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Discontinuous Galerkin for Stiff Hyperbolic Systems*

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A Discontinuous Galerkin (DG) method is applied to hyperbolic systems that contain stiff relaxation terms. We demonstrate that when the relaxation time is under-resolved, DG is accurate in the sense that the method accurately represents the system's Chapman-Enskog (or "diffusion") approximation. Moreover, we demonstrate that a high-resolution, finite-volume method using the same time-integration method as DG is very inaccurate in the diffusion limit. Results for DG are presented for the hyperbolic heat equation, the Broadwell model of gas kinetics, and coupled radiation-hydrodynamics.

Introduction

Many physical systems can be modeled by hyperbolic systems that contain relaxation terms, written in the form

$$\partial_t \mathbf{u} + \partial_x \mathbf{f}(\mathbf{u}) = \mathbf{s}(\mathbf{u})/\tau, \quad (1)$$

where $\mathbf{u}$, $\mathbf{f}(\mathbf{u})$, and $\mathbf{s}(\mathbf{u})$ are vectors of length p , and τ is a representative relaxation time. In general, there may be up to p -different relaxation times. For small τ , we assume that (1) can be accurately represented by the reduced system

$$\partial_t \mathbf{v} + \partial_x \mathbf{g}(\mathbf{v}) = \tau \partial_x [\mathbf{D}(\mathbf{v}) \partial_x \mathbf{v}], \quad (2)$$

where $\mathbf{v}$ and $\mathbf{g}(\mathbf{v})$ are vectors of length q , with $q < p$, and $\mathbf{D}(\mathbf{v})$ is a $q \times q$ matrix (positive semi-definite). This reduced system can be derived through a Chapman-Enskog expansion,¹ and is sometimes referred to as the continuum or equilibrium-diffusion limit. See also Liu² and Whitham³ for more discussion on these systems.

A classic example is from gas kinetics, where Eq. (1) is a moment closure of the Boltzmann equation,

Eq. (2) is the Navier-Stokes equations, and τ is proportional to the molecular mean-free-path. A much simpler example is the hyperbolic heat equation:^{4,5}

$$\partial_t u + \partial_x v = 0, \quad (3a)$$

$$\partial_t v + \partial_x u = -v/\tau. \quad (3b)$$

For small τ , this system can be accurately represented by the heat equation

$$\partial_t u = \tau \partial_{xx} u. \quad (4)$$

Although this system appears simple, it presents a challenge for many modern numerical methods.

Roughly speaking, we seek methods whose discretization of Eq. (1) reduce to an accurate discretization of Eq. (2) when $\Delta t/\tau$ is large, where Δt is the timestep. The difficulty is that a method that is, say, formally second-order accurate, in a practical sense may only be second-order when $\Delta t/\tau$ is small (so that all time scales are resolved). When $\Delta t/\tau$ is large, there is no guarantee that the method is even consistent with Eq. (2).

These same arguments also apply to the spatial discretization by defining the stiffness parameter as $\Delta x/(\lambda\tau)$, where Δx is the mesh spacing and λ a reference wavespeed. Given an accurate method for Eq. (1), when $\Delta x/(\lambda\tau)$ is large, again there no guarantee the method will reduce to an accurate method for Eq. (2).

Methods for stiff systems have recently been an active area of research.^{4,6-10} Such systems can be found, for example, in combustion, multi-phase flows, and rarefied gas dynamics. In addition, outside of typical aerospace applications, the equations that govern neutron and radiation transport are stiff. Over the past decade, much progress has been made in integrating transport methods. One method that has had good results for transport is Discontinuous Galerkin (DG) with linear elements.^{11,12} More generally, we denote $DG(k)$ as DG using k 'th order polynomials as a basis. We intend to show in this paper that DG may be an excellent choice for other stiff systems.

We stress that our interest is in methods that not only have the proper 'inviscid' (or 'Eulerian') limit, in that they asymptotically reduce to the left-hand side of Eq. (2), but also the method must accurately represent the diffusion term. Even if the interest is only

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in the inviscid limit, in this study we will demonstrate that if a method has the correct inviscid limit but the incorrect diffusion limit, it may be much too diffusive even when the exact diffusive length scale is sub-grid.

