau_{sl} - \tau_{ol})}{1 + \omega^2 \tau_{ol}^2}}, \quad (2.2.6)$$

Blanch et al. (1995) proposed the τ -method for the viscoelastic modeling. The τ

method is based on the simple observation that the level of attenuation caused by a SLS can be determined by a dimensionless (frequency scale independent) variable τ . If we defined τ as

$$\tau = \frac{\tau_{sl}}{\tau_{ol}} - 1 = \frac{\tau_{sl} - \tau_{ol}}{\tau_{ol}}, \quad (2.2.7)$$

the inverse of Q and the velocity $c(\omega)$ for one SLS can be written as,

$$Q^{-1} = \frac{\omega \tau_{sl} \tau}{1 + \omega^2 \tau_{ol}^2 (1 + \tau)}, \quad (2.2.8)$$

$$(c(\omega))^2 = \frac{M_U}{\rho} \left(1 + \left(\frac{M_U}{M_R} - 1 \right) \frac{1}{1 + \omega^2 \tau_{sl}^2} \right)^{-1}. \quad (2.2.9)$$

Using the parameter τ to tune an array of SLSs, and assuming that τ is small (i.e. $1 + \tau \approx 1$), equation (2.2.6) yields

$$Q^{-1} = \sum_{l=1}^L \frac{\omega \tau_{sl} \tau}{1 + \omega^2 \tau_{ol}^2}. \quad (2.2.10)$$

In this expression Q^{-1} is linear in τ . Therefore, we can easily find the best approximation in the least square sense over a predefined frequency range to any Q_0 by minimizing over τ the expression,

$$J = \int_{\omega_a}^{\omega_b} (Q^{-1}(\omega, \tau, \tau_l) - Q_0^{-1})^2 d\omega, \quad (2.2.11)$$

to zero and solve for τ . To find the minimum, we set the derivative of J with respect to τ to zero and solve for τ .

$$\frac{dJ}{d\tau} = 2 \int_{\omega_a}^{\omega_b} (Q^{-1}(\omega, \tau, \tau_l) - Q_0^{-1}) \times \frac{dQ^{-1}(\omega, \tau_{ol}, \tau)}{d\tau} d\omega = 0. \quad (2.2.12)$$

The final formula for τ is

$$\tau = \frac{1}{Q_0} \cdot \frac{\sum_{l=1}^L I_{0l}}{\sum_{l=1}^L I_{1l} + 2 \sum_{l=1}^{L-1} \sum_{k=l+1}^L I_{2kl}} \quad (2.2.13)$$

where,

$$I_{0l} = \frac{1}{2\tau_{\sigma l}} \left[\log (1 + \omega^2 \tau_{\sigma l}^2) \right]_{\omega_a}^{\omega_b} \quad (2.2.14)$$

$$I_{1l} = \frac{1}{2\tau_{\sigma l}} \left[\arctan (\omega \tau_{\sigma l}) - \frac{\omega \tau_{\sigma l}}{1 + \omega^2 \tau_{\sigma l}^2} \right]_{\omega_a}^{\omega_b} \quad (2.2.15)$$

$$I_{2kl} = \frac{\tau_{\sigma l} \tau_{\sigma k}}{\tau_{\sigma k}^2 - \tau_{\sigma l}^2} \left[\frac{\arctan (\omega \tau_{\sigma l})}{\tau_{\sigma l}} - \frac{\arctan (\omega \tau_{\sigma k})}{\tau_{\sigma k}} \right]_{\omega_a}^{\omega_b} \quad (2.2.16)$$

$$\tau_{\sigma l} = \frac{1}{\omega_l}, \quad (2.2.17)$$

where, ω_l is the frequency of interest. Figures 2.1 and 2.2 shows the example of $Q(\omega)$ and $c(\omega)$ defined by one SLS mechanism in which following parameters are used,

$$Q=10$$

$$\tau_{\epsilon}=0.003848$$

$$\tau_{\sigma}=0.003183$$

$$\text{density : } \rho=2000\text{kg/m}^3$$

$$\text{velocity : } c(\omega=\infty)=2000\text{m/sec (} M_U=8\text{GPa)}$$

$$\omega_l=50\text{Hz} \times 2\pi,$$

where τ_{σ} is given by equation (2.2.17), τ is given by equations (2.2.13) to (2.2.16), and τ_{ϵ} is given by equation (2.2.7). The Q shown in Figure 2.1 is given by equation (2.2.8), and the velocity shown in Figure 2.2 is given by equation (2.2.9).

2.2.2 Viscoelastic Staggered Grid Finite-difference Method

Robertsson et al. (1994) proposed the viscoelastic finite-difference modeling based on the τ method mentioned above. For simplicity, I will derive the equations for a one-dimensional case, where the viscoelastic equations are the same as the viscoacoustic. From the definition of pressure and dilatation,

$$\sigma = -p, \quad (2.2.18)$$

and

$$-\dot{\epsilon} = v_x, \quad (2.2.19)$$

where v is the particle velocity, and subscript x means spatial derivative. Taking the time derivative of equation (2.2.2) and using equations (2.2.18) and (2.2.19) leads to,

$$-\dot{p} = \dot{G} * v_x. \quad (2.2.20)$$

Substituting equation (2.2.3) into (2.2.20) yields,

$$-\dot{p} = M_R \left(1 - \sum_{l=1}^L \left(1 - \frac{\tau_{sl}}{\tau_{ol}} \right) \right) v_x + M_R \left(\sum_{l=1}^L \frac{1}{\tau_{ol}} \left(1 - \frac{\tau_{sl}}{\tau_{ol}} \right) e^{-\frac{t}{\tau_{ol}}} \right) \theta(t) * v_x. \quad (2.2.21)$$

The convolution terms in the equation (2.2.21) can be eliminated by introducing so-called memory variables, which will be denoted r_l (Carcione et al., 1988). Then, equation (2.2.21) reduces to,

$$-\dot{p} = M_R \left(1 - \sum_{l=1}^L \left(1 - \frac{\tau_{sl}}{\tau_{ol}} \right) \right) v_x + \sum_{l=1}^L r_l, \quad (2.2.22)$$

where

$$r_l = M_R \left(\frac{1}{\tau_{ol}} \left(1 - \frac{\tau_{sl}}{\tau_{ol}} \right) e^{-\frac{t}{\tau_{ol}}} \right) \theta(t) * v_x, \quad 1 \leq l \leq L. \quad (2.2.23)$$

From the equation (2.2.23), we see that r_l is governed by convolutions of v_x with exponential functions. A set of first-order linear differential equations can be obtained instead of the convolution as follows. First, by taking the time derivative of equation (2.2.23), we obtain,

$$\dot{r}_l = -\frac{1}{\tau_{sl}} M_R \left(\frac{1}{\tau_{sl}} \left(1 - \frac{\tau_{sl}}{\tau_{sl}} \right) e^{\frac{-t}{\tau_{sl}}} \right) \theta(t) * v_x + M_R \left(\frac{1}{\tau_{sl}} \left(1 - \frac{\tau_{sl}}{\tau_{sl}} \right) e^{\frac{-t}{\tau_{sl}}} \right) \delta(t) * v_x.$$

$$1 \leq l \leq L \quad (2.2.24)$$

From equation (2.2.23) we notice that equation (2.2.24) reduces to

$$\dot{r}_l = -\frac{1}{\tau_{sl}} r_l + M_R \left(\frac{1}{\tau_{sl}} \left(1 - \frac{\tau_{sl}}{\tau_{sl}} \right) e^{\frac{-t}{\tau_{sl}}} \right) v_x, \quad 1 \leq l \leq L. \quad (2.2.25)$$

We have derived a set of first-order linear differential equations for the memory variables. Newton's second law completes the full description of wave propagation in a viscoelastic medium. That is,

$$\rho \dot{v} = -p_x, \quad (2.2.26)$$

where ρ is the density. From equations (2.2.22), (2.2.25) and (2.2.26), we can derive staggered grid finite-difference equations (second-order accurate in time, fourth-order accurate in space) as follows,

$$p_j^{n+1/2} = p_j^{n-1/2} - M_R \frac{\tau_{sl}}{\tau_{sl}} \frac{\Delta t}{\Delta h} (c_1 (v_{j+3/2}^n - v_{j-3/2}^n) + c_2 (v_{j+1/2}^n - v_{j-1/2}^n))$$

$$- \frac{\Delta t}{2} (r_j^{n+1/2} + r_j^{n-1/2}) \quad (2.2.27)$$

$$r_j^{n+1/2} = \frac{1}{1 + \frac{\Delta t}{2\tau_{sl}}} \left(r_j^{n-1/2} \left(1 - \frac{\Delta t}{2\tau_{sl}} \right) - M_R \frac{1}{\tau_{sl}} \left(\frac{\tau_{sl}}{\tau_{sl}} - 1 \right) \right.$$

$$\left. \times \frac{\Delta t}{\Delta h} (c_1 (v_{j+3/2}^n - v_{j-3/2}^n) + c_2 (v_{j+1/2}^n - v_{j-1/2}^n)) \right) \quad (2.2.28)$$

$$v_j^{n+1/2} = v_j^{n-1/2} - \frac{\Delta t}{\Delta h \rho} (c_1 (p_{j+3/2}^n - p_{j-3/2}^n) + c_2 (p_{j+1/2}^n - p_{j-1/2}^n)) \quad (2.2.29)$$

$$c_1 = -\frac{1}{24}, \quad c_2 = \frac{9}{8} \quad (2.2.30)$$

where Δt is the time step and Δh is the spatial step. Indices n and j correspond to time and

space coordinates respectively ($p_j^n = p(n\Delta t, j\Delta h)$). In the equation (2.2.28), Crank-Nicolson scheme (Strang, 1986) is employed to ensure a stable solution. Figure 2.3 shows the one-dimensional staggered finite-difference grid.

The results from the one-dimensional case described above are easily generalized to higher dimensions (Appendix A). In two-dimensional viscoelastic staggered grid finite-difference modeling, the following three sets of equations are solved.

Equations governing stress:

$$\frac{\partial \sigma_{xx}}{\partial t} = \pi \frac{\tau_\epsilon^p}{\tau_\sigma} \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_z}{\partial z} \right) - 2\mu \frac{\tau_\epsilon^s}{\tau_\sigma} \frac{\partial v_z}{\partial z} + r_{xx} \quad (2.2.31)$$

$$\frac{\partial \sigma_{zz}}{\partial t} = \pi \frac{\tau_\epsilon^p}{\tau_\sigma} \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_z}{\partial z} \right) - 2\mu \frac{\tau_\epsilon^s}{\tau_\sigma} \frac{\partial v_x}{\partial x} + r_{zz} \quad (2.2.32)$$

$$\frac{\partial \sigma_{xz}}{\partial t} = \mu \frac{\tau_\epsilon^s}{\tau_\sigma} \left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} \right) + r_{xz} \quad (2.2.33)$$

Equations governing particle velocities:

$$\frac{\partial v_x}{\partial t} = \frac{1}{\rho} \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xz}}{\partial z} \right) \quad (2.2.34)$$

$$\frac{\partial v_z}{\partial t} = \frac{1}{\rho} \left(\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{zz}}{\partial z} \right) \quad (2.2.35)$$

Equations governing the so called memory variables, which are introduced to eliminate the numerically inconvenient convolution arising in the viscoelastic constitutive relation:

$$\frac{\partial r_{xx}}{\partial t} = -\frac{1}{\tau_\sigma} \left(r_{xx} + \pi \left(\frac{\tau_\epsilon^p}{\tau_\sigma} - 1 \right) \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_z}{\partial z} \right) - 2\mu \left(\frac{\tau_\epsilon^s}{\tau_\sigma} - 1 \right) \frac{\partial v_z}{\partial z} \right) \quad (2.2.36)$$

$$\frac{\partial r_{xz}}{\partial t} = -\frac{1}{\tau_\sigma} \left(r_{xz} + \pi \left(\frac{\tau_\epsilon^P}{\tau_\sigma} - 1 \right) \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_z}{\partial z} \right) - 2\mu \left(\frac{\tau_\epsilon^S}{\tau_\sigma} - 1 \right) \frac{\partial v_x}{\partial x} \right) \quad (2.2.37)$$

$$\frac{\partial r_{xz}}{\partial t} = -\frac{1}{\tau_\sigma} \left(r_{xz} + \mu \left(\frac{\tau_\epsilon^S}{\tau_\sigma} - 1 \right) \left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} \right) \right) \quad (2.2.38)$$

σ_{ij} : the ij th component of the symmetric stress tensor.

v_i : the i th component of the particle velocity.

r_{ij} : the memory variables.

τ_ϵ^P , τ_ϵ^S : the viscoelastic strain relaxation times for P- and SV-waves, respectively.

τ_σ : the viscoelastic stress relaxation time for both the P- and SV waves.

μ : the relaxation modulus corresponding to SV-waves, which is analogous to Lamé constant μ in the elastic case.

π : the relaxation modulus corresponding to P-wave, which is analogous to $\lambda + 2\mu$ in the elastic case.

ρ : the density.

Equation (2.2.31) to (2.2.35) corresponds to equations (2.1.19) in the elastic case. In order to calculate equations (2.2.31) to (2.2.38), I employed second-order accurate in time and forth-order accurate in space $O(2,4)$ scheme as I showed for the one-dimensional case. The differential equations are given in Appendix B. In this implementation, the finite-difference grid is staggered in space as shown in Figures 2.4 to 2.6.

In the two-dimensional case, the viscoelastic horizontal free-surface satisfies the following conditions:

$$\sigma_{iz} = 0 \quad i=x,z \quad (2.2.39)$$

$$r_{iz} = 0 \quad i=x,z \quad (2.2.40)$$

these conditions lead the following equations by using equations (2.2.31) to (2.2.33):

$$\frac{\partial v_x}{\partial z} = - \frac{\partial v_z}{\partial x} \quad (2.2.41)$$

$$\frac{\partial v_z}{\partial z} = - \left(1 - 2 \frac{\tau_s}{\tau_p} \frac{\mu}{\pi} \right) \frac{\partial v_x}{\partial x} \quad (2.2.42)$$

Vertical free-surface boundary can be implemented in the same way.

The stability criteria for the conditionally stable schemes are similar to elastic schemes (Levander, 1988). The Courant number ($c \Delta t / \Delta x$, where c is the velocity) for viscoelastic schemes has to be adjusted to the highest phase velocity, which is found at infinite frequency $c_{\max}$ as follows :

$$c_{\max} = \sqrt{\tau_s M_R / \tau_p \rho} \quad (2.2.43)$$

In order to minimize artificial reflections from the boundaries of a model, Higdon's (1986, 1987, 1990) absorbing boundary condition is used in the calculation.

2.3 Basic Test of the Accuracy of the Viscoelastic Finite-difference Method

In the next chapter, I will discuss the accuracy of finite-difference calculation in the presence of irregular free surface. To clarify the discussion about irregular free surface and show the reliability of the finite-difference code that I developed, I will show several fundamental benchmark tests in homogeneous case.

2.3.1 1-D Acoustic Case

First, I will show the accuracy of the forth-order approximation in space by comparison

with the second-order approximation in space in a one-dimensional case. An acoustic homogeneous model with velocity of 2000m/s, a density of 1500kg/m³ is used for this comparative study. The source is a point explosion represented with a 100 Hz Ricker wavelet. Nominal wavelength is 20m. The waveforms were collected at a 100m source receiver offset. The grid size is varied from 0.5m to 10m. Table 2.1 shows the results of the simulations as a function of grid size.

Table 2.1 Grid sizes and numerical dispersion in 1D simulation

Grid size (m)	Wavelength/grid size	Dispersion	
		2 nd order (Figure 2.7)	4 th order (Figure 2.8)
0.5	40	No	No
1	20	No	No
2	10	Small	No
2.5	8	Medium	Small
4	5	Large	Medium
5	4	Large	Large
10	2	Large	Large

In Figure 2.7 and Figure 2.8, particle velocities from the simulation with various grid sizes are plotted together with the analytical solution. The results of the simulation can be summarized as follows.

For the grid size of 0.5m and 1m (approximately 40 and 20 grid-points per wave length), the simulation yields sufficiently accurate results. However, for the grid size of 2m (approximately 10 grid-points per wave length), we can see obvious numerical dispersion in the calculation of second-order approximation. On the contrary, we can see that the fourth-order approximation has no numerical dispersion. Furthermore, in the case of the grid size of 2.5m (approximately 8 grid-points per wave length), the numerical dispersion is little in fourth-order approximation. These results confirm that the second-order approximation requires 20 grid-points per wave length and fourth-order approximation requires 10 grid-points per wave length in the case of a one-dimensional simulation.

2.3.2 2-D Elastic Case

In 2-D elastic medium, we have P- and S- waves which propagate with different velocity and wavelength. We also have surface waves in the presence of the free surface. Here, I will show the numerical dispersion due to the variation of grid sizes for these three waves. The model is an infinite elastic solid with P- and S-velocities of 2000 and 1000m/s, respectively, and a density of 1800kg/m³. The source is a point explosion for P-wave, and a vertical force for S-wave with a 50 Hz Ricker wavelet . Thus, the approximate P- and S-wave wavelength are 40 and 20m respectively. The waveforms were collected at the right side of the source (x direction). The source receiver offset is 100m. The grid sizes were varied from 0.5m to 10m. Table 2.2 shows the list of the grid sizes as well as the results of the simulation.

Table 2.2 Grid sizes and numerical dispersion in 2D simulation

Grid size (m)	P-wave			S-wave	
	Wavelength/ grid size	Dispersion		Wavelength/ grid size	Dispersion
		2 nd order (Figure 2.9)	4 th order (Figure 2.10)		4 th order (Figure 2.11)
0.5	80	No	No	40	No
1	40	No	No	20	No
2	20	No	No	10	No
2.5	16	No	No	8	Small
4	10	Small	No	5	Large
5	8	Large	Small	4	Large
10	4	Large	Large	2	Large

In Figure 2.9 to Figure 2.11, pressure from the simulation for the P-wave, and the vertical component of particle velocity for the S-wave with various grid sizes are plotted. In Figure 2.9 and Figure 2.10, analytical solutions are plotted together with the finite-difference solutions. In the case of the P-waves, for the grid size of 0.5m to 2.5m (approximately 80 to 16 grid-points per wave length), the simulation yields sufficiently accurate results. However, for the grid size of 4m (approximately 10 grid-points per wave length), we can see obvious numerical dispersion in the calculation of second-order approximation. On the contrary, we can see that the fourth-order approximation has no numerical dispersion. Furthermore, in

the case of the grid size of 5m (approximately 8 grid-points per wave length), the numerical dispersion is small in the fourth-order approximation. For S-waves in the grid size of 0.5m and 1m (approximately 40 and 20 grid-points per wave length), the simulation yield sufficiently accurate results. There is little dispersion in the grid size of 2m (approximately 10 grid-points per wave length), and obvious dispersion in the grid size of 2.5m (approximately 8 grid-points per wave length).

A flat semi-infinite elastic medium was used for the simulation of surface waves (Rayleigh waves). The model is a Poisson solid with P- and S-velocities of 3000 and 1730m/s, respectively, and a density of 2500kg/m³. A source and a receiver were placed on the surface. The source is a 15Hz Ricker wavelet that generates vertical component of particle velocity at a point on the free surface. The wavelength of Rayleigh waves is approximately 100m. The waveforms were collected at the 1000m source receiver offset. The grid sizes were varied from 5m to 25m. Table 2.3 shows the list of the grid sizes as well as the results of the simulation.

Table 2.3 Grid sizes and numerical dispersion for the simulation of Rayleigh waves
(Figure 2.12)

Grid size (m)	Wavelength/grid size	Dispersion
		4 th order
5	20	No
10	10	No
12.5	8	Small
20	5	Large
25	4	Large

In Figure 2.12, the vertical component of particle velocities from the simulation with various grid sizes are plotted. For the grid size of 5m and 10m (approximately 20 and 10 grid-points per wave length), the simulation yields sufficiently accurate results. There is little dispersion in the grid size of 12.5m (approximately 8 grid-points per wave length), and obvious dispersion in the grid size of 20m (approximately 5 grid-points per wave length).

The results of the simulation for two-dimensional elastic case can be summarized as follows. The second-order approximation requires 20 grid-points per wavelength and fourth-order approximation requires 10 grid-points per wavelength. It should be noted that we have to consider the wavelength of S-waves and surface waves with respect to S-velocity.

2.3.3 Viscoelastic Model

In this section, I will show three basic benchmark tests of viscoelastic modeling by comparison with analytical solutions.

The first example is a one-dimensional model that is an infinite viscoacoustic medium with velocity of 2000m/s and a density of 2000kg/m³. I will show three calculations in which a quality factor Q is 20, 50, 100 respectively. The receivers are located 0m to 190m from the source at 10m distance interval. For a source wavelet, 50Hz Ricker wavelet is employed. An analytical solution can be calculated in frequency domain by the use of complex velocity $c(\omega)$ given by the following equation (Lay and Wallace, 1995),

$$c(\omega) = c(\omega_{ref}) \left(1 + \frac{1}{\pi Q(\omega)} \ln \left(\frac{\omega}{\omega_{ref}} \right) + \frac{i}{2Q(\omega)} \right), \quad (2.2.44)$$

where, $Q(\omega)$ is a quality factor, ω_{ref} is some reference frequency, $c(\omega_{ref})$ is a velocity at the reference frequency. In this example, ω_{ref} of 50Hz, and elastic modulus M_0 of 8Gpa ($2000^2 \times 2000$) were used. $c(\omega_{ref})$ were calculated by equation (2.2.9). Table 2.4 shows the relaxation times for three Q models which were optimized by the least square method mentioned in the previous section and $c(\omega_{ref})$. These relaxation times are calculated for the τ_0 of 0.003183 that is obtained from equation (2.2.17), and optimized for a frequency range from 25 to 75Hz.

Table 2.4 Relaxation times (sec) and a velocity (m/sec) at the reference frequency for three different Q model.

	τ_{σ}	τ_{ϵ}	$c(\omega_{ref})$
Q=10	0.003183	0.003848	1920
Q=20	0.003183	0.003515	1954
Q=50	0.003183	0.003316	1980

In Figure 2.13 to 2.15, the particle velocity from the simulation and an analytical solution are plotted. We can see that the finite-difference solutions closely agree with analytical solutions without the slight difference in the phase velocity. I will comment about this difference later. In order to investigate the accuracy of the simulation, I compare the maximum amplitude of the finite-difference solution with theoretical amplitude. In Figure 2.16, the maximum amplitude of the finite-difference solution is plotted together with the theoretical amplitude $A(x)$ obtained from the following equation;

$$A(x) = A_0 \exp \left(- \frac{f\pi x}{cQ} \right), \quad (2.2.45)$$

Here, A_0 is the amplitude at the source, x is the distance between the source and a receiver, c is the velocity (2000m/sec), f is the frequency (50Hz). In Figure 2.16, we can see that the amplitude of the finite-difference solution agrees with the theoretical one very well. I also calculated a quality factor $Q(\omega)$ and phase velocity $c(\omega)$ from the finite-difference solution in the frequency domain using the amplitude ratio of two traces. Two traces used in the calculation are 90m and 190m. Figure 2.17 shows the amplitude ratio of two traces obtained by Fourier transform as well as the theoretical Q obtained from the equation (2.2.8). Figure 2.18 shows the phase velocity obtained by Fourier transform as well as the theoretical velocity $c(\omega)$ obtained from the equation (2.2.9). In Figure 2.17 and Figure 2.18, we can see that the finite-difference solution $Q(\omega)$ and $c(\omega)$ is very close to the analytical $Q(\omega)$ and $c(\omega)$.

The second example is the two-dimensional case. The model is an infinite viscoelastic solid with P- and S- wave velocities of 2000 and 1000 respectively, density of 1800kg/m³ and Q

of 20 (both Q_p and Q_s). The source is a point explosion with a 50Hz Ricker wavelet. The relaxation times optimized by the least square method are listed in Table 2.7. These relaxation times are calculated for the τ_σ of 0.003183 that is obtained from equation (2.2.21), and optimized for frequency range from 25 to 75Hz. Same relaxation times are used for both Q_p and Q_s .

Table 2.7. Relaxation times for two-dimensional example

	τ_σ	τ_ϵ
Q=20	0.003183	0.003515

The receivers are located at the right side of the shot (positive x direction) from 10m to 163m from the source at 9m distance interval. In figure 2.19, the pressure from the simulation is plotted together with the elastic case ($Q = \text{infinity}$). The dispersion due to the viscoelastic attenuation looks to be a correct tendency. The maximum amplitude of viscoelastic and elastic calculations are plotted in Figure 2.20, together with the theoretical amplitude. In a two dimensional case, the theoretical amplitude of body waves can be calculated by following equation,

$$A(x) = A_0 \exp\left(-\frac{f\pi(x-x_0)}{cQ}\right) \times \sqrt{\frac{x_0}{x}}, \quad (2.2.46)$$

here A_0 is the amplitude at the some reference receiver, x_0 is the location of the reference receiver, x is the location of each receiver, c is the velocity (2000m/sec), f is the frequency (50Hz). The amplitude of both the finite-difference and theoretical solutions closely agrees.

These numerical examples prove the accuracy and reliability of this viscoelastic modeling as well as my finite-difference code. The results also implies that the few pairs of relaxation mechanism can yield sufficiently accurate results for practical purposes.

