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A Short Sheffer Axiom for Boolean Algebra

by

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Abstract

A short Sheffer stroke identity is shown to be a single axiom for Boolean algebra. The axiom has length 15 and 3 variables. The proof shows that it is equivalent to Sheffer's original 3-basis for the theory. Automated deduction techniques were used to find the proof. The shortest single axiom previously known to us has length 105 and 6 variables.

1 Introduction

Logicians have long had an interest in finding simple axiom systems for various algebras and logics, where simplicity is characterized by the number of axioms in a system or by the lengths of the axioms in a system. In this report, we show that a short Sheffer stroke identity is a single axiom for Boolean algebra. The Sheffer stroke “|” can be interpreted as the “NOR” operation, $x|y = x' \wedge y'$.

In 1913, Sheffer [5] presented the following three-axiom equational basis (3-basis) for Boolean algebra.

$$(x|x)|(x|x) = x \quad (\text{Sheffer-1})$$

$$x|(y|(y|y)) = x|x \quad (\text{Sheffer-2})$$

$$(x|(y|z))|(x|(y|z)) = ((y|y)|x)|((z|z)|x) \quad (\text{Sheffer-3})$$

In 1969, Meredith [3] presented the following 2-basis for the same theory.

$$(x|x)|(y|x) = x \quad (\text{Meredith-1})$$

$$x|(y|(x|z)) = ((z|y)|y)|x \quad (\text{Meredith-2})$$

Researchers have known for some time that single equational axioms (i.e., 1-bases) exist for Boolean algebra, including presentation in terms of the Sheffer stroke. In 1973, Padmanabhan and Quackenbush [4] presented a method for constructing a single axiom for any finitely based theory that has certain distributive and permutable congruences, and Boolean algebra has these properties. However, straightforward application of the method usually yields single axioms of enormous length. McCune [2] used the construction method with a variety of automated deduction techniques to find single axioms of reasonable length for Boolean algebra under a variety of treatments. In particular, an axiom of length 105, with six variables, was found for the Sheffer stroke.

Stephen Wolfram recently brought to our attention [6] a set of 25 Sheffer identities, all of length 15 with 3 variables, that he was investigating as being possible single axioms. His interest in finding simple bases for Boolean algebra arose in work on his project *A New Kind of Science* [7].

In this note, we show that the equation

$$((x|z)|y)|((x|(x|y))|x) = y \quad (\text{BA-1})$$

is a single axiom for Boolean algebra in terms of the Sheffer stroke. As a simple corollary, we have that the mirror image of (BA-1),

$$(x|((y|x)|x))|(y|(z|x)) = y, \quad (\text{BA-2})$$

is also a single axiom, because the mirror image of a Sheffer basis is also a Sheffer basis.

Equation (BA-2) is a member of Wolfram's set of 25 Sheffer identities.

2 The Proof

The proof of Theorem 1 below is presented by way of derivations obtained with the automated deduction system Otter [1]. The Otter proofs are by contradiction. The justification $[m \rightarrow n]$ indicates paramodulation from equation m into equation n .

The methods we used to find the Otter proofs involved heavy use of automated deduction techniques. A subsequent report will contain details of the work.

THEOREM 1 *Equation (BA-1) is a basis for Boolean algebra in terms of the Sheffer stroke.*

Proof. We show that (BA-1) is equivalent to the Sheffer 3-basis $\{(\text{Sheffer-1}), (\text{Sheffer-2}), (\text{Sheffer-3})\}$ with two Otter proofs. First we show that (BA-1) is sound by proving it from the Sheffer 3-basis; then we show that it is complete by deriving the Sheffer 3-basis from it. The Sheffer stroke symbol is omitted to save space.

Part 1: $\{(\text{Sheffer-1}), (\text{Sheffer-2}), (\text{Sheffer-3})\} \implies (\text{BA-1})$.

