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A Singular-Perturbation Analysis of the Burning-Rate Eigenvalue for a Two- Temperature Model of Deflagrations in Confined Porous Energetic Materials

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A SINGULAR-PERTURBATION ANALYSIS OF THE
BURNING-RATE EIGENVALUE FOR A
TWO-TEMPERATURE MODEL OF DEFLAGRATIONS IN
CONFINED POROUS ENERGETIC MATERIALS

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ABSTRACT

Deflagrations in porous energetic materials are characterized by regions of two-phase flow, where, for sufficiently large flow velocities, temperature-nonequilibrium effects can significantly affect the overall burning rate. In the present work, we analyze a two-temperature model of deflagrations in confined porous propellants that exhibit a bubbling melt layer at their surfaces. For appropriately scaled rates of interphase heat transfer, the problem reduces to a nontrivial eigenvalue calculation in the thin reaction region where final conversion of the liquid to gaseous products occurs. For realistically small values of the ratio of gas-to-liquid thermal conductivity, solutions in the reaction zone take on a singular-perturbation character that can be exploited to derive an asymptotic expansion of the burning-rate eigenvalue. The resulting problem requires a rather sophisticated application of techniques in matched asymptotic expansions stemming from the appearance of an infinite number of logarithmic terms in the asymptotic development that must be summed to arrive at the desired level of approximation. The physical effects of temperature nonequilibrium, which decreases the rate of heat transfer from the reacting liquid phase to the gas-phase products and thus allows a greater amount of thermal energy to remain in the reacting phase, is to increase the burning rate relative to the single-temperature limit and to sharpen the transition from "conductive" to "convective" burning.

A SINGULAR-PERTURBATION ANALYSIS OF THE BURNING-RATE EIGENVALUE FOR A TWO-TEMPERATURE MODEL OF DEFLAGRATIONS IN POROUS ENERGETIC MATERIALS

1. Introduction

The combustion behavior of porous energetic materials is of increasing interest in predicting how such materials will perform after a period of long-term storage and/or exposure to abnormal thermal environments. Materials such as the nitramine propellants HMX and RDX, for example, find numerous applications in the fields of propulsion and pyrotechnics. Though manufactured and placed in a system while in a pristine state, partial decomposition can occur and significant void fractions can develop over time, either because of an inherent degree of metastability in the formulation or because of various accident scenarios that are associated with brief periods of elevated temperatures. During combustion, significant two-phase flow effects in such damaged materials play an important role, especially when the material is under confinement. Under these conditions, pressure-driven convective permeation of hot gases into the pores of the unburned solid occurs (*cf.* [1], [2]), leading to a superadiabatic preheating effect that accelerates the propagation speed and ultimately results in a convection-dominated mode of burning. During this transition from so-called conductive to convective burning, both conduction and convection play significant roles and must be considered in any model that describes this burning regime. In addition, as convective gas velocities increase in the two-phase regions, temperature-nonequilibrium effects associated with finite rates of heat transfer between the co-existing gaseous and condensed phases will also begin to influence the propagation speed. The purpose of this study is to analyze some of the influences associated with the latter.

Various effects that are associated with two-phase flow in porous energetic-material deflagration, and that are relevant to the present work, have been analyzed in several recent theoretical studies (*cf.* [2], [3], [4]). In particular, the porous-material models considered thus far, which represent specific formulations of more general reactive two-phase-flow descriptions (*cf.* [5]), have generally considered the case of a porous solid with a reactive bubbling melt layer at the surface, a feature that is frequently observed during combustion of nitramine propellants. Though much of this analysis has been restricted to the single-temperature limit associated with infinite rates of interphase heat transfer, it was previously shown for the unconfined problem that the first effects of large but finite rates of interphase heat transfer are felt primarily in the liquid/gas reaction zone ([3]). In the present work, we consider the more general confined problem previously analyzed in the single-temperature limit [2], at the same time allowing for more significant two-temperature effects arising from increased resistance to interphase heat transfer. In particular, we generalize the earlier unconfined and/or single-temperature results by allowing for an overpressure in the gaseous phase

external to the porous solid, and exploit the limit in which the gas-to-liquid thermal-conductivity ratio can be considered a small parameter. It is then shown that the leading-order effects on the burning rate associated with finite rates of interphase heat transfer again occur in the liquid/gas reaction region, but the asymptotic limit just described permits much more general results than those obtained previously. These are achieved through the application of nonstandard techniques in matched asymptotic expansions associated with the appearance of an infinite number of logarithmic terms in the asymptotic development. The final result is a modified expression for the leading-order burning-rate eigenvalue in which the finite rate of interphase heat transfer appears explicitly.

2. Model Formulation

A sketch of the physical problem is shown in Figure 1. In general, we consider a confined or partially confined geometry, in which case there is an overpressure-driven gas flow into the unburned porous material. It is assumed that melting of the solid occurs at a prescribed temperature and that subsequent to melting, a single-step exothermic reaction converts the liquid reactants to gaseous products, giving rise to a deflagration wave that propagates from right to left in the direction of the unburned propellant. The structure of the combustion wave consists of a solid/gas preheat region, the melting surface that marks the left boundary of a liquid/gas preheat region, and a relatively thin exothermic reaction zone in which chemical reaction occurs. Although it will not be necessary to consider the details in the present work, there also exists, for weak permeabilities, a gas-permeation boundary layer within the solid/gas preheat region in which the gas pressure decreases rapidly from its value at the surface of the solid, to values close to the ambient deep within the porous material ([2]). Since our focus in the present work is on the calculation of the burning-rate eigenvalue corresponding to quasi-steady, planar propagation of the combustion wave, we restrict attention to one spatial dimension ($\tilde{x}$), and use the subscripts s , l and g to denote solid, liquid and gas-phase quantities, respectively. The porous solid thus extends to $\tilde{x} = -\infty$, where conditions are denoted by a sub- or superscript u , while the product gases extend to $\tilde{x} = +\infty$, where conditions are identified by a sub- or superscript b . The appearance of a tilde over a symbol (e.g., $\tilde{x}$) will denote a dimensional quantity.

The governing system of equations consists of conservation equations for continuity, momentum and energy in the two-phase solid/gas and liquid/gas regions to the left and right of the melting surface $\tilde{x} = \tilde{x}_m$. Denoting the gas-phase volume fraction by α , continuity in the region $\tilde{x} > \tilde{x}_m$ is expressed overall and separately for the liquid-phase as

$$\frac{\partial}{\partial \tilde{t}} [(1 - \alpha)\tilde{\rho}_l + \alpha\tilde{\rho}_g] + \frac{\partial}{\partial \tilde{x}} [(1 - \alpha)\tilde{\rho}_l\tilde{u}_l + \alpha\tilde{\rho}_g\tilde{u}_g] = 0, \quad \tilde{x} > \tilde{x}_m, \quad (1)$$

$$\frac{\partial}{\partial \tilde{t}} [(1 - \alpha)\tilde{\rho}_l] + \frac{\partial}{\partial \tilde{x}} [(1 - \alpha)\tilde{\rho}_l\tilde{u}_l] = -\tilde{A}\tilde{\rho}_l(1 - \alpha) \exp\left(-\tilde{E}_l/\tilde{R}^\circ\tilde{T}_l\right), \quad \tilde{x} > \tilde{x}_m, \quad (2)$$

where $\tilde{\rho}$, $\tilde{u}$, $\tilde{T}$ and $\tilde{t}$ denote density, velocity, temperature and time, respectively. For simplicity, we assume a constant value for $\tilde{\rho}_l$, but not for $\tilde{\rho}_g$. As in previous work, the evaluation of the Arrhenius reaction rate is based constitutively on the local temperature of the liquid phase. In that expression, $\tilde{E}_l$ is the overall activation energy, $\tilde{R}^\circ$ is the universal gas constant, and $\tilde{A}$ is the exponential reciprocal-time prefactor which, for simplicity, will be assumed constant, although a pressure dependency can be readily incorporated into the present formulation. In the solid/gas region $\tilde{x} < \tilde{x}_m$, we assume a constant solid density $\tilde{\rho}_s$ and zero velocity ($\tilde{u}_s = 0$), with $\alpha \equiv \alpha_s$ also constant in this region. Gas-phase continuity for $\tilde{x} < \tilde{x}_m$ is thus independent of the solid phase and is given by

$$\frac{\partial \tilde{\rho}_g}{\partial \tilde{t}} + \frac{\partial}{\partial \tilde{x}} (\tilde{\rho}_g \tilde{u}_g) = 0, \quad \tilde{x} < \tilde{x}_m. \quad (3)$$

Overall conservation of energy and conservation of energy for the liquid phase separately in the liquid/gas region is given by

$$\begin{aligned} \frac{\partial}{\partial \tilde{t}} [\tilde{\rho}_l(1-\alpha)(\tilde{Q} + \tilde{c}_l \tilde{T}_l) + \tilde{\rho}_g \tilde{c}_g \alpha \tilde{T}_g] + \frac{\partial}{\partial \tilde{x}} [\tilde{\rho}_l \tilde{u}_l(1-\alpha)(\tilde{Q} + \tilde{c}_l \tilde{T}_l) + \tilde{\rho}_g \tilde{c}_g \tilde{u}_g \alpha \tilde{T}_g] \\ = \frac{\partial}{\partial \tilde{x}} \left[\tilde{\lambda}_l(1-\alpha) \frac{\partial \tilde{T}_l}{\partial \tilde{x}} + \tilde{\lambda}_g \alpha \frac{\partial \tilde{T}_g}{\partial \tilde{x}} \right] + \alpha \frac{\partial \tilde{p}_g}{\partial \tilde{t}}, \quad \tilde{x} > \tilde{x}_m, \end{aligned} \quad (4)$$

$$\begin{aligned} \frac{\partial}{\partial \tilde{t}} [\tilde{\rho}_l(1-\alpha)(\tilde{Q} + \tilde{c}_l \tilde{T}_l)] + \frac{\partial}{\partial \tilde{x}} [\tilde{\rho}_l \tilde{u}_l(1-\alpha)(\tilde{Q} + \tilde{c}_l \tilde{T}_l)] - \frac{\partial}{\partial \tilde{x}} \left[\tilde{\lambda}_l(1-\alpha) \frac{\partial \tilde{T}_l}{\partial \tilde{x}} \right] \\ = \tilde{K}_{lg} (\tilde{T}_g - \tilde{T}_l), \quad \tilde{x} > \tilde{x}_m, \end{aligned} \quad (5)$$

where liquid-phase continuity, Eq. (2), has been used to eliminate the reaction-rate terms in these equations. Here, $\tilde{p}_g$ is the gas pressure, assumed constant in the liquid/gas and burned-gas regions only. The quantities $\tilde{c}$ and $\tilde{\lambda}$, both of which are assumed constant, denote heat capacity (at constant volume for the liquid, and at constant pressure for the gas) and thermal conductivity, respectively, $\tilde{Q}$ (assumed constant) is the heat release for the exothermic liquid-to-gas reaction, assumed to be deposited in the reacting liquid phase, and $\tilde{K}_{lg}$ (assumed constant) is an interphase heat-transfer coefficient. We note that because of the small Mach number and the small ratio of gas-to-liquid densities in the problems to be considered, no terms involving the liquid pressure $\tilde{p}_l$ appear in Eq. (5). We also remark that the overall heat release $\tilde{Q}$ is, strictly speaking, not really a constant since it depends on the temperatures $\tilde{T}_l$ and $\tilde{T}_g$. In particular, $\tilde{Q}$ may be expressed as $\tilde{Q} = \tilde{Q}_0 + \tilde{c}_l(\tilde{T}_l - \tilde{T}_0) - \tilde{c}_g(\tilde{T}_g - \tilde{T}_0) = \tilde{Q}_{00} + (\tilde{c}_l - \tilde{c}_g)\tilde{T}_l + \tilde{c}_g(\tilde{T}_l - \tilde{T}_g)$, where $\tilde{Q}_0$ is the heat release at some common temperature $\tilde{T}_0$ and $\tilde{Q}_{00} = \tilde{Q}_0 - (\tilde{c}_l - \tilde{c}_g)\tilde{T}_0$ is the heat release at $\tilde{T} = 0$. Choosing $\tilde{T}_0 = \tilde{T}_b$, the still-to-be-determined burned temperature, $\tilde{Q}$ may be expressed as $\tilde{Q} = \tilde{Q}_b + \tilde{c}_l(\tilde{T}_l - \tilde{T}_b) - \tilde{c}_g(\tilde{T}_g - \tilde{T}_b)$, where $\tilde{Q}_b = \tilde{Q}_{00} + (\tilde{c}_l - \tilde{c}_g)\tilde{T}_b$. Since $\tilde{Q}_b$ is typically the dominant term in this last expression (especially when, as in the case here, $\tilde{T}_l$ and $\tilde{T}_g$ are close to $\tilde{T}_b$ in the reaction zone), it is a reasonable simplifying approximation to regard $\tilde{Q} \approx \tilde{Q}_b$ as constant,

which we have done. Finally, we remark that the term involving $\tilde{p}_g$ arises from the contribution to the rate of change of the internal energy of the gas from the sum of the rate of surface work $-\partial(\alpha\tilde{u}_g\tilde{p}_g)/\partial\tilde{x}$ and the rate of volume work $-\tilde{p}_g\partial\alpha/\partial\tilde{t}$ performed by the gas. Relating the internal energy ($\tilde{e}_g$) of the gas to its enthalpy ($\tilde{h}_g$) according to the thermodynamic identity $\tilde{e}_g = \tilde{h}_g - \tilde{p}_g/\tilde{\rho}_g$ then results in the last term on the right-hand side of Eq. (4).