The Discontinuous Galerkin Method

In this section we give a brief overview of our implementation of the semi-discrete version of DG(k). The discussion here is for only one space dimension. For an excellent description of DG, along with the extension to multiple dimensions, see the series of papers by Cockburn et. al.¹³⁻¹⁶ Also, all of the results we present in this study are for $k = 1$. We review the method for larger k for completeness and to show that our source-term treatment carries over to the general case.

Let the spatial mesh be made up of cells defined as the intervals $[x_{m-1/2}, x_{m+1/2}]$, $m = 1 \dots M$, and define $\Delta x_m = x_{m+1/2} - x_{m-1/2}$. For each cell m , linearly map the interval $[x_{m-1/2}, x_{m+1/2}]$ to $\xi \in [-1, 1]$. Then on each cell m , the semi-discrete version of DG for Eq. (1) can be written as

$$\int_{-1}^1 \phi_i \partial_t \mathbf{u} d\xi + \frac{[\phi_i \mathbf{F}]}{\Delta x_m} - \int_{-1}^1 \mathbf{f} \partial_\xi \phi_i d\xi = \int_{-1}^1 \phi_i \mathbf{s} d\xi, \quad (5)$$

where $\phi_i = \phi_i(\xi)$ is the i 'th basis function. In each cell, $\mathbf{u}$ is approximated as

$$\mathbf{u}|_m(\xi, t) = \sum_{j=1}^n \mathbf{U}_{m,j}(t) \phi_j(\xi), \quad (6)$$

where $n = k + 1$.

The basis set $\{\phi\}$ are Lagrange-Legendre polynomials and the set $\{\xi_1, \xi_2, \dots, \xi_n\}$ are the Gauss-Lobatto integration points.¹⁷ A property of this basis is that

$$\phi_i(\xi_j) = \delta_{i,j}, \quad (7)$$

so that $\mathbf{U}_{m,j}$ represents the value of the solution with respect to cell m at ξ_j . See Fig. 1. Our choice of basis is somewhat unconventional when compared with previous DG work, and as will be seen, is a consequence of our desire to treat the source term implicitly in an efficient manner.

For our choice of basis,

$$[\phi_i \mathbf{F}] = \delta_{i,n} \mathbf{F}_{m+1/2} - \delta_{i,1} \mathbf{F}_{m-1/2},$$

where $\mathbf{F}_{m+1/2} \equiv \mathbf{F}(\mathbf{U}_{m,n}, \mathbf{U}_{m+1,1})$ is any suitable flux function. At least in this study, the wave decomposition in the flux solver is based on the "frozen" flux $\mathbf{f}(\mathbf{u})$, as opposed to including any effects of the "equilibrium" waves defined by $\mathbf{g}(\mathbf{v})$. This issue will be discussed further in the results section for radiation hydrodynamics.

Following the "quadrature free" approach,¹⁸ if the integrals in Eq. (5) are replaced by quadrature, it can

Fig. 1 Example DG solution representation in two cells. Cell ' m ' uses the $k = 3$ (cubic) basis, while cell ' $m + 1$ ' uses the $k = 1$ (linear) basis. Note that in this study, k is a constant over the solution domain.

be written as

$$\mathbf{M}_U \dot{\mathbf{U}} + \frac{1}{\Delta x_m} \mathbf{M}_B \mathbf{B} - \mathbf{M}_V \mathbf{V} = \mathbf{M}_S \mathbf{S}, \quad (8)$$

where the subscript m is henceforth dropped unless needed, and

$$\mathbf{U} = (\mathbf{U}_1, \mathbf{U}_2, \dots, \mathbf{U}_n)^T, \quad \mathbf{B} = (\mathbf{F}_{m-1/2}, \mathbf{F}_{m+1/2})^T.$$