2	$(xx)(xx) = x$	[(Sheffer-1)]
3	$x(y(yy)) = xx$	[(Sheffer-2)]
4	$((yy)x)((zz)x) = (x(yz))(x(yz))$	[(Sheffer-3)]
5	$((AB)C)((A(AC))A) \neq C$	[denial of (BA-1)]
58	$x((yy)y) = xx$	[3 $\leftarrow$ 2]
60	$(xx)(y(yy)) = x$	[3 $\leftarrow$ 2]
67	$(xx)((yy)y) = x$	[58 $\leftarrow$ 2]
70	$(xx)(x((yy)y)) = x$	[58 $\rightarrow$ 2]
72	$x((yy)y)(z(zz)) = x$	[60 $\leftarrow$ 58]
80	$x((yy)y)(x((zz)z)) = x$	[70 $\leftarrow$ 58]
103	$x((yy)z)(x((yy)z)) = (yx)((zz)x)$	[4 $\leftarrow$ 2]
112	$((xx)(xx)y)((xx)y) = y$	[4 $\leftarrow$ 80]
116	$((x(yx))(x(yx)))(z(zz)) = (yy)x$	[4 $\rightarrow$ 72]
117	$((x(yy))(x(yy)))(zz)z = (yy)x$	[4 $\rightarrow$ 67]
143	$(xy)((xx)y) = y$	[112 $\leftarrow$ 2]
166	$((xx)y)(xy) = y$	[143 $\leftarrow$ 2]
198	$x(yx) = (yy)x$	[116 $\leftarrow$ 60]
209	$(xy)((xx)y) = (yy)(xy)$	[198 $\leftarrow$ 198]
264	$(xx)(yx) = x$	[209 $\leftarrow$ 143]
303	$(x(yx))x = yx$	[264 $\rightarrow$ 143]
372	$x(yy) = (yy)x$	[117 $\leftarrow$ 264]
386	$((xx)(xx))y = yx$	[372 $\leftarrow$ 264]
403	$(xy)(yy) = y$	[372 $\leftarrow$ 264]
435	$(x(yy))(yx) = x$	[372 $\rightarrow$ 166]
458	$(x(y(zu)))((z(z)y)((uu)y)) = y(zu)$	[403 $\leftarrow$ 4]

471	$(xy)((yy)x) = x$	[435 $\leftarrow$ 403]
518	$(xy)((y(z(zz)))x) = x$	[471 $\leftarrow$ 3]
558	$xy = yx$	[386 $\leftarrow$ 403]
594	$(xx)(xy) = x$	[386 $\rightarrow$ 264]
598	$(xy)(xx) = x$	[386 $\rightarrow$ 403]
602	$((xx)y)(yx) = y$	[386 $\rightarrow$ 143]
656	$(x(yz))(x(yz)) = ((yy)x)((zz)x)$	[558 $\leftarrow$ 4]
681	$((xy)(xy))x = xy$	[594 $\leftarrow$ 471]
713	$(xy)(x(z(zz))) = x$	[598 $\leftarrow$ 3]
730	$((x(y(yy)))z)(zx) = z$	[602 $\leftarrow$ 3]
1084	$((x(yz))(x(yz)))(((yy)x)((zz)x))$ $= (z((yy)x))((xx)((yy)x))$	[103 $\leftarrow$ 656]
1110	$((xy)((zz)y))y = y((xx)z)$	[103 $\rightarrow$ 681]
1239	$(x((yy)z))((zz)((yy)z)) = z(yx)$	[1084 $\leftarrow$ 458]
1259	$(x((yy)z))z = z(yx)$	[1239 $\leftarrow$ 264]
1362	$x((yy)z) = x(z(yx))$	[1259 $\leftarrow$ 1110]
1504	$x((yz)(yx)) = xy$	[1362 $\leftarrow$ 594]
1666	$(xy)((((y(z(zz)))x)u)x) = (xy)((y(z(zz)))x)$	[1504 $\leftarrow$ 730]
1842	$(xy)(x(((y(z(zz)))(y(z(zz))))u)) = (xy)((y(z(zz)))x)$	[1666 $\leftarrow$ 1110]
1847	$(xy)((y(z(zz)))x) = (xy)(x(yu))$	[1842 $\leftarrow$ 713]
1856	$(xy)(x(yz)) = x$	[1847 $\leftarrow$ 518]
1895	$(x(yz))(xy) = x$	[1856 $\leftarrow$ 1856]
1972	$((xy)z)(zx) = z$	[1895 $\leftarrow$ 558]
2142	$((xy)z)((x(zx))x) = z$	[1972 $\leftarrow$ 303]
3223	$((xy)z)((x(xz))x) = z$	[2142 $\leftarrow$ 558]
3224	$\square$	[3223, 5]