In a similar fashion, conservation of energy in the solid/gas region is expressed as

$$\begin{aligned} \frac{\partial}{\partial\tilde{t}} \left[\tilde{\rho}_s \tilde{c}_s (1 - \alpha_s) \tilde{T}_s + \tilde{\rho}_g \tilde{c}_g \alpha_s \tilde{T}_g \right] + \frac{\partial}{\partial\tilde{x}} \left(\tilde{\rho}_g \tilde{c}_g \tilde{u}_g \alpha_s \tilde{T}_g \right) \\ = \frac{\partial}{\partial\tilde{x}} \left[\tilde{\lambda}_s (1 - \alpha_s) \frac{\partial\tilde{T}_s}{\partial\tilde{x}} + \tilde{\lambda}_g \alpha_s \frac{\partial\tilde{T}_g}{\partial\tilde{x}} \right] + \alpha_s \frac{\partial\tilde{p}_g}{\partial\tilde{t}}, \quad \tilde{x} < \tilde{x}_m, \end{aligned} \quad (6)$$

$$\frac{\partial}{\partial\tilde{t}} \left[\tilde{\rho}_s \tilde{c}_s (1 - \alpha_s) \tilde{T}_s \right] - \frac{\partial}{\partial\tilde{x}} \left[\tilde{\lambda}_s (1 - \alpha_s) \frac{\partial\tilde{T}_s}{\partial\tilde{x}} \right] = \tilde{K}_{sg} (\tilde{T}_g - \tilde{T}_s), \quad \tilde{x} < \tilde{x}_m, \quad (7)$$

where Eq. (6) describes overall energy conservation and Eq. (7) describes conservation of energy in the solid phase separately.

Although analogous equations may be written for momentum conservation, we adopt instead certain simplifying approximations which are often used in the present types of problems. In particular, Darcy's law relating the gas velocity to the pressure gradient is adopted in the solid/gas region, while the assumption of small Mach number allows the gas pressure $\tilde{p}_g$ to be regarded as independent of the spatial coordinate in the liquid/gas and burned-gas regions to the right of the melting surface. We also assume that the gas pressure in the burned region is quasi-static, varying on a time scale that is longer than that associated with the flame structure itself. This latter assumption has been discussed previously ([2], [6]), and implies that the confining boundaries are sufficiently remote with respect to the combustion wave itself. It also allows one to seek a (quasi-) steadily propagating solution for the combustion wave as a function of gas-phase overpressure (i.e., as function of the difference in $\tilde{p}_g$ between its burned value $\tilde{p}_g^b$ at $\tilde{x} = +\infty$ and its ambient value $\tilde{p}_g^u$ at $\tilde{x} = -\infty$). Thus, in place of momentum conservation, we have the relations

$$\tilde{u}_g = -\frac{\tilde{\kappa}(\alpha_s)}{\alpha_s \tilde{\mu}_g} \frac{\partial\tilde{p}_g}{\partial\tilde{x}}, \quad \tilde{x} < \tilde{x}_m; \quad \tilde{p}_g = \tilde{p}_g^b, \quad \tilde{x} > \tilde{x}_m; \quad \tilde{u}_l = -\frac{d\tilde{x}_m}{d\tilde{t}} \left(\frac{\tilde{\rho}_s}{\tilde{\rho}_l} - 1 \right), \quad (8)$$

where $\tilde{\kappa}(\alpha_s)$ is the porosity-dependent permeability of the porous solid and $\tilde{\mu}_g$ is the gas viscosity. The last relation follows from an approximate consideration of liquid-phase momentum conservation ([7]) and $d\tilde{x}_m/d\tilde{t} < 0$ is the (unknown) propagation velocity of the melting surface $\tilde{x} = \tilde{x}_m(\tilde{t})$. Finally, the gas is assumed to be ideal, whereupon $\tilde{p}_g$ is coupled to the other field variables through the gas-phase equation of state,

$$\tilde{p}_g = \tilde{\rho}_g \tilde{R}^\circ \tilde{T}_g / \tilde{W}_g, \quad (9)$$

where $\tilde{W}_g$ is the molecular weight of the product gas.

The above equations now constitute a closed set for the variables α , $\tilde{u}_g$, $\tilde{T}_l$, $\tilde{T}_g$, $\tilde{T}_s$, $\tilde{\rho}_g$ and $\tilde{p}_g$. The problem is thus completely determined once initial and boundary conditions (including interface relations at $\tilde{x} = \tilde{x}_m$) are specified. In the present work, we will not be concerned with the initial-value problem, but only the long-time solution corresponding to a steadily propagating deflagration. Thus, the required boundary conditions are given by

$$\alpha = \alpha_s \text{ for } \tilde{x} < \tilde{x}_m; \quad \tilde{u}_g \rightarrow 0, \quad \tilde{T}_g \rightarrow \tilde{T}_s \rightarrow \tilde{T}_u, \quad \tilde{p}_g \rightarrow \tilde{p}_g^u \text{ as } \tilde{x} \rightarrow -\infty, \quad (10)$$

$$\tilde{p}_g = \tilde{p}_g^b \text{ for } \tilde{x} > \tilde{x}_m; \quad \alpha \rightarrow 1, \quad \tilde{T}_l \rightarrow \tilde{T}_g \rightarrow \tilde{T}_b, \quad \tilde{u}_g \rightarrow \tilde{u}_g^b \text{ as } \tilde{x} \rightarrow +\infty, \quad (11)$$

where the burned temperature $\tilde{T}_b$ and gas velocity $\tilde{u}_g^b$ are to be determined. Finally, the continuity and jump conditions across the melting surface are given by

$$\alpha|_{\tilde{x}=\tilde{x}_m^+} = \alpha_s, \quad \tilde{u}_g|_{\tilde{x}=\tilde{x}_m^+} = u_g|_{\tilde{x}=\tilde{x}_m^-} = -\frac{\tilde{\kappa}(\alpha_s)}{\alpha_s \tilde{\mu}_g} \frac{d\tilde{p}_g}{d\tilde{x}} \Big|_{\tilde{x}=\tilde{x}_m^-}, \quad \tilde{p}_g|_{\tilde{x}=\tilde{x}_m^-} = \tilde{p}_g|_{\tilde{x}=\tilde{x}_m^+} = \tilde{p}_g^b, \quad (12)$$

$$\tilde{T}_g|_{\tilde{x}=\tilde{x}_m^-} = \tilde{T}_g|_{\tilde{x}=\tilde{x}_m^+}, \quad \tilde{T}_s|_{\tilde{x}=\tilde{x}_m^-} = \tilde{T}_l|_{\tilde{x}=\tilde{x}_m^+} = \tilde{T}_m, \quad \frac{d\tilde{T}_g}{d\tilde{x}} \Big|_{\tilde{x}=\tilde{x}_m^+} = \frac{d\tilde{T}_g}{d\tilde{x}} \Big|_{\tilde{x}=\tilde{x}_m^-}, \quad (13)$$

$$\tilde{\lambda}_l \frac{d\tilde{T}_l}{d\tilde{x}} \Big|_{\tilde{x}=\tilde{x}_m^+} - \tilde{\lambda}_s \frac{d\tilde{T}_s}{d\tilde{x}} \Big|_{\tilde{x}=\tilde{x}_m^-} = \tilde{\rho}_s \frac{d\tilde{x}_m}{dt} \left[\gamma_s + (\tilde{c}_s - \tilde{c}_l) \tilde{T}_m \right], \quad (14)$$

where $\tilde{\gamma}_s$ is the heat of melting of the solid at temperature $\tilde{T} = 0$ ($\tilde{\gamma}_s$ being negative when melting is endothermic), and $\tilde{T}_m$ is the melting temperature of the solid.

3. Nondimensionalizations and the Quasi-Steady Problem

In the present work, we confine our attention to the case of a quasi-steady deflagration that propagates with the (unknown) speed $\tilde{U} = -d\tilde{x}_m/d\tilde{t}$, which is a convenient characteristic velocity for the problem. As is standard, we introduce the nondimensional variables

$$x = \frac{\tilde{\rho}_s \tilde{c}_s \tilde{U}}{\tilde{\lambda}_s} \tilde{x}, \quad t = \frac{\tilde{\rho}_s \tilde{c}_s \tilde{U}^2}{\tilde{\lambda}_s} \tilde{t}, \quad T_{s,l,g} = \frac{\tilde{T}_{s,l,g}}{\tilde{T}_u}, \quad u_{l,g} = \frac{\tilde{u}_{l,g}}{\tilde{U}}, \quad \rho_g = \frac{\tilde{\rho}_g}{\tilde{\rho}_g^u}, \quad (15)$$

where $\tilde{\rho}_g^u = \tilde{p}_g^u \tilde{W}_g / \tilde{R}^\circ \tilde{T}_u$ denotes the gas density at the unburned temperature $\tilde{T}_u$. In addition, the nondimensional parameters

$$\begin{aligned} r &= \frac{\tilde{\rho}_l}{\tilde{\rho}_s}, \quad \hat{r} = \frac{\tilde{\rho}_g^u}{\tilde{\rho}_s}, \quad l = \frac{\tilde{\lambda}_l}{\tilde{\lambda}_s}, \quad \hat{l} = \frac{\tilde{\lambda}_g}{\tilde{\lambda}_s}, \quad b = \frac{\tilde{c}_l}{\tilde{c}_s}, \quad \hat{b} = \frac{\tilde{c}_g}{\tilde{c}_s}, \quad \gamma_s = \frac{\tilde{\gamma}_s}{\tilde{c}_s \tilde{T}_u}, \quad Q = \frac{\tilde{Q}}{\tilde{c}_s \tilde{T}_u}, \\ K_{sg} &= \frac{\tilde{\lambda}_s \tilde{K}_{sg}}{\tilde{\rho}_s^2 \tilde{c}_s^2 \tilde{U}^2}, \quad K_{lg} = \frac{\tilde{\lambda}_s \tilde{K}_{lg}}{r b \tilde{\rho}_s^2 \tilde{c}_s^2 \tilde{U}^2}, \quad \kappa = \frac{\tilde{\rho}_s \tilde{c}_s \tilde{p}_g^u \tilde{\kappa}}{\tilde{\lambda}_s \tilde{\mu}_g}, \quad \hat{\pi} = \frac{\tilde{p}_g^u}{\tilde{\rho}_s \tilde{c}_s \tilde{T}_u} = \hat{r} \hat{b} \chi, \\ \chi &= \frac{\gamma - 1}{\gamma}, \quad N = \frac{\tilde{E}_l}{\tilde{R}^\circ \tilde{T}_b}, \quad \Lambda = \frac{\tilde{\lambda}_s \tilde{A}}{\tilde{\rho}_s \tilde{c}_s \tilde{U}^2} e^{-N} \end{aligned} \quad (16)$$

are defined, where γ is the ratio of specific heats for the gas and Λ is the burning-rate eigenvalue, the determination of which will yield the propagation speed $\tilde{U}$.

Transforming to the moving coordinate $\xi = x + t$ whose origin is defined to be at $x = x_m$, and introducing the above nondimensionalizations, quasi-steady deflagration for the problem formulated in Section 2 is described by the eigenvalue problem

$$\frac{d}{d\xi} [\rho_g(u_g + 1)] = 0, \quad \xi < 0, \quad (17)$$

$$\frac{d}{d\xi} [r(1 - \alpha)(u_l + 1) + \hat{r}\alpha\rho_g(u_g + 1)] = 0, \quad \xi > 0, \quad (18)$$

$$\frac{d}{d\xi} [(1 - \alpha)(u_l + 1)] = -\Lambda(1 - \alpha) \exp \left[N \left(1 - \frac{T_b}{T_l} \right) \right], \quad \xi > 0, \quad (19)$$

$$(1 - \alpha_s) \left(\frac{dT_s}{d\xi} - \frac{d^2T_s}{d\xi^2} \right) = K_{sg}(T_g - T_s), \quad \xi < 0, \quad (20)$$

$$(1 - \alpha_s) \frac{dT_s}{d\xi} + \hat{r}\hat{b}\alpha_s \frac{d}{d\xi} [\rho_g(u_g + 1)T_g] = \frac{d}{d\xi} \left[(1 - \alpha_s) \frac{dT_s}{d\xi} + \hat{l}\alpha_s \frac{dT_g}{d\xi} \right] + \hat{r}\hat{b}\chi\alpha_s \frac{dp_g}{d\xi}, \quad \xi < 0, \quad (21)$$

$$r \frac{d}{d\xi} [(1 - \alpha)(u_l + 1)(Q + bT_l)] = l \frac{d}{d\xi} \left[(1 - \alpha) \frac{dT_l}{d\xi} \right] + rbK_{lg}(T_g - T_l), \quad \xi > 0, \quad (22)$$

$$\frac{d}{d\xi} [r(1 - \alpha)(u_l + 1)(Q + bT_l) + \hat{r}\hat{b}\alpha(u_g + 1)\rho_gT_g] = \frac{d}{d\xi} \left[l(1 - \alpha) \frac{dT_l}{d\xi} + \hat{l}\alpha \frac{dT_g}{d\xi} \right], \quad \xi > 0, \quad (23)$$