The remaining quantities in Eq. (8) require additional explanation. The first integral in Eq. (5) is computed exactly, with $\mathbf{M}_U$ an $n \times n$ matrix and each component given by

$$(\mathbf{M}_U)_{i,j} = \int_{-1}^1 \phi_i(\xi) \phi_j(\xi) d\xi.$$

The matrix $\mathbf{M}_B$ is $n \times 2$, given by

$$(\mathbf{M}_B)_{i,j} = \delta_{i,n} \delta_{j,2} - \delta_{i,1} \delta_{j,1}.$$

To account for the possible nonlinearity of $\mathbf{f}(\mathbf{u})$, the second integral in Eq. (5) is approximated with n_V -quadrature points, with $n_V \geq n$. The quadrature rule has weights $\{w^V\}$ corresponding to the point locations, $\{\xi^V\}$. Also,

$$\mathbf{V} = (\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_{n_V})^T,$$

where $\mathbf{f}_j = \mathbf{f}(\mathbf{u}(\xi_j^V, t))$, and $\mathbf{M}_V$ is a $n \times n_V$ matrix:

$$(\mathbf{M}_V)_{i,j} = w_j^V \partial_\xi \phi_i|_{\xi=\xi_j^V}.$$

A similar approach could be used to account for the nonlinearities in the source term. However, it is sufficiently accurate to take

$$\mathbf{S} = (\mathbf{s}_1, \mathbf{s}_2, \dots, \mathbf{s}_n)^T,$$

where $\mathbf{s}_j = \mathbf{s}(\mathbf{U}_j)$, and $\mathbf{M}_S = \mathbf{M}_U$. The reason this works is that it is easy to show that there is no approximation (aside from time-integration errors) when $\mathbf{s}(\mathbf{u})$ is linear, or when $\mathbf{f}(\mathbf{u}) = 0$. The advantage of this simplification is that Eq. (8) can then be written as

$$\dot{\mathbf{U}} + \frac{1}{\Delta x_m} \mathbf{C}_B \mathbf{B} - \mathbf{C}_V \mathbf{V} = \mathbf{S}, \quad (9)$$

where $C_B = M_U^{-1}M_B$ and $C_V = M_U^{-1}M_V$. These matrices are independent of the particular element so that they can be computed once and stored. Moreover, coupled with our choice of basis, if B and V are treated explicitly, and S implicitly, the method is *point implicit*. By 'point' we mean a single U_j . For some other $M_S \neq M_U$, or for a nonlinear $s(u)$ used with a basis that does not satisfy Eq. (7), the method would be implicit over all n values of U_j within an element.

The ideas we have borrowed from the quadrature-free approach extend to multiple-space dimensions¹⁸ and the source-term treatment extends easily as well.

Time Integration

A common approach for DG is to use Runge-Kutta time-stepping.¹⁹ Runge-Kutta requires the entire residual to be evaluated at the same solution state. When the relaxation time scale is not important, an implicit treatment of the source term is warranted. On the other hand, so that the method is point-implicit, we prefer to treat the remaining terms explicitly. Therefore, instead of Runge-Kutta, we use a simple predictor-corrector approach.

For the predictor, the source term is treated fully-implicitly over a half-time step, while the remaining terms are explicit. Equation (9) then gives

$$\frac{U^{n+1/2} - U^n}{\Delta t/2} + \frac{1}{\Delta x_m} C_B B^n - C_V V^n = S^{n+1/2},$$

where the superscript- n denotes the time-level. The corrector-step is then

$$\frac{U^{n+1} - U^n}{\Delta t} + \frac{1}{\Delta x_m} C_B B^{n+1/2} - C_V V^{n+1/2} = \theta S^{n+1} + (1 - \theta) S^n,$$

where $0 \leq \theta \leq 1$. We typically use $\theta = 1$ (fully implicit), which in general is only first-order in time. An alternative is $\theta = 1/2$ (Crank-Nicholson), which is second-order when the relaxation time is resolved, but is well known to give oscillatory behavior if the relaxation time is unresolved. In practice, if the relaxation time is unresolved, a fully implicit treatment is sufficiently accurate since the remaining terms are treated with second-order accuracy.