Part 2: (BA-1) $\implies$ {(Sheffer-1),(Sheffer-2),(Sheffer-3)}.

2	$((xy)z)((x(xz))x) = z$	[(BA-1)]
3	$(AA)(AA) \neq A \quad \vee \quad A(B(BB)) \neq AA$ $\vee \quad ((BB)A)((CC)A) \neq (A(BC))(A(BC))$	[denial]
140	$(xy)((((zu)x)((zu)x)y))((zu)x) = y$	[2 $\leftarrow$ 2]

141	$((x(xy))x)z)((y(((xu)y)((x(xy))x)z))((xu)y)((x(xy))x)))$	
	$= z$	[140 $\leftarrow$ 2]
142	$(xy)((z)((zx)y)(zx)) = y$	[140 $\rightarrow$ 2]
143	$((x(xy))x)z)((y(yz))((xu)y)((x(xy))x)) = z$	[141 $\leftarrow$ 2]
144	$((x(xy))x)z)((y(yz))y) = z$	[143 $\leftarrow$ 2]
145	$x((y(y((x(xx))x)))y) = (x(xx))x$	[144 $\leftarrow$ 2]
146	$((x(xx))x)((x(xx))x)((x(xx))x))((x(xx))x)$	
	$= ((x(xx))x)((x((x(xx))x))x)$	[145 $\leftarrow$ 145]
147	$((x(xx))x)((x((x(xx))x))x) = (((x(xx))x)x)((x(xx))x)$	[146 $\leftarrow$ 2]
148	$((x(xx))x)((x((x(xx))x))x) = x$	[147 $\leftarrow$ 144]
149	$((x(xx))x)((x(xx))x)((x(xx))x))((x(xx))x) = x$	[148 $\leftarrow$ 146]
150	$((x(xx))x)((x(xx))x)((x(xx))x))((x(xx))x)$	
	$= ((x(xx))x)((y(yx))y)$	[149 $\rightarrow$ 145]
151	$(x(xx))x = xx$	[149 $\rightarrow$ 145]
152	$((xy)x)(xx) = x$	[151 $\rightarrow$ 2]
154	$(xx)(xx) = x$	[152 $\leftarrow$ 151]
155	$x((y(y(xx)))y) = xx$	[152 $\rightarrow$ 144]
156	$((x(xx))x)((y(yx))y) = x$	[150 $\leftarrow$ 149]
157	$(xx)((y(yx))y) = x$	[156 $\leftarrow$ 151]
158	$x((y(yx))y)((x(x)x)(xx)) = (y(yx))y$	[157 $\rightarrow$ 142]
159	$x((y(yx))y)x = (y(yx))y$	[158 $\leftarrow$ 152]
160	$((x(xy))x)((y(y)y(yy))((x(xy))x))$	
	$= ((yy)((yy)((x(xy))x)))(yy)$	[159 $\leftarrow$ 157]
161	$((x(xy))x)(yy) = y$	[159 $\rightarrow$ 152]
165	$(x(yy))(((x(xy))x)y((x(xy))x)) = yy$	[161 $\rightarrow$ 142]
166	$((xx)((xx)((y(yx))y))(xx) = (((y(yx))y)x)((y(yx))y)$	[160 $\leftarrow$ 152]
167	$((x(xy))x)y((x(xy))x) = ((yy)y)(yy)$	[166 $\leftarrow$ 157]
168	$((x(xy))x)y((x(xy))x) = y$	[167 $\leftarrow$ 152]
169	$(x(yy))y = yy$	[168 $\rightarrow$ 165]
170	$(xy)(yy) = (yy)(yy)$	[169 $\leftarrow$ 154]
177	$(xy)(yy) = y$	[170 $\leftarrow$ 154]
178	$x(((yx)x)(yx)) = xx$	[177 $\rightarrow$ 155]
179	$(x(xx))(((yx)x)(yx)) = xx$	[177 $\rightarrow$ 142]
180	$(xx)((x(x((yx)x)(yx)))x) = ((yx)x)(yx)$	[179 $\rightarrow$ 2]
181	$(xx)((x(xx))x) = ((yx)x)(yx)$	[180 $\leftarrow$ 178]