$$u_g = -\frac{\kappa}{\alpha_s} \frac{dp_g}{d\xi}, \quad \xi < 0, \quad (24)$$

$$\rho_gT_g = p_g, \quad (25)$$

$$u_l = \frac{1}{r}(1 - r), \quad (26)$$

subject to the boundary and melting-surface conditions

$$\alpha = \alpha_s \text{ for } \xi < 0; \quad u_g \rightarrow 0, \quad T_g \rightarrow T_s \rightarrow 1, \quad p_g \rightarrow 1 \text{ as } \xi \rightarrow -\infty, \quad (27)$$

$$p_g = p_g^b \text{ for } \xi > 0; \quad \alpha \rightarrow 1, \quad u_g \rightarrow u_g^b, \quad T_l \rightarrow T_g \rightarrow T_b \text{ as } \xi \rightarrow +\infty, \quad (28)$$

$$\alpha|_{\xi=0+} = \alpha_s, \quad \tilde{u}_g|_{\xi=0+} = u_g|_{\xi=0-} = -\frac{\kappa(\alpha_s)}{\alpha_s} \frac{dp_g}{d\xi} \Big|_{\xi=0-}, \quad p_g|_{\xi=0-} = p_g|_{\xi=0+} = p_g^b, \quad (29)$$

$$T_g|_{\xi=0-} = T_g|_{\xi=0+}, \quad \tilde{T}_s|_{\xi=0-} = T_l|_{\xi=0+} = T_m, \quad \frac{dT_g}{d\xi} \Big|_{\xi=0+} = \frac{dT_g}{d\xi} \Big|_{\xi=0-}, \quad (30)$$

$$l \frac{dT_l}{d\xi} \Big|_{\xi=0+} - \frac{dT_s}{d\xi} \Big|_{\xi=0-} = -\gamma_s + (b - 1)T_m. \quad (31)$$

We observe that in the limit that the interphase heat-transfer parameters K_{sg} and K_{lg} are infinitely large, the problem (17) - (30) collapses to our previous single-temperature model ([2]). In what

follows, we shall determine the burning-rate eigenvalue of the two-temperature model in the large activation-energy regime $N \gg 1$, with $K_{sg} \sim O(1)$ and $K_{lg} \sim O(N^{-2})$. The leading-order result is expressed in the form of an appropriate expansion with respect to the gas-to-solid conductivity ratio $\hat{l}$, which is also assumed small.

4. Preliminary Analysis

Independent of the magnitude of the interphase heat-transfer parameters K_{sg} and K_{lg} , one may derive expressions for the burned temperature T_b , as well as the burned gas velocity u_g^b and the gas velocity $u_g(0)$ at the melting surface. In particular, from the first integrals of Eqs. (17), (18), (21) and (23), using Eqs. (25), (26) and the corresponding boundary conditions (27) and (28), we obtain

$$\frac{p_g}{T_g}(u_g + 1) = 1, \quad \xi < 0, \quad (32)$$

$$1 - \alpha + \hat{r}\alpha \frac{p_g^b}{T_g}(u_g + 1) = \hat{r} \frac{p_g^b}{T_b}(u_g^b + 1), \quad \xi > 0, \quad (33)$$

$$(1 - \alpha_s)T_s + \hat{r}\hat{b}\alpha_s T_g - (1 - \alpha_s)\frac{dT_s}{d\xi} - \hat{l}\alpha_s \frac{dT_g}{d\xi} - \hat{r}\hat{b}\chi\alpha_s p_g = 1 - \alpha_s + \hat{r}\hat{\alpha}_s - \hat{r}\hat{b}\chi\alpha_s, \quad \xi < 0, \quad (34)$$

$$(1 - \alpha)(Q + bT_l) + \hat{r}\hat{b}\alpha p_g^b(u_g + 1) - l(1 - \alpha)\frac{dT_l}{d\xi} - \hat{l}\alpha \frac{dT_g}{d\xi} = \hat{r}\hat{b}p_g^b(u_g^b + 1), \quad \xi > 0. \quad (35)$$

Evaluation of Eqs. (32) – (35) at $\xi = 0$ using the melting-surface conditions (29) – (31) then gives

$$u_g(0) = \frac{T_g(0)}{p_g^b} - 1, \quad (35)$$

$$u_g^b = \frac{1 - \alpha_s + \hat{r}\alpha_s}{\hat{r}} \left(\frac{T_b}{p_g^b} \right) - 1, \quad (36)$$

$$T_b = \frac{(1 - \alpha_s)(Q + 1 + \gamma_s) + \hat{r}\hat{b}\alpha_s[1 + \chi(p_g^b - 1)]}{\hat{b}(1 - \alpha_s + \hat{r}\alpha_s)}. \quad (37)$$

The results (36) and (37) for u_g^b and T_b are independent of K_{sg} and K_{lg} and are thus identical to the single-temperature result ([2]), whereas Eq. (35) for $u_g(0)$ collapses to the single-temperature result $u_g(0) = T_m/p_g^b - 1$ only in the limit $K_{lg} \rightarrow \infty$. The behavior of these quantities as a function of overpressure $p_g^b - 1$ has been discussed previously ([2]), but the main qualitative result is that the gas velocity decreases and, in fact, becomes negative as the overpressure increases, leading to convective permeation of the burned gas into the pores of the unburned solid. This in turn results in a superadiabatic effect that increases the combustion temperature beyond its adiabatic value corresponding to $p_g^b = 1$ according to Eq. (37).

At this point, if one assumes that K_{lg} is infinite, then $T_l = T_g$ and, since all conditions at $\xi = 0$ are then known according to Eqs. (35) – (37), the problem collapses to an analysis of

the single-temperature region $\xi > 0$, independent of K_{sg} and the solution in the two-temperature region $\xi < 0$. Thus, in that case, the expression for the burning-rate eigenvalue is the same as that obtained in the single-temperature limit corresponding to infinite values for both K_{sg} and K_{lg} ([2]). In the present work, we will obtain nontrivial modifications of that result, which will be reintroduced shortly, by considering a case in which K_{lg} is large, but finite. In particular, we shall consider the asymptotic limit in which the Zel'dovich number β , defined by $\beta = (1 - T_b^{-1})N$, is large, and consider the parameter regime $K_{lg} \sim O(\beta^2)$. In that limit, the leading-order approximation for the burning-rate eigenvalue will be independent of K_{sg} , which, for the sake of definiteness, is assumed to be $O(1)$.

In the large activation-energy limit just described, all chemical activity is concentrated in a very thin region where T_l is within $O(1/\beta)$ of T_b . Denoting the location of this thin zone by $\xi_r > 0$, we see that the semi-infinite liquid/gas region is comprised of a preheat zone ($0 < \xi < \xi_r$) where chemical activity is exponentially small, the thin reaction zone where the chemical reaction goes to completion, and a burned region $\xi > \xi_r$. Thus, we conclude from Eq. (19) that in the outer regions

$$\alpha = \begin{cases} \alpha_s, & \xi < \xi_r \\ 1, & \xi > \xi_r, \end{cases} \quad (38)$$

and from Eq. (33),

$$u_g = \begin{cases} T_g/p_g^b - 1, & \xi < \xi_r \\ u_g^b, & \xi > \xi_r, \end{cases} \quad (39)$$

where the expression for u_g^b was given by Eq. (36). Finally, based on the assumed scaling for K_{lg} , T_l and T_g in the outer regions are expressed as

$$T_l \sim \begin{cases} T^{(0)} + \beta^{-2}T_l^{(2)} + \dots, & 0 < \xi < \xi_r \\ T_b, & \xi > \xi_r, \end{cases} \quad (40)$$

$$T_g \sim \begin{cases} T^{(0)} + \beta^{-2}T_g^{(2)} + \dots, & 0 < \xi < \xi_r \\ T_b, & \xi > \xi_r. \end{cases} \quad (41)$$

From Eq. (35), the common leading-order outer temperature $T^{(0)}$, which has the value T_m at $\xi = 0$ and is continuous at $\xi = \xi_r$, is given by the single-temperature result

$$T^{(0)}(\xi) = \begin{cases} B + (T_b - B) \exp \left[\frac{b(1 - \alpha_s) + \hat{r}\hat{b}\alpha_s}{l(1 - \alpha_s) + \hat{l}\alpha_s} (\xi - \xi_r) \right], & 0 < \xi < \xi_r \\ T_b, & \xi > \xi_r, \end{cases} \quad (42)$$

with

$$B \equiv \frac{(1 - \alpha_s)(1 + \gamma_s) + \hat{r}\hat{b}\alpha_s [1 + \chi(p_g^b - 1)]}{b(1 - \alpha_s) + \hat{r}\hat{b}\alpha_s}, \quad \xi_r = \frac{l(1 - \alpha_s) + \hat{l}\alpha_s}{b(1 - \alpha_s) + \hat{r}\hat{b}\alpha_s} \ln \left(\frac{T_b - B}{T_m - B} \right). \quad (43)$$

We observe that both u_g and $dT^{(0)}/d\xi$ are discontinuous across the reaction layer at $\xi = \xi_r$, which is the location, on the outer scale, of a source for both gas and heat production.

The outer analysis described thus far is not sufficient to determine the burning-rate eigenvalue, which requires an analysis of the thin reaction zone. In particular, it is necessary at this stage to construct inner solutions in the reaction layer, which is where temperature-nonequilibrium effects first emerge in the present model. The requirement that the reaction-layer solutions match with the outer profiles obtained above will then determine the propagation speed.

4. Reaction-Zone Analysis

Based on the assumed largeness of the nondimensional activation energy, we formally introducing the scaled interphase heat-transfer parameter k and the stretched reaction-zone coordinate η according to

$$K_{lg} = \beta^2 k, \quad \eta = \beta(\xi - \xi_r), \quad \beta = (1 - T_b^{-1})N \gg 1. \quad (44)$$

Since, according to Eq. (19), chemical reaction is negligible until T_l is within $O(\beta^{-1})$ of T_b , and K_{lg} is small, solutions in the liquid/gas region are sought in the form of the expansions

$$\Theta = \frac{T_l - 1}{T_b - 1} \sim 1 + \beta^{-1}\theta_1 + \beta^{-2}\theta_2 + \dots, \quad \Psi = \frac{T_g - 1}{T_b - 1} \sim 1 + \beta^{-1}\psi_1 + \beta^{-2}\psi_2 + \dots, \quad (45)$$

$$\alpha \sim \alpha_0 + \beta^{-1}\alpha_1 + \beta^{-2}\alpha_2 + \dots, \quad \Lambda \sim \beta(\Lambda_0 + \beta^{-1}\Lambda_1 + \beta^{-2}\Lambda_2 + \dots), \quad (46)$$

where the expansion for the burning-rate eigenvalue Λ follows from Eq. (19) when the above scalings and expansions are introduced, and an expansion for u_g in terms of the above expansion coefficients follows from Eq. (33).

Substituting these expansions into Eqs. (19), (22), (33) and (35), collecting coefficients of like powers of β , and requiring that the inner reaction-zone solution match with the outer solutions for $\xi < \xi_r$ and $\xi > \xi_r$, leads to a sequence of problems for the recursive determination of the coefficients in Eqs. (44) - (46). In particular, the leading-order inner problem is given by

$$\frac{d\alpha_0}{d\eta} = r\Lambda_0(1 - \alpha_0)e^{\theta_1}, \quad (47)$$

$$l(1 - \alpha_0)\frac{d\theta_1}{d\eta} + \hat{l}\alpha_0\frac{d\psi_1}{d\eta} = q_1(1 - \alpha_0), \quad (48)$$

$$l\frac{d}{d\eta}\left[(1 - \alpha_0)\frac{d\theta_1}{d\eta}\right] = -q_2\frac{d\alpha_0}{d\eta} + rbk(\theta_1 - \psi_1), \quad (49)$$

subject to the matching conditions

$$\alpha_0 \rightarrow \alpha_s, \quad \psi_1 \rightarrow \theta_1 \sim E\eta \text{ as } \eta \rightarrow -\infty; \quad \alpha_0 \rightarrow 1, \quad \theta_1 \rightarrow \psi_1 \rightarrow 0 \text{ as } \eta \rightarrow +\infty, \quad (50)$$

where q_1 , q_2 and E are defined as

$$q_1 = \frac{Q + (b - \hat{b})T_b}{T_b - 1}, \quad q_2 = \frac{Q + bT_b}{T_b - 1}, \quad E = \frac{1}{T_b - 1} \frac{dT^{(0)}}{d\xi} \Big|_{\xi=\xi_r^-} = \frac{T_b - B}{T_b - 1} \cdot \frac{b(1 - \alpha_s) + \hat{r}\hat{b}\alpha_s}{l(1 - \alpha_s) + \hat{l}\alpha_s}. \quad (51)$$

We observe that the effects of interphase heat transfer, while absent from the leading-order outer problem, are felt at leading order in the inner problem through Eq. (49).