Hyperbolic Heat Equation

In this section we present results for Eq. (3) on the domain $x \in [0, 1]$, with periodic boundary conditions, and the initial condition $u = v = \cos(2\pi x)$. For reference, we compare with a high-resolution (HR) method that uses a central-difference slope reconstruction and the same time-integrator presented in the previous section. By no means do we intend to imply that this is a state-of-the-art high-resolution method for stiff systems; instead, HR is meant to represent what happens when a standard method for hyperbolic systems is naively applied to a stiff system.

The DG results were run at a Courant number of 0.3 (stability limit is $1/3$), based on the "frozen" wavespeeds of ± 1 . The HR results used a Courant number of 0.8 (stability limit is 1). The time-integrator used $\theta = 1/2$, although little difference was observed in the values of u with $\theta = 1$. No slope-limiting was applied in either method.

Figure 2 shows the results for two values of τ , and the final time set to $0.01/\tau$. The exact total amount of damping is the same for both cases. The results show that DG(1) is fairly independent of τ , whereas for $\tau = 1 \times 10^{-5}$, the HR results are significantly over-damped, even for 80-mesh cells. Note that on the same mesh, DG(1) has twice the unknowns per cell as the HR method.

The difficulties for the HR method cannot be overcome by simply lowering the timestep. For example, for the 80-cell case in Fig. 2b, lowering the Courant number from 0.8 to 0.08 *decreased* the amplitude from 0.46142 to 0.46122; that is, the error increased.

Figure 3 shows the error convergence of DG(1) for three values of τ and a fixed Courant number. A comparison is made between the two values of θ used in the corrector. For Crank-Nicholson ($\theta = 1/2$), the method is uniformly second order. The $\theta = 1$ results initially converge second-order, until the mesh is refined to a point that the error drops below $\mathcal{O}(\tau)$. With further refinement the rate slows to *below* first-order, until finally the relaxation time-scale is resolved and the method slowly asymptotes to first-order accuracy. The initial second-order behavior is encouraging, in that it demonstrates that the source-term treatment need not be second-order to obtain a second-order representation for the diffusion approximation, Eq. (2).

For strongly nonlinear problems such as radiation hydrodynamics, Crank-Nicholson is not an option because of its oscillatory behavior. For the remainder of this study, $\theta = 1$ is used, which still gives good results, particularly when the diffusion approximation is valid. More accurate time-integration methods are left for future study.

Broadwell Equations

A simple gas kinetics model that is often studied is the Broadwell equations:

$$\partial_t \rho + \partial_x m = 0, \quad (10a)$$

$$\partial_t m + \partial_x z = 0, \quad (10b)$$

$$\partial_t z + \partial_x m = (m^2 + \rho^2 - 2\rho z)/(2\tau), \quad (10c)$$

where ρ is the density, m the momentum, and z the pressure. As $\tau \rightarrow 0$, the last equation gives that $z \rightarrow (m^2/\rho + \rho)/2$ and the first two equations take on a form that resembles the isothermal Euler equations.

Figure 4 shows results for similar conditions as those run by Arora & Roe.⁴ Jin¹⁰ has also run the problem shown in Fig. 4d. The initial condition used

Fig. 2 Results for the hyperbolic heat equation. DG(1) and HR used the same time integrator.

Fig. 3 DG(1) error convergence for a fixed Courant number of 0.3. Final time set as $0.01/\tau$.

in Figs. 4a-c is $(\rho, m, z) = (2, 1, 1)$ for $x < 0$, and $(\rho, m, z) = (1, 0.13962, 1)$ for $x > 0$. Note that Arora & Roe put the initial discontinuity at $x = 0.2$. The initial condition for Fig. 4d is $(\rho, m, z) = (1, 0, 1)$ for $x < 0.5$ and $(\rho, m, z) = (0.2, 0, 1)$ for $x > 0.5$. The results are compared with 2000-cell, mesh-converged results.

These problems required the use of Van Leer's double-minmod limiter,²⁰ applied to the characteristic variables of the frozen system after both the predictor

and corrector.