183	$((xy)y)(xy) = y$	[181 $\leftarrow$ 157]
184	$(x(yx))((yx)x)(yx) = yx$	[183 $\leftarrow$ 183]
185	$(x(yx))x = yx$	[184 $\leftarrow$ 183]
187	$(xx)((x(yx))(yx))(x(yx)) = x$	[185 $\rightarrow$ 157]
192	$(xx)(yx) = x$	[187 $\leftarrow$ 183]
193	$x(y(xx)) = xx$	[192 $\leftarrow$ 192]
195	$((x(yy))y)(zy) = y$	[192 $\leftarrow$ 169]
196	$((xx)y)(zx)((xx)x)(xx) = zx$	[192 $\rightarrow$ 2]
198	$((xy)((z(yy))y)(xy))(xy) = y$	[195 $\leftarrow$ 185]
201	$((xx)y)(zx)x = zx$	[196 $\leftarrow$ 183]
202	$(x(xy))x = yx$	[201 $\leftarrow$ 2]
205	$(xx)(xy) = x$	[202 $\rightarrow$ 157]
206	$(xy)(xx) = x$	[202 $\rightarrow$ 161]
207	$(xy)(y(zx)) = y$	[202 $\rightarrow$ 142]
208	$((xy)z)(zx) = z$	[202 $\rightarrow$ 2]
211	$x((xy)(z(xx))) = xy$	[207 $\leftarrow$ 205]
212	$((xy)z)(zy) = z$	[207 $\leftarrow$ 198]
213	$(x(xy))x = xy$	[207 $\leftarrow$ 206]
214	$(xy)(y(xz)) = y$	[208 $\leftarrow$ 206]
216	$xy = yx$	[213 $\leftarrow$ 202]
221	$((xy)z)(xz) = z$	[213 $\rightarrow$ 2]
222	$(x(yz))(zx) = x$	[216 $\leftarrow$ 207]
224	$((x(xy))x)((xz)y) = y$	[216 $\leftarrow$ 2]
225	$(x(yz))(xz) = x$	[216 $\rightarrow$ 212]
226	$(x(yz))(xy) = x$	[216 $\rightarrow$ 208]
227	$(xy)(x(zy)) = x$	[216 $\rightarrow$ 207]
228	$(xy)((xz)y) = y$	[214 $\leftarrow$ 216]
229	$((xy)(xz))z = xz$	[214 $\rightarrow$ 207]
230	$x(y(x(yz))) = x(yz)$	[221 $\leftarrow$ 214]
231	$x(y(x(zy))) = x(zy)$	[221 $\leftarrow$ 207]
232	$(x(y(zx)))y = y(zx)$	[222 $\rightarrow$ 214]
233	$x((yx)(zy)) = yx$	[222 $\rightarrow$ 212]
234	$(x(y(xz)))y = y(xz)$	[226 $\rightarrow$ 207]
235	$((xy)(zy))x = xy$	[227 $\leftarrow$ 227]
236	$((x(yz))z)x = x(yz)$	[227 $\leftarrow$ 225]