The problem (47) – (50), which does not possess a closed-form solution, was briefly considered in the context of the unconfined problem ([5]). In that study, it was further assumed that the scaled heat-transfer coefficient k itself was large, and a regular perturbation expansion in powers of k^{-1} was developed. In that case, θ_1 and ψ_1 differ only by an $O(k^{-1})$ amount, and the leading-order versions of Eqs. (47) and (48), with θ_1 and ψ_1 replaced by their common leading-order coefficient θ , decouple from Eq. (49). The leading-order inner problem then becomes identical to the single-temperature case, for which a closed-form solution is readily obtainable, and modifications to the burning-rate eigenvalue due to finite rates of interphase heat transfer are obtained as an $O(k^{-1})$ correction to the single-temperature result.

In the present work, we wish to allow for more significant influences of temperature nonequilibrium. Therefore, as an alternative to the previous approach, we consider the realistic limit in which the gas-to-solid conductivity ratio $\hat{l}$ is small, treating l , the liquid-to-solid conductivity ratio, and k as $O(1)$ quantities. Though we are ultimately rewarded with an expression for the Λ_0 that is more significantly affected by temperature-nonequilibrium effects, the asymptotic treatment of Eqs. (47) – (50) is considerably more complex. In particular, it is readily observed that, owing to the second term in Eq. (48), the perturbation expansion in powers of $\hat{l}$ is singular, reflecting a boundary-layer character of the solution.

4.1. Asymptotic Formulation of the Reaction-Zone Problem

Before proceeding, it is advantageous to first recast Eqs. (47) – (50) by employing α_0 , which is a monotonic function of η , as the independent variable. Using Eq. (47), Eqs. (48) – (50) are transformed as

$$l(1 - \alpha_0)e^{\theta_1} \frac{d\theta_1}{d\alpha_0} + \hat{l}\alpha_0 e^{\theta_1} \frac{d\psi_1}{d\alpha_0} = \frac{q_1}{r\Lambda_0}, \quad (52)$$

$$l(1 - \alpha_0)e^{\theta_1} \frac{d}{d\alpha_0} \left[(1 - \alpha_0)^2 e^{\theta_1} \frac{d\theta_1}{d\alpha_0} \right] = -\frac{q_2}{r\Lambda_0} (1 - \alpha_0)e^{\theta_1} + \frac{rbk}{r^2\Lambda_0^2} (\theta_1 - \psi_1), \quad (53)$$

$$\psi_1 \rightarrow \theta_1 \rightarrow -\infty \text{ as } \alpha_0 \rightarrow \alpha_s; \quad \theta_1 \rightarrow \psi_1 \rightarrow 0 \text{ as } \alpha_0 \rightarrow 1. \quad (54)$$

Next, we define new variables u , w and χ and parameters $\bar{q}_1$, $\bar{q}_2$, $\bar{k}$ and $\bar{l}$ according to

$$u = e^{\theta_1}, \quad w = \theta_1 - \psi_1, \quad \chi = \alpha_0, \quad \bar{q}_1 = \frac{q_1}{rl}, \quad \bar{q}_2 = \frac{q_2}{rl}, \quad \bar{k} = \frac{b}{rl}k, \quad \bar{l} = \frac{\hat{l}}{l}. \quad (55)$$

In terms of these quantities, Eqs. (52) – (54) are written more simply as

$$[(1 - \chi) + \bar{l}\chi] \frac{du}{d\chi} - \bar{l}\chi u \frac{dw}{d\chi} = \frac{\bar{q}_1}{\Lambda_0}, \quad (56)$$

$$(1 - \chi)u \frac{d}{d\chi} \left[(1 - \chi)^2 \frac{du}{d\chi} \right] = -\frac{\bar{q}_2}{\Lambda_0} (1 - \chi)u + \frac{\bar{k}}{\Lambda_0^2} w, \quad (57)$$

$$u \rightarrow 0, \quad w \rightarrow 0 \text{ as } \chi \rightarrow \alpha_s; \quad u \rightarrow 1, \quad w \rightarrow 0 \text{ as } \chi \rightarrow 1. \quad (58)$$

In what follows, Eqs. (56) – (58) are solved in the asymptotic limit $\bar{l} \ll 1$, the main result being a series expansion of Λ_0 in appropriate functions of $\bar{l}$.

Identification of the correct form of the expansions for $\bar{l} \ll 1$ is facilitated by a consideration of the simpler single-temperature limit ($\bar{k} \rightarrow \infty, w \rightarrow 0$) of Eqs. (56) – (58), as described in the Appendix. However, Eq. (56) immediately suggests that u and w should have, for small $\bar{l}$, a boundary-layer character with respect to the coordinate χ in the neighborhood of $\chi = 1$. Consequently, we introduce a stretched coordinate γ defined by

$$\gamma = \frac{1}{\bar{l}}(1 - \chi), \quad (59)$$

in terms of which Eqs. (56) and (57) become

$$[1 + \gamma(1 - \bar{l})] \frac{du}{d\gamma} - (1 - \bar{l}\gamma)u \frac{dw}{d\gamma} = -\frac{\bar{q}_1}{\Lambda_0}, \quad (60)$$

$$\bar{l}\gamma u \frac{d}{d\gamma} \left(\gamma^2 \frac{du}{d\gamma} \right) = -\bar{l} \frac{\bar{q}_2}{\Lambda_0} \gamma u + \frac{\bar{k}}{\Lambda_0^2} w. \quad (61)$$

Guided in part by the analysis of the single-temperature model, inner and outer expansions for u and w , denoted respectively by $u^{(i)}, w^{(i)}$ and $u^{(o)}, w^{(o)}$, along with a corresponding expansion for the eigenvalue Λ_0 , are sought in the form

$$\Lambda_0 \sim (-\ln \bar{l}) \left[\mu_0 + (-\ln \bar{l})^{-1} \mu_1 \right] + \bar{l}(-\ln \bar{l}) \left[\mu_2 + (-\ln \bar{l})^{-1} \mu_3 + (-\ln \bar{l})^{-2} \mu_4 + \dots \right] + O[\bar{l}^2(-\ln \bar{l})], \quad (62)$$

$$u^{(o)}(\chi) \sim (-\ln \bar{l})^{-1} \left[u_{0,0}(\chi) + (-\ln \bar{l})^{-1} u_{0,1}(\chi) + (-\ln \bar{l})^{-2} u_{0,2}(\chi) + \dots \right] + \bar{l}(-\ln \bar{l})^{-1} \left[u_{1,0}(\chi) + (-\ln \bar{l})^{-1} u_{1,1}(\chi) + (-\ln \bar{l})^{-2} u_{1,2}(\chi) + \dots \right] + O[\bar{l}^2(-\ln \bar{l})^{-1}], \quad (63)$$

$$w^{(o)}(\chi) \sim \bar{l} \left[w_{1,0}(\chi) + (-\ln \bar{l})^{-1} w_{1,1}(\chi) + (-\ln \bar{l})^{-2} w_{1,2}(\chi) + \dots \right] + O(\bar{l}^2), \quad (64)$$

$$u^{(i)}(\gamma) \sim 1 + (-\ln \bar{l})^{-1} \hat{u}_{0,1}(\gamma) + (-\ln \bar{l})^{-2} \hat{u}_{0,2}(\gamma) + \dots + \bar{l}(-\ln \bar{l})^{-1} \left[\hat{u}_{1,0}(\gamma) + (-\ln \bar{l})^{-1} \hat{u}_{1,1}(\gamma) + (-\ln \bar{l})^{-2} \hat{u}_{1,2}(\gamma) + \dots \right] + O[\bar{l}^2(-\ln \bar{l})^{-1}], \quad (65)$$

$$w^{(i)}(\gamma) \sim \bar{l}(-\ln \bar{l}) \left[\hat{w}_{0,0}(\gamma) + (-\ln \bar{l})^{-1} \hat{w}_{0,1}(\gamma) + (-\ln \bar{l})^{-2} \hat{w}_{0,2}(\gamma) + \dots \right] + O[\bar{l}^2(-\ln \bar{l})]. \quad (66)$$

The coefficients in the outer expansions are governed by Eqs. (56) and (57) along with the boundary conditions at $\chi = \alpha_s$, whereas the coefficients in the inner expansions are determined from Eqs. (60)

and (61) along with the boundary conditions at $\gamma = 0$ ($\chi = 1$). The coefficients in the eigenvalue expansion are then obtained from an appropriate matching of the inner and outer solutions. We observe that in order to calculate a term that is $O(\bar{l})$ smaller than a preceding term, it is necessary to compute (and sum) all preceding terms that differ from one another by at most a power of $(-\ln \bar{l})$. Indeed, it turns out (*cf.* [8]) that the asymptotic matching procedure works best when all such series of terms are computed and summed, and we thus find it convenient to introduce the notation

$$\lambda_0(\bar{l}) = (-\ln \bar{l})\mu_0 + \mu_1, \quad \lambda_1(\bar{l}) = \sum_{n=0}^{\infty} (-\ln \bar{l})^{1-n} \mu_{2+n}, \quad \dots, \quad (67)$$

$$u_j(\chi; \bar{l}) = \sum_{n=0}^{\infty} (-\ln \bar{l})^{-(n+1)} u_{j,n}(\chi), \quad w_j(\chi; \bar{l}) = \sum_{n=0}^{\infty} (-\ln \bar{l})^{-n} w_{j,n}(\chi), \quad j = 0, 1, 2, \dots, \quad (68)$$

$$\hat{u}_0(\gamma; \bar{l}) = 1 + \sum_{n=1}^{\infty} (-\ln \bar{l})^{-n} \hat{u}_{0,n}(\gamma), \quad \hat{u}_j(\gamma; \bar{l}) = \sum_{n=0}^{\infty} (-\ln \bar{l})^{-(n+1)} \hat{u}_{j,n}(\gamma), \quad j = 1, 2, \dots, \quad (69)$$

$$\hat{w}_{j+1}(\gamma; \bar{l}) = \sum_{n=0}^{\infty} (-\ln \bar{l})^{n+1} \hat{w}_{j,n}(\gamma), \quad j = 0, 1, 2, \dots. \quad (70)$$

In terms of these quantities, Eqs. (62) – (66) may be written in a more compact form as generalized asymptotic expansions given by

$$\Lambda_0 \sim \lambda_0(\bar{l}) + \bar{l}\lambda_1(\bar{l}) + \dots, \quad (71)$$

$$u^{(o)} \sim u_0(\chi; \bar{l}) + \bar{l}u_1(\chi; \bar{l}) + \dots, \quad w^{(o)} \sim w_0(\chi; \bar{l}) + \bar{l}w_1(\chi; \bar{l}) + \dots, \quad (72)$$

$$u^{(i)} \sim \hat{u}_0(\gamma; \bar{l}) + \bar{l}\hat{u}_1(\gamma; \bar{l}) + \dots, \quad w^{(i)} \sim \bar{l}\hat{w}_1(\gamma; \bar{l}) + \dots, \quad (73)$$

where any coefficient, denoted generically by $c_j(\bar{l})$, simultaneously satisfies the conditions $\bar{l}c_j(\bar{l}) \ll 1 \ll \bar{l}^{-1}c_j(\bar{l})$. In particular, the c_j possess only logarithmic dependencies on $\bar{l}$ according to Eqs. (67) – (70) and are thus consistent with the form of the generalized asymptotic expansions (72) – (73). In what follows, it will frequently prove convenient to work with these summed forms directly. We also observe in Eq. (67) that although $\lambda_0(\bar{l})$ is postulated to have, based on the single-temperature analysis described in the Appendix, a finite expansion in negative powers of $(-\ln \bar{l})$, it will turn out that the application of the matching conditions requires an infinite expansion of $\lambda_1(\bar{l})$ for the two-temperature problem.

4.2. Outer Solution

Proceeding with a formal analysis of the outer reaction-zone problem, we substitute the expansions (62) – (64) into Eqs. (56) and (57), and equate coefficients of like orders with respect to

$\bar{l}$. At $O[(-\ln \bar{l})^{-n-1}]$, $n = 0, 1, 2, \dots$, Eq. (56) determines a sequence of equations for the $u_{0,n}$ as

$$(1 - \chi) \frac{du_{0,n}}{d\chi} = (-1)^n \frac{\bar{q}_1}{\mu_0} \left(\frac{\mu_1}{\mu_0} \right)^n, \quad n = 0, 1, 2, \dots, \quad (74)$$

where the right hand side stems from the Taylor expansion of Λ_0^{-1} according to Eq. (62). Equations (74) are solved subject to $u_{0,n}(\alpha_s) = 0$, thus determining the $u_{0,n}$ as

$$u_{0,n}(\chi) = (-1)^{n+1} \frac{\bar{q}_1}{\mu_0} \left(\frac{\mu_1}{\mu_0} \right)^n \ln \left(\frac{1 - \chi}{1 - \alpha_s} \right). \quad (75)$$

Consequently, the sum represented by $u_0(\chi; \bar{l})$ can be calculated from its definition in Eqs. (68) to yield

$$u_0(\chi; \bar{l}) = -\frac{\bar{q}_1}{\mu_0(-\ln \bar{l})} \left[1 + \frac{\mu_1/\mu_0}{(-\ln \bar{l})} \right]^{-1} \ln \left(\frac{1 - \chi}{1 - \alpha_s} \right) = -\frac{\bar{q}_1}{\lambda_0(\bar{l})} \ln \left(\frac{1 - \chi}{1 - \alpha_s} \right), \quad (76)$$

where we have used the definition $\lambda_0(\bar{l}) = (-\ln \bar{l})\mu_0 + \mu_1$ in the final equality. Substituting this last result into Eq. (57) allows $w_0(\chi; \bar{l})$ to be determined directly as

$$w_0(\chi; \bar{l}) = w_{0,0}(\chi) = \frac{\bar{q}_1}{\bar{k}} (\bar{q}_1 - \bar{q}_2) (1 - \chi) \ln \left(\frac{1 - \chi}{1 - \alpha_s} \right), \quad (77)$$

which implies that $w_{0,n} = 0$ for $n \geq 1$. We observe that since the outer solution $w_0(\chi; \bar{l})$ satisfies both the outer and inner boundary conditions at $\chi = \alpha_s$ and at $\chi = 1$, respectively, Eq. (77) represents a uniformly valid, leading-order approximation for the temperature difference w . Indeed, the $O(1)$ inner expansion for w is identically zero, as indicated in the second of Eqs. (73) and demonstrated below. This approximation for w is displayed in Figure 2. The local maximum occurs at $\chi = 1 - (1 - \alpha_s)/e$, where w_0 achieves the value $\bar{k}^{-1} \bar{q}_1 (\bar{q}_2 - \bar{q}_1) (1 - \alpha_s)/e$. Clearly, the degree of temperature nonequilibrium is inversely proportional to the heat-transfer coefficient $\bar{k}$, where increased resistance to heat transfer (smaller $\bar{k}$) gives rise to a liquid temperature that is increasingly greater than that of the co-existing gas.