Figure 2.1: Approximation to a constant Q of 10 by one SLS mechanism using the equation (2.2.8). $\tau_e=0.003848$, $\tau_o=0.003183$. density : $\rho=2000\text{kg/m}^3$, velocity : $c(\omega=\infty)=2000\text{m/sec}$, ($M_0=8\text{GPa}$), $\omega_1=50\text{Hz} \times 2\pi$.

Figure 2.2: Velocity defined by one SLS mechanism using the equation (2.2.9). $\tau_e=0.003848$, $\tau_o=0.003183$. density : $\rho=2000\text{kg/m}^3$, velocity : $c(\omega=\infty)=2000\text{m/sec}$, ($M_0=8\text{GPa}$), $\omega_1=50\text{Hz} \times 2\pi$.

Figure 2.3: The one dimensional staggered finite-difference grid. σ_{xx} is a normal stress, and v_x is a velocity in x-direction.

Figure 2.4: The two dimensional staggered finite-difference grid. σ_{xx} and σ_{zz} are normal stresses, v_x and v_z are particle velocities. r_{xx} , r_{zz} and r_{xz} are memory variables.

a) A stencil for a shear stress, σ_{xz} , and a memory variable r_{xz} .

b) A stencil for normal stresses, σ_{xx} and σ_{zz} and memory variables r_{xx} and r_{zz} .

Figure 2.5: Fourth-order spatial stencils for stress update. Within the grid-cells, the solid squares represent the σ_{xx} , σ_{zz} , r_{xx} , r_{zz} components, the light squares the σ_{xz} , r_{xz} components, the solid circles the v_x components, the light circles represent the v_z components (σ : stress, r : memory variable, v : velocity).

a) A stencil for a particle velocity, v_z .

b) A stencil for a particle velocity, v_x .

Figure 2.6: Fourth-order spatial stencils for velocity update. Within the grid-cells, the solid squares represent the σ_{xx} , σ_{zz} , r_{xx} , r_{zz} components, the light squares the σ_{xz} , r_{xz} components, the solid circles the v_x components, the light circles represent the v_z components (σ : stress, r : memory variable, v : velocity).

Figure 2.7: Particle velocities from the second-order approximation with various grid sizes (one-dimensional). Δx means grid size and $\lambda / \Delta x$ means wavelength per grid size.

Figure 2.8: Particle velocities from the fourth-order approximation with various grid sizes (one-dimensional). Δx means grid size and $\lambda / \Delta x$ means wavelength per grid size.

Figure 2.9: P-waves (pressure) from the second-order approximation with various grid sizes (two-dimensional). Δx means grid size and $\lambda/\Delta x$ means wavelength per grid size.

Figure 2.10: P-waves (pressure) from the fourth-order approximation with various grid sizes (two-dimensional). Δx means grid size and $\lambda/\Delta x$ means wavelength per grid size.

Figure 2.11: S-waves (particle velocity) from the fourth-order approximation with various grid sizes (two-dimensional). Δx means grid size and $\lambda / \Delta x$ means wavelength per grid size.

Figure 2.12: Rayleigh-waves (vertical component of particle velocity) from the fourth-order approximation with various grid sizes (two-dimensional). Δx means grid size and $\lambda / \Delta x$ means wavelength per grid size.

Figure 2.13: Comparison of a finite-difference solution with a analytical solution. $Q=10$.

Figure 2.14: Comparison of a finite-difference solution with a analytical solution. $Q=20$.

Figure 2.15: Comparison of a finite-difference solution with a analytical solution. $Q=50$.

Figure 2.16: Maximum amplitude of finite-difference solutions (Figure 2.13 to 2.15) in comparison to analytical solutions.

Figure 2.17: $Q(\omega)$ obtained from finite-difference solution in comparison to a analytical solution.

a) $Q=10$

b) $Q=20$

c) $Q=50$

Figure 2.18: Phase velocity $c(\omega)$ obtained from finite-difference solution in comparison to a analytical solution.

Figure 2.19: Comparison of viscoelastic model ($Q=20$) with elastic model ($Q=\text{infinity}$).

Figure 2.20: Maximum amplitude of finite-difference solutions (Figure 2.19) in comparison to analytical solutions.

Chapter 3

A Free Surface Approximation in the Presence of Topography

3.1 Introduction

Robertsson (1996) proposed a generalized image method for the free surface condition in which stress-tensor components are imaged and the irregular free surface boundary condition is enforced to ensure that the shear and normal stresses perpendicular to the boundary are zero. This method is based on a robust theoretical derivation and the criteria for stability and accuracy are well established. An alternative method to model surface topography is to let $V_p, V_s \rightarrow 0$ in the region above the free surface (the density in the region above the free surface is not 0 to avoid numerical instability). This method is called the vacuum formulation (Graves, 1996). This approach is attractive because it can be implemented with the same difference equations used in the interior of the model, and thus, the effects of surface topography are modeled in the same manner as internal media interfaces. However, the method is unstable and could be inaccurate.

In order to combine the simplicity of the vacuum method as well as the accuracy of the generalized image method, I propose an approach based on a combination of these methods.

3.2 Finite-difference Approximation

In the proposed method, stresses are calculated so that the shear and normal stresses

perpendicular to the boundary are zero, just as in Robertsson's generalized image method. However, the calculation of particle velocities does not involve any specific boundary calculation. Table 1 shows the comparison of the three different methods for dealing with free surface conditions.

Table 3.1 Three methods of free-surface approximation

	Calculation of particle velocities	Calculation of stresses
Generalized image method	Imaging stresses.	Free surface condition (the shear and normal stresses perpendicular to the boundary are zero). Set particle velocities to zero above the free surface.
Vacuum formulation	No specific calculation.	No specific calculation. Set P and S wave velocities to zero above the free surface.
Proposed method	No specific calculation.	Free surface condition (the shear and normal stresses perpendicular to the boundary are zero). Set particle velocities to zero above the free surface.

The implementation of three approximation methods can be summarized as follows.

3.2.1 Generalized Image Method

In this method, the free surface is located exactly through the upper part of the staggered grid points $z=0$, (in the case of horizontal surface) as shown in Figure 3.1. Then, the normal stresses σ_{xx} , σ_{zz} , the memory variables (see the Chapter 2) r_{xx} , r_{zz} , and the particle velocity v_x are located on the free surface. At $z=0$, the σ_{zz} and the r_{zz} are equal to zero. To honor the free surface condition, the σ_{xx} and r_{xx} are updated using only horizontal derivatives of the velocities through equations (2.2.41) and (2.2.42). If we employ fourth-order approximation in space, further consideration is required. To calculate the shear stress σ_{xz} and the memory variable r_{xz} at $z=0$, and stresses and memory variables at $z=1$, the particle velocities are set to zero on all rows above the free surface. To calculate particle velocities, stresses above the free surface are set to the anti-symmetric value of under the free surface (imaging), so that the shear and normal stresses perpendicular to the boundary on the free surface are zero.

Figure 3.2 shows the staggered finite-difference grid in the vicinity of the free surface used in the generalized image method. In order to account for the irregular surface, the following calculation is performed at vicinities of the corners. The normal stresses σ_{xx} , σ_{zz} and the memory variables r_{xx} , r_{zz} are set to zero at the inner and outer corner. The shear stresses σ_{xz} and the memory variables r_{xz} are set to zero at the outer corner. Imaging is performed in both directions to respect the vertical and horizontal derivatives of the stress components. The updating of velocities is calculated separately for the vertical and horizontal derivatives.

3.2.2 Vacuum Method

The vacuum method does not involve any specific boundary calculation. Free surface condition is implicitly calculated by P- and S-wave velocities which are set to zero in the region above the free surface. To avoid numerical instability, the density in the region above the free surface is not zero (usually, the same density as the interior of the model is used).

In order to assure the symmetry of the left and right sides of the vertical boundary, the following special consideration is needed. On the vertical boundary whose right side is air, P- and S-wave velocities are set to zero only for the calculation of the shear stress (Figure 3.3).

3.2.3 Proposed Method

The proposed method is a combination of the image method and the vacuum method. In the proposed method, stresses are calculated so that the shear and normal stresses perpendicular to the boundary are zero, just as in the generalized image method. However, the calculation of particle velocities is similar to the vacuum method and does not involve any specific boundary calculation. Figure 3.4 shows the staggered finite-difference grid in the vicinity of

the free surface used in the proposed method.

In the calculation of stresses for the irregular surface, there is a difference between the generalized image method and the proposed method. In the generalized image method, the σ_{xx} , σ_{zz} and r_{xx} , r_{zz} are set to zero at the inner and outer corner, and the σ_{xz} and r_{xz} are set to zero at the outer corner. However, in the proposed method, the grid on the inner corner and the σ_{xz} and r_{xz} on the outer corner are calculated in the same manner as the interior of the model. Thus, only the σ_{xx} , σ_{zz} and r_{xx} , r_{zz} of the grid on the outer corner are set to zero. The calculation of particle velocities does not involve any specific calculation even if the surface topography is not flat.

3.3 Comparison of Accuracy

To examine the validity of the method, I have carried out a large number of numerical tests to compare the three methods (Generalized image method, Vacuum formulation, Proposed method). Figure 3.5 shows the example of the computations including irregular surface. Here, I show two examples of the results of numerical tests.

3.3.1 Flat Surface with Various Slopes

A flat, semi-infinite, elastic medium was chosen to be the first numerical test for the free surface with topography. This model is similar to the model presented by Robertsson (1996). The model is a Poisson solid with P- and S-velocities of 3000 and 1730m/s, respectively, and a density of 2500kg/m³. In this test, the slope of the free surface was varied. If wave propagation is modeled accurately, independent of the slope of the surface, then the algorithm should also allow for accurate modeling of free surfaces with more general shapes. To avoid problems due to the uncertainty of the exact location of the source, a P-wave source (15Hz Ricker wavelet) located 30 m below the surface was employed. The

slope of the free surface was varied from -90° to 90° at 15° intervals, and the waveforms were collected at 1000m source receiver offset in a direction parallel to the free surface. The receivers were located 50 m below the surface. The recorded particle velocities were rotated to the coordinate system of the 0° case, so that all waveforms can be compared to each other directly. The solution for the 0° slope can be regarded as the exact solution. Figure 3.6 and 3.7 show the examples of particle velocity wavefields (parallel to the free surface).

Before showing slope results, I show some basic comparison of different methods. Figure 3.8 shows the particle velocities from the 0° case with various offsets. The first arrival is body waves (P-waves) and the second arrival is surface waves (Rayleigh waves). We can see horizontal component of particle velocity component dominates P-waves and vertical velocity dominates Rayleigh waves. Figure 3.9 shows the comparison of three methods in the 0° case against a solution of a discrete wave-number integral method that can be considered as an exact solution. In this calculation, grid size was set to 2m (approximately 50 grid points per wavelength) which is sufficiently small, and we can assume there is no numerical error due to the grid size. In Figure 3.9, there is a phase shift of Rayleigh waves in the proposed method compared with the boundary element method. However, the difference is so small that it may be neglected if we are mainly concerned with body waves. Furthermore, other numerical tests showed this difference of phase velocity only appears when the source and receivers are placed at very shallow depth below the surface compared with the wave lengths. For most applications, I consider this difference not to be significant.

In Figures 3.10 to 3.12, particle velocity from the simulations with various slopes and the methods are plotted, for the grid sizes of 2m and 5m. For the grid size of 2m (approximately 50 grid points per wavelength), all methods yield sufficiently accurate results. However, for the grid size of 5m (approximately 20 grid points per wavelength), we can see a significant numerical dispersion in the image method and the vacuum method. On

the contrary, the dispersion of the proposed method is relatively small.

3.3.2 Presence of Irregular Surface

In this example, three methods of free surface approximation were compared with a finite-difference method (generalized image method) using Dr. Johan O. Robertsson's code. Figure 3.13 shows the model used in the second numerical test. The model is a elastic Poisson solid with P- and S-velocities of 2000 and 1155m/s, respectively, and a density of 1000kg/m³. A P-wave source (110Hz Ricker wavelet) located at a distance of 30m and 5m below the surface was employed.

At first, I will compare three methods of free surface approximation with the various grid sizes. In Figure 3.14, the particle velocity (vertical component) from the simulations with the various grid sizes is plotted. The receivers were located at a distance of 120m and 4m below the surface. All methods yield sufficiently accurate results when using a large number of grid-points per wavelength (40 or 80). However, in the case of a smaller number of grid points (10 or 20), we can see that the proposed method is most accurate.

Next, I compared the following four methods using the same model.

Table 3.2 Benchmark test

Method	Developer
FD (Image method)	Hayashi, K.
FD (Proposed method)	Hayashi, K.
FD (Vacuum method)	Hayashi, K.
FD (Image method)	Robertsson, J. O.

The receivers were placed on two lines shown in Figure 3.15. The grid size of 0.25m (approximately 40 grid points per wavelength) is used in my three calculations. Figures 3.16 to 3.19 show the particle velocities from the simulations. Figure 3.16 and 3.17 show the waveforms collected at the receivers located before the ramp. We can consider that these waveforms have little effect from irregular surface. Before the ramp, five calculations

show excellent agreement, and it means that four methods yield the same result if the surface is flat. Figure 3.18 and 3.19 show the waveforms collected at the receivers located behind the ramp. I plotted the waveforms at the shallowest receiver (distance of 120m and 4m below the surface) in Figure 3.20 with a large scale. We can see that the four finite-difference solutions agree sufficiently. This result proves the reliability of the finite-difference code that I developed, even if the surface is irregular.

The results of the above two numerical examples imply that the proposed method is most accurate, when using at least 20 to 40 grid-points per wavelength. Therefore, I will use the propose method as free surface approximation in the following discussion.

3.4 Investigation about Stair-shaped Boundary

Robertsson (1996) mentioned that the generalized free surface condition that we employed yields a good representation of a "staircase-shaped" function, whereas a smooth boundary must be discretized in terms of such steps. Fuyuki and Matsumoto (1980) found that the scattering of Rayleigh waves can be substantial from relatively small steps compared to the wavelength. This scattering from the stair-shaped boundary should be avoided when we model a smooth boundary. In addition to that, our free surface numerical tests did not reveal whether the observed Rayleigh wave dispersion was an actual wave phenomenon or a numerical error. Thus, I performed the following tests to reveal the effect of a stair-shaped discretization on a smooth free surface.

Figure 3.21 shows the 45° slope model used in the numerical test. The model is an elastic solid with P- and S-velocities of 5082 and 3000m/s, respectively, and a density of 2000kg/m³. A P-wave source (200Hz Ricker wavelet) located at a distance of 25m and 6m below the surface was employed. In this test, not only the grid size but also the step size of the slope was varied. The step size and the grid size are defined in Figure 3.22. In Figure 3.23, particle velocity (vertical component) from the simulations with the various grids and

step sizes are plotted. The receivers were located 4m below the surface and at 6m distance intervals. Table 3.3 shows the list of the grid and step sizes as well as the results of the computation.

Table 3.3 Grid and step sizes and the resulting dispersion

Model	Grid size (m)	Step size (m)	Wavelength/ grid size	Wavelength/ step size	Dispersion	Figure
A	0.5	0.5	30	30	No	Figure9 (a)
B	0.5	1.0	30	15	No	(b)
C	1.0	1.0	15	15	No	(c)
D	0.5	1.5	30	10	No	(d)
E	1.5	1.5	10	10	Small	(e)
F	0.5	2.0	30	7.5	Small	(f)
G	2.0	2.0	7.5	7.5	Medium	(g)
H	0.5	3.0	30	5	Medium	(h)
I	3.0	3.0	5	5	Large	(i)

The waveforms are also ordered as a matrix in Figure 3.24. The results of the computation can be summarized as follows. The results from model A to C (Figure 3.23 (a) to (c)) have no large difference and no dispersion; we can consider these waveforms to be accurate. The model D (Figure 3.23 (d)) and the model E (Figure 3.23 (e)) are the same step size, however, the dispersion of the model E is larger than the one of the model D. We can see a similar tendency in the models F, G (Figure 3.23 (f) and (g)) and the models H, I (Figure 3.23 (h) and (i)) in which the step sizes are equivalent. This result implies that the numerical errors due to large grid sizes are larger than the Rayleigh wave scattering due to the stair-shaped boundary.

The above numerical tests were performed on a model with a constant slope of 45° . However, we can expect that the constant slope is accurate compared to the surface with arbitrary slope. To confirm that the accuracy does not depend on the curvature of the free surface boundary, I performed other numerical tests using the model shown in Figure 3.25 (model A) and Figure 3.26 (model B). In these tests, the free surface boundary is curved smoothly. The model A has a relatively gentle slope, and the model B has a steep one. The

models are elastic solids with P- and S-velocities of 6000 and 3000m/s, respectively, and a density of 1800kg/m³. A P-wave source (100Hz Ricker wavelet) located at a distance of 24m and 12m below the surface was employed. In this test, the grid sizes were set to 0.5m, 1.0m, 2.0m and 3.0m, and the shortest wavelength is approximately 30m (corresponding to grid-points per wavelength of 60, 30, 15 and 10 respectively). The grid size and step size are equivalent in this test. In Figure 3.27 (model A) and Figure 3.28 (model B), particle velocity (vertical component) from the simulations with the various grid sizes are plotted. The waveforms were obtained at receivers located 12m below the surface and at 12m distance intervals. With 60 grid-points per wave length (Figure 3.27 (a) and Figure 3.28 (a)), there is no dispersion and it can be considered as accurate. As the grid size increases, the dispersion due to the stair-shaped surface boundary increases. However, we can see from Figure 3.27 that the dispersion is mainly generated at the distance between 200 and 250m where the slope of surface is relatively gentle. To compare this result with constant 45° case mentioned before, we can see that the dispersion of the latter is larger than the former.

3.5 Conclusions

The results of numerical tests imply that the proposed method requires at least 30 grid-points per wavelength. In particular, relatively gentle slopes require a large number of grid points per wavelength. The cause of the dispersion is mainly a numerical error due to the large grid sizes rather than the actual Rayleigh wave scattering due to the stair-shaped free surface boundary.

Figure 3.1: Staggered finite-difference grid in the vicinity of the horizontal free-surface boundary (image method). Within the grid-cells, the solid squares represent the σ_{xx} , σ_{zz} , r_{xx} , r_{zz} components, the light squares the σ_{xz} , r_{xz} components, the solid circles the v_x components, the light circles represent the v_z components (σ : stress, r : memory variable, v : velocity).

Figure 3.2: Staggered finite-difference grid of generalized image method (Robertsson, 1996) in the vicinity of free surface boundary. On the boundary, free surface condition(F.C.) is that the normal and shear stresses perpendicular to the boundary are zero. Within the grid-cells, the solid squares represent the σ_{xx} , σ_{zz} , r_{xx} , r_{zz} components, the light squares the σ_{xz} , r_{xz} components, the solid circles the v_x components, the light circles represent the v_z components(σ : stress, r : memory variable, v : velocity).

Figure 3.3: Staggered finite-difference grid of the vacuum method in the vicinity of free surface boundary. On the boundary, free surface condition(F.C.) is that the normal and shear stresses perpendicular to the boundary are zero. Within the grid-cells, the solid squares represent the σ_{xx} , σ_{zz} , r_{xx} , r_{zz} components, the light squares the σ_{xz} , r_{xz} components, the solid circles the v_x components, the light circles represent the v_z components(σ : stress, r : memory variable, v : velocity).

Figure 3.4: Staggered finite-difference grid of the proposed method in the vicinity of free surface boundary. On the boundary, free surface condition(F.C.) is that the normal and shear stresses perpendicular to the boundary are zero. Within the grid-cells, the solid squares represent the σ_{xx} , σ_{zz} , r_{xx} , r_{zz} components, the light squares the σ_{xz} , r_{xz} components, the solid circles the v_x components, the light circles represent the v_z components(σ : stress, r : memory variable, v : velocity).

Figure 3.5: Example of the computation including irregular free surface (proposed method). a) to d) : wave field of pressure. e) : pressure collected at receivers. A model is a homogeneous elastic solid with P- and S-velocities of 3000 and 1500m/sec, respectively, and a density of 1800kg/m³. A source is a P-wave source with 100Hz Ricker wavelet.

Figure 3.5, continued: e) pressure collected at receivers. 0° corresponds to the leftmost receiver, and 180° corresponds to the rightmost receiver.

Figure 3.6: Example of particle velocity wavefield (parallel to the free surface) at time 400msec. 0° case.

Figure 3.7: Example of particle velocity wavefield (parallel to the free surface) at time 400msec. 30° case.

a) Horizontal component of particle velocity.

b) Vertical component of particle velocity.

Figure 3.8: Particle velocities from the 0° case with various offset. Waveforms are normalized by each trace. a) Horizontal component of particle velocity. b) Vertical component of particle velocity.

Figure 3.9: Comparison of a three methods in the 0° case against a solution of a discrete wave-number integral method. Grid size is 2m. Particle velocity parallel to the surface is plotted.

a) Generalized image method. Grid size = 2 m ($\lambda / \Delta x = 50$).

b) Generalized image method. Grid size = 5 m ($\lambda / \Delta x = 20$).

Figure 3.10: Generalized image method. Time series collected at 50m below the free surface and 1000m source-receiver offset. Particle velocity perpendicular to the free surface is plotted. The angles correspond to the dip of the flat free surface (0° is horizontal).

a) Grid size = 2 m (λ (wavelength)/ Δx (grid size) = 50). b) Grid size = 5 m ($\lambda / \Delta x = 20$).

a) Vacuum method. Grid size = 2 m ($\lambda / \Delta x = 50$).

b) Vacuum method. Grid size = 5 m ($\lambda / \Delta x = 20$).

Figure 3.11: Vacuum method. Time series collected at 50m below the free surface and 1000m source-receiver offset. Particle velocity perpendicular to the free surface is plotted. The angles correspond to the dip of the flat free surface (0° is horizontal).

a) Grid size = 2 m ($\lambda / \Delta x = 50$). b) Grid size = 5 m ($\lambda / \Delta x = 20$).

a) Proposed method. Grid size = 2 m ($\lambda / \Delta x = 50$).

b) Proposed method. Grid size = 5 m ($\lambda / \Delta x = 20$).

Figure 3.12: Proposed method. Time series collected at 50m below the free surface and 1000m source-receiver offset. Particle velocity perpendicular to the free surface is plotted. The angles correspond to the dip of the flat free surface (0° is horizontal).
a) Grid size = 2 m ($\lambda / \Delta x = 50$). b) Grid size = 5 m ($\lambda / \Delta x = 20$).

Figure 3.13: The model used in the second numerical test.
P-velocity=2000m/s, S-velocity=1155m/s,
Dencity=1000kg/m³, Qp=10000, Qs=10000,
Source=Ricker wavelet (110Hz)
Minimum wavelength (S-wave) is approximately 10m.

Figure 3.14: Time series of particle velocity perpendicular to the free surface with various grid sizes for the model given in Figure 5.
 Δx : grid size. λ : minimum wave length (approximately 10m) .
Image : Generalized image method. Vacuum : Vacuum formulation.
Proposed : Proposed method.

Figure 3.15: Receiver locations with a wave field of particle velocity (vertical component) at tome 45msec.

Figure 3.16: Particle velocity (horizontal component) collected at the receivers located before the ramp.

Figure 3.17: Particle velocity (vertical component) collected at the receivers located before the ramp.

Figure 3.18: Particle velocity (horizontal component) collected at the receivers located behind the ramp.

Figure 3.19: Particle velocity (vertical component) collected at the receivers located behind the ramp.

Figure 3.20: Particle velocity (horizontal component) collected at the shallowest receiver located behind the ramp.

a) Model A. Grid size = 0.5m, Step size = 0.5m.

b) Model F. Grid size = 0.5m, Step size = 2.0m.

Figure 3.21: Two models from table 2 used in the numerical test, with a wavefield of particle velocity (vertical component) at time 25msec.
P-velocity=5082m/s, S-velocity=3000m/s, Density=2000kg/m³

Figure 3.22: The definition of the grid size and the step size.

a) Model A. Grid size = 0.5m, Stepsize=0.5m.

b) Model B. Grid size = 0.5m, Step size = 1m.

c) Model C. Grid size = 1.0m, Step size = 1.0m.

Figure 3.23: The Particle velocity (vertical component) from the simulation with the various grid and step sizes for the models given in Table 3.3.

d) Model D. Grid size = 0.5m, Step size = 1.5m.

e) Model E. Grid size = 1.5m, Step size = 1.5m.

Figure 3.23, continued:

f) Model F. Grid size = 0.5m, Step size = 2.0m.

g) Model G. Grid size = 2.0m, Step size = 2.0m.

Figure 3.23, continued:

h) Model H. Grid size = 0.5m, Step size = 3.0m.

i) Model I. Grid size = 3.0m, Step size = 3.0m.

Figure 3.23, continued:

Figure 3.24: The Particle velocity (vertical component) from the simulation with the various grid and step sizes for the models given in Table 3.3.

Model A

Time= 100.0

Figure 3.25: The model (A) used in the numerical test, with a wavefield of particle velocity (vertical component) at time 100msec.
P-velocity=6000m/s, S-velocity=3000m/s, Density=1800kg/m³.