239	$(x(yz))(x(u(yx))) = (yx)(x(yz))$	[233 $\leftarrow$ 214]
243	$(x(xy))y = yy$	[224 $\rightarrow$ 229]
244	$(x(yx))y = yy$	[243 $\leftarrow$ 216]
245	$x(y(yx)) = xx$	[243 $\leftarrow$ 216]
251	$x(y(xy)) = xx$	[244 $\leftarrow$ 216]
257	$x((yx)y) = xx$	[245 $\leftarrow$ 216]
258	$(xx)((xy)(z(xz))) = z(xz)$	[251 $\rightarrow$ 228]
259	$(xy)(x(x(zy))) = (xy)(xy)$	[257 $\leftarrow$ 225]
266	$x(yy) = x(yx)$	[230 $\leftarrow$ 251]
267	$x(yy) = x(xy)$	[266 $\leftarrow$ 216]
268	$(xy)y = y(xx)$	[266 $\leftarrow$ 216]
269	$x((xy)(z(xz))) = xy$	[266 $\rightarrow$ 211]
270	$x(yx) = (yy)x$	[266 $\rightarrow$ 216]
273	$(x(yy))x = yx$	[266 $\rightarrow$ 185]
274	$x(x(yy)) = xy$	[267 $\leftarrow$ 206]
275	$(xy)x = x(yy)$	[267 $\leftarrow$ 216]
279	$((xy)(xy))(xx) = (xx)x$	[270 $\leftarrow$ 206]
280	$x(x(yy)) = yx$	[273 $\leftarrow$ 216]
286	$x(xy) = x(yy)$	[274 $\leftarrow$ 206]
288	$((xy)(zy))(xx) = (xy)((xy)(zy))$	[275 $\leftarrow$ 235]
290	$(x(y(xz)))y = y(zx)$	[232 $\leftarrow$ 216]
291	$x(yz) = x(zy)$	[290 $\leftarrow$ 234]
293	$(xy)z = z(yx)$	[291 $\leftarrow$ 216]
294	$x(yz) = (zy)x$	[291 $\leftarrow$ 216]
296	$(xy)(zu) = (uz)(yx)$	[293 $\leftarrow$ 291]
297	$((xy)(xy))(xx) = x(xx)$	[279 $\leftarrow$ 294]
298	$((xx)(xx))((xy)(z(xz))) = ((xy)(z(xz)))(z(xz))$	[258 $\rightarrow$ 270]
299	$(x(y(zy)))(x(x(zz))) = (x(y(zy)))(x(y(zy)))$	[259 $\leftarrow$ 251]
300	$(x(yz))(x(u(yx))) = x$	[239 $\leftarrow$ 214]
302	$x(y(x(z(yx)))) = xx$	[300 $\rightarrow$ 236]
303	$x(y(z(xy))) = x(yy)$	[302 $\rightarrow$ 230]
304	$x((yy)(z(x(xy)))) = x((yy)(yy))$	[303 $\leftarrow$ 286]
305	$x(y((yx)z)) = x(yy)$	[303 $\leftarrow$ 294]
306	$(xy)((xy)(xy)) = x(xx)$	[288 $\leftarrow$ 297]
307	$((x(xx))y)((xz)y) = y$	[306 $\rightarrow$ 221]