At the next order with respect to a power of $\bar{l}$, by which we mean, in the present context, terms that are $O[(-\ln \bar{l})^{-n-1}]$, Eq. (56) provides a set of equations for the $u_{1,n}(\chi)$. These may be summed to yield an equation for $u_1(\chi; \bar{l})$ given by

$$(1 - \chi) \frac{du_1}{d\chi} + \chi \frac{du_0}{d\chi} - \chi u_0 \frac{dw_0}{d\chi} = -\bar{q}_1 \frac{\lambda_1(\bar{l})}{\lambda_0^2(\bar{l})}, \quad (78)$$

subject to $u_1(\alpha_s) = 0$. Equation (78) is integrable and thus determines $u_1(\chi; \bar{l})$ as

$$\begin{aligned} u_1(\chi; \bar{l}) = & \frac{\bar{q}_1}{\lambda_0(\bar{l})} \left[\frac{\lambda_1(\bar{l})}{\lambda_0(\bar{l})} - 1 \right] \ln \left(\frac{1 - \chi}{1 - \alpha_s} \right) - \frac{\bar{q}_1}{\lambda_0(\bar{l})} \cdot \frac{\chi - \alpha_s}{(1 - \chi)(1 - \alpha_s)} \\ & + \frac{\bar{q}_1^2 (\bar{q}_1 - \bar{q}_2)}{\bar{k} \lambda_0(\bar{l})} \left[\left(\frac{1}{2} - \chi \right) \ln^2 \left(\frac{1 - \chi}{1 - \alpha_s} \right) - \frac{1}{3} \ln^3 \left(\frac{1 - \chi}{1 - \alpha_s} \right) \right. \\ & \left. - (1 - \chi) \ln \left(\frac{1 - \chi}{1 - \alpha_s} \right) - (\chi - \alpha_s) \right]. \end{aligned} \quad (79)$$

We may now proceed to determine $w_1(\chi; \bar{l})$, but it turns out not to be needed in the matching condition that determines $\lambda_1(\bar{l})$.

We emphasize that, although we have introduced the generalized expansions (67) – (70) into the present analysis, Eq. (76), and the inner results that are obtained below, represent strictly formal results. Indeed, $u_1(\chi; \bar{l})$ could have been constructed in precisely the same fashion as $u_0(\chi; \bar{l})$ above. In that case, the individual problems associated with each of the $u_{1,n}$ are identical in form to Eq. (78), the only difference being that the factor $\lambda_1(\bar{l})/\lambda_0^2(\bar{l})$ is expanded in appropriate powers of $(-\ln \bar{l})$ to determine the proper inhomogeneous term corresponding to each n . Summing according to the first of Eqs. (68) then gives the same result as determining $u_1(\chi; \bar{l})$ directly from Eq. (78). Conversely, expanding $u_1(\chi; \bar{l})$ in powers of $(-\ln \bar{l})$ yields the coefficients $u_{1,n}$. Since we ultimately desire the summed forms in lieu of an infinite number of logarithmic terms, it is sometimes simpler to compute the parameters and functions defined in Eqs. (67) – (70) directly, and to later expand these quantities when applying the matching conditions.

4.3. Inner Solution

We now consider solutions of Eqs. (60) and (61) in the inner region defined by the scaled coordinate γ . Substituting the inner generalized expansions (73) into Eqs. (60) – (61) and equating terms of like orders of magnitude, we obtain at leading order [i.e., at $O(1)$ with respect to algebraic powers of $(-\ln \bar{l})$] a problem for $\hat{u}_0(\gamma; \bar{l})$ given by

$$(\gamma + 1) \frac{d\hat{u}_0}{d\gamma} = -\frac{\bar{q}_1}{\lambda_0(\bar{l})}; \quad \hat{u}_0(0) = 1. \quad (80)$$

Here, we have used the fact, based on Eq. (61), that there is no $O(1)$ term in the expansion (73) of $w^{(i)}$. Hence, we obtain

$$\hat{u}_0(\gamma; \bar{l}) = 1 - \frac{\bar{q}_1}{\lambda_0(\bar{l})} \ln(\gamma + 1). \quad (81)$$

At $O(\bar{l})$ in the generalized expansion of the inner problem, Eq. (61) yields a strictly algebraic equation for $\hat{w}_1(\gamma; \bar{l})$ given by

$$\gamma \hat{u}_0 \frac{d}{d\gamma} \left(\gamma^2 \frac{d\hat{u}_0}{d\gamma} \right) = -\frac{\bar{q}_2}{\lambda_0(\bar{l})} \gamma \hat{u}_0 + \frac{\bar{k}}{\lambda_0^2(\bar{l})} \hat{w}_1. \quad (82)$$

Substitution of the result (81) for $\hat{u}_0$ thus determines $\hat{w}_1$ as

$$\hat{w}_1(\gamma; \bar{l}) = \frac{\gamma}{\bar{k}} [\lambda_0(\bar{l}) - \bar{q}_1 \ln(\gamma + 1)] [\bar{q}_2 - \bar{q}_1 + \bar{q}_1(\gamma + 1)^{-2}], \quad (83)$$

which, we observe, satisfies the boundary condition $\hat{w}_1(0) = 0$. Also at this order, Eq. (60) gives a first-order problem for $\hat{u}_1(\gamma; \bar{l})$ in terms of $\hat{u}_0$ and $\hat{w}_1$ as

$$-(\gamma + 1) \frac{d\hat{u}_1}{d\gamma} + \gamma \frac{d\hat{u}_0}{d\gamma} + \hat{u}_0 \frac{d\hat{w}_1}{d\gamma} = -\bar{q}_1 \frac{\lambda_1(\bar{l})}{\lambda_0^2(\bar{l})}; \quad \hat{u}_1(0) = 0. \quad (84)$$

Using the results (81) and (83), Equation (84) may be integrated to yield, after some algebraic rearrangement, a closed-form expression for $\hat{u}_1$ as

$$\begin{aligned}\hat{u}_1(\gamma; \bar{l}) = & c_1(\bar{l}) + c_2(\bar{l})(\gamma + 1)^{-1} + c_3(\bar{l})(\gamma + 1)^{-2} + c_4(\bar{l})(\gamma + 1)^{-3} + c_5(\bar{l}) \ln(\gamma + 1) \\ & + c_6(\bar{l}) \ln^2(\gamma + 1) + c_7(\bar{l}) \ln^3(\gamma + 1) + c_8(\bar{l})(\gamma + 1)^{-1} \ln(\gamma + 1) \\ & + c_9(\bar{l})(\gamma + 1)^{-2} \ln(\gamma + 1) + c_{10}(\bar{l})(\gamma + 1)^{-3} \ln(\gamma + 1) \\ & + c_{11}(\bar{l})(\gamma + 1)^{-2} \ln^2(\gamma + 1) + c_{12}(\bar{l})(\gamma + 1)^{-3} \ln^2(\gamma + 1),\end{aligned}\quad (85)$$

where the coefficients $c_i(\bar{l})$ are defined as

$$\begin{aligned}c_1(\bar{l}) &= \frac{\bar{q}_1}{\lambda_0(\bar{l})} + \frac{\bar{q}_1}{k} \left[\frac{1}{6} \lambda_0(\bar{l}) + \bar{q}_2 - \frac{10}{9} \bar{q}_1 - \frac{\bar{q}_1}{\lambda_0(\bar{l})} \left(\bar{q}_2 - \frac{28}{27} \bar{q}_1 \right) \right], \\ c_2(\bar{l}) &= -\frac{\bar{q}_1}{\lambda_0(\bar{l})} - \frac{\bar{q}_1(\bar{q}_2 - \bar{q}_1)}{k} \left[1 - \frac{\bar{q}_1}{\lambda_0(\bar{l})} \right], \quad c_3(\bar{l}) = \frac{\bar{q}_1}{2k} \lambda_0(\bar{l}), \\ c_4(\bar{l}) &= \frac{\bar{q}_1}{k} \left[-\frac{2}{3} \lambda_0(\bar{l}) + \frac{\bar{q}_1}{9} - \frac{\bar{q}_1^2}{27 \lambda_0(\bar{l})} \right], \quad c_5(\bar{l}) = \frac{\bar{q}_1}{\lambda_0(\bar{l})} \left[\frac{\lambda_1(\bar{l})}{\lambda_0(\bar{l})} - 1 \right] + \frac{\bar{q}_2 - \bar{q}_1}{k} [\lambda_0(\bar{l}) - \bar{q}_1], \\ c_6(\bar{l}) &= \frac{\bar{q}_1(\bar{q}_2 - \bar{q}_1)}{k} \left[\frac{\bar{q}_1}{2 \lambda_0(\bar{l})} - 1 \right], \quad c_7(\bar{l}) = \frac{\bar{q}_1^2(\bar{q}_2 - \bar{q}_1)}{3k \lambda_0(\bar{l})}, \quad c_8(\bar{l}) = 3c_7(\bar{l}), \quad c_9(\bar{l}) = -\frac{\bar{q}_1^2}{k}, \\ c_{10}(\bar{l}) &= \frac{\bar{q}_1^2}{k} \left[\frac{4}{3} - \frac{\bar{q}_1}{9 \lambda_0(\bar{l})} \right], \quad c_{11}(\bar{l}) = \frac{\bar{q}_1^3}{2k \lambda_0(\bar{l})}, \quad c_{12}(\bar{l}) = -\frac{4}{3} c_{11}(\bar{l}).\end{aligned}\quad (86)$$

We again remark that Eqs. (81), (83) and (85) are strictly formal results that are consistent with the generalized expansions (71) and (73) of the inner problem. As indicated above, the same results are obtained using the more primitive expansions (62), (65) and (66), and then summing the logarithmic terms according to Eqs. (69) and (70).

4.4. Asymptotic Matching

Owing to the appearance of an infinite number of logarithmic terms in the inner and outer expansions (63) – (66), specification of appropriate matching conditions, and the subsequent determination of the coefficients in the expansion (62) of the burning-rate eigenvalue, are not entirely standard. In particular, it is often found that, when logarithmic terms appear in the asymptotic development of boundary-layer expansions such as these, the formulation of a proper set of matching conditions is best accomplished by including all terms that differ by no more than a logarithmic power of the small expansion parameter at each level of the matching procedure (cf. [8]). We thus consider the application of Van Dyke's matching rule (cf. [9]) to the generalized expansions (71) – (73), rather than the primitive expansions (62) – (66). This effectively interprets an “ n ”-term expansion as one that includes n terms with respect to powers of $\bar{l}$ times any power of $(-\ln \bar{l})$, where the quotes are henceforth used to denote this particular definition of ordering. Specifically, we shall demonstrate a formal (“2,2”) generalized matching of the inner and outer solutions $u^{(o)}$

and $u^{(i)}$, thereby determining all coefficients in the "2"-term expansion (71) of the burning-rate eigenvalue Λ_0 .