Model B

Time = 100.0

Figure 3.26: The model (B) used in the numerical test, with a wavefield of particle velocity (vertical component) at time 100msec.
P-velocity=6000m/s, S-velocity=3000m/s, Density=1800kg/m³.

a) Grid size = 0.5m ($\lambda / \Delta x = 60$).

b) Grid size = 1.0m ($\lambda / \Delta x = 30$).

Figure 3.27: The Particle velocity (vertical component) from the simulation with the various grid sizes for the model (A) in Figure 3.25.

c) Grid size = 2.0m ($\lambda / \Delta x = 15$).

d) Grid size = 3.0m ($\lambda / \Delta x = 10$).

Figure 3.27, continued:

a) Grid size = 0.5m ($\lambda / \Delta x = 60$).

b) Grid size = 1.0m ($\lambda / \Delta x = 30$).

Figure 3.28: The Particle velocity (vertical component) from the simulation with the various grid sizes for the model (B) in Figure 3.26.

c) Grid size = 2.0m ($\lambda / \Delta x = 15$).

d) Grid size = 3.0m ($\lambda / \Delta x = 10$).

Figure 3.28, continued:

Chapter4

Variable Grid Finite-difference Method

4.1 Introduction

A uniformly spaced grid used to model an irregular surface and low velocities in the shallow subsurface require large portions of the model to be over sampled. The resulting memory requirements would severely limit the size of the models. An obvious solution to this dilemma is to use a finer grid in the vicinity of the free surface compared to the deeper parts of the model. The variable grid approach allows us to vary the discretization of the model and the wavefield as required by the velocity structure. Compared to a standard uniform finite-difference grid approach, this method saves a considerable amount of memory and computations. Therefore, we combined the variable grid method with our irregular free-surface modeling.

In the variable grid method, one has to decide a particular size ratio of fine grids to coarse grids (grid ratio). The selection of the appropriate grid ratio is important. The large grid ratio can improve the efficiency of the computation. However, as the grid ratio increases, the numerical inaccuracy due to the change of the grid sizes may increase (De Lilla, 1997). In this study, I employ the grid ratio of three for the following reasons: First, the smaller grid ratio is better for the stability and accuracy. Second, the previous chapter has revealed that the calculation of irregular surface required at least 30 grid-points per wavelength for sufficiently accurate result. Thus, if we use a three times finer grid in the near surface region, other region can be calculated by using ten grid-points per wavelength that is the optimum number of O(2,4) staggered grid scheme (see the Chapter 2). Finally, an odd value for grid ratio is preferable, as it will be shown later.

4.2 One-dimensional Variable Grid Method

4.2.1 Computation Procedure

At first, I will describe an one-dimensional case for the sake of simplicity. Figure 4.1 shows the one-dimensional staggered grid and a computation procedure. If we choose an odd number for the grid ratio, the particle velocities and the stresses on both coarse and fine grids can be placed in exactly the same place. This character of the odd grid ratio can significantly reduce the complication of the computation. Therefore, we can say that the odd grid ratio is preferable for the velocity-stress staggered grid. The computation procedure is the following (Figure 4.1).

1. Update the stresses on the coarse grid.
2. At the boundary on the fine grid (Figure 4.1 A), replace the stresses on the fine grid by the stresses on the coarse grid, calculated in the first step.
3. Update the stresses on the fine grid without a boundary. At the next boundary (Figure 4.1 B), a fourth-order stencil cannot be used. Therefore, we use a second-order approximation there.
4. Replace coarse grid stresses within the fine grid region by the fine grid. Subsequently, the same procedure is applied to the velocity fields, and the second step can be omitted.

4.2.2 Numerical Example

I will show two, one-dimensional, numerical examples to demonstrate the accuracy and efficiency of this variable grid method. The first model is an acoustic homogeneous model with velocity of 2000m/s and a density of 2000kg/ m³. The source is a point explosion with a 100 Hz Ricker wavelet. Then, the approximate wavelength is 20m. The receivers are

located 0m to 190m from the source at 10m distance interval. The three times finer grids were placed at the distance between 100m and 145m. The grid size was 1m in the coarse grid region and 1/3m in the fine grid region. Figure 4.2 shows the particle velocity from the simulation. Waveforms seem not to be affected by the change of the grid sizes. Figure 4.3 shows the same waveforms with large gain. In Figure 4.3, we can see reflection from the boundary between the two grid sizes. However, the amplitude of the reflection is sufficiently small. Figure 4.4 shows the maximum amplitude of waveforms. From Figure 4.4, we can see that the effect of the change of grid sizes on the amplitude is less than 0.03%. Table 4.1 shows the amplitude of the direct and the reflected waves at the distance of 50m. The amplitudes of the reflected waves from the boundary between different grid sizes are less than 0.01% of the incident waves. The amplitude ratio of the reflected waves to the incident waves is sufficiently small compared with the reflection from the absorbing boundary (Peng and Toksöz, 1995). Therefore, I consider that these numerical reflection can be neglected in the practical simulation.

Table 4.1 The amplitude of the direct and the reflected waves at the distance of 50m.

	Amplitude	Amplitude ratio
Direct wave (40msec)	2.500231	100
First reflection (90msec)	0.000197	0.00788
Second reflection (130msec)	0.000117	0.00468

The second model is an heterogeneous acoustic model. P-velocities and densities are shown in Table 4.2. The source is a point explosion with a 100 Hz Ricker wavelet. The approximate wavelength in each velocity region is also shown in Table 4.2.

Table 4.2 Velocity model and corresponding wave length

Distance	Velocity(m/sec)	Density(kg/m ³)	Wavelength(m)	Wavelength/grid size	
				Coarse(1m)	Fine(1/3m)
~105m	2000	1000	20	20	60
105~140m	500	1000	5	5	15
140m~	4000	2000	40	40	120

The receivers are located 10m to 190m from the source at 10m distance intervals. Two

simulations were performed. The first simulation employed the uniform grid of 1m and second model employed the variable grid in which the coarse grid of 1m and the fine grid of 1/3m were used. In the variable grid model, three times finer grids were placed at the distance between 100m to 145m. Figure 4.5 and Figure 4.6 shows the particle velocity from the simulations. From the simulation of the uniform grid (Figure 4.5), we can see obvious numerical dispersion due to the small grid-points per wavelength in the low velocity zone (105 to 140m). We can also see that the result of the variable grid calculation (Figure 4.6) has no numerical dispersion and yields accurate waveforms.

4.3 Two-dimensional Variable Grid Method

4.3.1 Computation Procedure

Figure 4.7 shows the two-dimensional, staggered, finite-difference grid in the vicinity of the boundary between different grid sizes. We use a three times finer grid (both horizontally and vertically) in the near surface or low velocity region compared to the rest of the model. Figure 4.8 shows the computation procedure of the proposed variable grid method. The computation procedure can be summarized as follows:

1. Update the stresses on the coarse grid.
2. At the fine grid boundary, interpolate the stresses on the coarse grid calculated in the first step. Use a simple linear interpolation.
3. Update the stresses on the fine grid without the boundary. Next to the boundary, a fourth-order stencil can not be used. Therefore, a second-order approximation is used.
4. Coarse grid stresses within the fine grid region by the fine grid values. Apply the same procedure to the velocity fields. For the viscoelastic case, memory variables can be updated with the stress field simultaneously. Only the velocity field needs to be updated separately.

4.3.2 Improvement of Stability

The simple implementation of this variable grid method is not stable if we perform a large number of time steps. Figure 4.9 shows an example of the instability due to the variable grid computation. In order to reduce the instability of the computation, I applied averaging or weighting to the replacement of the coarse grid components within the fine grid (fourth step in Figure 4.8). Figure 4.10 shows several methods of averaging or weighting applied to the replacement components. Figure 4.11 shows the energy of the waveforms (shown in Figure 4.9) integrated over time for various methods. As a result of this comparison, I conclude that 5 point averaging or 9 point weighting is most stable. This result implies that the weight of the center node of the stencil is crucial for a stable computation.

4.3.3 Numerical Example

I will show several two-dimensional numerical examples to demonstrate the accuracy and efficiency of the variable grid method. The first model is an elastic Poisson solid with P- and S-velocities of 3000 and 1730m/sec respectively, and a density of 1800kg/m³. Figure 4.12 shows the location of the source and receivers, and the fine grid region. A vertical line source (z direction) is placed at the distance of 20m with the horizontal component (x-direction) of velocity. Thus the source generates a plane wave propagating in x direction. A 50 Hz Ricker wavelet is used as a source wavelet. The receivers were located at the right side of the source, 10 to 190m offset at 10m distance interval. The area of fine grid is located between distance of 100 and 150m as shown in Figure 4.12. The grid size is 3m in the coarse grid region and 1m in the fine grid region. Figure 4.13 shows the pressure (the sum of normal stresses: $\sigma_{xx} + \sigma_{zz}$) from the simulation. Waveforms seem not to be affected by the change of the grid sizes. I performed the simulation for the same velocity model but using the uniform grid of 3m. Figure 4.14 shows the difference between the variable grid and the uniform grid. We can see clear numerical reflection from the boundary between

different grid sizes. However, the amplitude ratio of the reflected waves to the incident waves is less than 0.3%. This ratio is larger than the one-dimensional case, but still sufficiently small compared with the reflection from the absorbing boundary (Peng and Toksöz, 1995). Figure 4.15 shows the comparison of maximum amplitude of the variable grid simulation with the uniform grid simulation. The increase of amplitude in large offset (source-receiver distance) may be due to the reflection from absorbing boundary. This figure implies that the effect of the change of grid sizes is much less than the effect of the reflected waves from the absorbing boundary. Therefore, I assume that these numerical reflection can be neglected in practical simulations.

In order to investigate the stability and accuracy in more complex structure, I performed two more numerical tests. The second model has a fine grid area that is square as shown in Figure 4.16. The model is an elastic solid with P- and S-velocities of 3000 and 1500m/sec respectively and density of 1800kg/m^3 . Two simulations with different source locations were performed. The location of the source and receivers are shown in Figure 4.16. The source is a point explosion with a 200Hz Ricker wavelet. The grid size was 1m in the coarse grid region and 1/3m in the fine grid region. Figure 4.17 and Figure 4.18 show the pressure collected at the receivers, together with the simulations in which the uniform grid of 1m was used. We can see the difference between the variable grid calculation and uniform grid calculation is sufficiently small wherever the source is placed.

The third model has the low velocity area which is covered by the fine grid. The velocity model and the location of a source and receivers are shown in Figure 4.19. The source is a P-wave source with a 50Hz Ricker wavelet. Velocities, densities and corresponding wave lengths are listed in Table 4.3.

Table 4.3 Velocities, density and corresponding wavelength for third model

	P-velocity (m/sec)	S-velocity (m/sec)	Density (kg/m ³)	Shortest wavelength (m)	Wavelength/grid size	
					Coarse (3m)	Fine(1m)
Background	3000	1500	1800	30	10	30
Low velocity area	1000	500	1500	10	3.333	10

Three calculations (uniform grid of 3m, uniform grid of 1m, variable grid of 3m in coarse grid and 1m in fine grid) were performed. Figure 4.20 shows the discretization of the velocity model at the vicinity of a velocity boundary. I would like to mention that the uniform grid of 1m and the variable grid use the different discretization of the velocity models. An example of the snapshots are shown in Figure 4.21. The snapshots of the uniform grid (3m) computation shows clear numerical dispersion due to the low velocity area (see the snapshot of 200msec). However, one of the variable grid computation yields an accurate result compared with the uniform grid of 1m. Furthermore, the boundary numerical reflections from different grid sizes are small in the variable grid computation. In the snapshots of variable grid computation, we can see a slight scattering that cannot be seen in the uniform grid of 1m. This scattering may be due to the difference of the discretization of the velocity model shown in Figure 4.20.

Figure 4.22 shows the pressure collected at the receivers. We can see the clear numerical dispersion in the waveforms from the uniform grid of 3m. However, the waveforms from the uniform grid of 1m and the variable grid (3m in coarse grid and 1m in fine grid) yield sufficiently accurate results. This example shows the efficiency of the variable grid method. There is a slight difference between the uniform grid of 1m and the variable grid. This difference may be due to the difference of discretization of velocity model.

4.4 The Calculation of the Irregular Free Surface Combined with Variable Grid Method

I applied the variable grid method to the calculation of irregular topography mentioned in

the Section 3.3. The first model is the flat surface with a slope. The area of the fine grid is shown in Figure 4.23. Here, I will show the case of 45° . Figure 4.24 shows the comparison of the uniform grid of 10m with the variable grid in which the coarse grid of 10m and the fine grid of 10/3m are used. Plotted waveforms are particle velocity parallel to the surface. We can see the waveforms in the variable grid have no numerical dispersion that can be seen in the waveforms of the uniform grid of 10m. Figure 4.25 shows the waveforms collected at 1000m source receiver offset with the various grid sizes. The result shows that a coarse grid size of 10 m combined with a fine grid size of 10/3m gives a more accurate solution than using a uniform grid of 5m.

The second model has the irregular surface. The area of the fine grid is shown in Figure 4.26. The calculated waveforms with the various grid sizes are shown in Figure 4.27. The result shows that the waveforms of a coarse grid size of 1 m combined with a fine grid size of 1/3m are as accurate as the solution using a uniform grid of 0.5m.

These two results suggest that approximately ten grid-points per wavelength, combined with the variable grid method, will give an accurate solution to a model with an irregular surface.

1-D Variable grid

Figure 4.1: The one-dimensional staggered grid and a computation procedure.

- ① Updating the stresses on the coarse grid.
- ② At the boundary on the fine grid (A), the stresses on the fine grid are replaced by the stresses on the coarse grid calculated in the first step.
- ③ Updating the stresses on the fine grid without a boundary. At the next boundary (B), a fourth-order stencil can not be used. Therefore, a second-order approximation is used at there.
- ④ Coarse grid stresses within the fine grid region are replaced by the fine grid. Subsequently, the same procedure is applied to the velocity fields, and the second step can be omitted.

Figure 4.2: Time series of particle velocity from the 1-D simulation. A model is an acoustic homogeneous model, with velocity of 2000m/sec and a density of 2000kg/m³. The three times finer grids are placed between 100m and 145m.

Figure 4.3: The same waveforms as Figure 4.2 with large gain.

Figure 4.4: The maximum amplitude of waveforms shown in Figure 4.2 and 4.3. Theoretically, the amplitude should be constant. The effect of the change of grid sizes on the amplitude is less than 0.03%.

Figure 4.5: The particle velocity from the simulation with the uniform grid of 1m.

Figure 4.6: The particle velocity from the simulation with the variable grid in which the coarse grid of 1m and the fine grid of 1/3m. Three times finer grids are placed at the distance between 100m to 145m.

Figure 4.7: The two-dimensional staggered finite-difference grid in the vicinity of the boundary between different grid sizes.

Figure 4.8: The computation procedure for the variable grid method.

- ① Updating the stresses on the coarse grid.
- ② At the fine grid boundary, the stresses are interpolated by the stresses on the coarse grid calculated in the first step. Simple linear interpolation is used.
- ③ Updating the stresses on the fine grid without the boundary. Next to the boundary, a fourth-order stencil can not be used. Therefore, a second-order approximation is used.
- ④ Coarse grid stresses within the fine grid region are replaced by the fine grid. Subsequently, the same procedure is applied to the velocity fields. For the viscoelastic case, memory variables can be updated with the stress field simultaneously. Only the velocity field is updated separately.

a) A source and receivers locations, and the fine grid region. A model is a homogeneous elastic solid with P- and S-velocities of 3000 and 1500m/sec, respectively, and a density of 1800kg/m^3 .

b) Pressure collected at the receivers.

Figure 4.9: Instability due to the variable grid implementation. a) A model used in the simulation. b) Pressure collected at the receivers. A source is a P-wave source with 100Hz Ricker wavelet. Δt (time step) is 0.05msec. 16384 time steps are performed.

Figure 4.10: The method of averaging and weighting applied to the replacement of the coarse grid components within the fine grid (Figure 4.8. Step④).

Figure 4.11: The energy of waveforms shown in Figure 4.9 with various weighting methods.

Figure 4.12: A source and receivers locations, and the fine grid region. A model is a homogeneous elastic solid with P- and S-velocities of 3000 and 1730m/sec, respectively, and a density of 1800kg/m³. The area of fine grid is located between distance of 100 and 150m. The grid size is 3m in the coarse grid region, and 1m in the fine grid region.

Figure 4.13: Pressure from the simulation. A 50 Hz Ricker wavelet is used as a source wavelet.

Figure 4.14: Difference between the variable grid and the uniform grid of 3m.

Figure 4.15: The maximum amplitude of waveforms shown in Figure 4.13 in comparison to the uniform grid of 3m. Theoretically, the amplitude should be constant. The effect of the change of grid sizes on the amplitude is less than 0.02%.

Figure 4.16: Sources and receivers locations, and the fine grid region. A model is a homogeneous elastic solid with P- and S-velocities of 3000 and 1500m/sec, respectively, and a density of 1800kg/m³. The source is a point explosion with a 200Hz Ricker wavelet. The grid size was 1m in the coarse grid region and 1/3m in the fine grid region.

Figure 4.17: Simulation with the source of 100m. Pressure collected at the receivers is plotted. a) small gain. b) large gain. Solid line: uniform grid of 1m. Dashed line: variable grid of 1/3 and 1m.

a) small gain.

b) large gain.

Figure 4.18: Simulation with the source of 40m. Pressure collected at the receivers is plotted. a) small gain. b) large gain. Solid line: uniform grid of 1m. Dashed line: variable grid of 1/3 and 1m.

Figure 4.19: Velocity model and the location of a source and receivers. The low velocity area (a inner circle) is covered by the fine grid (a outer circle). The source is a P-wave source with a 50Hz Ricker wavelet. Velocities, densities and corresponding wave lengths are listed in Table 4.3.

Figure 4.20: The different discretization of velocity model.

Uniform grid of 3m.

Variable grid.

Grid size is 3m in the coarse grid region and 1m in the fine grid region.

Uniform grid of 1m.

Figure 4.21: Wavefield snapshots (pressure) across a variable grid area and a low velocity area. The model is shown in Figure 4.19.

a) 40msec

b) 60msec

c) 80msec

Figure 4.21, continued.

d) 100msec

e) 120msec

Figure 4.21, continued.

f) 140msec

g) 160msec

Figure 4.21, continued.

h) 180msec

i) 200msec.

Figure 4.21, continued.

a) Solid line: variable grid of 1m and 3m. Dashed line: uniform grid of 3m.

b) Solid line: variable grid of 1m and 3m. Dashed line: uniform grid of 1m.

Figure 4.22: Pressure collected at the receivers. a) The variable grid of 1m and 3m and the uniform grid of 3m. b) The variable grid of 1m and 3m, and the uniform grid of 1m.

Figure 4.23: Fine grid area, and a source and receivers locations. The model is a Poisson solid with P- and S-velocities of 3000 and 1730m/s, respectively, and a density of 2500kg/m³. A P-wave source (15Hz Ricker wavelet) located 30 m below the surface is employed. Receivers are located 50 m below the surface.

Figure 4.24: Comparison of the uniform grid of 10m with the variable grid in which the coarse grid of 10m and the fine grid of 10/3m are used. Plotted waveforms are particle velocity parallel to the surface.

Figure 4.25: Waveforms collected at 1000m source receiver offset with the various grid sizes.

Figure 4.26: The area of the fine grid. The grid size is $1/3\text{m}$ in the fine grid, and 1m in the coarse grid. The model is a elastic Poisson solid with P- and S-velocities of 2000 and 1155m/s , respectively, and a density of 1000kg/m^3 . A P-wave source (110Hz Ricker wavelet) located at a distance of 30m and 5m below the surface is employed.

Figure 4.27: A comparison of waveforms with the various grid sizes. Particle velocity (vertical component) is plotted.

Chapter 5

Wave Propagation in Near Surface Region

5.1 Introduction

The wave propagation in the near surface region is complicated by many factors, such as large velocity contrasts, strong heterogeneity, topographic relief and large attenuation. In some cases, such complications can contaminate the seismic data in shallow seismic surveys. With careful analysis and interpretation, these complications can provide valuable information of the near surface region. The surface seismic methods can be classified into three main groups; the reflection method, the refraction method and the methods using surface waves. The effect of a complicated near surface structure on these methods are different as summarized below.

When seismic reflection method is applied to great depths, such as those in the exploration of petroleum, the contamination of data from near surface scattering can be removed by using the differences between the data character (such as frequency, apparent velocity, etc.) of the signal and the noise contaminant. In the shallow seismic reflection, it is difficult to remove this contamination, since the structures of interest are close to the structures that cause the contamination (Figure 5.1). Furthermore, wavelengths in the shallow reflection seismic method are relatively long compared to the structures of interest. This character strongly complicates the analysis of the shallow reflection data compared with the seismic data used in petroleum exploration. In spite of the significant differences between the shallow reflection and the petroleum exploration methods, analysis methods used for petroleum exploration are applied to the shallow reflection data. This inappropriate

approach may result in the poor accuracy and reliability. For example, Steeples and Miller (1998) referred to this difficulty and pointed out some pitfalls in the shallow reflection method.

The seismic refraction method may be applied to delineation of shallow subsurface structures better than the reflection method, since it only concerns with first arrival traveltimes. The recent application of tomographic reconstruction technique to refraction analysis has greatly improved the accuracy and reliability of the method (Hayashi and Saito, 1996; Lanz et al. 1998; Zhang and Toksöz, 1998; Hayashi, 1999). The obvious limitation in the seismic refraction method is the non-uniqueness of its analysis. Lay and Wallace (1995) shows the difficulty in a traveltime inversion (Figure 5.2). A velocity model that has clear layer boundaries and a model in which velocity increases smoothly, can yield the same traveltime curves. As long as we only use the first arrival traveltimes, the non-uniqueness can not be resolved. However, waveforms should have some differences between a layered model and a smooth model.

In the early 1980s, a spectral analysis of surface waves (so called SASW) was introduced (Nazarian et al. 1983). Park et al (1999a, b) improve the SASW to the multi-channel analysis of surface waves (MASW) and Xia et al. (1999a) generated two-dimensional S-wave velocity map by MASW. The SASW or MASW can estimate S-wave velocity structures that are extremely valuable in geotechnical work (Imai and Tonouchi, 1982). Therefore, the SASW (MASW) has been increasingly used in various civil engineering investigation. The SASW (MASW) uses the dispersive character of Rayleigh waves. Body waves can contaminate the data in the SASW (MASW), and the separation of the surface waves and the body waves may be one of the difficulties in the SASW (MASW) analysis. The SASW (MASW) analysis relies on an one-dimensional analysis of dispersion curves (Xia et al., 1999b). However, the analysis of dispersion curves may have the non-uniqueness that is similar to that of the refraction method. The number of theoretical studies about surface wave propagation in the heterogeneous near-surface region is relatively small.

For example, the effect of two and three-dimensional structures on the method has not been investigated.

In order to improve the accuracy and reliability of the shallow seismic methods, we have to understand the fundamental wave propagation in the near surface region, and establish a new analysis approach optimized for the shallow seismic methods. I believe that finite-difference modeling can be a valuable tool for this objective. In this chapter, I will perform several numerical simulations to show the wave propagation characters in a near surface region. Then, I will compare observed waveform data with theoretical waveforms calculated by the finite-difference method.

5.2 Near Surface Modeling

A viscoelastic two-layer structure is chosen as the model for numerical simulations. The objective of the simulations is to evaluate the effect of the following parameters on surface seismic data.

- * Thickness of the first layer (low velocity layer).
- * Attenuation of the first layer.
- * Presence of velocity gradient.
- * Presence of a step on the surface.
- * Presence of a step on the layer boundary.

Figure 5.3 shows the model. Velocities, densities and Q are listed in Table 5.1.

Table 5.1 Velocities, density and Q of model

	P-velocity (m/sec)	S-velocity (m/sec)	Density (kg/m ³)	Q _p and Q _s	Wavelength(m)	
					P-waves	S-waves
First layer	1000	500	1500	10 or 10000	10	5
Second layer	3000	1500	2000	10000	30	15

A P-wave source with a 100Hz Ricker wavelet is placed at the distance of 7.5m and the depth of 2m. It is obvious that the effect of near surface structures on the surface seismic record is frequency dependent. I will use a particular frequency, 100Hz, that is a common frequency in the shallow seismic methods. Receivers are placed on the surface between the distance of 10 to 95m at 5m distance intervals. In all the following figures, the vertical component of particle velocity collected at the receivers will be plotted, and waveforms will be normalized by each trace as long as no particular notes.

Figure 5.4 shows the waveforms for the homogeneous model with a flat surface and Q_p and Q_s of 10000. The first arrival is the body waves (P-waves) and the second arrival is the surface waves (Rayleigh waves). It should be noted that the surface seismic data is simple if the model is homogeneous half space.

5.2.1 Thickness of the First Layer

The first example is an investigation about the change of waveforms due to the thickness of the low velocity first layer. In this example, the Q_p and Q_s of the first layer are set to 10000, and it can be considered an elastic case. The surface topography and the subsurface layer boundary are flat. Figure 5.5 shows the wave forms with various thicknesses. The thickness of the first layer was set to 2.5m, 5.0m, 10.0m, 20.0m and 30.0m.