In each problem, DG(1) used the same number of cells as Arora & Roe's cell-centered scheme, which translates into DG(1) using twice the number of unknowns. In our experience, to resolve discontinuities with equal resolution as cell-centered methods, even for non-stiff problems DG(1) requires the same number of cells. With this in mind, our results compare very well with those presented by Arora & Roe and by Jin.

A final comment needs to be made about Fig. 4b. For those familiar with shock-capturing methods, the bump near $x = -0.2$ appear to be an unphysical starting error. However, we believe it is the correct solution. To begin with, both the bump's amplitude and width do not change with increasing mesh size. Moreover, if τ is changed slowly from $\tau = 1$ to $\tau = 1 \times 10^{-2}$, one sees (although not shown here) a smooth change in the solution from Fig. 4a to 4b.

Radiation Hydrodynamics

The non-relativistic Euler equations of gas dynamics, coupled with a gray P_1 -model of radiation transport, can be written in non-dimensional form as^{21,22}

$$\partial_t \rho + \partial_x (\rho v) = 0, \quad (11a)$$

$$\partial_t (\rho v) + \partial_x (\rho v^2 + p) = -\mathcal{P} S_F, \quad (11b)$$

$$\partial_t (\rho E) + \partial_x [(\rho E + p)v] = -\mathcal{P} C S_E, \quad (11c)$$

$$\partial_t E_r + C \partial_x E_r = C S_E, \quad (11d)$$

$$\partial_t F_r + \frac{1}{3} C \partial_x E_r = C S_F, \quad (11e)$$

where ρ is the material density, v the velocity, E the total material specific energy, p the material pressure,

a) $\tau = 1$, $t = 0.5$, 200 cells

b) $\tau = 1 \times 10^{-2}$, $t = 0.5$, 100 cells

c) $\tau = 1 \times 10^{-8}$, $t = 0.5$, 200 cells

d) $\tau = 1 \times 10^{-8}$, $t = 0.25$, 200 cells

Fig. 4 DG(1) cell averages (symbols) for the Broadwell equations, Courant number 0.3, $\theta = 1$. The solid lines are the results for 2000 cells, which are mesh-converged to the plotting scale shown. Note that d) used a different initial condition than the other solutions; see the text for a description.

E_r the radiation energy density, and F_r the radiation flux. Assuming negligible scattering and Planckian emission, the energy and momentum depositions are given by, respectively,

$$S_E = \sigma_t(T^4 - E_r) + \sigma_t \frac{v}{c} (F_r - \frac{4}{3} \frac{v}{c} E_r), \quad (11f)$$

$$S_F = -\sigma_t(F_r - \frac{4}{3} \frac{v}{c} E_r) + \sigma_t \frac{v}{c} (T^4 - E_r), \quad (11g)$$

where σ_t is the non-dimensional cross-section (flow-length scale over the photon mean-free-path) and T is the material temperature. There are two non-dimensional constants in these equations:

$$\mathcal{C} = \frac{c}{a_\infty}, \quad \mathcal{P} = \frac{a_R T_\infty^4}{\rho_\infty a_\infty^2},$$

where c is the lightspeed, a_R the radiation constant, and ' ∞ ' denotes reference conditions, with a_∞ the reference soundspeed. The parameter $\mathcal{C}$ is a measure of relativistic effects, while $\mathcal{P}$ is proportional to the equilibrium-radiation pressure over the material pressure. Unlike the majority of engineering applications, where $\mathcal{P}$ is very small, we are interested in regimes where $\mathcal{P}$ can become order unity or higher.