According to the generalized approach just described, we wish to equate the "2"-term outer expansion (of the "2"-term inner solution) with the "2"-term inner expansion (of the "2"-term outer solution). The requirement that this equality hold at each order with respect to $\bar{l}$, including now all logarithmic orders, will then uniquely determine the coefficients μ_i in the expansions (62) of the generalized eigenvalue coefficients λ_0 and λ_1 . Thus, we consider the "2"-term inner solution $\hat{u}_0(\gamma; \bar{l}) + \bar{l} \hat{u}_1(\gamma; \bar{l})$, rewrite it in terms of the outer coordinate $\chi = 1 - \bar{l}\gamma$, and expand the result for small $\bar{l}$ through $O(\bar{l})$. Denoting the first "2" terms of this outer expansion of the inner solution by $\hat{u}^{(2,2)}$, we find it convenient for future use, owing to the appearance of inverse powers of $\lambda_0 = (-\ln \bar{l})[1 + (\mu_1/\mu_0)(-\ln \bar{l})^{-1}]$ in various terms, to multiply $\hat{u}^{(2,2)}$ by $[1 + (\mu_1/\mu_0)(-\ln \bar{l})^{-1}]^2$ to obtain

$$\begin{aligned}
& \left[1 + \frac{\mu_1/\mu_0}{(-\ln \bar{l})}\right]^2 \hat{u}^{(2,2)} = \\
& 1 + 2 \frac{\mu_1/\mu_0}{(-\ln \bar{l})} + \frac{\mu_1^2/\mu_0^2}{(-\ln \bar{l})^2} - \frac{\bar{q}_1}{\mu_0} \left[1 + \frac{\mu_1/\mu_0}{(-\ln \bar{l})}\right] \left[1 + \frac{\ln(1-\chi)}{(-\ln \bar{l})} + \frac{\bar{l}}{(-\ln \bar{l})}(1-\chi)^{-1}\right] \\
& + \bar{l} \left\{ \frac{\bar{q}_1 \mu_0}{6k} (-\ln \bar{l}) \left[1 + 3 \frac{\mu_1/\mu_0}{(-\ln \bar{l})} + 3 \frac{\mu_1^2/\mu_0^2}{(-\ln \bar{l})^2} + \frac{\mu_1^3/\mu_0^3}{(-\ln \bar{l})^3}\right] \right. \\
& \quad + \frac{\bar{q}_1}{k} \left(\bar{q}_2 - \frac{10}{9} \bar{q}_1\right) \left[1 + 2 \frac{\mu_1/\mu_0}{(-\ln \bar{l})} + \frac{\mu_1^2/\mu_0^2}{(-\ln \bar{l})^2}\right] \\
& \quad + \frac{\bar{q}_1}{\mu_0(-\ln \bar{l})} \left[1 - \frac{\bar{q}_1}{k} \left(\bar{q}_2 - \frac{28}{27} \bar{q}_1\right)\right] \left[1 + \frac{\mu_1/\mu_0}{(-\ln \bar{l})}\right] \Big\} \\
& + \bar{l} \left\{ \frac{\mu_0(\bar{q}_2 - \bar{q}_1)}{k} (-\ln \bar{l}) \left[1 + 3 \frac{\mu_1/\mu_0}{(-\ln \bar{l})} + 3 \frac{\mu_1^2/\mu_0^2}{(-\ln \bar{l})^2} + \frac{\mu_1^3/\mu_0^3}{(-\ln \bar{l})^3}\right] \right. \\
& \quad - \frac{\bar{q}_1(\bar{q}_2 - \bar{q}_1)}{k} \left[1 + 2 \frac{\mu_1/\mu_0}{(-\ln \bar{l})} + \frac{\mu_1^2/\mu_0^2}{(-\ln \bar{l})^2}\right] - \frac{\bar{q}_1}{\mu_0(-\ln \bar{l})} \left[1 + \frac{\mu_1/\mu_0}{(-\ln \bar{l})}\right] \\
& \quad + \frac{\bar{q}_1}{\mu_0^2(-\ln \bar{l})} \left[\mu_2 + \frac{\mu_3}{(-\ln \bar{l})} + \frac{\mu_4}{(-\ln \bar{l})^2} + \frac{\mu_5}{(-\ln \bar{l})^3} + \dots\right] \Big\} [-\ln \bar{l} + \ln(1-\chi)] \\
& + \bar{l} \frac{\bar{q}_1(\bar{q}_2 - \bar{q}_1)}{k} \left\{ - \left[1 + 2 \frac{\mu_1/\mu_0}{(-\ln \bar{l})} + \frac{\mu_1^2/\mu_0^2}{(-\ln \bar{l})^2}\right] + \frac{\bar{q}_1}{2\mu_0(-\ln \bar{l})} \left[1 + \frac{\mu_1/\mu_0}{(-\ln \bar{l})}\right] \right\} \\
& \quad \times [(-\ln \bar{l})^2 + 2(-\ln \bar{l}) \ln(1-\chi) + \ln^2(1-\chi)] \\
& + \bar{l} \frac{\bar{q}_1^2(\bar{q}_2 - \bar{q}_1)}{3k\mu_0(-\ln \bar{l})} \left[1 + \frac{\mu_1/\mu_0}{(-\ln \bar{l})}\right] [(-\ln \bar{l})^3 + 3(-\ln \bar{l})^2 \ln(1-\chi) \\
& \quad + 3(-\ln \bar{l}) \ln^2(1-\chi) + \ln^3(1-\chi)] .
\end{aligned} \tag{87}$$

Similarly, we consider the "2"-term outer solution $u_0(\chi; \bar{l}) + \bar{l} u_1(\chi; \bar{l})$, rewrite it in terms of the

inner coordinate $\gamma = (1 - \chi)/\bar{l}$, and expand the result for small $\bar{l}$ through $O(\bar{l})$. Denoting the first "2" terms of this inner expansion of the outer solution by $u^{(2,2)}$ and multiplying it by $[1 + (\mu_1/\mu_0)(-\ln \bar{l})^{-1}]^2$, we obtain

$$\begin{aligned}
\left[1 + \frac{\mu_1/\mu_0}{(-\ln \bar{l})}\right]^2 u^{(2,2)} = & \frac{\bar{q}_1}{\mu_0(-\ln \bar{l})} \left[1 + \frac{\mu_1/\mu_0}{(-\ln \bar{l})}\right] \left[-\ln \bar{l} - \frac{1}{\gamma} - \ln \gamma + \ln(1 - \alpha_s)\right] \\
& + \bar{l} \frac{\bar{q}_1}{\mu_0(-\ln \bar{l})} \left\{1 + \frac{\mu_1/\mu_0}{(-\ln \bar{l})} - \frac{\mu_2}{\mu_0} \left[1 + \frac{\mu_3/\mu_2}{(-\ln \bar{l})} + \frac{\mu_4/\mu_2}{(-\ln \bar{l})^2} + \frac{\mu_5/\mu_2}{(-\ln \bar{l})^3} + \dots\right]\right\} \\
& \quad \times [-\ln \bar{l} - \ln \gamma + \ln(1 - \alpha_s)] \\
& + \bar{l} \frac{\bar{q}_1}{\mu_0(-\ln \bar{l})} \left[1 + \frac{\mu_1/\mu_0}{(-\ln \bar{l})}\right] \left[\frac{1}{1 - \alpha_s} - \frac{\bar{q}_1(\bar{q}_1 - \bar{q}_2)}{\bar{k}}(1 - \alpha_s)\right] \\
& - \bar{l} \frac{\bar{q}_1(\bar{q}_1 - \bar{q}_2)}{2\bar{k}\mu_0(-\ln \bar{l})} \left[1 + \frac{\mu_1/\mu_0}{(-\ln \bar{l})}\right] \left\{(-\ln \bar{l})^2 + \ln^2 \gamma + \ln^2(1 - \alpha_s)\right. \\
& \quad \left.+ 2(-\ln \bar{l})[-\ln \gamma + \ln(1 - \alpha_s)] - 2\ln \gamma \ln(1 - \alpha_s)\right\} \\
& - \bar{l} \frac{\bar{q}_1(\bar{q}_1 - \bar{q}_2)}{3\bar{k}\mu_0(-\ln \bar{l})} \left[1 + \frac{\mu_1/\mu_0}{(-\ln \bar{l})}\right] \left\{-(-\ln \bar{l})^3 + \ln^3 \gamma - \ln^3(1 - \alpha_s)\right. \\
& \quad \left.+ 3(-\ln \bar{l})^2[\ln \gamma - \ln(1 - \alpha_s)] - 3(-\ln \bar{l})[\ln^2 \gamma + \ln^2(1 - \alpha_s) - 2\ln \gamma \ln(1 - \alpha_s)]\right. \\
& \quad \left.- 3\ln^2 \gamma \ln(1 - \alpha_s) + 3\ln \gamma \ln^2(1 - \alpha_s)\right\}.
\end{aligned} \tag{88}$$

In order to compare $\hat{u}^{(2,2)}$ and $u^{(2,2)}$, we may either express Eq. (87) in terms of γ , or Eq. (88) in terms of χ . Thus, in the latter instance, Eq. (88) is rewritten in terms of χ as

$$\begin{aligned}
\left[1 + \frac{\mu_1/\mu_0}{(-\ln \bar{l})}\right]^2 u^{(2,2)} = & -\frac{\bar{q}_1}{\mu_0(-\ln \bar{l})} \left[1 + \frac{\mu_1/\mu_0}{(-\ln \bar{l})}\right] \left[\ln \left(\frac{1 - \chi}{1 - \alpha_s}\right) + \frac{\bar{l}}{1 - \chi}\right] \\
& - \bar{l} \frac{\bar{q}_1}{\mu_0(-\ln \bar{l})} \left\{1 + \frac{\mu_1/\mu_0}{(-\ln \bar{l})} - \frac{\mu_2}{\mu_0} \left[1 + \frac{\mu_3/\mu_2}{(-\ln \bar{l})} + \frac{\mu_4/\mu_2}{(-\ln \bar{l})^2} + \frac{\mu_5/\mu_2}{(-\ln \bar{l})^3} + \dots\right]\right\} \ln \left(\frac{1 - \chi}{1 - \alpha_s}\right) \\
& + \bar{l} \frac{\bar{q}_1}{\mu_0(-\ln \bar{l})} \left[1 + \frac{\mu_1/\mu_0}{(-\ln \bar{l})}\right] \left[\frac{1}{1 - \alpha_s} - \frac{\bar{q}_1(\bar{q}_1 - \bar{q}_2)}{\bar{k}}(1 - \alpha_s)\right] \\
& - \bar{l} \frac{\bar{q}_1^2(\bar{q}_1 - \bar{q}_2)}{\bar{k}\mu_0(-\ln \bar{l})} \left[1 + \frac{\mu_1/\mu_0}{(-\ln \bar{l})}\right] \left[\frac{1}{2} \ln^2 \left(\frac{1 - \chi}{1 - \alpha_s}\right) + \frac{1}{3} \ln^3 \left(\frac{1 - \chi}{1 - \alpha_s}\right)\right].
\end{aligned} \tag{89}$$

The matching requirement that $\hat{u}^{(2,2)} - u^{(2,2)} = 0$ then implies that coefficients of like functions (orders) of the arbitrarily-small expansion parameter $\bar{l}$ must individually sum to zero. Thus,

considering the individual orders

$$1, (-\ln \bar{l})^{-1}, \bar{l}, \bar{l}(-\ln \bar{l})^{-1}, \bar{l}(-\ln \bar{l})^{-2}, \bar{l}(-\ln \bar{l})^{-3}, \dots$$

that appear in Eqs. (87) and (89), we determine the “zeroth” order coefficients μ_0 and μ_1 from equality at the two “zeroth” orders 1 and $(-\ln \bar{l})^{-1}$, and the “first”-order coefficients μ_i , $i = 2, 3, 4, \dots$, from equality at the “first” orders $\bar{l}(-\ln \bar{l})^{2-i}$. In particular, we obtain

$$\mu_0 = \bar{q}_1, \quad \mu_1 = \mu_0 \ln(1 - \alpha_s), \quad (90)$$

$$\mu_2 = \bar{q}_1 - \frac{\bar{q}_1^2}{\bar{k}} \left\{ \bar{q}_2 - \frac{10}{9} \bar{q}_1 + \frac{1}{2} \bar{q}_1 \ln(1 - \alpha_s) + (\bar{q}_2 - \bar{q}_1) \left[2 \ln^2(1 - \alpha_s) - \frac{3}{2} \ln(1 - \alpha_s) \right] \right\}, \quad (91)$$

$$\begin{aligned} \mu_3 = \bar{q}_1 \left[\frac{\alpha_s}{1 - \alpha_s} + \ln(1 - \alpha_s) \right] + \frac{\bar{q}_1^2}{\bar{k}} \left\{ \bar{q}_2 - \frac{28}{27} \bar{q}_1 - \left(\bar{q}_2 - \frac{10}{9} \bar{q}_1 \right) \ln(1 - \alpha_s) \right. \\ \left. + (\bar{q}_2 - \bar{q}_1) \left[1 - \alpha_s + \frac{2}{3} \ln^3(1 - \alpha_s) \right] \right\}, \end{aligned} \quad (92)$$

$$\mu_4 = \frac{\bar{q}_1^2}{\bar{k}} \ln^3(1 - \alpha_s) \left[\frac{1}{6} (3\bar{q}_2 - 4\bar{q}_1) - (\bar{q}_2 - \bar{q}_1) \ln(1 - \alpha_s) \right], \quad (93)$$

$$\mu_n = \mu_4 \ln^{n-4}(1 - \alpha_s), \quad n > 4. \quad (94)$$

We observe, based on Eq. (94), that the partial sum $\sum_{n=2}^{\infty} (-\ln \bar{l})^{1-n} \mu_{2+n}$ in Eq. (67) for $\lambda_1(\bar{l})$ can be computed to obtain the finite expansion

$$\lambda_1(\bar{l}) = (-\ln \bar{l}) \mu_2 + \mu_3 + \frac{\mu_4}{(-\ln \bar{l}) - \ln(1 - \alpha_s)}, \quad (95)$$

and thus the burning-rate eigenvalue Λ_0 given by Eq. (71) is expressed as

$$\Lambda_0 = (-\ln \bar{l}) \mu_0 + \mu_1 + \bar{l} \left[(-\ln \bar{l}) \mu_2 + \mu_3 + \frac{\mu_4}{(-\ln \bar{l}) - \ln(1 - \alpha_s)} \right] + O[\bar{l}^2(-\ln \bar{l})]. \quad (96)$$

The appearance of terms proportional to $\bar{k}^{-1}$ in the expression for μ_2 (as well as in the expressions for $\mu_3, \mu_4, \dots$) indicates that temperature-nonequilibrium effects lead, for the scalings adopted here, to a significant $O[\bar{l}(-\ln \bar{l})]$ modification in the expression for the burning-rate eigenvalue compared with the single-temperature ($\bar{k} \rightarrow \infty$) result. The “1”-term eigenvalue approximation $\Lambda_0 \approx \lambda_0(\bar{l}) = (-\ln \bar{l}) \mu_0 + \mu_1$, which is independent of $\bar{k}$, and the $\bar{k}$ -dependent, “2”-term approximation given by Eq. (96) are shown in Figure 3. We note, since the dimensional burning rate is inversely proportional to $\sqrt{\Lambda_0}$, that decreasing values of the latter with increasing values of $\bar{k}$ corresponds to an increase in the burning rate associated with greater retention of the heat of reaction by the reacting liquid phase. We remark that the approximation (96) is valid for values of $\bar{k}$ that are $O(“1”)$ relative to $\bar{l}$ such that the $O(“\bar{l}”)$ correction term in Eq. (96) retains its order.