For the thickness of 2.5m, we can see clear dispersive waves. However, for the thickness of 5.0m, the appearance of waveforms change drastically. The coherent dispersive waves disappear and scattered waves dominate. For the thickness of 10.0m, multiple reflection or refraction starts to dominate waveforms. For the thickness of 20.0m, we can recognize reflected waves clearly. For the thickness of 30.0m, we can see the primary reflection and two multiple reflections. In the large source-receiver offset (more than a distance of 30m), we can recognize clear P-S_v converted waves. It should be noted that the amplitude of P-S_v

reflections are larger than the P-P reflections.

5.2.2 Attenuation of the First Layer

Low velocity layers just below the surface generally have high attenuation. In this example, I use the same model as the first example but set the Q_p and Q_s of the first layer to 10. Figure 5.6 shows the waveforms with various thickness in comparison to the first example (where Q_p and Q_s of the first layer are 10,000). For the thicknesses of 2.5m and 5.0m, the appearance of waveforms changes drastically from the first example. For the thickness of 2.5m, low velocity surface waves disappear and multiple reflections and refraction dominate. For the thickness of 5.0m, scattered waves decrease amplitude and frequency. As the thickness increases, the difference between two simulations decreases. However, for the thickness of 30m, the effect of attenuation is still large. This example clearly shows the effect of the anelastic attenuation on the surface seismic data.

5.2.3 Presence of Velocity Gradient

Generally, seismic velocities in the near surface region increase with depth. In some sites, velocities increase with velocity boundaries and layers, but in other sites, velocities smoothly increase with depth. However, surface seismic methods, such as the refraction and reflection methods, can not be easily applied to both structures as mentioned before. The conventional refraction method that assume discrete refractors, faces difficulties in the presence of a velocity gradient. Likewise, the reflection method needs reflectors. In the near-surface seismic surveys, sites that have no clear layer boundaries fall difficulties with both the refraction and reflection methods.

In order to overcome these difficulties, we need to investigate wave propagation with a velocity gradient. As a preliminary study, I show several simulations for the models that have velocity gradients. Figure 5.7 defines the thicknesses of the first layer, in the models

with velocity gradient. In the calculations, Q_p and Q_s are a constant 10000. Figure 5.8 shows the waveforms with various thickness in comparison to layered models.. We can see clear distinctions between the layered models and the smooth models. This implies that using waveforms may solve the non-uniqueness in the refraction method. For the thickness of 2.5m, we can see the typical dispersion of Rayleigh waves. As the thickness increases, dispersive Rayleigh waves converge on two waves, S-waves and Rayleigh waves.

Next, I show the transition of waveforms from a layered model to a smooth model. Figure 5.9 defines models. N is the number of layers. $N=2$ indicates a two-layer model and $N=\infty$ indicates a smooth model. Figure 5.10 shows the waveforms with various N . We can only see clear reflection in the case of $N=2$ (two layer model). For the $N=3$ or $N=4$, the appearance of waveforms are too complicated to distinguish the reflections from layer boundaries. For N greater than 6, the appearance of waveforms looks like the waveforms from the smooth model.

5.2.4 Presence of a Step on the Surface

So far, I have been concerned with the models which have a flat surface and a layer boundary. Next, I will show the effect of an irregular surface on a seismic record. To avoid complexities, I limit the study to one small step on the surface (Figure 5.3). At first, I show the simulation with an elastic homogeneous model (V_p of 3000m/sec and V_s of 1500m/sec). Figure 5.11 shows the waveforms with various step sizes. We can see Rayleigh waves scattered from the step, and converted from the body waves at the step. As the step size increases, the amplitude of scattered waves increases. However, the magnitude of scattered waves in the step size of 10m and 20m, is almost the same. In this calculation, the wavelength is approximately 30m for P-waves and 15m for Rayleigh waves. It seems that if the step size is larger than the wavelength, the step size no longer affects the amplitude of scattered waves.

Next, I show the simulation for a two-layer model. Figure 5.12 shows the waveforms for the two-layer model together with the homogeneous case. The model has only the low velocity layer behind the step. We can see that the appearance of waveforms changes drastically at the step. Figure 5.13 shows the waveforms with various thicknesses of the first layer. The step size is fixed at 10m. We can see that the appearance of waveforms changes drastically with the thickness of the first layer. The result implies that the effect of the thickness of the low velocity first layer is much larger than the effect of the step on the surface.

5.2.5 Presence of a Step on the Layer Boundary

Here, I will show the effect of an irregular layer boundary on a surface seismic record. Figure 5.14 shows the waveforms from the simulation for the flat surface model with the low velocity layer. The low velocity layer is placed only in the left half of the model. The dispersive surface waves due to the thin low velocity layer are clearly blocked at the edge of the low velocity layer. Figure 5.15 shows the waveforms from the step model on the layer boundary together with the previous one. This model is almost the same as the previous model, but the low velocity layer continues to the right side of the model. We can see that the surface waves are not blocked as in the previous calculation. This example clearly shows the significant effect of a thin low velocity layer on the surface seismic record.

Next, I will show the effect of the step shape on the layer boundary. Figure 5.16 shows models used in the simulations and corresponding waveforms. We can see the change of the appearance of waveforms due to the step shape. However, this change is relatively small in comparison to the effect of the first layer thickness (see Figure 5.11 to 5.15). The results imply that in the surface seismic methods applied to near surface region, the effect of the low velocity layer thickness is much larger than the effect of the step shape on the layer boundary.

5.2.6 Application of the Reflection Method to Active Faults Survey

Finally, I show the calculations that simulate the reflection method applied to an active faults survey. Figure 5.17 shows the velocity models used in the simulations together with the source locations. Material properties, the receiver locations and a source wavelet are identical to the previous simulations. Two models are used in the simulation as shown in Figure 5.17. The first model has a flat surface and the second model has a step on the surface. The deformation of bed rock (second layer) is 10m. The finite-difference calculation is performed for four different sources 7.5m, 37.5m, 67.5m and 97.5m. The fault is located at a distance of 52.5m in the flat model and 50.0m in the step model.

Figure 5.18 shows the waveforms from the simulation for the first model (flat surface model). In this calculation the Q_p and Q_s of the first layer are set to 10000. We can compare Figure 5.18 with Figure 5.5 showing the waveforms from the flat layer boundary models. Using a source of 7.5m, we can see the obvious complication due to the fault, at the larger distances (60 to 95m). However, it is difficult to detect the presence of the fault from the waveforms. Using a source of 37.5m, we can see the obvious discontinuity of the reflected waves behind the fault (distance of 60m). It is possible to detect the presence of the fault from the waveforms. The sources at the distance of 67.5m and 97.5m are located at the upper side of the fault. It is difficult to detect the presence of the fault from the waveforms of these two sources.

Figure 5.19 shows the waveforms from the second model simulation (a step on the surface). The figures above show the simulations in which the Q_p and Q_s of the first layer are set to 10000. The figures below show the simulations in which the Q_p and Q_s of the first layer are set to 10. The waveforms in Figure 5.19 are more complicated in comparison to the flat surface case (Figure 5.18). In the flat surface case, we can detect the fault presence in the seismic record of 37.5m. It is more difficult to detect the fault presence from the waveforms

in Figure 5.19. In particular, it is difficult to detect the fault from the simulation for the anelastic case. It is obvious that the real subsurface structures are more complicated, and the real surface seismic record is more complicated. Detecting the subsurface structure from the field record is sometimes difficult in the actual reflection surveys.

5.2.7 Conclusions from simulation

As the result of the simulations, the effects of the five subsurface condition on the surface seismic records may be ordered (from significant to slight) as follows:

1. Thickness of the first layer (low velocity layer).
2. Presence of velocity gradient.
3. Attenuation of the first layer.
4. Presence of a step on the surface.
5. Presence of a step on the layer boundary.

Figure 5.5 clearly shows the effect of low velocity layers just below the surface. Figure 5.8 and 5.10 show velocity gradient has large effect on the character of the surface seismic record. These results imply that reflection and surface waves can be used to obtain information about near surface structure. Figure 5.6 also shows the large effect of anelastic attenuation. Generally, the anelastic attenuation is large in the near surface region, and its effects should be considered in near surface wave propagation studies. Figure 5.16, 5.18 and 5.19 show that the layer boundary shape has a small effect on the surface seismic record in comparison to the presence of low velocity layers.

5.3 Comparison of Observed and Theoretical Data in the Refraction Method

5.3.1 2-D Seismic Refraction Data

A two-dimensional seismic refraction method was performed in the mountainous area of Japan. The dynamite, placed at the depth of about 1m, was used as source. The quantity of dynamite was approximately 100 to 200g. The waveform data were collected at the receivers placed on the surface. The velocity seismometers (geo-phone : OYO SMC70) that have the natural period of 28Hz were used as the receivers, and a OYO DAS-1 was used as the data acquisition system. The receivers collected the vertical component of particle velocities.

First arrival traveltimes were picked and the non-linear traveltimes tomography (Hayashi and Saito, 1996) was applied to the traveltimes data. The result of the tomography (P-wave velocity structure) is shown in Figure 5.20. The traveltimes tomography of refraction data has non-uniqueness, and therefore, we can show several velocity models that satisfy the observed traveltimes data. Figure 5.20 (a) shows a velocity model with the assumption that a velocity structure is a three-layer model. Figure 5.20 (b) shows a velocity model with the assumption that a velocity increases smoothly with depth. Figure 5.21 shows the comparison of the observed and theoretical traveltimes data. We can see that both velocity models can satisfy the observed data with a smaller error than an error due to traveltimes picking. We can not decide which velocity model is true, as long as we use only the first arrival travel times.

In the previous section, I showed the large difference of waveforms between layered and smooth models based on my simulations. In this section, I will calculate the theoretical waveforms for both layered and smooth velocity models, and compare the theoretical and observed waveforms.

5.3.2 Model Building for Viscoelastic Finite-difference Calculation

Since I have a only P-velocity model, I have to assume other material properties. S-velocity is assumed to be the $1/\sqrt{3}$ of P-velocity (Poisson solid). A density is assumed to be constant 2000kg/m^3 . P-wave sources are placed at the depth of 1m, and a 50Hz Ricker wavelet is used as a source wavelet. The center frequency of the source wavelet is chosen from the frequency of the observed data. The simulations are performed for two cases. The first simulation uses a model in which the Q_p and Q_s are set to constant 10000. The second simulation uses a model in which the Q_p and Q_s are set to 5 in the region where the P-velocity is slower than 2000m/sec, and 100 in the region where the P-velocity is faster than 2000m/sec. The Q is generally very small in the near surface region of a mountainous area, and the later model is more realistic than the first model.

In the smooth model, velocity model is represented by 30 by 15 (horizontal by vertical) arbitrary quadrilateral cells. In the finite-difference calculations, the grid size of 0.125m is used and the model (150m by 100m) is gridded into 1200 by 800 grids. The variable grid method is not used in this simulation.

Since the absolute amplitude of waveforms highly depends on the coupling of geo-phones and the ground, I will not be concerned with absolute amplitude of waveforms in this comparison, but will be concerned only with the general appearance.

5.3.3 Comparison of Layered and Smooth Models Using the Source of 122.5m

First, I will compare the layered and smooth models. Figure 5.22 shows the observed common source data with the source at the distance of 122.5m. In Figure 5.22, waveforms are normalized by each trace. We can clearly see first arrival (P-waves) and the later low

frequency arrivals. The continuity of the later arrival implies that the model has no sharp layer boundary. Figure 5.23 shows theoretical waveforms for the layered model. Regardless of the attenuation, the appearance of waveforms differs from the observed data. The theoretical waveforms are much scattered in comparison to the observed data. Figure 5.24 shows the theoretical waveforms for the smooth model. The waveforms for a small Q model (c) are more similar to the observed data in comparison to those of the layered model. This result strongly suggests that the true model is a smooth one. We can see several obvious characters in the observed data, such as the first arrival traveltimes, the frequency difference between the first arrival and the later arrival, the amplitude ratio of the first arrival and the later arrival, the apparent velocity of the later arrival, and complicated waveforms at smaller distances (0 to 40 m). We can see that the theoretical waveforms of the smooth model with small Q have similar characters.

5.3.4 Comparison of Observed and Theoretical Data for Other Sources

Next, I calculated theoretical waveforms for different source locations. The source locations are shown in Figure 5.25 together with the P-velocity model (identical with Figure 5.20 (b)). Based on the result of the source at the 122.5m, I use the smooth model with small Q in a low velocity area, Q_p and Q_s were set to 5 in the region where the P-velocity is slower than 2000m/sec, and 100 in the region where the P-velocity is faster than 2000m/sec. The sources are placed at the depth of 1m and a 50Hz Ricker wavelet is used as a source wavelet. Figure 5.26 shows the theoretical waveforms in comparison to the observed data. The comparison of the theoretical waveforms with observed data can be summarized as follows.

1. 3.5m (Figure 5.26 (a))

The first arrival traveltimes of the theoretical waveforms agrees with the observed data. In the observed data, there is a clear first arrival and a relatively small later arrival (100 to 170msec) which shows good coherency and the same amplitude as the first arrival. In the

theoretical waveforms, we can see the clear first arrival. The later arrival is not clear in comparison to the observed data.

2. 27.5m (Figure 5.26 (b))

The observed data shows a obvious later arrival that has a larger amplitude than the first arrival. The theoretical data also shows the obvious later arrival with a large amplitude. A different appearance may somewhat be due to the difference of source frequency. The first arrival travelttime of the theoretical waveforms agrees with the observed data.

3. 57.0m (Figure 5.26 (c))

There is a clear later arrival that shows large amplitude and good coherency in the observed data. The coherency of the later arrival is better in the large distance than in the small distance. We can see that the theoretical waves have similar characters. A different appearance may somewhat be due to the difference of source frequency. The first arrival travelttime of the theoretical waveforms agrees with the observed data.

4. 77.5m (Figure 5.26 (d))

In the observed data, the later arrival is not clear and coherent in comparison to the other sources. The theoretical waveforms show a similar character. The first arrival travelttime of the theoretical waveforms agrees with the observed data.

5. 102.5m (Figure 5.26 (e))

There is a clear later arrival that shows large amplitude and good coherency in the observed data. Theoretical waveforms also show the clear later arrival that has large amplitude in comparison to the first arrival. The frequency of the later arrival in the theoretical waveforms is higher than the observed data. The first arrival travelttime of the theoretical waveforms agrees with the observed data.

6. 147.5m (Figure 5.26 (f))

We can see a clear first arrival and low frequency later arrival in the observed data. Theoretical waveforms also show the clear first arrival and the later arrival. The frequency and apparent velocity of the later arrival have a clear difference between the theoretical and observed data. The first arrival traveltime of the theoretical waveforms agrees with the observed data.

5.3.5 Discussion

With my modeling, it was possible to obtain synthetic seismograms that were close to observed field data. We can say that the model used in the simulation was not far from the true structure. However, the theoretical waveforms did not completely agree with the observed data. The reason for this disagreement may be due to following factors:

1. Q and S-velocity structures

Since a P-velocity model was constructed by traveltime tomography, the accuracy and reliability of the P-velocity model are relatively high. However, Q and S-velocity models were assumed to be simple models based on the P-velocity model. The quantity of Q and Poisson's ratio were also assumed. A real structure should be more complicated. In order to perform a complete simulation, we have to obtain the Q, S-velocity and density models by based on observation.

2. Source wavelet

I used the Ricker wavelet as a source wavelet. It is obvious that a real source wavelet is more complicated. I also used a 50Hz Ricker wavelet for all sources. It is clear that each source has a different wavelet and frequency in the observed data. Furthermore, I assumed that the sources are isotropic. However, the emission of energy from real sources may not be isotropic.

3. Three-dimensional structure

Although the survey line is almost perpendicular to topography and geological structure, a three-dimensional structure may have an effect on the seismic record. In order to perform a complete simulation, we have to build a three-dimensional model and apply a three-dimensional calculation.

This simulation should be considered as a preliminary step. In order to perform complete modeling, we need to avoid the assumptions, build a three-dimensional model, and calculate three-dimensional wave propagation. Such a complete modeling is possible, but extremely expensive and time consuming at this time. However, the comparison of the waveforms for the layered model with the smooth model shows that the finite-difference modeling could be a strong analysis tool for shallow seismic methods, in spite of the assumptions.

Figure 5.1. The comparison of the exploration of petroleum and the shallow seismic methods

Figure 5.2: First-arrival traveltime curve that is exactly matched by the three velocity profiles shown below. (from Lay and Wallace (1995))

Figure 5.3: Description of the near surface model

Figure 5.4: Waveforms for the homogeneous model with flat surface and Q_p and Q_s of 10000.

a) Thickness of the first layer is 2.5m.

b) Thickness of the first layer is 5m.

c) Thickness of the first layer is 10m.

Figure 5.5: Waveforms with various thicknesses. Q_p and Q_s of the first layer are set to 10000.

d) Thickness of the first layer is 20m.

e) Thickness of the first layer is 30m.

Figure 5.5, continued:

a) Thickness of the first layer is 5m. Q_p and Q_s of the first layer are 10000.

	Q_p	Q_s
1st layer	10000	10000
2nd layer	10000	10000

b) Thickness of the first layer is 2.5m. Q_p and Q_s of the first layer are 10.

	Q_p	Q_s
1st layer	10	10
2nd layer	10000	10000

Figure 5.6: Comparison of elastic models with varying Q values. Q_p and Q_s of the first layer are set to 10000. Thickness of the first layer is 2.5m.

	Q_p	Q_s
1st layer	10000	10000
2nd layer	10000	10000

c) Thickness of the first layer is 5m. Q_p and Q_s of the first layer are 10000.

	Q_p	Q_s
1st layer	10	10
2nd layer	10000	10000

d) Thickness of the first layer is 5m. Q_p and Q_s of the first layer are 10.

Figure 5.6, continued: Thickness of the first layer is 5m.

	Q_p	Q_s
1st layer	10000	10000
2nd layer	10000	10000

e) Thickness of the first layer is 10m. Q_p and Q_s of the first layer are 10000.

	Q_p	Q_s
1st layer	10	10
2nd layer	10000	10000

f) Thickness of the first layer is 10m. Q_p and Q_s of the first layer are 10.

Figure 5.6, continued: Thickness of the first layer is 10m.

	Q_p	Q_s
1st layer	10000	10000
2nd layer	10000	10000

g) Thickness of the first layer is 20m. Q_p and Q_s of the first layer are 10000.

	Q_p	Q_s
1st layer	10	10
2nd layer	10000	10000

h) Thickness of the first layer is 20m. Q_p and Q_s of the first layer are 10.

Figure 5.6, continued: Thickness of the first layer is 20m.

	Q_p	Q_s
1st layer	10000	10000
2nd layer	10000	10000

i) Thickness of the first layer is 30m. Q_p and Q_s of the first layer are 10000.

	Q_p	Q_s
1st layer	10	10
2nd layer	10000	10000

j) Thickness of the first layer is 30m. Q_p and Q_s of the first layer are 10.

Figure 5.6, continued: Thickness of the first layer is 30m.

Figure 5.7 Description of thickness of first layer for the layered model and the smooth model.

a) Layered model. Thickness of the first layer is 2.5m.

b) Smooth model. Thickness of the first layer is 2.5m.

Figure 5.8: Comparison of a layered model and a smooth model. Thickness of the first layer is 2.5m

c) Layered model. Thickness of the first layer is 5m.

d) Smooth model. Thickness of the first layer is 5m.

Figure 5.8, continued: Thickness of the first layer is 5m

e) Layered model. Thickness of the first layer is 10m.

f) Smooth model. Thickness of the first layer is 10m.

Figure 5.8, continued: Thickness of the first layer is 10m

g) Layered model. Thickness of the first layer is 20m.

h) Smooth model. Thickness of the first layer is 20m.

Figure 5.8, continued: Thickness of the first layer is 20m

Figure 5.9: Description of N.

a) N=2.

b) N=3.

c) N=4.

Figure 5.10: Waveforms from a layered model to smooth a model.

d) N=6.

e) N=10.

f) N=20.

Figure 5.10, continued:

a) flat surface.

b) step size=2.5m.

c) step size=5m.

Figure 5.11: Waveforms with various step sizes.

d) step size=10m.

e) step size=20m.

Figure 5.11, continued:

a) Two-layer model.

b) Two-layer model.

Figure 5.12: Comparison of a homogeneous model and a two-layer model with the presence of a step.

a) Thickness of the first layer is 2.5m.

b) Thickness of the first layer is 10m.

Figure 5.13: Waveforms for a two-layer model with the presence of a step.

c) Thickness of the first layer is 20m.

d) Thickness of the first layer is 30m.

Figure 5.13, continued:

a) Normalized by each trace.

b) No normalized.

Figure 5.14: Waveforms for a two-layer model with the low velocity layer which placed only in the left half of the model.

a) Low velocity layer is placed only in the left half of the model.

b) Low velocity layer continues to right side of the model.

Figure 5.15: Waveforms for a two-layer model with the low velocity layer which has a step on the boundary. The waveforms are normalized by each trace.

Figure 5.16: Waveforms from the simulation with various step shapes.

Figure5.16: continued

a) Flat surface model. The fault is located at a distance of 52.5m.

b) Step model. The fault is located at a distance of 50.0m.

Figure 5.17: Active fault models. The first model (a) has a flat surface and the second model (b) has a step on the surface. The deformation of bed rock (second layer) is 10m. The finite-difference calculation is performed for four different sources 7.5m, 37.5m, 67.5m and 97.5m. The fault is located at a distance of 52.5m in the flat model and 50.0m in the step model.

a) source=7.5m.

b) source=37.5m.

Figure 5.18: Waveforms from the flat surface model.

c) source=97.5m.

d) source=97.5m.

Figure 5.18, continued:

a) elastic model. source=7.5m.

b) small Q model. source=7.5m.

Figure 5.19: Waveforms from the step model. Source = 7.5m

c) elastic model. source=37.5m.

d) small Q model. source=37.5m.

Figure 5.19, continued: Source =37.5m

e) elastic model. source=67.5m.

f) small Q model. source=67.5m.

Figure 5.19, continued: Source = 67.5m

g) elastic model. source=97.5m.

h) small Q model. source=97.5m.

Figure 5.19, continued: Source =97.5m

a) Three-layer model

b) Smooth model

Figure 5.20 : P-velocity tomogram obtained by the refraction traveltimes tomography. a) A velocity model with the assumption that the velocity structure is a three-layer model. b) A velocity model with the assumption that the velocity is increasing with depth smoothly.

a) Three-layer model (Figure 5.20 a)

b) Smooth model (Figure 5.20 b)

Figure 5.21: Comparison of the observe and theoretical traveltime data.

Figure 5.22: Example of observed common source data. A source location is 122.5m. A source depth is about 1m. The quantity of is 100g. A particle velocity (vertical component) is plotted.

a) Observed data

b) $Q=10000$

c) $Q=5$ ($V_p < 2000$ m/sec) and $Q=100$ ($V_p \geq 2000$ m/sec)

Figure 5.23 Comparison of theoretical data with observed data (layered model)

a) Observed data

b) $Q=10000$

c) $Q=5$ ($V_p < 2000\text{m/sec}$) and $Q=100$ ($V_p \geq 2000\text{m/sec}$)

Figure 5.24 Comparison of theoretical data with observed data (smooth model)

Figure 5.25: Source locations and P-velocity model. The sources are placed at the depth of 1m and a 50Hz Ricker wavelet is used as a source wavelet.

a) Observed data (Source=3.5m)

b) Theoretical data (Source=3.5m)

Figure 5.26: Comparison of observed data and theoretical data.
(Source=3.5m)

c) Observed data (Source=27.5m)

d) Theoretical data (Source=27.5m)

Figure 5.26 continued: Comparison of observed data and theoretical data.
(Source=27.5m)

e) Observed data (Source=57.0m)

f) Theoretical data (Source=57.0m)

Figure 5.26 continued: Comparison of observed data and theoretical data.
(Source=57.0m)

g) Observed data (Source=77.5m)

h) Theoretical data (Source=77.5m)

Figure 5.26 continued: Comparison of observed data and theoretical data.
(Source=77.5m)

i) Observed data (Source=102.5m)

j) Theoretical data (Source=102.5m)

Figure 5.26 continued: Comparison of observed data and theoretical data.
(Source=102.5m)

k) Observed data (Source=147.5m)

l) Theoretical data (Source=147.5m)

Figure 5.26 continued: Comparison of observed data and theoretical data.
(Source=147.5m)

Chapter 6

Conclusions

In this thesis, a variable grid, viscoelastic, 2-D finite-difference method is developed for supporting shallow seismic field survey. The new method is simple to implement for staggered grid finite-difference schemes, is computationally efficient, and enables modeling of highly irregular topography.

In order to demonstrate the accuracy and reliability of the code, benchmark tests were performed using simple models. The results confirm that the second-order approximation requires 20 grid-points per wavelength, and the fourth-order approximation requires 10 grid-points per wavelength.

An accurate free-surface boundary condition for irregular topography, and a stable and efficient variable grid implementation were presented. The results of numerical tests imply that the method presented in this thesis requires at least 30 grid-points per wavelength. In particular, relatively gentle slopes require a large number of grid-points per wavelength. Implementation of variable grid, with high grid density near the surface and low velocity zone, ensures accuracy in the calculation with improved computational efficiency. The method allows us to handle complex structures in finite-difference modeling. Although the method was implemented in 2-D, it can be easily extended to 3-D wave propagation problems.

Several numerical simulations were performed to show the characters of wave propagation in the near surface region. The simulations show that the low velocity thin layers just below the surface and anelastic attenuation have a significant effect on the surface seismic record.