For σ_t large, Eqs. (11) can be approximated by

$$\partial_t \rho + \partial_x(\rho v) = 0, \quad (12a)$$

$$\partial_t(\rho v) + \partial_x(\rho v^2 + p^*) = 0, \quad (12b)$$

$$\partial_t(\rho E^*) + \partial_x[(\rho E^* + p^*)v] = \partial_x \left[\frac{\mathcal{P}\mathcal{C}}{3\sigma_t} \partial_x T^4 \right], \quad (12c)$$

where

$$p^* = p + \frac{1}{3} \mathcal{P} T^4, \quad E^* = E + \mathcal{P} T^4 / \rho.$$

This limit is often referred to as the *equilibrium-diffusion limit*.²¹ Note that the left-hand side of Eqs. (12) is in the form of the Euler equations. Equilibrium radiation effectively modifies the equation-of-state. It can be shown that the system (11) satisfies the sub-characteristic condition.²²

A flux solver for the frozen flux in Eqs. (11) is straightforward, since in the absence of the source terms, the radiation and hydrodynamics decouple. The hydrodynamic flux is computed with Roe's approximate Riemann solver with entropy correction,²³ while the radiation component (Eqs. (11d-e)) is simply a linear first-order wave equation with wavespeeds $\pm \mathcal{C}/\sqrt{3}$ and is easily upwinded. Although the frozen flux is used in this study, there is a substantial difference in the wave structures of Eqs. (11) and Eqs. (12).²²

The results in this section are generated with $\mathcal{C} = 100$, which is too small, but allows an explicit treatment of the radiation flux at a reasonable cost. An implicit treatment of the radiation waves should not affect the accuracy properties¹² and is left for future study. All results used the Van Leer limiter applied to the characteristic variables of the system (11).

Fig. 5 Temperature profiles for stationary Mach 10 shock, $\sigma_t = 200$. Denoted is the number of cells used in the range $0.35 \leq x \leq 0.55$. The HR method solved the equilibrium-diffusion equations, while DG(1) solved the P_1 equations.

We first consider a steady, Mach 10 shock with the following conditions: $\gamma = 5/3$, $\mathcal{P} = 0.1288852$, $\mathcal{C} = 100$. The initial condition is $(\rho, u, p) = (1.0, 10.0, 0.6)$ for $x < 0.5$ and $(\rho, u, p) = (6.02005, 1.661115, 22.243715)$ for $x > 0.5$, with E_r and F_r set to their equilibrium values. These conditions result in a shock profile that separates two equilibrium states; that is, the Rankine-Hugoniot conditions are defined through Eqs. (12).

Figure 5 compares temperature profiles for DG(1) solutions of Eqs. (11) and an HR solution of Eqs. (12). Also plotted is the exact equilibrium shock-profile, determined by numerically integrating an ordinary differential equation that results from Eq. (12c). The HR solution is a second-order method that uses a central-difference treatment for the diffusion operator and a general equation-of-state implementation (to account for the radiation effects) of Roe's approximate Riemann solver.^{24,25} For the same number of cells, the DG(1) results compare well with the HR results.

Also interesting are results when the shock profile is unresolved. Figure 6 shows results for a large enough σ_t such that the exact shock profile is much less than any of the mesh spacings used. For a stationary shock, DG(1) is able to capture the shock over a single mesh cell. We considered two different initial conditions:

1. Continuous: The shock jump is initialized as a linear profile across a single cell.
2. Discontinuous: The shock jump is placed at a cell interface. This is the method used for all calcula-

Fig. 6 Density profiles for stationary Mach 10 shock, $\sigma_t = 5 \times 10^6$.

tions.

Both initial conditions converge to the solution shown in Fig. 6. However, as one might expect, the continuous method does not deviate from its initial condition. The discontinuous method generates small wiggles at the interface which are then transported out of the domain.

Ideally, the discontinuous initialization should also be a steady solution for this example. In the present implementation, the discontinuous initialization cannot be maintained because the "wrong" Riemann problem is solved at the cell interface. As $\sigma_t \Delta x$ increases (i.e., the cell-optical thickness increases), the Riemann problem should transition from that for Eqs. (11) to that for Eqs. (12). This conclusion has also been reached by other researchers for various stiff systems,^{4,7-9} and at this point there is no clear approach to resolve the issue so we leave it for future study.