For the sake of completeness, we also show in Figure 4 the composite solutions $u_c^{("1,1")} = u_0 + \hat{u}_1 - u^{("1,1")}$, which, being independent of $\bar{k}$, is identical to the single-temperature result (A.25), and $u_c^{("2,2")} = u_0 + \bar{l}u_1 + \hat{u}_1 + \bar{l}\hat{u}_1 - u^{("2,2")}$. It is clear from the profile of the latter that the liquid-phase temperature increases with decreasing $\bar{k}$ since, again, increased resistance to interphase heat transfer allows the reactive liquid to retain the heat of reaction longer than would be the case if interphase heat transfer were instantaneous. Particularly noteworthy is the fact that for sufficiently small values of $\bar{k}$, the liquid temperature profile becomes non-monotonic, rising beyond the final burned temperature and then approaching that value from above as the reaction (and the amount of remaining liquid) approaches completion.

5. Discussion of the Burning-Rate Eigenvalue

From the definition of Λ given by the last of Eqs. (16) and (46), the leading-order expression for the dimensional propagation speed $\tilde{U}$ is given by

$$\tilde{U}^2 \sim \frac{\tilde{A}(p_g^b)}{\tilde{\rho}_s \tilde{c}_s} \cdot \frac{e^{-N}}{\beta \bar{q}_1} \cdot \frac{\tilde{\lambda}_s}{\bar{\Lambda}_0(\bar{l}, T_b)} = \frac{rT_b^2}{Q + (b - \hat{b})T_b} \cdot \frac{\tilde{\lambda}_l}{\bar{\Lambda}_0(\bar{l}, T_b)} \cdot \frac{\tilde{A}(p_g^b)}{\tilde{\rho}_s \tilde{c}_s N_u} \cdot e^{-N_u/T_b}, \quad (97)$$

where, in the second equality, we have used the definitions of β and $\bar{q}_1$ given in Eqs. (44), (51) and (55), and have defined $N_u = NT_b = \tilde{E}_l/\tilde{R}^\circ \tilde{T}_u$, which is independent of T_b , and $\bar{\Lambda}_0 = \Lambda_0/\bar{q}_1$, which does depend on T_b , and hence p_g^b , through $\bar{q}_1$ and $\bar{q}_2$. The second factor in the last equality, which contains the complete dependence of the burning rate on the thermal conductivities, may be expressed as

$$\frac{\tilde{\lambda}_l}{\bar{\Lambda}_0(\bar{l}, \bar{k}, T_b)} \sim \frac{\tilde{\lambda}_l}{\ln[(\tilde{\lambda}_g/\tilde{\lambda}_l)/(1 - \alpha_s)]} \left\{ -1 + \frac{\tilde{\lambda}_g}{\tilde{\lambda}_l} \left[1 - \frac{\alpha_s/(1 - \alpha_s)}{\ln[(\tilde{\lambda}_g/\tilde{\lambda}_l)/(1 - \alpha_s)]} \right] + \bar{l}g[\bar{l}, \bar{k}, T_b(p_g^b)] + O(\bar{l}^2) \right\}, \quad (98)$$

where we have used the definition $\bar{l} = \tilde{\lambda}_g/\tilde{\lambda}_l$ and have defined $g(\lambda_l, T_b)$, which contains the effects of temperature nonequilibrium, as

$$g[\bar{l}, \bar{k}, T_b(p_g^b)] = \frac{(-\ln \bar{l})(\mu_2/\bar{q}_1 - 1) + \mu_3/\bar{q}_1 - \alpha_s/(1 - \alpha_s) - \ln(1 - \alpha_s)}{(-\ln \bar{l}) + \ln(1 - \alpha_s)} + \frac{\mu_4/\bar{q}_1}{\ln^2 \bar{l} - \ln^2(1 - \alpha_s)}. \quad (99)$$

In the equilibrium limit $\bar{k} \rightarrow \infty$, the result (98) is recognized as the "two"-term expansion (with respect to $\bar{l}$) of the exact single-temperature result³ given by

$$\frac{\tilde{\lambda}_l}{\bar{\Lambda}_0} \bigg|_{\bar{k} \rightarrow \infty} = \tilde{f}(\tilde{\lambda}_g, \tilde{\lambda}_l) = \frac{\tilde{\lambda}_g - \tilde{\lambda}_l}{\ln(\tilde{\lambda}_g/[\tilde{\lambda}_l + (\tilde{\lambda}_g - \tilde{\lambda}_l)\alpha_s])}. \quad (100)$$

It is readily seen that $\partial \tilde{f}/\partial \tilde{\lambda}_g > 0$ and $\partial \tilde{f}/\partial \tilde{\lambda}_l > 0$, corresponding to an increase in the propagation speed as the thermal conductivity of either the liquid or the gas phase increases. It is also observed that at this leading order of approximation, $\tilde{U}$ does not depend on either the thermal conductivity or permeability of the solid, and thus, in the parameter regime considered, it is the solution in the liquid/gas region that plays the dominant role in determining the propagation speed. This conclusion would likely be modified for smaller magnitudes of K_{lg} [see the first of Eqs. (44)] that correspond to the extension of temperature nonequilibrium to the preheat portion of the liquid/gas region, and hence to the melting surface itself. However, such a limit may be less realistic for the present model since it also corresponds to larger-magnitude gas velocities, and hence much larger gas overpressures, that would tend to eliminate the presence of the melt layer.

Of particular interest is the dependence of the burning rate on overpressure, which enters into the expression for $\tilde{U}$ through the dependence of the burned temperature T_b on $p_g^b - 1$. For this purpose, it is convenient to define the nondimensional burning rate $U^* = \tilde{U}(p_g^b)/\tilde{U}(1)$, which is the ratio of the burning rate at an overpressure $p_g^b - 1$ to the burning rate in the unconfined limit $p_g^b = 1$. Consequently, $U^* = U_n(\bar{l}, \bar{k}, T_b) [\tilde{A}(p_g^b)/\tilde{A}(1)]^{1/2}$, where the coefficient U_n , which contains the entire thermal effect of overpressure on U^* , is given by

$$U_n(\bar{l}, \bar{k}, T_b) \approx \exp \left\{ \frac{N_u}{2} \left[\frac{1}{T_b(1)} - \frac{1}{T_b(p_g^b)} \right] \right\} \cdot \frac{T_b(p_g^b)}{T_b(1)} \left[\frac{Q + (b - \hat{b})T_b(1)}{Q + (b - \hat{b})T_b(p_g^b)} \right]^{1/2} \quad (101)$$

$$\times \left[\frac{1 - \bar{l} \{ 1 - \alpha_s(1 - \alpha_s)^{-1} \ln^{-1}[\bar{l}/(1 - \alpha_s)] + g[\bar{l}, \bar{k}, T_b(p_g^b)] \}}{1 - \bar{l} \{ 1 - \alpha_s(1 - \alpha_s)^{-1} \ln^{-1}[\bar{l}/(1 - \alpha_s)] + g[\bar{l}, \bar{k}, T_b(1)] \}} \right]^{1/2}.$$

Owing to the largeness of the activation-energy parameter N_u , the influence of overpressure on the burning rate is dominated initially by its linear effect on T_b through the exponential factor in Eq. (101), whereas the other factors in Eq. (101) contribute at most an algebraic dependence on p_g^b to the burning rate. However, as the overpressure increases, this exponential factor, which approaches a constant value in the limit $p_g^b \rightarrow \infty$, becomes less sensitive to further changes in p_g^b . Consequently, for relatively large overpressures, the overall pressure sensitivity of the propagation speed is primarily determined by the other factors in Eq. (101). This rapid increase in the burning rate with increasing overpressure reflects a transition from a low-pressure, conduction-dominated mode of burning to a regime in which pressure-driven convective permeation of the burned gas into the porous unburned solid plays a significant role. This transition in the burning-rate dependence on p_g^b , illustrated in Figure 5, is true even in the single-temperature limit ([5]). Such behavior is also consistent with confined experiments (cf. [1]) that indicate a rapid change in the burning rate over critical pressure ranges from a less pressure-sensitive, "conductive" mode of deflagration to a more pressure-sensitive, "convective" form of burning characterized by an algebraic pressure sensitivity that is usually represented empirically in the form Ap^n . However, at least in the transition regime described here, conduction is never uniformly negligible throughout

the combustion wave, as indicated by the preceding reaction-zone analysis in which diffusion terms, which account for the highest-order derivatives, clearly play a crucial role.

The effects of temperature nonequilibrium associated with finite values of $\bar{k}$ are reflected in the last factor in Eq. (101), which is unity in the single-temperature limit. Based on the behavior of the burning-rate eigenvalue shown in Figure 3 with respect to increased resistance to interphase heat transfer, the effect of decreasing $\bar{k}$ is to increase U_n , as shown in Figure 5. This enhancement in the burning rate is especially significant once the influence of the exponential factor has diminished for larger overpressures, whereupon the algebraic modification in the burning rate associated with the last factor in Eq. (101) is as important as the other factors in the expression for U_n . The increased propagation speed and increased pressure-sensitivity of the burning-rate response that are associated with finite rates of interphase heat transfer are attributable to greater retention of the heat of reaction in the reactive liquid phase and a greater amount of convective preheating, both of which translate into higher liquid temperatures and hence a faster reaction rate.

As a final note, we remark that since, according to its definition given by Eqs. (44) and (55), the dimensionless heat-transfer parameter $\bar{k} = bK_{lg}/rl\beta^2 = \bar{\lambda}_s \bar{K}_{lg}/r^2 l \beta^2 \bar{\rho}_s^2 \bar{c}_s^2 \bar{U}^2$, where the last equality follows from the definition of K_{lg} in Eqs. (16), $\bar{k}$ itself is inversely proportional to $\bar{U}^2$. This means that as p_g^b increases, $\bar{k}$ itself is not constant for a given value of the dimensional heat-transfer coefficient $\bar{K}_{lg}$, but in fact decreases with increasing propagation speed. Thus, the dependence of the burning rate on overpressure is actually steeper than that shown in Figure 5 for a constant value of $\bar{k}$. Using the preceding definition of $\bar{k}$ in Eqs. (97) – (99) and solving for $\bar{U}^2$, we obtain a final explicit expression for the dimensional burning velocity as

$$\bar{U}^2 \approx \frac{rT_b^2 \bar{A}(p_g^b) e^{-N_u/T_b}}{\bar{\rho}_s \bar{c}_s N_u [Q + (b - \hat{b})T_b]} \cdot \frac{\bar{\lambda}_l}{\ln [(\bar{\lambda}_g/\bar{\lambda}_l)/(1 - \alpha_s)]} \left\{ -1 + \frac{\bar{\lambda}_g}{\bar{\lambda}_l} \left[1 - \frac{\alpha_s/(1 - \alpha_s)}{\ln [(\bar{\lambda}_g/\bar{\lambda}_l)/(1 - \alpha_s)]} \right] \right\} \\ \div \left\{ 1 - \frac{rT_b^2 \bar{A}(p_g^b) e^{-N_u/T_b}}{\bar{\rho}_s \bar{c}_s N_u [Q + (b - \hat{b})T_b]} \cdot \frac{\bar{\lambda}_g \bar{\lambda}_l / \bar{\lambda}_s^2}{\ln [(\bar{\lambda}_g/\bar{\lambda}_l)/(1 - \alpha_s)]} \cdot \frac{r^2 \beta^2 \bar{\rho}_s^2 \bar{c}_s^2}{\bar{K}_{lg}} g^*[\bar{l}, T_b(p_g^b)] \right\}, \quad (102)$$

where $g^*[\bar{l}, T_b(p_g^b)] = \bar{k} g[\bar{l}, \bar{k}, T_b(p_g^b)]$, determined from Eq. (99), is independent of $\bar{k}$. The corresponding expression for the normalized coefficient U_n defined above, using the definitions of r and β , is thus given by

$$U_n \approx \exp \left\{ \frac{N_u}{2} \left[\frac{1}{T_b(1)} - \frac{1}{T_b(p_g^b)} \right] \right\} \cdot \frac{T_b(p_g^b)}{T_b(1)} \left[\frac{Q + (b - \hat{b})T_b(1)}{Q + (b - \hat{b})T_b(p_g^b)} \right]^{1/2} \\ \times \left\{ 1 - \frac{\bar{\rho}_l^3 \bar{c}_s \bar{\lambda}_g \bar{\lambda}_l [T_b(1) - 1]^2 \bar{A}(1) g^*[\bar{l}, T_b(1)] N_u e^{-N_u/T_b(1)}}{\bar{\rho}_s^2 \bar{\lambda}_s^2 T_b^2(1) [Q + (b - \hat{b})T_b(1)] \bar{K}_{lg} \ln [(\bar{\lambda}_g/\bar{\lambda}_l)/(1 - \alpha_s)]} \right\}^{1/2} \\ \times \left\{ 1 - \frac{\bar{\rho}_l^3 \bar{c}_s \bar{\lambda}_g \bar{\lambda}_l [T_b(p_g^b) - 1]^2 \bar{A}(p_g^b) g^*[\bar{l}, T_b(p_g^b)] N_u e^{-N_u/T_b(p_g^b)}}{\bar{\rho}_s^2 \bar{\lambda}_s^2 T_b^2(p_g^b) [Q + (b - \hat{b})T_b(p_g^b)] \bar{K}_{lg} \ln [(\bar{\lambda}_g/\bar{\lambda}_l)/(1 - \alpha_s)]} \right\}^{-1/2}, \quad (101)$$

where the last two factors contain the effects of temperature nonequilibrium arising from finite values of the interphase heat-transfer coefficient $\tilde{K}_{lg}$.