The algorithm was also applied to 2-D modeling of a near surface structure beneath a 2-D refraction survey line. The characters in the observed data can be explained by a subsurface model constructed by P-velocity traveltimes tomography.

There are several improvements needed for the finite-difference method developed in this thesis for broader applications. The inclusion of anisotropy is an important extension. We used the grid size ratio of three. The development of a stable calculation using a larger grid size ratio is another task. The most important improvement is the development of a three-dimensional code. It is currently possible to apply realistic three-dimensional modeling that includes attenuation and irregular free surface topography to practical problems. This extension also requires overcoming difficulties of generating 3-D earth model. The finite-difference method can be used as a forward modeling method for full wave tomography. The development of practical full wave tomography is one of our ultimate goals in near surface seismic studies.

Appendix A

Derivation of the 2-D and 3-D Viscoelastic Wave Equation

The constitutive relation for a linear viscoelastic 2-D ($i,j,k = x,y$) or 3-D ($i,j,k = x,y,z$) homogeneous isotropic solid is,

$$\sigma_{ij} = \dot{\Lambda} * \delta_{ij} \varepsilon_{kk} + 2\dot{M} * \varepsilon_{ij}, \quad (\text{A-1})$$

according to Christensen (1982). It is possible to express the time derivative of ε_{ij} as,

$$\dot{\varepsilon}_{ij} = \frac{1}{2}(\partial_i v_j + \partial_j v_i). \quad (\text{A-2})$$

Let us define

$$\Pi = \Lambda + 2M \quad (\text{A-3})$$

and use the standard linear solid model for Π and M , i.e.,

$$\Pi = \pi \left(1 - \sum_{l=1}^L \left(1 - \frac{\tau_{sl}^p}{\tau_{sl}} \right) e^{\frac{-t}{\tau_{sl}}} \right) \theta(t) \quad (\text{A-4})$$

and

$$M = \mu \left(1 - \sum_{l=1}^L \left(1 - \frac{\tau_{sl}^s}{\tau_{sl}} \right) e^{\frac{-t}{\tau_{sl}}} \right) \theta(t). \quad (\text{A-5})$$

The definition of Π allows us to define Q independently for P-waves and S-waves through the τ_{sl}^p (P-waves) and τ_{sl}^s (S-waves). Using equations (A-1) through (A-3), we find for a diagonal element of $\dot{\sigma}_{ij}$, ($i=j$),

$$\dot{\sigma}_{ij} = (\dot{\Pi} - 2\dot{M}) * \partial_k v_k + 2\dot{M} * \partial_i v_j \quad (\text{A-6})$$

and for an off-diagonal element ($i \neq j$)

$$\dot{\sigma}_{ij} = \dot{M} * (\partial_i v_j + \partial_j v_i). \quad (\text{A-7})$$

By evaluating parts of convolution and defining the memory variables r_{ijl} , we can write equation (A-6) as

$$\begin{aligned}\dot{\sigma}_{ij} = & \left(\pi \left(1 - \sum_{l=1}^L \left(1 - \frac{\tau_{sl}^p}{\tau_{sl}} \right) \right) - 2\mu \left(1 - \sum_{l=1}^L \left(1 - \frac{\tau_{sl}^s}{\tau_{sl}} \right) \right) \right) \partial_k v_k \\ & + 2\mu \left(1 - \sum_{l=1}^L \left(1 - \frac{\tau_{sl}^s}{\tau_{sl}} \right) \right) \partial_i v_j + \sum_{l=1}^L r_{ijl}\end{aligned}\quad (\text{A-8})$$

and equation (A-7) as

$$\dot{\sigma}_{ij} = \mu \left(1 - \sum_{l=1}^L \left(1 - \frac{\tau_{sl}^s}{\tau_{sl}} \right) \right) (\partial_i v_j + \partial_j v_i) + \sum_{l=1}^L r_{ijl} \quad (\text{A-9})$$

Applying the same technique as was used to derive equation (2.2.25) from equation (2.2.23) yields for diagonal r_{ijl} ($i=j$)

$$\begin{aligned}\dot{r}_{ijl} = & -\frac{1}{\tau_{sl}} \left(r_{ijl} + \left(\pi \left(\frac{\tau_{sl}^p}{\tau_{sl}} \right) - 2\mu \left(\frac{\tau_{sl}^s}{\tau_{sl}} \right) \right) \partial_k v_k + 2\mu \left(\frac{\tau_{sl}^s}{\tau_{sl}} \right) \partial_i v_j \right) \\ & 1 \leq l \leq L.\end{aligned}\quad (\text{A-10})$$

and off-diagonal ($i \neq j$)

$$\dot{r}_{ijl} = -\frac{1}{\tau_{sl}} \left(r_{ijl} + \mu \left(\frac{\tau_{sl}^s}{\tau_{sl}} \right) (\partial_i v_j + \partial_j v_i) \right) \quad 1 \leq l \leq L. \quad (\text{A-11})$$

Adding Newton's second law,

$$\rho \partial_t v_i = \partial_j \sigma_{ij}, \quad (\text{A-12})$$

yields the viscoelastic wave equation. Specializing to two dimensions ($i,j,k=x,y$) and one standard linear solid ($L=1$) results in equations (2.2.31) to (2.2.38).

Appendix B

2-D Finite-difference Equations

I show the finite-differential equations derived from equations (2.2.31) to (2.2.38). For the sake of simplicity, Clank-Nicolson scheme is not employed in the following finite-differential equations.

Equations governing stress:

$$\begin{aligned}\sigma_{x,j}^{n+1} = & \sigma_{x,j}^n + \frac{\Delta t}{\Delta x} \left(\frac{\pi \tau_{\epsilon,j}^p}{\tau_{\sigma}} (c_1 (v_{x,j+3/2}^n - v_{x,j-3/2}^n) + c_2 (v_{x,j+1/2}^n - v_{x,j-1/2}^n)) \right. \\ & + \frac{1}{\tau_{\sigma}} (\pi \tau_{\epsilon,j}^p - 2\mu \tau_{\epsilon,j}^s) (c_1 (v_{x,j+3/2}^n - v_{x,j-3/2}^n) + c_2 (v_{x,j+1/2}^n - v_{x,j-1/2}^n)) \Big) \\ & + \frac{\Delta t}{2} (r_{x,j}^{n+1} + r_{x,j}^n) \quad (B-1)\end{aligned}$$

$$\begin{aligned}\sigma_{x,j}^{n+1} = & \sigma_{x,j}^n + \frac{\Delta t}{\Delta x} \left(\frac{\pi \tau_{\epsilon,j}^p}{\tau_{\sigma}} (c_1 (v_{x,j+3/2}^n - v_{x,j-3/2}^n) + c_2 (v_{x,j+1/2}^n - v_{x,j-1/2}^n)) \right. \\ & + \frac{1}{\tau_{\sigma}} (\pi \tau_{\epsilon,j}^p - 2\mu \tau_{\epsilon,j}^s) (c_1 (v_{x,j+3/2}^n - v_{x,j-3/2}^n) + c_2 (v_{x,j+1/2}^n - v_{x,j-1/2}^n)) \Big) \\ & + \frac{\Delta t}{2} (r_{x,j}^{n+1} + r_{x,j}^n) \quad (B-2)\end{aligned}$$

$$\begin{aligned}\sigma_{x,j}^{n+1} = & \sigma_{x,j}^n + \mu \frac{\Delta t \tau_{\epsilon,j}^s}{\Delta x \tau_{\sigma}} (c_1 (v_{x,j+3/2}^n - v_{x,j-3/2}^n) + c_2 (v_{x,j+1/2}^n - v_{x,j-1/2}^n)) \\ & + c_1 (v_{x,j+3/2}^n - v_{x,j-3/2}^n) + c_2 (v_{x,j+1/2}^n - v_{x,j-1/2}^n) \\ & + \frac{\Delta t}{2} (r_{x,j}^{n+1} + r_{x,j}^n) \quad (B-3)\end{aligned}$$

Equations governing particle velocities:

$$\begin{aligned}v_{x,j}^{n+1} = & v_{x,j}^n + \rho \frac{\Delta t}{\Delta x} (c_1 (\sigma_{x,j+3/2}^n - \sigma_{x,j-3/2}^n) + c_2 (\sigma_{x,j+1/2}^n - \sigma_{x,j-1/2}^n)) \\ & + c_1 (\sigma_{x,j+3/2}^n - \sigma_{x,j-3/2}^n) + c_2 (\sigma_{x,j+1/2}^n - \sigma_{x,j-1/2}^n) \quad (B-4) \\ v_{x,j}^{n+1} = & v_{x,j}^n + \rho \frac{\Delta t}{\Delta x} (c_1 (\sigma_{x,j+3/2}^n - \sigma_{x,j-3/2}^n) + c_2 (\sigma_{x,j+1/2}^n - \sigma_{x,j-1/2}^n))\end{aligned}$$

$$+ c_1 (\sigma_{xx,i,j+3/2}^n - \sigma_{xx,i,j-3/2}^n) + c_2 (\sigma_{xx,i,j+1/2}^n - \sigma_{xx,i,j-1/2}^n) \quad (B-5)$$

Equations governing memory variables:

$$\begin{aligned} r_{xx,i,j}^{n+1} = & r_{xx,i,j}^n - \frac{\Delta t}{\tau_\sigma} \left(r_{xx,i,j}^n + \frac{\pi}{\Delta x} \left(\frac{\tau_\epsilon^p}{\tau_\sigma} - 1 \right) \left(c_1 (v_{x,i+3/2,j}^n - v_{x,i-3/2,j}^n) + c_2 (v_{x,i+1/2,j}^n - v_{x,i-1/2,j}^n) \right) \right. \\ & \left. + \frac{1}{\Delta x} \left(\pi \left(\frac{\tau_\epsilon^p}{\tau_\sigma} - 1 \right) - 2\mu \left(\frac{\tau_\epsilon^s}{\tau_\sigma} - 1 \right) \right) \left(c_1 (v_{x,i,j+3/2}^n - v_{x,i,j-3/2}^n) + c_2 (v_{x,i,j+1/2}^n - v_{x,i,j-1/2}^n) \right) \right) \end{aligned} \quad (B-6)$$

$$\begin{aligned} r_{zz,i,j}^{n+1} = & r_{zz,i,j}^n - \frac{\Delta t}{\tau_\sigma} \left(r_{zz,i,j}^n + \frac{\pi}{\Delta x} \left(\frac{\tau_\epsilon^p}{\tau_\sigma} - 1 \right) \left(c_1 (v_{x,i,j+3/2}^n - v_{x,i,j-3/2}^n) + c_2 (v_{x,i,j+1/2}^n - v_{x,i,j-1/2}^n) \right) \right. \\ & \left. + \frac{1}{\Delta x} \left(\pi \left(\frac{\tau_\epsilon^p}{\tau_\sigma} - 1 \right) - 2\mu \left(\frac{\tau_\epsilon^s}{\tau_\sigma} - 1 \right) \right) \left(c_1 (v_{x,i+3/2,j}^n - v_{x,i-3/2,j}^n) + c_2 (v_{x,i+1/2,j}^n - v_{x,i-1/2,j}^n) \right) \right) \end{aligned} \quad (B-7)$$

$$\begin{aligned} r_{xz,i,j}^{n+1} = & r_{xz,i,j}^n - \frac{\Delta t}{\tau_\sigma} \left(r_{xz,i,j}^n + \frac{\mu}{\Delta x} \left(\frac{\tau_\epsilon^s}{\tau_\sigma} - 1 \right) \left(c_1 (v_{x,i,j+3/2}^n - v_{x,i,j-3/2}^n) + c_2 (v_{x,i,j+1/2}^n - v_{x,i,j-1/2}^n) \right) \right. \\ & \left. + c_1 (v_{x,i+3/2,j}^n - v_{x,i-3/2,j}^n) + c_2 (v_{x,i+1/2,j}^n - v_{x,i-1/2,j}^n) \right) \end{aligned} \quad (B-8)$$

i : x-axes discretization.

j : y-axes discretization.

n : time discretization.

Δt : time step (sec).

Δx : grid size(m).

$\sigma_{xx}, \sigma_{zz}, \sigma_{xz}$: the symmetric stress tensor.

v_x, v_z : the particle velocities.

r_{xx}, r_{zz}, r_{xz} : the memory variables.

$\tau_\epsilon^p, \tau_\epsilon^s$: the viscoelastic strain relaxation times for P- and SV-waves, respectively.

τ_σ : the viscoelastic stress relaxation time for both the P- and SV waves.

μ : the relaxation modulus corresponding to SV-waves, $\mu = \rho\beta^2$.

π : the relaxation modulus corresponding to P-wave, $\pi = \lambda + 2\mu = \rho\alpha^2$.

ρ : the density (kg/m³).

α : P-velocity (m/sec).

β : S-velocity (m/sec).

c1 : -1/24.

c2 : 9/8.

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STRATIGRAPHIC IMAGING ATTRIBUTES: MONOSCALE ANALYSIS

Relating seismic reflection images to the sedimentary record in the subsurface is the focus of stratigraphic imaging. We have developed a seismic attribute that can capture and categorize the main reflector features without being sensitive to amplitudes. The sharpness of a reflector is defined by a scale exponent, which expresses singularity order and determines the reflection signature/waveform. These local scale exponents are obtained by using a monoscale analysis method. Unlike multi-scale analysis, this method measures transitions in the subsurface at a fixed scale by using fractional differentiation/integration to estimate the scale exponents. This attribute provides a local singularity measure along each seismic interface, which can be interpreted in terms of facies variations in the subsurface. This method has been applied to 2-D seismic reflection data with very promising results. Variations in the attribute along major reflectors is consistent with expected lithologic variations related to the environment of deposition. This attribute also facilitates well ties with the seismic data. The following two abstracts and preprint by Felix Herrmann explain the methodology in more detail.

A scale attribute for texture in well- and seismic data

Felix Herrmann, ERL, MIT and Colin Stark, Lamont-Doherty Earth Observatory, Columbia University

Summary

For many years people have been struggling to integrate well and seismic data. As main reasons for this struggle one may list the inherent bandwidth limitation of seismic data; the problem of seismic amplitudes and finally the apparent inability to delineate and characterize those transitions in wells that can be linked to and held responsible for the major reflection events and their signatures.

As a means of approaching these problems an alternative approach is presented and partly tested in this paper. The method is based on the introduction of a new attribute which, at a *fixed scale*, characterizes the location and sharpness of the transitions in the well and the location and signature of the corresponding reflection events. As such the method entails an approach where the *variations* in the magnitude of the well properties and reflection amplitudes are characterized by their *order of magnitudes*. These orders of magnitude are represented by scale exponents, which express the local *texture* (sharpness) of the interfaces and the nature of the reflection signature. It appears that these exponents are robust and localized and truly independent of the actual point values in the log and seismic data. This independence makes these exponents rightful candidates for a new trace attribute, complementing existing attributes such as instantaneous phase and frequency. Because sharpness characterizes the local texture the attribute promises to serve as a key quantity for the integration of well and seismic data on the *level of texture*, with the additional benefit of being able to generate pseudo wells and seismic data.

The proposed method to obtain the scale exponents was first introduced in Herrmann and Stark (1999; 2000; 2000). There it was shown that, as opposed to multiple scale wavelet methods, it is possible to estimate fractional scale exponents at a *fixed scale* by the monoscale β -transform. Similar attempts (Dessing, 1997) have been made, using the instantaneous phase, which is difficult to compute from well and/or seismic data. As shown below the application of the monoscale method to multi-trace post-stack migrated data may be regarded as highly successful in the sense that the attribute assigns only one value to a reflector, in a manner that is lateral consistent and insensitive to amplitude variations along the interface. This lateral consistency makes the scale attribute useful for revealing the (singularity) structure of the subsurface.

Besides the strict locality additional advantages are physical interpretation; scale-invariance; insensitivity to the seismic wavelet and reconstruction capability. The locality opposes smeared attributes such as instantaneous

frequency/phase while the scale-invariance refers to the scale-invariant manner in which the sharpness of the reflectors is characterized. This sharpness characterization has the advantage that it also applies to the seismic wavelet and is additive under convolution. This latter property facilitates both the interpretation and seismic wavelet deconvolution on the level of the attribute. Finally, the reconstruction capability allows for the generation of pseudo wells and reflectivity, based on the location and sharpness characterization of the attribute.

The setup of this paper is as follows. Generalized layer transitions of varying sharpness are introduced first, followed by a brief outline of the fixed scale analysis method. Subsequently, attention is paid to the directionality of fractional order transitions and the construction of pseudo wells and seismic data from the scale attributes. Finally, the method will be put to the test on the Mobil AVO dataset and some reflection traces from the Gulf of Mexico. The well of the first data set is "blocked", post stack migrated data is analyzed and reconstructed.

Basic concepts and methodology

Seismic waves pick up localized information on regions where the earth's interior varies rapidly on the scale of the seismic wavelet. At those regions, seismic waves are reflected and mode converted. Either jump discontinuities or ramp functions are usually used to represent these regions. Multiscale analysis by the wavelet transform applied to well- and seismic data (Herrmann, 1998; Herrmann and Stark, 1999) has demonstrated that these zero and first order transitions are too restricted to represent the wide variety of transitions that are found in the Earth. Instead, fractional order transitions are introduced to generalize the representation for the major reflectors. The order of the transitions characterizes the sharpness in a scale invariant manner and depends on the variations in the magnitudes only.

The onsets

The basic building block for media profiles with transitions of varying order are the causal and anti-causal fractional order onset functions given by

$$\chi_+^\alpha(x) \triangleq \begin{cases} 0 & x \leq 0 \\ \frac{x^\alpha}{\Gamma(\alpha+1)} & x > 0, \end{cases} \quad (1)$$

$$\chi_-^\alpha(x) \triangleq \begin{cases} \frac{-x^\alpha}{\Gamma(\alpha+1)} & x \leq 0 \\ 0 & x > 0, \end{cases} \quad (2)$$

A scale attribute for texture in well- and seismic data

Fig. 1: Generalization of edges from a jump on the top left to fractional edges with $\alpha \in (0, 1)$ on the top right. The signature of the normal incident amplitude normalized reflections (Ricker wavelet) clearly depend on the order. The direction of the onsets also plays a role as seen in the second half of this figure.

where the $\chi_{\pm}^{\alpha}(x)$'s are the order α onset functions (Herrmann, 1998). The degree α rules the *sharpness* (regularity) of the transitions which is linked to its scaling (Herrmann, 1998). For $\alpha = 0$ one has a jump discontinuity and for $\alpha = 1$ the onset function of a ramp function. The α describes the sharpness irrespective of the scale. In fact the exponent quantifies the *order of magnitude* of the variations at the singular onsets. Fig. 1 illustrates the significance of the sharpness for the description of both the transitions and the signature of the reflection events.

Generalized blocked well profiles

The onset functions defined in Eq. 1-2 are used to construct "blocked" profiles with varying order transitions

$$f(x) = \sum_{i=1}^N c_{\pm}^i \chi_{\pm}^{\alpha_i}(x - x_i) + P_n(x), \quad (3)$$

where the index i runs over the number of transitions, $\pm_i$, $c_{\pm}^i$, α_i and x_i determine the direction (causal/anti-causal), magnitude, sharpness and location respectively. The $P_n(x)$ is a n^{th} order polynomial representing the

background, e.g. a constant or gradient. Cases where $c_{+} \neq 0 \wedge c_{-} = 0$ or $c_{+} = 0 \wedge c_{-} \neq 0$ are being considered only. See Fig. 1 bottom for the 4 different directions.

Monoscale attribute analysis

Traditionally, multiscale wavelet transforms are used to characterize the order of transitions/edges (Herrmann, 1998; Herrmann and Stark, 1999). Unfortunately these multiscale methods are not applicable in cases where either the scale/bandwidth content of the data is too limited or where the data contains too many interfering transitions. Unfortunately, both situations are typical for seismic and well data, withstanding a truly successful and unambiguous estimation of the scale exponents.

By extending the wavelet transform to a transform with the scale fixed and wavelet order varied (Stark and Herrmann, 2000; Herrmann and Stark, 2000), i.e.

$$\mathcal{W}_{\beta}\{f, \phi\}(\sigma, x) \triangleq \sigma^{\beta} \frac{d^{\beta}}{dx^{\beta}}(f * \phi_{\sigma})(x), \quad (4)$$

one should also be able to analyze different types of transitions from both broadband well and single band seismic data. The convolution with ϕ , a Gaussian bell shape function with a width proportional to the scale σ , smoothes the data to the fixed scale, while the β controls the amount of sharpening $\beta > 0$ or de-sharpening $\beta < 0$. By varying this β fractionally, one is able to detect and estimate the order of the transitions via the following two on-off criteria (Herrmann and Stark, 1999; Stark and Herrmann, 2000; Herrmann and Stark, 2000):

for transitions with $\alpha \geq 0$ and $\beta \in \mathbb{R}^{+}$,

$$\alpha(\sigma, x) = \inf_{\beta} \{ \partial_x \mathcal{W}_{\beta}\{f, \phi\}(\sigma, x) = 0 \}. \quad (5)$$

for reflection events with $\alpha < 0$ and $\beta \in \mathbb{R}_0^{-}$,

$$\alpha(\sigma, x) = \sup_{\beta} \{ \partial_x \mathcal{W}_{\beta}\{f, \phi\}(\sigma, x) = 0 \}. \quad (6)$$

These criteria represent the property that for a positive order onset function ($\alpha \geq 0$) a local modulus maxima emerges when the order of fractional differentiation ($\beta > 0$) in Eq. 4 infinitesimally exceeds the order of the transition α . Conversely, for a differentiated transition, i.e. a reflection event with $\alpha < 0$, the local modulus maxima disappear when the order of fractional integration ($\beta \leq 0$) infinitesimally exceeds the negative order α .

As shown in Fig.'s 1 and 2 the onset functions can be right- or left-handed and flipped in sign. The onset-criteria as depicted in Eq. 5 and 6 are affected by this directivity. To circumvent this problem the β -transform (Eq. 4-6) is conducted using causal and anti-causal derivatives or integrals. In Fig. 2 an example is given on the directional analysis of a synthetic example which contains smoothed causal and anti-causal onset functions as depicted on the top. The location and direction of the singularities are color coded in the third plot from the top.

A scale attribute for texture in well- and seismic data

Reconstruction

Given the location, direction and orders one is able to construct pseudo wells and reflectivity. The relative magnitude of the blocks in the pseudo wells are either determined by the magnitude of the derivative at the location and scale of the attribute or by the seismic reflection amplitudes. The term pseudo refers to a reconstruction which is *modulo* polynomials, i.e. the smooth/trend is not reconstructed. The data is reconstructed via Eq. 3 without the P_n and with the $c_{\pm}^i$'s, $\pm_i$'s, α_i 's and x_i 's, set to their estimated values. Fig. 2 contains a synthetic reconstruction example where the estimates are taken from the second and third plot to generate the reconstruction on the bottom. The position and order of the transitions are well reconstructed. The smoothing of the original function is removed by setting the smoothing of the reconstruction to zero. Possible deviations in the reconstruction between the transitions (e.g. between the first and the second transition) are due to fact that the method is only sensitive to variations that are singular.

Application to well data

Fig. 3 contains a pseudo well (middle) and reflectivity (bottom) obtained from a coarse scale attribute analysis applied to the impedance log on the top. Although the reconstruction is not perfect this result is very encouraging because the pseudo reflectivity matches the reflectivity from the original well. Clearly the proposed reconstruction scheme corresponds in this context to a *generalized blocking*, including fractional transitions.

Application to seismic data

The results of the monoscale method, applied to a selection of 3 traces of the Mobil post-stack migrated data set, are depicted in Fig. 5. In this plot the location, color and size of the balls refer to the location of the reflector, its order and relative magnitude. Clearly, there are different order singularities present in the reflection data. Fig 4 contains a comparison of the original *versus* pseudo reflectivity, generated from a trace of the time migrated Gulf of Mexico data set (top) together with its tie (bottom). Finally, Fig. 6 contains an example of the application of the method to a section of a time migrated Mobil data set. The position and order of the reflections are depicted by the location and color of the dots. The corresponding orders for the reflectors, the transitions causing the reflections, can readily be obtained when the order of the seismic wavelet is known, e.g. for a Ricker wavelet one simply adds 3. The results of Fig. 6 demonstrates that the proposed method captures the location and sharpness of the reflectors in a consistent manner unaffected by lateral variations in amplitude.

Fig. 2: Example of the analysis and reconstruction of a synthetic well with different order singularities and directions. (top) the synthetic well with 5 smoothed singularities; (second plot) "reflectivity"; (third plot) position, order and direction singularities. Blue is causal positive, green anti-causal positive, red causal negative sign, magenta anti-causal negative sign. (bottom) the reconstructed profile.

The migrated section on the top can be reconstructed, using the scale attributes depicted in the middle. The pseudo reflectivity is plotted on the bottom and shows a beautiful agreement with the section on the top.

Conclusions

A powerful new tool has been developed which allows for the introduction of a new attribute. This attribute characterizes the sharpness of the reflectors and allows for the construction of pseudo wells and seismic data. Both the location and the value of the attribute are estimated in a lateral consistent way, revealing a structural image. Because sharpness describes the local texture a possibility is created to infer information on the depositional environment underlying the stratigraphy.