Results for DG(0) are also shown in Fig. 6. DG(0) is equivalent to a first-order, finite volume method, and is also known in the transport literature as the "step method." Larsen, Morel, and Miller¹¹ prove that steady solutions using the step method for linear transport do not have the proper equilibrium-diffusion limit. This proof can be extended to any high-resolution method whose reconstruction for each dependent variable is based *solely* on the cell averages of that variable.²⁶ The overly-diffusive DG(0) results demonstrate that the lack of the correct diffusion limit extends to radiation hydrodynamics. Note that DG(0) does obtain the correct shock-jump (it is conservative), which indicates that it obtains the correct inviscid limit. For a different system of equations that also satisfy the sub-characteristic condition, Pem-

ber⁶ showed that Godunov's method coupled with an implicit treatment of the source term converges to the correct inviscid limit. Unfortunately, as shown in our results for radiation hydrodynamics, the correct inviscid limit does not mean the method is practical. The consequences of the improper diffusion limit may be dramatic even in cases where the exact diffusion effects are sub-grid.

We next consider an unsteady case with a larger value of $\mathcal{P}$. Figures 7 to 9 show results for various σ_t . The conditions for these cases are $\gamma = 5/3$, $\mathcal{P} = 44.930910$, $C = 100$. The initial condition is $(\rho, u, p) = (1.0, 0.0, 0.6)$ for $x < 0.75$ and $(\rho, u, p) = (3.122689, -6.797632, 2.874136)$ for $x > 0.75$, with E_r and F_r set to their equilibrium values. For $\sigma_t = 0$, radiation and hydrodynamics decouple, and the exact hydrodynamic solution is two shocks moving to the left separated by a contact discontinuity. Results for $\sigma_t = 100$ show that the remnants of the double-shock structure are still apparent, but the contact is diffused away.

As $\sigma_t \rightarrow \infty$, the exact solution approaches a single shock moving to the left at Mach 10. These results are shown Fig. 9 and indicate that DG(1) is able to resolve the equilibrium shock in 2 to 3 cells. Starting errors are also apparent at $x = 0.4$ and $x = 0.7$. The bump at $x = 0.4$ corresponds to an entropy wave, even though there is a bump in the hydrodynamic pressure. The effective pressure, $p + E_r/3$, does not show a bump at $x = 0.4$. We suspect these starting errors would be smaller if the Riemann solver included source-term effects.

Conclusions

Promising results using DG have been presented for a variety of systems with stiff relaxation terms. The time-integration method is simple and point-implicit, and therefore the addition of stiff-source terms does not greatly increase the cost. It is possible to obtain second-order results in the diffusion limit even with a first-order treatment for the source term. To obtain a stable, uniformly second-order method over all values of the stiffness parameter, we are currently investigating other time-integrators.

There remains the question as to whether the Riemann problem should include the source term effects. We believe that for some problems, our current approach of using the frozen limit is sufficiently accurate. For example, DG(1) solutions of the the hyperbolic heat equation appear continuous in the diffusion limit, so that any consistent Riemann solver will give the same flux. Note that for pure transport problems, the continuity of the DG(1) solution in the diffusion limit has been proven to leading order.^{11,12} For radiation transport, this means that the temperature is continuous. The continuity can also be viewed as a consequence of the diffusion terms in Eqs. (4) and

Fig. 7 DG(1) results for the unsteady radiation hydrodynamics problem, $\sigma_t = 0$, Courant number 0.3. Symbols are the cell averages of the 200-cell solution. The solid line is the exact solution.

Fig. 8 Same problem as Fig. 7, but with $\sigma_t = 100$. Symbols are the cell averages of the 200-cell solution. The solid line is the 2000-cell solution, which is mesh-converged to the plotting scale shown..

Fig. 9 Same problem as Fig. 7, but with $\sigma_t = 1 \times 10^8$. Symbols are the cell averages of the 200-cell solution. The solid line is the exact solution for $\sigma_t \rightarrow \infty$.

(12c). However, when radiation transport is coupled to hydrodynamics, discontinuities can occur in density and velocity, even if the diffusion layer is resolved. The correct wave decomposition may then become critical, although for the results presented here, DG is still able to obtain a reasonably accurate result.

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