317	$(x(xx))y = yy$	[307 $\rightarrow$ 229]
319	$x(y(yy)) = xx$	[317 $\leftarrow$ 216]
326	$(xy)(x(z(zz))) = x$	[319 $\rightarrow$ 206]
327	$x((yy)(z(x(xy)))) = xy$	[304 $\leftarrow$ 206]
328	$(x(y(zy)))(x(y(zy))) = (x(y(zy)))(xz)$	[299 $\leftarrow$ 274]
329	$((xx)(xx))((xy)(z(xz))) = (z(xz))((xy)(xy))$	[298 $\leftarrow$ 268]
330	$(x(yx))((yz)(yz)) = y((yz)(x(yx)))$	[329 $\leftarrow$ 206]
331	$(x(yx))((yz)(yz)) = yz$	[330 $\leftarrow$ 269]
332	$(x((yz)x))(y((yz)(y(u(uu)))))) = (yz)(y(u(uu)))$	[331 $\leftarrow$ 326]
335	$(x((yz)x))(yy) = (yz)(y(u(uu)))$	[332 $\leftarrow$ 326]
336	$(x((yz)x))(yy) = y$	[335 $\leftarrow$ 326]
337	$(x(x(yz)))(zz) = z$	[336 $\leftarrow$ 294]
344	$x((y(y(zx)))x) = y(y(zx))$	[337 $\rightarrow$ 226]
345	$x(y(z(z(u(yx)))))) = x(yy)$	[344 $\rightarrow$ 305]
346	$x(y(y(z(xy)))) = x(y(xx))$	[345 $\rightarrow$ 230]
347	$x(y(y(z(xy)))) = xx$	[346 $\leftarrow$ 193]
348	$(xx)(y(y(z(y(xy)))))) = (xx)(xx)$	[347 $\leftarrow$ 270]
350	$(xx)(y(y(z(y(xy)))))) = x$	[348 $\leftarrow$ 206]
351	$x(((y(x(zx)))(y(x(zx))))z) = x(y(x(zx)))$	[350 $\rightarrow$ 327]
352	$x(((y(x(zx)))(yz))z) = x(y(x(zx)))$	[351 $\leftarrow$ 328]
353	$x(y(x(zx))) = x(yz)$	[352 $\leftarrow$ 229]
354	$x(y(x(xz))) = x(yz)$	[353 $\leftarrow$ 216]
355	$x(y(x(zz))) = x(yz)$	[353 $\leftarrow$ 266]
358	$x(y(zx)) = x(y(zz))$	[354 $\leftarrow$ 280]
359	$(x(yz))(x(yz)) = (x(yz))(x(zz))$	[358 $\leftarrow$ 231]
360	$(x(y(x(zz))))(x(zz)) = (x(yz))(x(yz))$	[359 $\leftarrow$ 355]
361	$(x(yy))(x(z(x(yy)))) = (x(zy))(x(zy))$	[360 $\leftarrow$ 216]
362	$(x(yy))(x(zz)) = (x(zy))(x(zy))$	[361 $\leftarrow$ 358]
363	$((xx)y)((zz)y) = (y(xz))(y(xz))$	[362 $\leftarrow$ 296]
364	$\square$	[3, 206, 319, 363]

Web Resources

www.mcs.anl.gov/~mccune/ba-sheffer-15 contains Otter input files for the Otter proofs in Section 2.

www.mcs.anl.gov/~mccune/ba-axioms contains Otter input files, proofs, and other data related to paper [2].

www.mcs.anl.gov/AR/otter/ contains information on Otter, including freely downloadable versions of Otter for Unix-like systems, Macintoshes, and Microsoft Windows systems.

References

- [1] W. McCune. Otter 3.0 Reference Manual and Guide. Tech. Report ANL-94/6, Argonne National Laboratory, Argonne, IL, 1994.
- [2] W. McCune. Single Axioms for Boolean Algebra. Tech. Memo ANL/MCS-TM-243, Mathematics and Computer Science Division, Argonne National Laboratory, Argonne, IL, June 2000.
- [3] C. A. Meredith. Equational postulates for the Sheffer stroke. *Notre Dame J. Formal Logic*, 10(3):266–270, 1969.
- [4] R. Padmanabhan and R. W. Quackenbush. Equational theories of algebras with distributive congruences. *Proc. of AMS*, 41(2):373–377, 1973.
- [5] H. Sheffer. A set of five independent postulates for Boolean algebras, with application to logical constants. *Trans. AMS*, 14(4):481–488, 1913.
- [6] S. Wolfram, Feb 4 2000. Correspondence by electronic mail.
- [7] S. Wolfram. A New Kind of Science. <http://wolframscience.com>, 2000.