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Fig. 3: Application of the monoscale method to a Mobil well-data set. (top) acoustic impedances; (middle) reconstructed pseudo well, based on the scale attribute and estimated at the scale of the reflectivity (bottom). The pseudo well contains different order transitions and yields a pseudo reflectivity that is close to the one of the well on the top.

Fig. 4: Pseudo versus original time migrated traces from the Gulf of Mexico, tied with a synthetic. On the top comparison for the synthetic on the bottom for the migrated data (courtesy Schlumberger).

Fig. 5: Application of the monoscale method to 3 traces of the Mobil time migrate data (see Fig. 6). The color of balls refers to the order and the size to the reflection magnitude. The estimated exponents are mostly negative because reflection entails differentiation.

Fig. 6: Time migrated section from the Mobil data set (top). Attribute (middle) and Reconstruction (bottom).

Stratigraphy and lithology classification by singularity analysis

Felix J. Herrmann*, William J. Lyons†

* Earth Resources Laboratory, EAPS, MIT

† MIT, EAPS, MIT-WHOI Joint Program in Oceanography

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Abstract

Relating sedimentary records to seismic data is a major challenge. By shifting focus to a scale-invariant sharpness characterization for the reflectors, we develop an attribute that can capture and categorize the main reflector features, without being sensitive to amplitudes. Sharpness is defined by a scale exponent, which expresses singularity order and determines the reflection signature/waveform. Local scale exponent estimates are obtained with a new monoscale method. Compared to multiscale wavelet analysis, our method has the advantage of measuring transition exponents at a single fixed scale using a simple on-off criterion. The exponents contain amplitude variation information and describe lithological onsets. We create an image of the earth's local singularity structure by applying our method to seismic traces and well-log data. The singularity map facilitates interpretation, facies characterization, and integration of well and seismic data on the level of local texture.

Introduction

Seismic reflections describe regions where the earth's properties (e.g. acoustic impedance) vary significantly on the length scale of seismic waves. Mathematically, these regions are represented by zero or first-order transitions, which are singular in their derivatives. Multifractal analysis on well-log and seismic data demonstrates that variations in the sedimentary upper crust consist of intertwined fractal sets of singularities, implying the following. First, seismic wavefields inherit the global singularity structure of the earth (Muller et al., 1992; Herrmann, 1998; Herrmann, 1998), albeit in a bandwidth-limited fashion. Second, traditional transition models are restrictive and must be generalized to allow fractional order singularities. Third, the multifractality suggests an accumulation of the singularities.

The effectiveness of multifractals in seismic imaging is limited because local characteristics of the singularity structure are lost. This loss makes sedimentary transition characterization based on local scaling impossible. Recent results (Holschneider, 1995; Mallat, 1997; Herrmann, 1998; Dessing, 1997; Herrmann, 2001) show that local Hölder exponents can be estimated from the localized decay/growth rate of the wavelet coefficients, along the wavelet transform modulus maxima lines. These Hölder exponents, or scale exponents, estimate the *order of magnitudes* of the scaling in sedimentary records and reflectivity variations.

Because seismic waves are bandwidth-limited, the "multifractal" earth is effectively observed at the scale of one "wavelet" only. Both the bandwidth limitation and accumulation (the mutual interference of the singularities) make it difficult to estimate exponents locally. Dessing (1997) proposes a method based on the instantaneous phase (Payton, 1977). Our method differs because it computes the local scale exponents at the fixed scale of the seismic wavelet using the "on-off" property that the β^{th} derivative of a degree α differentiable function diverges when $\beta > \alpha$. This property manifests itself in the emergence of local maxima for the modulus of a smoothed observation at the location of the transitions.

Besides strict locality, additional advantages of the method are the geological interpretation of the exponents, their scale-invariance, and their insensitivity to the seismic wavelet and reconstruction capability (Herrmann, 2001). More importantly, the sharpness characterization quantifies a seismic stratigraphic description of the reflected waveforms depending on the depositional environments. Classical seismic stratigraphical models (Payton, 1977; Verhelst, 2000), based on aggregates of multiple subwavelength zero- and/or first-order transitions, are simplified by fractional order transitions. This sharpness characterization eliminates the necessity to explain the observed waveforms via a superposition of multiple reflectors.

We demonstrate the applied techniques by introducing a generalized transition model and an method for analyzing transition sharpness from seismic data. We discuss the relevance of the sharpness characterization to seismic facies and apply the proposed method to two datasets, with emphasis on integration of well and seismic data and facies characterization. We hope to establish this integration with exponents rather than with pointwise values for reflection amplitudes and medium variations.

Basic concepts and methodology

Seismic reflectors are generally represented by either zero-order jump discontinuities or first-order ramp functions. Multiscale analysis by the wavelet transform, applied to well and seismic data (Herrmann, 1997; Dessing, 1997; Herrmann, 1998; Herrmann, 2001), show that these representations are too restrictive for the wide variety of transitions found in the data. We introduce fractional order transitions as a general representation for the major reflectors. The order of the transitions depends on the amplitude variations only and characterizes sharpness in a scale-invariant manner.

A monoscale analysis is introduced to measure the sharpness from seismic data. Maps of the singularity structure of the sedimentary upper crust are applied to the integration of well and seismic data and the interpretation of facies transitions.

Generalized transitions

Transitions of varying order are defined by causal (left-right), anti-causal (right-left) and sign-flipped onset functions (Holschneider, 1995; Dessing, 1997; Herrmann, 1997; Herrmann, 1998)

$$\chi_{\pm}^{\alpha}(t) = \begin{cases} 0 & \pm t \leq 0 \\ \frac{\pm t^{\alpha}}{\Gamma(\alpha+1)} & \pm t > 0, \end{cases} \quad (1)$$

where the real valued exponent α determines the sharpness of the transition, linked to a local scale-invariance of the type, $\sigma^{-\alpha} \chi_{\pm}^{\alpha}(\sigma z) = \chi_{\pm}^{\alpha}(z)$, $\sigma > 0$, and Γ is the Gamma function. For integer $\alpha = \{-1, 0, 1\}$ the transition corresponds to the Dirac delta distribution, the jump discontinuity and the ramp function, respectively. Irrespective of any particular scale, the exponent α quantifies the sharpness and the *order of magnitude* of the variations. Figure 1 illustrates both the sharpness and directionality of fractional order transitions and their relation to the signature/waveform of the reflection events. Figure 1 contains an example of medium profiles (left column) with causal (top) and anti-causal (bottom) onset functions with $\alpha \in [-0.9, -0.1]$. To give the transitions extent, the transitions are smoothed with a Gaussian bell-shape function. The right column contains the corresponding seismic signatures obtained after convolution with a Ricker wavelet. The colored Δ (causal) and ∇ (anti-causal) are used to indicate the direction and order (colorbar). As the order α increases, the sharpness decreases and the color changes from blue to red.

Monoscale sharpness analysis

Multiscale wavelet transforms are normally used to characterize the order of transitions (Holschneider, 1995; Mallat, 1997; Herrmann, 1997; Herrmann, 1998). These characterizations are based on asymptotic growth/decay rates of wavelet coefficients along the transform's extrema and as a function of the scale. The transform itself is defined by convolutions with a multitude of (Ricker) wavelets with decreasing scale (increasing central frequency). Compared to analyzing reflection magnitudes, estimating the exponents has the advantage of being less prone to measurement, model, and calibration errors. However, seismic data have the disadvantage of being bandwidth-limited, restricting the available scale range for the analysis.

To leading order, imaged seismic reflectivity can be written as

$$R(t) \approx \frac{d}{dt} (\zeta * \varphi_s^{\gamma})(t) = s^{\gamma} \frac{d^{\gamma+1}}{dt^{\gamma+1}} (\zeta * \phi_s)(t). \quad (2)$$

ζ is the time-parametrized acoustic impedance and φ_s^{γ} the γ -order/ s -scale parametrized wavelet. This wavelet is defined as the γ^{th} -derivative (e.g. $\gamma = 2$ for a Ricker wavelet) of a (Gaussian) smoothing function at seismic scale s , i.e. $\varphi_s^{\gamma} = s^{\gamma} \partial_t^{\gamma} \phi_s$ with $\gamma \in \mathbb{R}^+$.

By extending the standard continuous wavelet transform (Holschneider, 1995; Mallat, 1997) to,

$$\mathcal{W}_{\beta}\{f, \phi\}(\sigma, t) \stackrel{\text{(de-) sharpening}}{=} \sigma^{\beta} \frac{d^{\beta}}{dt^{\beta}} \underbrace{(f * \phi_{\sigma})(t)}_{\text{smoothing}} \quad (3)$$

for $\beta \in \mathbb{R}$ (denoting β^{th} -order differentiation/integration for $\beta > 0/\beta < 0$) and scale ($\sigma > 0$) fixed, we are not only able to generate reflectivity (by setting $f(t) = \zeta(t)$, $\beta = \gamma > 0$ and $\sigma = s$), but we can also build a monoscale sharpness estimator. This estimator measures from the imaged reflectivity (setting $f(t) = R(t)$) and using the additional smoothing (to reduce the noise) relative sharpness changes in impedance transitions of the type given in Eq. 1. The estimator derives its action from the property that the local extrema of the $(1 - \alpha + \gamma + 1)^{\text{th}}$ -derivative of ϕ_s are within a window of size $\sigma \geq s$ removed, when in Eq. 3 the degree of fractional integration ($\beta < 0$) infinitesimally exceeds this differentiation, yielding the estimate

$$\hat{\alpha}(\sigma, t) = \sup_{\beta} \{\partial_z \mathcal{W}_{\beta}\{f, \phi\}(\sigma, t) = 0\} + \gamma + 1. \quad (4)$$

The degree of fractional integration needed to remove the extrema provides a relative measure for the combined action of the reflectivity (1), the wavelet (γ) and the generalized transitions $\alpha(t)$. The estimator provides an on-off criterion with which we can find locations and relative orders of lithological boundaries. Due to the non-linear nature of the estimator, no precise results exist on bias, confidence limits and the effects of noise. However, tests on synthetic reflectivity have shown good results (Herrmann, 2001). Finally, knowledge on the seismic wavelet's order (γ) can be used to calibrate the estimate.

Facies transition categorization

Following (Harms and Tackenberg, 1972) we propose two simplified models for lithologic boundaries. The boundaries are related to the contrast in the sand/mud ratio. As the sand content increases so does the velocity. The two facies types under consideration are (1) coarsening upwards, where the contact is a transition from fine-grained sediments below to coarser sediments above, see Fig. 1 (top); and (2) fining upwards, where the contact is a transition from coarse-grained sediments below to finer sediments above, see Fig. 1 (bottom).

Onsets defined in Eq. 1 provide a model for these two types of transition zones. Setting the sharpness to a negative value $-1 < \alpha < 0$, asymmetric profiles are generated with varying sharpness and with characteristics similar to the nearshore regressive and marine transgressive facies.

Data and results

Monoscale singularity analysis may be applied directly to two seismic data sets. Fig. 2 focuses on migrated data including synthetic reflectivity computed from a well. Fig. 2 (top) displays both the synthetic reflection amplitudes (middle) and real earth imaged reflectivity (sides). Besides the wiggle traces, this plot also contains the *locii*, order and relative magnitudes of the singularities. The position of the circles refers to the location of the singularities, while the gray-scale and size refer to the singularity strength and relative magnitude. There is a reasonable lateral consistency for the gray-scale along the major reflection horizons, persisting across the synthetic traces (around trace # 15). As an ultimate test, reconstruction results (see (Herrmann, 2001) for detail) for both the pseudo reflectivity and well are presented in the bottom two plots. These results clearly demonstrate that the proposed singularity inversion method works quite well, despite the severe bandwidth limitation of the seismic data (middle), compared to the well (bottom).

In the second example, we analyze an extracted line from a three-dimensional data set. The data set includes prograding clinoforms and we focus our attention to the output of the method analysis along these features. The results are presented in Fig. 3.

There is a general trend in the singularity order as we move from up-dip to down-dip along the clinoform features. For example, if we consider the surface located immediately above the purple line in Fig. 3, we see that the up-dip extent of this bounding surface is composed of singularities of consistently higher order (less negative) than those of the down-dip extent. A comparable trend is apparent in other clinoform surfaces in the data set. These singularity orders suggest that the transition is less-sharp up-dip than down-dip. In terms of grain size variations, we may interpret this to represent an increase in grain size contrast across the surface as we move down-dip along the clinoform.

Conclusions

A powerful new tool has been developed which allows for the introduction of a new attribute. This attribute characterizes the sharpness of the reflectors and allows for improved geological interpretation of the stratigraphy. Both the location and the value of the attribute are estimated in a laterally consistent way, revealing a structural image and possible lithological changes along the major reflector horizons. Because sharpness describes the local texture, a possibility is created to infer information on the depositional environment underlying the stratigraphy. Finally, the method facilitates the integration of well and seismic data.

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Figure 1: Generalization of onsets for geologic boundaries. Left column the causal (top) and anti-causal (bottom) onsets as defined in Eq. 1 for $\alpha \in [-0.9, -0.1]$. The onsets model the coarsening upwards (top) and fining upwards (bottom) lithologic boundaries. As α increases the transition changes from a sharp, slightly skewed and smeared delta distribution to a less sharp jump discontinuity. The Δ and ∇ denote causal/anti-causal and the order is color coded (blue for sharp and red for less sharp). The reflection signatures (right column) for a Ricker wavelet (second derivative Gaussian) become more and more integrated as α increases. The colors for the reflections are corrected for the wavelet by $\alpha_{ref} \mapsto \alpha_s + 2$, with the 2 corresponding to the amount of differentiation of the Ricker wavelet.

Figure 2: Details of reflectivity and pseudo well-log reconstruction, based on singularity characterization. Top: wiggle plot of the selected migration amplitudes (within the white box in Fig 3, top) together with the estimated singularity orders, depicted by the gray-colored circles whose size depend on the relative amplitudes. Middle: the seismic reflectivity (dashed) at the well (around trace # 15 in top plot) and the reconstructed reflectivity (solid). Bottom: reconstructed well (solid) and the original well (dashed). Notice, the remarkable reconstruction despite the severe bandwidth limitation of the seismic (compare reflectivity to the detailed well).

Figure 3: Singularity attributes for clinoforms and depicted by color-coded dots. Both the position and order of the reflectors are recovered. Note consistent down-dip decrease in singularity order along clinoform boundaries. The singularity structure aligns nicely with the migration amplitudes variations.

Singularity Characterization by Monoscale Analysis: Application to Seismic Imaging

Felix J. Herrmann

*Earth Resources Laboratory
Massachusetts Institute of Technology
42 Carleton Street, Bldg. E34-366
Cambridge, MA, 02142-1324, USA
E-mail: felix@erl.mit.edu*

Seismological images represent maps of the earth's structure. Apparent bandwidth limitation of seismic data prevents successful estimation by the multiscale wavelet transform of Lipschitz/Hölder regularity of non-oscillating singularities. To overcome this fundamental problem, a new method is proposed which provides local estimates from information, essentially residing at only one single scale. Within this method, the exponents are no longer calculated from the decay or growth rate of the wavelet coefficients. Instead, the estimates are obtained by transforming the data with respect to a family of generalized "wavelets" of fractional order. This generalized family is defined in terms of causal and anti-causal fractional integro-differentiations of a fixed-scale Gaussian smoothing function. Supplementing this transform with criteria that predict the onset or disappearance of modulus maxima as a function of the wavelet order, provide exponent estimates at the scale of the smoothing function. The estimated exponents are equivalent to Hölder exponents when the scale of the smoothing function approaches zero.

Key Words: singularity detection, Lipschitz/Hölder regularity, fractional calculus, multifractals, seismic imaging, wave propagation, migration, inversion, reconstruction

1. INTRODUCTION

Localized detection and characterization of singularities forms a crucial step in the analysis of various signals and images. Within the geosciences, signals and images in particular, contain a wide variety of singularities and edges. For broadband signals, multifractal analysis, based on the wavelet transform, has successfully been applied to globally characterize the singularity structure by means of the singularity spectrum [1, 22, 13, 15, 16]. Estimates for the singularity spectrum suggest that

the observed data may be considered as a sampled multifractal function [22], with an accumulation of singularities.

In seismology, the usefulness of the multifractal framework is limited because information on the local characteristics of the singularity structure is lost. Results by [21, 29, 24] show that local Hölder exponents can be estimated from the localized decay/growth rate of the wavelet coefficients, along the wavelet transform modulus maxima lines [29, 22, 13, 15]. Unfortunately, the applicability of this technique is also limited because of bandwidth-limited data and "accumulation" of singularities. Seismic waves, for example, observe the multifractal earth at the scale of only one wavelet because seismic waves, as are many other remotely sensed data, are bandwidth limited.

Both accumulation and bandwidth limitation prevent a useful characterization of the local regularity by scale exponents. To tackle this problem an analysis tool is proposed which, under particular conditions on the data, locally detects and characterizes singularity strengths [18, 20, 19, 17]. The method amounts to a generalization of the wavelet transform to a transform where the "wavelets" are fractional integro-differentiations of the Gaussian bell shape. By varying the order of the wavelets for a fixed scale, coarse-grained scale exponents are found via the emergence or disappearance of local modulus maxima as a function of the order. In the small-scale limit these exponents are equivalent to Hölder exponents, as shown by [39], who applies fractional calculus to regularity estimation.

The primary goals of this paper are: (1) to establish a direct link between the wavelet transform and seismic reflectivity; (2) to introduce a new measure, which expresses local orders of magnitude for variations in both the medium and wavefield; and (3) to present a monoscale analysis method, based on fractional order wavelets, which estimates singularity orders at a fixed scale. Attempts to estimate the local scale exponents at a fixed scale have been made by [8] and [33]. The former uses the instantaneous phase. The latter computes an effective local Hölder exponent by differentiating the logarithm of the wavelet coefficients, along the modulus maxima lines, with respect to the logarithm of the scale. The disadvantage of the instantaneous phase method is that it is difficult to compute for strongly varying well and/or seismic data, while the scale derivative method lacks on-off criteria.

The paper is organized as follows. First, we present a review on the essentials of the seismic method in the simplified case where the medium is considered to vary along the vertical only. For small medium variations, the forward and inverse mappings of the medium properties to the seismic wavefield and back are linearized and written in terms of temporal and spatial convolutions. Next, we demonstrate the relation of these convolutions to particular instances of limited scale range, continuous wavelet transforms of the medium fluctuations. In section 4, we introduce a method to detect and characterize singularities at essentially the fixed scale of the seismic wavelet. Given the singularity characterization, singularity maps of the earth are generated. These maps are used for interpretation and reconstruction of pseudo-medium profiles and reflection images.

2. SEISMIC REFLECTIVITY IMAGING METHOD

In the seismic reflection method, recorded surface reflectivity data are converted to an image of the earth's subsurface structure through a process called "migra-

tion". Seismic data itself consists of measurements of the wavefield, acquired by geo/hydrophones which measure the earth's response to a source located at the surface. The process of pre-stack depth migration [9, 2] is based on a time-reversed map of the recorded surface seismic data to a depth-parametrized, reflectivity function. The time-reversal operators are obtained by solving a linear wave equation for the up- and downward wave constituents, and require *a priori* knowledge of the smooth part of the velocity structure. By applying time-reversal operators to the data, information is obtained on the singularities in the earth's medium properties. This information is contained in the reflection density function. When performed correctly, i.e. when the *a priori* velocity information is accurate enough, migration operators are pseudo-differential, mapping the singularities one to one, from the recorded seismic data to the imaged reflectivity function [34, 5]. For the purpose of this paper it suffices to consider acoustic media, which vary along the vertical coordinate only. First, the one-way representation for the forward and inverse maps will be presented, followed by a linearization in terms of small medium variations.

2.1. The forward map: one-way wavefield representation

Medium variations of the earth exhibit a distinct directional preference along the vertical. Therefore, a formulation in terms of a one-way wave equation is beneficial. This formulation distinguishes between up- and downward traveling waves. As a consequence, natural separation can be made between the propagation and reflection of seismic waves. Following [2, 37, 36, 12, 6, 7, 38], we can write, for acoustic one-dimensional media, the upgoing acoustic pressure at the surface ($z = 0$) as

$$\hat{p}(p, z = 0; \omega) = \int_{z=0}^{\infty} \hat{w}^-(p, 0, z; \omega) r(p, z) \hat{w}^+(p, z, 0; \omega) \hat{\phi}(\omega) dz. \quad (1)$$

In the above temporal frequency ray-parameter (p, ω)-domain formulation:

- Symbol $\hat{\cdot}$ indicates temporal Fourier transformed quantities, i.e. $\hat{f}(\omega) = \mathcal{F}\{f\}(\omega)$, with the inverse Fourier transform $f(t) = \mathcal{F}^{-1}\{\hat{f}\}(t)$ and where t stands for the intercept time.

- z is the vertical coordinate (positive z -direction points downward), $p = \frac{\sin \theta(z)}{c(z)}$ the ray-parameter with θ the angle of incidence and $c(z)$ the depth-dependent wavespeed, and $\omega = 2\pi f$ the temporal angular frequency.

- $\hat{p}(p, z = 0; \omega)$ represents the decomposed plane-wave reflection data at the surface, i.e. a monochromatic plane upward-traveling pressure wave.

- $\hat{w}^+(p, z, 0; \omega)$ and $\hat{w}^-(p, 0, z; \omega)$ are the flux-normalized single-scatter propagation operators given by

$$\hat{w}^+(p, z, 0; \omega) = \hat{w}^-(p, 0, z; \omega) = \exp[j\omega \int_0^z q(p, z') dz'], \quad (2)$$

with $q(p, z) = \sqrt{\frac{1}{c^2(z)} - p^2}$ the local vertical slowness.

- $r(p, z)$ is the p -dependent reflection density,

$$r(p, z) = \frac{1}{2Y} \partial_z Y(p, z) \quad \text{with} \quad Y(p, z) = q(p, z)/\rho(z) \quad (3)$$

with $Y(p, z)$ the acoustic admittance and $\rho(z)$ the volume density of mass.

• $\hat{\varphi}(\omega)$ represents the frequency characteristics of a bandwidth-limited downward source function.

The propagation operators carry the source wavefield from the surface down to the reflector and back up to a detector at the surface. The reflection density couples the down- and upward traveling waves at depth levels where the acoustic admittance varies. The one-way wavefield representation of Eq. 1 is approximate because it is based on the first term in the Bremmer series expansion [7], yielding single scattered waves only.

Despite the approximation, Eq. 1 captures leading order terms, describing the forward mapping of the medium singularities to the wavefield. For a constant background velocity model ($\bar{c} = \text{constant}$), Eq. 1 can be written as a temporal convolution:

$$p(p, z = 0; t) = (r_t(p, \cdot) *_{\text{t}} \varphi)(p; t) \quad \text{with} \quad r_t(p, t) = \frac{1}{2\bar{q}(p)} r(p, \frac{t}{2\bar{q}(p)}), \quad (4)$$

where $*_{\text{t}}$ denotes time convolution. The $r_t(p, t)$ represents the depth to two-way traveltimes-converted reflection density. Eq. 4 defines after linearization the functional form for the forward map, which describes how the medium variations are mapped to the wavefield.

2.2. The inverse map: migration and stacking

After applying the downward extrapolation with time-reversed extrapolation operators [7] and neglecting multiple scattering we find, by imposing the imaging condition, the following expression for the imaged reflectivity [38]:

$$\langle R(p, z) \rangle = \frac{2\bar{q}(p)}{\pi} \Re \int_0^\infty \hat{p}(p, z; \omega) d\omega. \quad (5)$$

In this expression the imaged reflectivity equals the downward continued pressure wavefield evaluated at time zero, $\langle R(p, z) \rangle \triangleq p(p, z; t = 0)$ with $p(p, z; t)$ the inverse Fourier transformed downward continued pressure. Eq. 5 contains the source contribution and the angular brackets $\langle \rangle$ are used to denote the resulting bandwidth limitation.

Likewise the surface reflectivity, the imaged reflectivity (cf. Eq. 5), can also be formulated as a convolution. Substitution of Eq. 5 into the one-way wave representation (cf. Eq. 1) yields,

$$\langle R(p, z) \rangle = \frac{2\bar{q}(p)}{\pi} (r_+(p, \cdot) *_{\text{z}} \varphi_z)(p, z), \quad (6)$$

where $*_{\text{z}}$ denotes depth convolution. The source function and reflection density are now given by

$$\varphi_z(p, z) = \varphi(2\bar{q}(p))(z) \quad \text{and} \quad r_+(p, \cdot) = H(z - \cdot) r(p, \cdot), \quad (7)$$

with $H(\cdot)$ the Heaviside distribution. Equation 7 corresponds to a p -dependent rescaling of the temporal source function. To complete the inverse map that relates

the surface reflectivity back to the medium properties, we propose the following simplified stacking:

$$\langle \Delta(z) \rangle = \int_{p_0}^{p_1} \langle R(p, z) \rangle dp. \quad (8)$$

Compared to commonly used stacking, Eq. 8 neglects frequency weighting normally found in the inverse Radon transform. Later, this choice will allow us to identify stacking as an inverse wavelet transform of the medium. Eq.'s 6 and 8 combined form the basis for the inverse map. Both expressions for the forward and inverse maps are still non-linear in the medium properties. In the next section we will linearize the expression for reflection density.

2.3. Linearized forward and inverse maps

Following Ostrander84, Castagna93, Wijngaarden98, the reflection density can be linearized in the normalized acoustic impedance ($\Delta Z(z)$) and compressional wavespeed (Δc) fluctuations, yielding,

$$r(p, z) \approx \left[\frac{1}{2} \quad \frac{1}{2} \frac{c^2 p^2}{\cos \bar{\theta}(p)} \right] \partial_z [\Delta Z \quad \Delta c]^T = \bar{M}(p) \partial_z \Delta(z). \quad (9)$$

Here, $\cos \bar{\theta}(p) = \sqrt{1 - \bar{c}^2 p^2}$ and $\Delta Z \approx \Delta c + \Delta \rho$. This expression is linear in the normalized medium fluctuations, which are assumed to be small and given by $\Delta f(z) = \frac{f(z) - \bar{f}}{\bar{f}} \ll 1$ with $f(z)$ and $\bar{f}(z)$ being either the actual/background acoustic impedance or the actual/background wavespeeds. Notice that the p -dependent factor of Eq. 9 remains non-linear in the background velocity, $\bar{c}$.

Given the linearized expression for the reflection density we are able to derive linear functionals for the forward mapping, imaging step and inverse mapping. Using Eq.'s 4 and 9 we find

$$p(p, z=0; t) \approx L\{\bar{c}; \Delta, \varphi\}(p, t) = \frac{\bar{M}(p)}{2\bar{q}}(p) (\partial_z \Delta(\frac{\cdot}{2\bar{q}(p)}) *_t \varphi)(t), \quad (10)$$

for the approximate linearized forward map ($L\{\bar{c}; \Delta, \varphi\}$). This forward map depends on the seismic wavelet and linearly maps the medium fluctuations to variations in the pressure wavefield. After imaging, we obtain a similar expression

$$\langle R \rangle(p, z) = \frac{2\bar{q}(p)}{\pi} \bar{M}(p) (\partial_z \Delta_+ *_z \phi_z)(z), \quad (11)$$

for the imaged reflectivity. Finally, after stacking, the inverse mapping equals

$$\langle \Delta \rangle(z) = L^*\{\bar{c}; \langle R \rangle\}(z) = (L^* \circ L)(z) \approx \int_{p_0}^{p_1} \langle R \rangle(p, z) dp, \quad (12)$$

where L^* refers to the approximate formal adjoint [34] of the one-way forward map L and $\circ$ denotes functional composition. Depending on the application, the inverse mapping can be supplemented with an additional pre- or post-stack deconvolution for the source wavelet.

Both linearized maps in Eq.'s 10 and 12 describe the reflection and subsequent imaging and inversion to leading order. Following [34], one can show that the

operator $(L^* \circ L)(z)$ is approximately pseudo-differential when the reference velocity model $(\bar{c}(z))$ is close enough to the smooth part of the actual velocity model. Consequently, singularities in the acoustic medium are preserved during the seismic "acquisition process" (L) and processing (L^*) . During processing, migration primarily concerns itself with finding the *locii* of the singularities, whereas during inversion, information on the magnitude of the acoustic medium variations are targeted.

Despite significant progress [34] in migration and inversion, acquiring information on the actual medium fluctuations remains difficult. Primary reasons are (1) the requirement of true amplitude data, not necessarily met in the field; (2) the lack of an accurate velocity reference model; and (3) the theoretical requirement of a *separation of scales*, underlying the wave theoretical formulation [34]. These difficulties form the primary motivation for this paper, where we will direct our attention towards finding the *locii* and orders of singularities instead of inverting for actual medium properties. Since the operators are approximately pseudo-differential, we may expect the singularity orders to represent a robust quantity during the seismic data processing flow. In the next section, we explore relationships between the forward/inverse maps and the forward and inverse wavelet transform.

3. WAVELETS AND SEISMIC REFLECTIVITY

Singular functions with isolated [21, 29, 24] or accumulated [1, 21, 24, 22, 23, 29, 22, 23] singularities have successfully been analyzed by the continuous wavelet transform (CWT). The singularities in these functions refer to edges/transitions in the medium properties [13, 15, 19, 17, 16] and to singularities in the wavefront set [19, 17, 16]. For isolated singularities, Hölder/Lipschitz exponents can be assigned to locally describe the regularity. Global regularity information is obtained by multifractal analysis, which produces a global but useful characterization, containing information on relationships between Besov norms of the medium and reflectivity [16].

As shown by [30, 32, 13, 15, 16], well-log data, as well as seismic reflectivity [13, 15, 16] behave multifractally which means that they display heterogeneous scaling behavior. Well-log data are *in situ* measurements of the earth subsurface, taken with a tool that is lowered into the borehole. The scaling heterogeneity means that the medium behaves as

$$|f(z + \Delta) - f(z)| \leq C \Delta^{\alpha(z)} \quad \text{as } \Delta \rightarrow 0, \quad (13)$$

where C is a finite constant and α the depth dependent Lipschitz/Hölder exponent. These exponents vary discontinuously [31], an indication of multifractality.

First, we introduce the continuous wavelet transform, followed by techniques to locally detect, measure and characterize singularities. Then, a direct link between seismic reflectivity and the wavelet transform will be established. Finally, a generalized transition model will be presented, yielding an effective parameterization by means of homogeneous distributions [10, 40, 21], which go beyond the commonly used zero or first order discontinuities [13, 15, 17, 16].

3.1. The continuous wavelet transform

Multiscale analysis by the continuous wavelet transform (CWT) can be seen as the interplay of a *smoothing* and a *de-smoothing* operation. This property becomes apparent when we define the CWT [29] as a multiscale derivative operator acting on the function f ,

$$\mathcal{W}\{f, \psi^M\}(\sigma, z) \triangleq \overbrace{\sigma^M \frac{d^M}{dz^M} (f * \phi_\sigma)(z)}^{\text{de-smoothing}} = (f * \psi_\sigma^{(M)})(z). \quad (14)$$

smoothing

In this expression, $\mathcal{W}\{f, \psi^M\}(\sigma, z)$ denote the wavelet coefficients of f with $\psi_\sigma^{(M)}$ the M^{th} -order scale-indexed analyzing wavelets. We define the wavelets as derivatives of dilated, real and symmetric smoothing functions ($\phi_\sigma(z)$):

$$\phi_\sigma(z) \triangleq \frac{1}{\sigma} \phi\left(\frac{z}{\sigma}\right) \quad \text{and} \quad \psi_\sigma^M(z) = (-1)^M \sigma^M \frac{d^M}{dz^M} \phi_\sigma(z). \quad (15)$$

The wavelets are L^1 -normalized and have an effective support proportional to the scale σ , which is related to the reciprocal of the dominant wavenumber, $\sigma \sim 1/\zeta_0$.

For the specific case of a Gaussian bell-shape smoothing function and $M = 2$, the wavelet $\psi_\sigma^2(z)$ becomes a Ricker wavelet, also known as a Mexican hat. Now, the wavelet transform of Eq. 14 can be written as [21]:

$$\mathcal{W}\{f, \psi^2\}(\sigma, z) \triangleq -\sigma \frac{d}{d\sigma} (f * \phi_\sigma)(z), \quad (16)$$

yielding wavelet coefficients that are given by the log σ -derivative of the smoothings. Notice that these smoothings contain *all* scales up to σ , while the wavelet coefficients contain only the *details* of f at σ .

By computing the inverse wavelet transform (ICWT, [4, 28, 21, 24, 29, 3]) defined as

$$f(z) = \int_0^\infty \int_{-\infty}^{+\infty} \mathcal{W}\{f, \psi\}(\sigma, z') \tilde{\psi}\left(\frac{z - z'}{\sigma}\right) dz' \frac{d\sigma}{\sigma} \quad (17)$$

f can be reconstructed. In this inverse transform, the wavelet and its "dual", $\tilde{\psi}$, are related by $1 = \int_0^\infty \tilde{\psi}(\zeta) \bar{\psi}(\zeta) \frac{d\zeta}{\zeta}$ where $\bar{\cdot}$ is the complex conjugate. The reconstruction holds *modulo* polynomials when f is a tempered distribution, $f \in S'(\mathbb{R})$, e.g. f is a delta distribution. Under the restriction $1 = \int_0^\infty \tilde{\psi}(\sigma\zeta) \frac{d\sigma}{\sigma}$ we can, following [3], derive a simpler reconstruction formula

$$f(z) = \int_0^\infty \mathcal{W}\{f, \psi\}(\sigma, z) \frac{d\sigma}{\sigma}, \quad (18)$$

which is possible because of redundancy within the CWT.

3.2. Local singularity characterization by scale exponents

Using multiscale analysis, the singularity strength can be calculated by following the wavelet coefficients along modulus maxima lines as a function of the scale.

The singularity strength is expressed by scale exponents, measuring the local Lipschitz/Hölder regularity [21, 24, 29]. Wavelets with M vanishing moments,

$$\int_{-\infty}^{+\infty} z^m \psi^M(z) dz = 0 \quad \text{for } m \leq M \quad (19)$$

are "blind", that is, their coefficients vanish with respect to polynomials of order $M - 1$. As a result, wavelets detect and characterize singularities in the M^{th} -order derivative of $f(z)$. The wavelet coefficients of a $n \leq M$ times differentiable function, $f(z)$, measure the remainder of the n^{th} order Taylor approximation of $f(z)$ around the point $z = z_0$, i.e.

$$\mathcal{W}\{f, \psi\}(\sigma, z_0) = \mathcal{W}\{\varepsilon, \psi\}(\sigma, z_0) \quad (20)$$

with the remainder

$$|\varepsilon(z)| = |f(z) - P_n(z)| \leq K|z - z_0|^\alpha. \quad (21)$$

Here, $P_n(z)$ is the $n = [\alpha]$ -order polynomial approximation of f around the point $z = z_0$. The remainder equals the difference between f and its approximation. Hölder exponents, α , measure the order of magnitude for the remainder. A function is called local Lipschitz/Hölder regular when the above inequality holds for a finite constant K . For technical details refer to [21, 24, 29].

Following [29, 1], the local Hölder regularity can be measured by the asymptotic decay/growth rate for the modulus maxima of the wavelet transform,

$$|\mathcal{W}\{f, \psi\}(\sigma, z)| \leq A\sigma^\alpha \quad \text{for } z = Z(\sigma) \quad \text{as } \sigma \rightarrow 0 \quad (22)$$

with A a finite scale, independent constant. The asymptotic behavior is studied along wavelet transform modulus maxima WTMM, that lie within the cone of influence given by $|z - z_0| \leq C\sigma$ and with C being the wavelet's support. Wavelet transform modulus maxima lines (WTMML, with $z = Z(\sigma)$) are found by connecting a line within this cone between the maxima across the different scales. The WTMML's are defined by

$$Z(\sigma) = \{z : \partial_z \mathcal{W}\{f, \psi\}(\sigma, z) = 0\}. \quad (23)$$

At each scale the local maximum should be a strict local maximum in either the right or left neighborhood of $z = z_0$ (see also definition 4.1). These local maxima [28, 29] delineate the major points of interest within the scale plane from which the original function can be reconstructed.

3.3. Seismic reflectivity as the continuous wavelet transform

Given the expression for the CWT (cf. Eq. 14), can we establish a relationship between this transform and the (imaged) seismic reflectivity? Comparing Eq. 14 with expressions for reflectivity we can show that for $M = 1$ both the linearized reflectivity at the surface and the imaged reflectivity can be expressed in terms of wavelet transforms. By replacing the source function by an analyzing wavelet,

$\varphi(\cdot) \mapsto \frac{1}{\sigma} \psi(\frac{\cdot}{\sigma})$, and using [21]'s scale-covariance relations, we find

$$p(p, z = 0; t) = \frac{\bar{M}(p)}{\sigma 2\bar{q}(p)} \mathcal{W}\{\Delta, \psi\}(\frac{\sigma}{2\bar{q}(p)}, \frac{t}{2\bar{q}(p)}), \quad (24)$$

and

$$\langle R(p, z) \rangle = \frac{\bar{M}(p)}{\sigma} \mathcal{W}\{\Delta_+, \psi\}(\frac{\sigma}{2\bar{q}(p)}, z). \quad (25)$$

For the reflected pressure wavefield (cf. Eq. 24), the response is identical to a CWT with rescalings for both the scale and time axis, while the imaged reflectivity (cf. Eq. 25) involves only a rescaling of the scale axis. Stacking of the imaged reflectivity can, by comparing Eq. 's 8 and 18, be associated with an inverse wavelet transform:

$$\langle \Delta \rangle(z) \sim \mathcal{W}^{-1}\{\langle R(p, z) \rangle, \delta\}(z), \quad (26)$$

where the integration runs over a non-linear p -dependent "scale".

EXAMPLE 3.1. Fig. 1 (top) contains the forward map of a medium profile with constant velocity and varying density. Top left, the acoustic impedance is plotted with five typical transitions (to be defined in section 3.4). Top right, the corresponding Radon-domain shot record, $p(p, z = 0; t)$, is displayed with the intercept time on the vertical and the ray-parameter on the horizontal axis. The source function is a Ricker wavelet (Mexican hat). Notice the time axis compressions and the wavelet dilations for the reflection data. Because the velocity is constant, the observed (AVP) behavior is caused solely by the scaling contribution (see Eq.'s 24-25). Also, notice the varying waveforms pertaining to the different reflectors. These different waveforms are also present in the pre-stack imaged reflectivity, depicted on the bottom right. In this plot, the reflections are aligned by the migration and a p -dependent stretch may be observed. Finally, the stacked trace (bottom left) qualitatively confirms the partial reconstruction of the singularities in the medium fluctuations (cf. Eq. 26).

From Eq.'s 24-26 and Example 3.1, we conclude that (i) amplitude *versus* ray-parameter (AVP)-behavior depends on both a *geometric* (by the $M(p)$) and a *scaling* contribution (by the $\bar{q}(p)$ in the $\mathcal{W}$'s); (ii) waveforms are preserved (besides dilations); and (iii) singularities are partially reconstructed by stacking. These observations clearly suggest that seismic data contain information on the singularities in the medium fluctuations. Even without source deconvolution, the stacking reconstructs the fluctuations for a scale range, limited by both the data's temporal frequency content and the available p -range.

3.4. Generalized reflector model: algebraic onset functions

To leading order seismic waves are sensitive to singularities in the velocity and density variations. When significant variations occur over length scales of the order of the seismic wavelength, waves are reflected. Correspondingly, wavelet coefficients are large at the location of singularities. Thus far, in seismology zero- or first-order discontinuities are mostly used to represent transitional regions in the

FIG. 1. Illustration of the forward and inverse mappings (cf. Eq.'s 10 and 12) with a Ricker (Mexican hat) source function. Top (left): the acoustic impedance (c is kept constant); Top (right): the reflection data, $p(p, z = 0; t)$; Bottom (left): stacked imaged reflectivity; Bottom (right): imaged reflectivity, $\langle R(p, z) \rangle$. Notice the $\Delta V P$ in both reflectivities, the compression in time and dilation of the wavelet for the reflection data and the wavelet dilations for imaged reflectivity. Also notice the different signatures for the reflection events, preserved throughout the forward, inverse map and the stacking. Refer to section 3.4 for detail on how the transitions were defined.

earth's medium properties. Given the singularity detection and characterization capabilities of wavelets, why limit ourselves to these singularities? Why not consider the wider class of algebraic non-oscillatory singularities [13, 15, 14, 19, 17] defined by

$$\chi_+^\alpha(z) \triangleq \begin{cases} 0 & z \leq 0 \\ \frac{z^\alpha}{\Gamma(\alpha+1)} & z > 0, \end{cases} \quad \text{and} \quad \chi_-^\alpha(z) \triangleq \begin{cases} \frac{-z^\alpha}{\Gamma(\alpha+1)} & z \leq 0 \\ 0 & z > 0. \end{cases} \quad (27)$$

These singularities can be seen as building blocks of heterogeneously-scaling singular functions (cf. Eq. 13, [1, 22]), and are used to define the medium variations as follows

$$f(z) = \bar{f}(z)[1 + \Delta f(z)] \quad \text{where} \quad \Delta f(z) = c_- \chi_-^\alpha(z - z_0) + c_+ \chi_+^\alpha(z - z_0). \quad (28)$$

Again, the barred quantity refers to a smoothly varying or constant function, while the fluctuations $\Delta f(z) \ll 1$ contain a generalized reflector, located at $z = z_0$. This reflector is given by a transition defined by the $\chi_-^\alpha(z)$ and $\chi_+^\alpha(z)$ -functions. These functions are known as the causal/anti-causal homogeneous distributions, or onset functions [10, 40, 21], and are parametrized by the singularity-order exponent $\alpha \in \mathbb{R}$. These onset functions constitute only the principal part of the asymptotic expansion found in [21], pg. 294. For $\alpha \in \mathbb{C}$ the singularities become oscillatory, a case not being considered in this paper. The $c_\pm$'s define the real valued bounded magnitude of the generalized reflector.

For $\alpha \geq 1$ the onset functions are continuous and differentiable; for $0 < \alpha < 1$ they are continuous but non-differentiable; and for $-1 < \alpha \leq 0$ they are discontinuous and non-differentiable. Finally, for $\alpha \leq -1$ the functions are no longer locally integrable and become singular tempered distributions. Besides the local regularity, the order parameter α also rules the homogeneity property,

$$\chi_\pm^\alpha(\sigma z) = \sigma^\alpha \chi_\pm^\alpha(z) \quad \sigma > 0. \quad (29)$$

This homogeneity property expresses the scale-invariance of the onset functions after appropriate rescalings. Scale-invariance is reflected into the following renormalization property for the wavelet coefficients

$$\mathcal{W}\{f, \psi\}(\sigma_0, z) = \left(\frac{\sigma_1}{\sigma_0}\right)^\alpha \mathcal{W}\{f, \psi\}(\sigma_1, z), \quad (30)$$

where f is defined as in Eq. 28. As a result, differences in scale of observation are interchangeable with the magnitude of the transition. This ambiguity makes it difficult to issue precise statements on amplitudes without knowledge of the (seismic) wavelet.

When $f(z)$ is defined according to Eq. 28, the wavelet coefficients display a power-law behavior for their moduli along WTML's and in the log-log plane. Using

$$\log |\mathcal{W}\{f, \psi\}(\sigma, z)| \leq \log A + \alpha \log \sigma \quad \text{for} \quad z = Z(\sigma) \quad \text{as} \quad \sigma \rightarrow 0, \quad (31)$$

we invert for the order of the transition, α , by estimating the slope of the wavelet coefficients.

4. MONOSCALE ANALYSIS BY FRACTIONAL ORDER WAVELETS

Within (exploration) seismology, the data are always bandwidth limited due to a combination of the finite-frequency range of the source, aperture limitation and possible dispersion effects. Consequently, the lack of available scale range may impede the asymptotics required to characterize the scaling by a single-scale exponent α . As a result, asymptotic techniques attempting to fit power-law dependence in the seismic data are not applicable. At this point, we may argue that the data do not provide adequate information on the singularities. However, the waveform variations observed in the idealized, but bandwidth limited examples of Fig. 1, suggest that reflection events contain regularity/sharpness information on the reflector [20, 19, 17].

As possible remedies for the apparent lack of information, [8] and [33] propose to either compute the instantaneous phase or the local derivative of the wavelet coefficients in the log-log plane. The former author limits himself to isolated singularities while the latter aims to estimate coarse-grained Hölder exponents for singularities by constraining the range of possible local slopes. In this way [33] is able to extend his method to non-isolated singularities without suffering from instabilities, arising from vanishing wavelet coefficients caused by mutual inferences of singularities. Both methods have the disadvantage that they (i) have to examine all the WTMML's pertaining to a single singularity; and (ii) lack an on/off criterion to predict the presence of a coarse-grained singularity.

To overcome some of these issues, a monoscale analysis method is introduced where the order of the wavelet is adapted to the singularity. The method is based on the heuristic that for the particular choice of singular functions as defined in Eq. 28, occurrences of modulus maxima depend on the wavelet order. By fractionally varying the amount of sharpening or de-sharpening in Eq. 14, we will examine the properties of convolution with a series of "wavelets"¹ consisting of fractional integro-differentiations of the Gaussian bell-shape smoothing function. Even though the Gaussian is not of compact support, it has, for integer² order moments, the advantage of propagating the maxima correctly towards the finer scales [29].

First, we introduce a framework to calculate the fractional integro-differentiations, followed by the definition of a transformation with respect to fractional integro-differential orders of the Gaussian. Second, we devise a criterion that estimates, for a particular subclass of functions, the local coarse grained scale exponents as a function of causal/anti-causal fractional integro-differentiations. Finally, we present a simple reconstruction scheme that reconstructs the singularity structure of the original function, using the location, direction, local order, sign and magnitude of the singularities.

4.1. Fractional calculus

Following [39, 11, 35], define Liouville's generalization of differentiation of fractional orders as the left inverse of the action of the onset functions, as defined in

¹A wavelet is no longer a wavelet when admissibility condition no longer holds, i.e. $\int \psi \neq 0$.

²Without proof it is assumed that this result also holds for fractional order moments.

Eq. 27. The causal fractional α^{th} derivative is defined according to

$$(\mathcal{D}_+^\alpha f)(z) \triangleq (\mathcal{D}^m \mathcal{I}_+^{m-\alpha} f)(z), \quad m-1 < \alpha \leq m, \quad m \in \mathbb{N}, \quad (32)$$

where

$$(\mathcal{I}_+^\alpha f)(z) \triangleq (\chi_+^{\alpha-1} * f)(z), \quad \alpha > 0 \quad (33)$$

is the causal α^{th} -order fractional integration of f . For specific conditions on f , refer to [39]. Similarly, the anti-causal fractional α^{th} -order derivative is defined according to

$$(\mathcal{D}_-^\alpha f)(z) \triangleq (\mathcal{D}^m \mathcal{I}_-^{m-\alpha} f)(z), \quad m-1 < \alpha \leq m, \quad m \in \mathbb{N} \quad (34)$$

with

$$(\mathcal{I}_-^\alpha f)(z) \triangleq (\chi_-^{\alpha-1} * f)(z), \quad \alpha > 0. \quad (35)$$

For simplicity denote fractional differentiation as $\mathcal{I}^{-\alpha}$. Moreover,

$$(\mathcal{D}_+^{\alpha_1} \mathcal{D}_+^{\alpha_2} f)(z) = (\mathcal{D}_+^{\alpha_1+\alpha_2} f)(z) \quad (36)$$

$$(\mathcal{D}_-^{\alpha_1} \mathcal{D}_-^{\alpha_2} f)(z) = (\mathcal{D}_-^{\alpha_1+\alpha_2} f)(z). \quad (37)$$

Unlike for integer exponents, the operations of fractional integro-differentiation generally do not commute. See [11] and references (to the Law of Exponents) therein.

Using the Fourier transform, fractional differentiation can be defined as

$$(\mathcal{D}_\pm^\alpha f)(z) = \mathcal{F}^{-1}\{(\pm j\zeta)^\alpha \hat{f}\}(z) \quad (38)$$

with $(\pm j\zeta)^\alpha \triangleq |\zeta|^\alpha \exp(j \text{sign}(\pm\zeta) \frac{\alpha}{2} \pi)$.

Finally, notice that by generalizing differentiation to fractional differentiation, the operator kernels no longer decay fast and depend on direction.

4.2. The β -transform

Given the definition for fractional differentiation, we are ready to generalize the definition of the CWT to a transform defined by multiscale causal/anti-causal differentiations,

$$\mathcal{W}\{f, \psi_\pm^\beta\}(\sigma, z) \triangleq \sigma^\beta \mathcal{D}_\pm^\beta (f * \phi_\sigma)(z) = (f * \psi_{\pm/\sigma}^\beta)(z) \quad \text{with } \beta \in \mathbb{R}^+ \quad (39)$$

and $\psi_{\pm/\sigma}^\beta(z) = \sigma^\beta \mathcal{D}_\pm^\beta \phi_\sigma(z)$. The smoothing function is taken to be the Gaussian yielding,

$$\mathcal{W}\{f, \psi_\pm^\beta\}(\sigma, z) = \mathcal{F}^{-1}\{[(\pm j\sigma\zeta)^\beta e^{-(\sigma\zeta)^2}] \hat{f}(\zeta)\}(z), \quad (40)$$

where the action of the fractional-order wavelet is contained within the square brackets. Fig. 2 contains examples of causal and anti-causal $\psi_\pm^\beta(z)$'s with $\beta \in (0, 3]$

FIG. 2. Generalized "wavelets". The wavelets are generated in the Fourier domain, which explains their periodicity. Top row (left): anti-causal, $\psi_-^\beta(z)$, and causal (right) wavelets $\psi_+^\beta(z)$ for $\beta \in (0, 3]$. Bottom row (left): anti-causal, $\psi_-^\beta(z)$, and causal (right) "wavelets" $\psi_+^\beta(z)$ for $\beta \in [-1, 0]$.

(top row) and $\beta \in [-1, 0]$ (bottom row). For the first β -interval the $\psi_\pm^\beta(z)$'s are still wavelets, but for the second (bottom row) the $\psi_\pm^\beta(z)$'s can no longer be considered as wavelets since $\int \psi_\pm^\beta(z) \neq 0$. The "wavelets" are computed by inverse Fourier transforming the expression between the brackets in Eq. 40, which explains their periodicity. Finally, as $\sigma \rightarrow 0$ $\mathcal{W}\{f, \psi_\pm^\beta\}(\sigma, z) \rightarrow (\mathcal{D}_\pm^\beta f)(z)$ for $f \in C^\alpha$ with $\beta > \alpha$.

Following [21], the conventional CWT of an algebraic onset function can be written as

$$\mathcal{W}\{\chi_\pm^\alpha, \psi\}(\sigma, z) \simeq \sigma^\alpha U_\pm^\alpha\left(\frac{z}{\sigma}\right) \quad (41)$$

with

$$U_\pm^\alpha\left(\frac{z}{\sigma}\right) = \mathcal{M}\{\psi(\mp \cdot \pm \frac{z}{\sigma})\}(\alpha + 1). \quad (42)$$

The Mellin transform is given by

$$\mathcal{M}\{f\}(q) = \int_0^\infty \frac{dt}{t} t^q f(t), \quad (43)$$

which is defined for a causal f (defined for $z \geq 0$) if

$$|f(z)| \leq c_1 z^{\alpha_1} \quad \text{for } 0 < z \leq 1, \quad (44)$$

$$|f(z)| \leq c_2 z^{-\alpha_2} \quad \text{for } 1 < z < \infty, \quad (45)$$

for some $c_1, c_2 > 0$, and $\alpha_1 < \alpha_2$. In this case the transform is analytic in the strip $\{z \in \mathbb{C} : \alpha_1 \leq \Re z \leq \alpha_2\}$. In case f itself an onset function, i.e. $f(z) = \chi_{\pm}^{\alpha}(z)$, the region of convergence for Eq. 43 reduces to an empty strip [25], for which the Mellin transform still exists as a generalized function.

The number of vanishing moments, $M \in \mathbb{N}$, of the CWT are chosen such that $M \geq [\alpha]$, implying the order $\alpha \geq 0$ singularities to lie in the singularity observation window of the wavelet transform. The window is bounded from below (for negative α) by the wavelet's smoothness and from above by the number of vanishing moments. By generalizing the CWT to a transform with fractional order wavelets, $\psi \mapsto \psi_{\pm}^{\beta}$, the region of convergence is shifted, $\Re q = \alpha \mapsto \alpha + \beta$, implying

$$\mathcal{W}\{\chi_{\pm}^{\alpha}, \psi_{\pm}^{\beta}\}(\sigma, z) = \sigma^{\alpha} U_{\pm}^{\alpha+\beta}\left(\frac{z}{\sigma}\right). \quad (46)$$

This equation holds when $\psi_{\pm}^{\beta} \in C^{\infty}$, $\beta > \alpha$, such that $z_{\pm}^{\alpha} \psi_{\pm}^{\beta} \in L^1$. For $\beta \leq \alpha$ Eq. 43 no longer converges. In the Fourier domain this behavior corresponds to a divergence around the origin, which occurs when $(j\zeta)^{\beta-\alpha-1}$ exceeds $(j\zeta)^{-1}$. In that case the inverse Fourier transform is no longer locally integrable. By shifting towards fractional orders, a precise condition has been obtained for which the β -transform converges or diverges.

4.3. The β mml

As with the multiscale wavelet transform [28, 29], β -transform modulus maxima (β MML) can be defined as follows:

DEFINITION 4.1. [β -transform modulus maxima] Let $\mathcal{W}\{f_{\pm}, \psi_{\pm}^{\beta}\}(\sigma, z)$ be the β -transform (cf. Eq. 39) of a causal/anti-causal function $f(z)$ defined for $\pm z \geq 0$ with $\beta > \alpha$.

- A local extremum is any point (σ_0, z_0, β_0) for which $\partial_z \mathcal{W}\{f_{\pm}, \psi_{\pm}^{\beta_0}\}(\sigma_0, z)$ has a zero-crossing at $z = z_0$, when z varies.
- Call a β -transform modulus maximum, a β MM, any point (σ_0, z_0, β_0) such that $|\mathcal{W}\{f_{\pm}, \psi_{\pm}^{\beta_0}\}(\sigma_0, z)| < |\mathcal{W}\{f_{\pm}, \psi_{\pm}^{\beta_0}\}(\sigma_0, z_0)|$ when z belongs to either the right or the left neighborhood of z_0 , and $|\mathcal{W}\{f_{\pm}, \psi_{\pm}^{\beta_0}\}(\sigma_0, z)| \leq |\mathcal{W}\{f_{\pm}, \psi_{\pm}^{\beta_0}\}(\sigma_0, z_0)|$ when z belongs to the other side of the neighborhood of z_0 .
- Call a β -transform modulus maxima line, a β MML, any connected curve in the β space (β, z) , σ fixed, along which all points are modulus maxima.

4.4. The on/off criteria

Similar to WTMM's, β MML's contain information on the local behavior of f , within a cone given by $z/\sigma = \text{constant}$. In this paper, we are only interested in obtaining estimates for the unknown coarse-grained, local scale exponent α at a

fixed scale. For this purpose it suffices to look at the onset of a β MML as a function of position and β within a fixed-scale of the zoom. With a slight abuse of notation, the following criterion provides the fixed scale estimate for $\alpha, \beta \geq 0$

$$\alpha(\sigma_0, z_0) = \inf_{\beta} \{ \beta : \partial_z \mathcal{W}\{f_{\pm}, \psi_{\pm}^{\beta}\}(\sigma_0, z = z_0) = 0 \}, \quad (47)$$

for a causal/anti-causal function $f_{\pm}(z)$ containing an (coarse-grained) algebraic singularity at $z = z_0$. This criterion is inspired by the work of [39], and argues that under certain conditions on $f_{\pm}(z)$,

$$(\mathcal{D}_{\pm}^{\beta} f_{\pm})(z) = \begin{cases} 0 & \text{for } \beta < \alpha(z) \\ \infty & \text{for } \beta > \alpha(z), \end{cases} \quad (48)$$

which in the coarse-grained case corresponds to the emergence of a local maximum. By taking $\sigma \rightarrow 0$ in Eq. 47, the criterion becomes equivalent to Eq. 48. But notice the necessary distinction in direction. For $\alpha < 0$ a similar criterion can be derived by reversing the argument, stating

$$\alpha(\sigma_0, z_0) = \sup_{\beta} \{ \beta : \partial_z \mathcal{W}\{f_{\pm}, \psi_{\pm}^{\beta}\}(\sigma_0, z = z_0) = 0 \} \quad (49)$$

with $\alpha, \beta < 0$. Here, the β -transform as defined in Eq. 39 is extended to include β -order fractional integrations.

The first criterion (cf. Eq. 47) is based on the property that when $\alpha > 0$, a local maximum appears if the order of fractional differentiation exceeds the order of the transition infinitesimally. Conversely, the second criterion (cf. Eq. 49) uses the property that a local maximum disappears when the fractional integration exceeds the negative scaling exponent. Differentiation, in the distributional sense, of onset functions $\alpha \geq 0$ reduces the exponent by the order of differentiation. For example, during reflection the deconvolved³ reflectivity involves a single differentiation of the medium fluctuations, yielding negative α in case of medium fluctuations with $0 < \alpha < 1$.

At the point of onset/disappearance, the location of the singularity is not well-resolved. Depending on direction, the location tends to be biased towards the left or right. To circumvent this problem the β MML is followed until the order of differentiation exceeds the estimated exponent by at least 1.

Notice that both criteria of Eq.'s 47 and 49 are based mainly on heuristical arguments valid for singular functions of the type

$$\Delta f(z) \triangleq \sum_{n \in \mathbb{N}} c_+^n \chi_+^{\alpha_n}(z - z_n) + c_-^n \chi_-^{\alpha_n}(z - z_n), \quad (50)$$

with $(c_+^n \neq 0 \wedge c_-^n = 0) \vee (c_+^n = 0 \wedge c_-^n \neq 0)$ and $c_+^n c_-^{n+1} > 0$ the magnitudes such that $\Delta f(z_n - \epsilon) = \Delta f(z_n)$, and z_n 's the abscissa. Singularities in these singular functions can, for scales exceeding the inter-singularity distance, no longer be treated as

³Under the assumption that deconvolution removes a possible additional differentiation by the seismic wavelet, e.g. the twofold differentiation of a Ricker wavelet.

isolated. This implies that the smoothing function should be of compact support. By limiting the number of modulus maxima per cone of influence to one, we are partially able to eliminate the effects of the non-vanishing support for the Gaussian.

The method we propose can now be formulated as follows:

PROCEDURE 4.1 (Monoscale measurement and detection of singularities).

1. Select wavelets with decreasing β such that $\beta_0 < \alpha < \beta_1$.
2. Fix σ such that possible high frequency noise is reduced.
3. Compute the generalized wavelet transform with Eq. 39 for $\beta \in [\beta_0, \beta_1]$.
4. Locate β MM's with definition 4.1, working from large to small β 's.
5. Create the set $\beta_{\pm} = \{1, 2, \dots, l\}$ of l curves, parameterized by $\{Z(\sigma, \beta)\}_{m \in \beta_{\pm}}$.
6. Check $\{Z(\sigma, \beta)\}_{m \in \beta_{\pm}}$'s lie inside the cone of influence ($|z - z_m| \leq K\sigma$).
7. Apply Eq.'s 47-49, yielding $\alpha_m = \alpha(z_m)$ as the $\inf_{\beta}$ along the m^{th} β MML.
8. Remove from α_m those estimates for which α_m lies too close to $[\alpha]$ or $\inf \beta$.
9. Keep α_m for which one of the two criteria holds for both directions.
10. Determine z_m by following the β MML until $\beta = \alpha_m + 1$ or $\sup \beta$.
11. Determine $c_{\pm}^m = \mathcal{W}\{f, \psi_{\pm}^m\}(\sigma, z_m)$ with $\gamma_m = \alpha_m + 1$ or $\sup \beta$.
12. Determine the direction for each singularity.
13. Remove those $c_{\pm}^m c_{\pm}^{m+1} < 0$ with $\pm_m = + \wedge \pm_{m+1} = -$.

Items 8, 9 and 13 are necessary to eliminate β MM's caused by either causal/anti-causal singularities, analyzed by the wrong anti-causal/causal analyzing wavelets, or by cusp-like singularities. In the first case, the generalized wavelet transform gives rise to an extra erroneous extremum. This extremum is an artifact caused by a response which, in case of an opposite direction wavelet, is given by the Hilbert transform. Extrema caused by cusp-like singularities can be removed by extending the search to left and right neighborhoods with an asymmetric cone. Locations and magnitudes are obtained at values for β such that the singularity becomes locally delta-like, which corresponds to a local pre-whitening. Finally, we note that the items in the above procedure are for a large part based on heuristics and require further justification and proof. For example, it remains an open problem how to deal with the directional problem of cusp-like singularities (not strictly (anti)-causal).

4.5. Reconstruction

Reconstruction of functions with fluctuations, defined by Eq. 50, is possible given estimates for the location, direction, regularity and relative magnitude for the observed singularities. For instance (see e.g. Example 4.1), the fluctuations can be reconstructed from estimates obtained from the smoothed first derivative (read reflectivity) of the original function. We base the reconstruction procedure on Eq. 50 with the parameters set by the estimated values, supplemented with additional conditions. The reconstruction procedure can be defined as follows:

PROCEDURE 4.2 (Reconstruction).

1. Normalize the magnitudes, $c_{\pm}^m$ such that $\sum_{m \in \beta_{\pm}} c_{\pm}^m = 1$.
2. α -correction (optional), $\alpha_m \mapsto \alpha_m + \alpha_{\text{cor}}$.
3. For each $m \in \beta_{\pm}$ take the estimated $z_m, \pm_m, \alpha_m, c_{\pm}^m$.

FIG. 3. Directional local regularity analysis and reconstruction of a synthetic post-stack migrated reflectivity trace (obtained by the stacking procedure illustrated in Fig. 1). Post-stack reflectivity for a Ricker wavelet corresponds to the smoothed third derivative of the acoustic impedance function. For reference the smoothed impedance profile is plotted in the first plot. Second plot: imaged reflectivity; Third plot: source wavelet corrected ($\alpha \mapsto \alpha + 3$) local regularity, directivity and sign estimates. Solid $\triangleright$'s are causal positive, $\triangleleft$ anti-causal positive, dotted $\triangleright$ causal negative sign, dotted $\triangleleft$ anti-causal negative sign. Bottom plot: reconstructed acoustic impedance profile.

if $(\pm_m = \pm_{m+1})$ then

$$f(z) = \chi_{\pm_m}^{\alpha_m}(z - z_m)1_{[z_m, z_{m+1})} \quad (51)$$

else

$$f(z) = \chi_{\pm_{m+1}}^{\alpha_{m+1}}(z - z_{m+1})1_{(z_m, z_{m+1}]} \quad (52)$$

if $(\pm_m \neq \pm_{m+1}) \wedge (\pm_m = +)$ then

$$f(z) = \inf\{\chi_+^{\alpha_m}(z - z_m)1_{[z_m, z_{m+1})}, \chi_-^{\alpha_{m+1}}(z - z_{m+1})1_{(z_m, z_{m+1}]} \} \quad (53)$$

for $c_+^m, c_-^{m+1} > 0$ and

$$f(z) = \sup\{\chi_+^{\alpha_m}(z - z_m)1_{[z_m, z_{m+1})}, \chi_-^{\alpha_{m+1}}(z - z_{m+1})1_{(z_m, z_{m+1}]} \} \quad (54)$$

for $c_+^m, c_-^{m+1} < 0$.

4. Smoothing (optional), $f(z) \mapsto (f * \phi_\sigma)(z)$.

Reconstructions using procedure 4.2 are *modulo* polynomials, which is consistent with the behavior of the inverse wavelet transform. The optional α -correction takes care of a possible additional differentiation, which would be ($\alpha_{\text{cor}} = 1$) when reconstructing the medium from the imaged, deconvolved reflectivity. For cases where causal singularities are followed by anti-causal ones, reconstruction creates an additional singularity at the location where the two reconstructions meet. Generally, this singularity will not be important because of its relative large regularity.

EXAMPLE 4.1. Fig. 3 demonstrates the application of the monoscale analysis and reconstruction to a single post-stack imaged reflection trace (see Fig. 1 (bottom left)). The medium consists of 5 transitions with $\pm \in \{1, -1, 1, -1, 1\}$ and $\alpha \in \{0.25, 0.39, 0.15, 0.52, 0.60\}$. For a Ricker wavelet, the stacked reflectivity (Fig. 3 (second row)) corresponds to the smoothed derivative of the acoustic impedance which is displayed in Fig. 3 (top). Singularity estimates, after correction for the differentiation, are plotted in the third row of Fig. 3. Solid $\triangleright$'s symbols are used for causal positive sign reflectors, $\triangleleft$ for anti-causal positive reflectors, dotted $\triangleright$ for causal negative sign reflectors, and dotted $\triangleleft$ for anti-causal negative signs. Clearly, the method finds and characterizes the singularities accurately. Given the location, direction and order of the singularities, a pseudo-acoustic impedance profile is generated using the reconstruction, as described in procedure 4.2. The magnitude of the reflectivity at the location of the singularities is used to define the relative magnitudes of the transitions (the $c_{\pm}^T$'s). Fig. 3 (bottom) displays the reconstructed profile. As expected, the method is not able to retrieve the actual impedances. Instead, the reconstruction recovers the singularities in the fluctuations. Deviations occur when causal singularities are followed by non-causal ones. In that case the method lacks information on where to connect the two functions (e.g. between the first and second transitions). Finally, notice that the reconstruction is not smoothed which, under the assumption that coarse-grained singularities persist to finer scales, corresponds to an effective deconvolution (together with the exponent correction for the differentiation by the Ricker).

FIG. 4. Application of the monoscale method to a smoothed Mobil well-data set [26]. Top: singularity characterization with $\beta \in [-1, 3]$, the gray-scale bar defines singularity orders and the circle size the relative magnitude. Second plot: smoothed ($\sigma = 6$), thick line) and fine-grained (thin line) acoustic impedances. Third plot: reconstructed pseudo well, based on the singularity characterization, plotted in the first plot. The pseudo well contains different order reflectors and yields a reflectivity (solid line bottom plot) close to that of the smoothed original well (dashed line).

5. APPLICATION TO SEISMOLOGY

During the seismic imaging process amplitudes remain difficult to control. Consequently, full inversion toward the earth's medium properties has not always been successful. We know that the composition of the forward and inverse map $(L^* \circ L)(z)$ is approximately pseudo-differential and as a result, we can be confident that major singularities are likely to be preserved during the seismic imaging process. For the purpose of geological horizon delineation, interpretation and reconstruction, it suffices to characterize singularities by finding their location and measuring their order.

5.1. Well Data

Fig. 4 demonstrates the application of our method to a real earth acoustic impedance profile obtained from a well-log measurement. The sampled (sample rate is 1 m) data were analyzed using procedure 4.1. Fig. 4 contains both the monoscale analysis for $\beta \in [-0.75, 3]$ and the reconstruction of a pseudo well from a coarse-grained singularity map. This coarse-grained map is obtained by smoothing the fine-grained (see Fig. 4, second plot thin line) acoustic impedance profile with a ($\sigma = 6$) Gaussian kernel. The estimated regularity and magnitude estimates are depicted by gray-scale circles (Fig. 4, top). Positions of the gray-filled circles point to the location of the singularities. The gray-scales themselves refer to the order, and the size to the magnitudes of the derivative of the smoothed well at the locations of the singularities. The gray-scale bar on the right defines the orders corresponding to the different gray-values. Dark shades are used to delineate the sharp, i.e. irregular, transitions which in the Fourier domain tend to cause "blue divergences". The color white, on the other hand, indicates smooth transitions, which in the Fourier domain correspond to "infrared divergences". Clearly, one can see that the irregular gray-value changes support the observation that well data behave multifractally [30, 32, 13, 15, 16]. Moreover, the emergence of circles almost everywhere is an indication of an accumulation of singularities. The third plot contains the reconstructed profile using singularity characterization. Note the sharpness variations of the reconstructed transitions. At the bottom, comparison is made between the first derivative of the original smoothed acoustic impedance and the derivative (read reflectivity), obtained by smoothing and differentiating the reconstructed profile. Although the reconstruction is not perfect, the result is encouraging because the pseudo-reflectivity matches the reflectivity from the original quite well.

5.2. Seismic Images

The results of applying the monoscale analysis method to a post-stack, time-migrated section are summarized in Fig.'s 5 and 6. Simply speaking, time migration differs from depth migration by staying in time rather than converting to depth. Hence, the vertical axis of Fig. 5 corresponds to the two-way traveltime. The horizontal axis measures the lateral direction. Migration amplitudes are depicted in gray-scale and reveal part of the reflector structure in the earth subsurface.

FIG. 5. Time migrated Troll dataset and its reconstruction, using the singularity characterization with $\beta \in [-4, 1]$ and $\sigma = 1$. Top: in gray-scale the imaged reflection amplitudes. Middle: scatter plot of the estimated singularities; gray-values of the dots refer to the corrected singularity orders (darker means more irregular/sharper). Bottom: trace by trace reconstruction of the reflectivity, based on the location, order and relative seismic amplitude. Both the position and order of the reflectors are well recovered. The correction for the singularity order was $\alpha_{cor} = 3$. The singularity structure aligns nicely with the imaged reflectors. Good lateral consistency is obtained along the major reflectors and across the wells located at trace #'s 205 and 365. These wells are tied to the seismic by generating synthetic reflectivity using Eq. 1 and a source wavelet estimate. The trace by trace basis of the method is responsible for the apparent horizontal differences in the reconstruction.

By conducting the monoscale analysis on each individual vertical trace of the migrated data (Fig. 5, top), we create a map of the earth's singularity structure. This map is depicted in Fig. 5 (middle) and was generated using $\beta \in [-4, 1]$ and $\sigma = 1$, respectively. Gray-scales are used to display the singularity map by a scatter plot. The gray-scales of this scatter plot are related to the local regularity of the vertical variations in reflection amplitude. The dark shades refer to relatively sharp reflectors, while the white shades mark the relative smooth transitions. Notice the location and, to a lesser extent the gray-scale, to be captured in a laterally-consistent manner, even though the amplitudes show relative large variations along the major reflector horizons. Refer to [19, 27] for a discussion on the geological interpretation of the singularity map. Finally, Fig. 5 (bottom) contains the reconstructed pseudo reflectivity. The reconstruction is based on the location and gray-scales of the singularity scatter plot (Fig. 5, middle) in conjunction with the relative magnitude of the reflectivity at the location of the singularities. Clearly, the reconstruction captures the major features of the original reflection data quite well, although there are trace to trace variations. Finally, notice that both the singularity map and the reconstruction display a nice consistency across the wells located at trace #'s 205 and 365. The wells are tied to the seismic data by forward modeling a synthetic reflectivity. This reflectivity is computed using Eq. 1, with the medium given by the well-log profile and the source wavelet estimated so as to minimize the difference between the synthetic and the imaged surrounding reflectivity. For clarity, the synthetic trace is repeatedly plotted.

Finally, Fig. 6 illustrates more details of the monoscale analysis and reconstruction. For this purpose the area surrounding the second well (white box in Fig. 5, top) is examined. Fig. 6 (top) displays both the synthetic reflection amplitudes (middle) and real earth imaged reflectivity (sides). Besides the wiggle traces, this plot also contains the *loci*, order and relative magnitudes of the singularities. Again, the position of the circles refers to the location of the singularities, while the gray-scale and size refer to the singularity strength and relative magnitude. As one can see there is a reasonable lateral consistency for the gray-scale along the major reflection horizons, persisting across the synthetic traces (around trace # 15). As an ultimate test, reconstruction results for both the pseudo reflectivity and well are depicted in the bottom two plots of Fig. 6. These results are compared to data we actually know from the well-log. What these results clearly demonstrate is that the proposed singularity inversion method works quite well, despite the severe bandwidth limitation of the seismic data (middle), compared to the well (bottom). The reconstructed impedance (solid line) follows the original well for most of the major transitions. The estimated transitions vary in regularity/sharpness, but are limited in number because of the limited resolution of the seismic data. Still, the transitions capture the leading order scaling behavior of the transitions at the seismic length scale.

6. DISCUSSION AND CONCLUSIONS

In this paper, we presented a case where both the advantages and shortcomings of the continuous wavelet transform in seismology are revealed. Surely, as an analysis tool, wavelets are a natural analysis technique for seismological data because the wavelet transform is intrinsically linked to the physics of reflection

FIG. 6. Details of reflectivity and pseudo well-log reconstruction, based on singularity characterization. Top: wiggle plot of the selected migration amplitudes (within the white box in Fig 5, top) together with the estimated singularity orders, depicted by the gray-colored circles whose size depend on the relative amplitudes. Middle: the seismic reflectivity (dashed) at the well (around trace # 15 in top plot) and the reconstructed reflectivity (solid). Bottom: reconstructed well (solid) and the original well (dashed). Notice, the remarkable reconstruction despite the severe bandwidth limitation of the seismic (compare reflectivity to the detailed well).

seismology. Also, we show that seismic reflectivity to leading order behaves as a rescaled wavelet transform of the earth medium properties. Consequently, seismic waves share the property of triggering on singularities with wavelets, which opens the wavelet singularity characterization framework to seismology. When imaging is performed correctly, seismic processing is pseudo-differential, preserving the singularities, something which cannot necessarily be said about the seismic amplitudes. Therefore, introducing a singularity order map seems to be the appropriate way to infer information from seismic data, describing the earth's structure. As shown such a singularity map represents an image with localized information on the orders of magnitude of variations in the medium properties. These orders of magnitude are described by scale exponents, which capture the local regularity, local scaling and sharpness of the reflectors.

Besides the advantages of using the wavelet local regularity estimation framework, there is one important downside to wavelets — the bandwidth limitation of seismic data. As a result, seismic images contain information essentially residing at the fixed scale of the seismic wavelet. Clearly, this bandwidth limitation prevents successful application of the standard small-scale, asymptotic wavelet techniques. To overcome this fundamental difficulty a monoscale analysis technique was presented which allows the estimation of coarse-grained local regularity estimates at the fixed scale. The method is based on a transform with respect to a family of integro-differentiations of the Gaussian bell-shape function, supplemented by two criteria. These two criteria predict the appearance or disappearance of modulus maxima as a function of the order of integro-differentiations for a special class of functions. The computational cost of this method is comparable to that of the ordinary scale asymptotic WTML methods. The number of convolutions and maxima searching are of the same order. However, compared to [33]'s constraint local Hölder exponent estimation method, our approach has the advantage that it only assigns one single exponent per singularity. Because we limit the β -range, our method also automatically constrains the local slopes. Theoretically, there remain many issues to be resolved. For example, the directionality of the parametrized singularities is not easily resolved. Application of techniques from micro-local analysis may help to resolve this issue. Despite the lack of a formal theoretical justification of our method, the application of our transform and criteria to both well and seismic data demonstrate that singularity maps of well and seismic data (1) display heterogeneous scaling behavior, consistent with their observed multifractal behavior; (2) can be used to generate reconstructions; (3) can be used to interpret angle *versus* off-set behavior which, in part, is due to scaling; and (4) contain useful (geological) information on the location and local sharpness/regularity of the reflectors. Application of the essential one-dimensional monoscale analysis technique to seismic images shows a remarkable performance even though the data are two-dimensional.

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