6. Conclusion

Previous work describing the quasi-steady deflagration of confined porous energetic materials in the single-temperature limit has been extended to allow for temperature-nonequilibrium effects associated with finite rates of interphase heat transfer. Such effects become increasingly significant as the burning rate increases owing to the inverse relationship between the nondimensional heat-transfer coefficients and the square of the propagation speed, reflecting the reduced characteristic time scale associated with higher pressure-driven flow velocities. For the present model that assumes a bubbling melt layer on the surface of the porous solid, the first effects of temperature nonequilibrium are felt in the liquid/gas reaction zone. Consequently, if the rate of interphase heat transfer between the liquid and gas phases is still relatively large, temperature nonequilibrium is confined, to a first approximation, to the reactive portion of the melt layer. As a result, the leading-order expression for the burning rate is independent of the rate of interphase heat transfer in the solid/gas portion of the preheat zone. This is true even if the rate of interphase heat transfer in the solid/gas region is significantly smaller than that in the melt layer and stems from the fact that in this parameter regime, the temperature of all co-existing phases are known to be equal to the melting temperature at the solid/liquid interface. This result is valid until either temperature nonequilibrium extends to the melting surface or the melt layer disappears. In those cases, the gas temperature at the solid surface is not necessarily equal to the melting temperature and, consequently, the solution in the liquid/gas region and that in the solid/gas preheat zone, where temperature nonequilibrium is likely to be present to a greater degree, are more tightly coupled.

Thus, in the parameter regime considered here, the problem reduces to a nontrivial eigenvalue calculation in the thin reaction region where final conversion of the liquid to gaseous products occurs. Although a closed-form solution to that problem is not generally available, it turns out that for realistically small gas-to-liquid thermal-conductivity ratios, solutions in the reaction zone take on a singular-perturbation character that can be exploited to derive an asymptotic expansion of the burning-rate eigenvalue. However, because there appear an infinite number of logarithmic terms in the subsequent asymptotic development that must be included in order to arrive at the desired level of approximation, the resulting problem requires a generalized, nonstandard approach to the usual formalism associated with the method of matched asymptotic expansions.

The physical effects of temperature nonequilibrium, which decreases the rate of heat transfer from the reacting liquid phase to the gas-phase products, and thus allows a greater amount of thermal energy to remain in the reacting phase, is to increase both the propagation speed and the sharpness of the transition to "convective" burning relative to the single-temperature limit. Indeed,

for sufficiently small values of the scaled interphase heat-transfer rate, the temperature profile of the reactive liquid phase becomes non-monotonic, rising above the final burned temperature and thus supporting a faster reaction rate, before decreasing, in a thin equilibration sublayer, to approach the gas temperature as the reaction goes to completion. The end result arising from this thermal nonequilibrium is a factor in the expression for the normalized burning rate that reflects finite values of the scaled interphase heat-transfer coefficient and collapses to unity in the single-temperature limit.

Appendix A. Asymptotic Treatment of the Single-Temperature Limit

In the limit $\bar{k} \rightarrow \infty$, Eq. (57) implies that $w = 0$ (i.e., equal temperatures), and Eqs. (56) and (58) collapse to the reduced problem

$$[(1 - \chi) + \bar{l}\chi] \frac{du}{d\chi} = \frac{\bar{q}_1}{\Lambda_0}; \quad u(\alpha_s) = 0, \quad u(1) = 1. \quad (\text{A.1})$$

The solution is expressible in closed-form, for arbitrary $\bar{l}$, as

$$u(\chi; \bar{l}) = \frac{\bar{q}_1}{\Lambda_0} \int_{\alpha_s}^{\chi} \frac{d\chi}{1 - (1 - \bar{l})\chi}, \quad (\text{A.2})$$

where, through evaluation of Eq. (A.2) at $\chi = 1$, the eigenvalue coefficient Λ_0 is determined as either

$$\Lambda_0 = -\frac{\bar{q}_1}{(1 - \bar{l})} \ln \left[\frac{\bar{l}}{1 - (1 - \bar{l})\alpha_s} \right], \quad \bar{l} \neq 1, \quad (\text{A.3})$$

or $\Lambda_0 = \bar{q}_1(1 - \alpha_s)$ for the special case $\bar{l} = 1$. Substitution of these expressions for Λ_0 into Eq. (A.2) for arbitrary χ then completely determines $u(\chi; \bar{l})$ as

$$u(\chi; \bar{l}) = \frac{\ln [1 - (1 - \bar{l})\chi] - \ln [1 - (1 - \bar{l})\alpha_s]}{\ln \bar{l} - \ln [1 - (1 - \bar{l})\alpha_s]}, \quad \bar{l} \neq 1, \quad (\text{A.4})$$

and $u(\chi) = (\chi - \alpha_s)/(1 - \alpha_s)$ for $\bar{l} = 1$.

Since the gas-to-liquid conductivity ratio $\bar{l}$ is typically small, it is of interest to consider the solution behavior in that limit. With respect to the eigenvalue coefficient Λ_0 , this may be accomplished by simply expanding the exact solution (A.3) for small $\bar{l}$, which gives

$$\Lambda_0 = (-\ln \bar{l})\mu_0 + \mu_1 + \bar{l}[(-\ln \bar{l})\mu_2 + \mu_3] + O[\bar{l}^2(-\ln \bar{l})], \quad \bar{l} \ll 1, \quad (\text{A.5})$$

where

$$\mu_0 = \bar{q}_1, \quad \mu_1 = \mu_0 \ln(1 - \alpha_s), \quad \mu_2 = \mu_0, \quad \mu_3 = \mu_0 \left[\frac{\alpha_s}{1 - \alpha_s} + \ln(1 - \alpha_s) \right]. \quad (\text{A.6})$$

On the other hand, the expansion of the right-hand side of Eq. (A.4) is clearly nonuniform, since it depends on the proximity of χ to unity. Indeed, a plot of $u(\chi; \bar{l})$ for various $\bar{l}$ is shown in

Figure A.1, illustrating the development of a boundary-layer structure for $\bar{l} \ll 1$. However, rather than proceed with a further investigation of the exact solution, which is not available for the two-temperature ($\bar{k} < \infty$) problem, it is instructive, in order to develop the methodology required to treat the more general case, to consider the formal asymptotic development of the problem (A.1) directly. In particular, it is immediately clear from Eq. (A.1) that a perturbation analysis based on $\bar{l} \ll 1$ is singular, since the coefficient of the derivative vanishes to leading order as χ approaches unity. Hence, one may anticipate the boundary layer exhibited in Figure A.1 in the neighborhood of $\chi = 1$, but the appearance of logarithms in both the expansion (A.5) and the outer [$\chi \sim O(1)$] and inner ($1 - \chi \ll 1$) expansions of the exact solution (A.4) suggests that the appropriate asymptotic matching procedure needed to fully determine an approximate solution will involve some nonstandard considerations.

Guided by the expansion of Eq. (A.4) for $\chi \sim O(1)$, we seek an outer expansion $u = u^{(o)}$ in the form

$$u^{(o)}(\chi) \sim (-\ln \bar{l})^{-1} \left[u_{0,0}(\chi) + (-\ln \bar{l})^{-1} u_{0,1}(\chi) + (-\ln \bar{l})^{-2} u_{0,2}(\chi) + \dots \right] \\ + \bar{l}(-\ln \bar{l})^{-1} \left[u_{1,0}(\chi) + (-\ln \bar{l})^{-1} u_{1,1}(\chi) + (-\ln \bar{l})^{-2} u_{1,2}(\chi) + \dots \right] + O[\bar{l}^2(-\ln \bar{l})^{-1}]. \quad (\text{A.7})$$

Substituting this expansion, along with the expansion (A.5) for Λ_0 , into the problem (A.1) and equating coefficients of like orders of magnitude with respect to $\bar{l}$, we obtain a sequence of problems for the coefficients in Eq. (A.7). In particular, at $O[(-\ln \bar{l})^{-n-1}]$ for $n = 0, 1, 2, \dots$, we obtain

$$(1 - \chi) \frac{du_{0,n}}{d\chi} = (-1)^n \frac{\bar{q}_1}{\mu_0} \left(\frac{\mu_1}{\mu_0} \right)^n; \quad u_{0,n}(\alpha_s) = 0, \quad (\text{A.8})$$

the solution of which is

$$u_{0,n}(\chi) = (-1)^{n+1} \frac{\bar{q}_1}{\mu_0} \left(\frac{\mu_1}{\mu_0} \right)^n \ln \left(\frac{1 - \chi}{1 - \alpha_s} \right). \quad (\text{A.9})$$

Defining now inner variables $u = u^{(i)}$ and $\gamma = (1 - \chi)/\bar{l}$, where the latter stretches the boundary layer in the vicinity of $\chi = 1$, the boundary-layer problem, from Eq. (A.1), becomes

$$(\gamma + 1 - \bar{l}\gamma) \frac{du^{(i)}}{d\gamma} = -\frac{\bar{q}_1}{\Lambda_0}; \quad u^{(i)}(0) = 1. \quad (\text{A.10})$$

Consistent with the expansion of the exact solution (A.4) for $\chi = 1 - \bar{l}\gamma$, the inner solution $u^{(i)}$ is sought in the form

$$u^{(i)}(\gamma) \sim 1 + (-\ln \bar{l})^{-1} \hat{u}_{0,1}(\gamma) + (-\ln \bar{l})^{-2} \hat{u}_{0,2}(\gamma) + \dots \\ + \bar{l}(-\ln \bar{l})^{-1} \left[\hat{u}_{1,0}(\gamma) + (-\ln \bar{l})^{-1} \hat{u}_{1,1}(\gamma) + (-\ln \bar{l})^{-2} \hat{u}_{1,2}(\gamma) + \dots \right] + O[\bar{l}^2(-\ln \bar{l})^{-1}], \quad (\text{A.11})$$

where the leading term satisfies the inner boundary condition at $\gamma = 0$. Substituting Eqs. (A.5) and (A.11) into the inner problem defined by Eq. (A.10), we obtain at $O[(-\ln \bar{l})^{-n}]$, for $n = 1, 2, 3, \dots$, the sequence of problems

$$(\gamma + 1) \frac{d\hat{u}_{0,n}}{d\gamma} = (-1)^n \frac{\bar{q}_1}{\mu_0} \left(\frac{\mu_1}{\mu_0} \right)^{n-1}; \quad \hat{u}_{0,n}(0) = 0, \quad (\text{A.12})$$

which yields

$$\hat{u}_{0,n} = (-1)^n \frac{\bar{q}_1}{\mu_0} \left(\frac{\mu_1}{\mu_0} \right)^{n-1} \ln(\gamma + 1). \quad (\text{A.13})$$

Before proceeding with the calculation of the next higher-order group of terms [those proportional to $\bar{l}$ times some power of $(-\ln \bar{l})$], we first investigate various matching strategies involving the terms computed thus far. In particular, we consider an (m, m) match, whereby we attempt to equate, according to Van Dyke's rule (*cf.* [9]), the m -term inner expansion (of the m -term outer solution) to the m -term outer expansion (of the m -term inner solution). Thus, for $m = 1$, we take the one-term outer solution $(-\ln \bar{l})^{-1} u_{0,0}(\chi)$, rewrite it in terms of the inner variable γ and expand the result for $\bar{l} \ll 1$, retaining one term. Denoting the result by $u^{(1,1)}$, we obtain $u^{(1,1)} = \bar{q}_1/\mu_0$. Similarly, we take the one-term inner solution, rewrite it in terms of the outer variable χ and expand the result for $\bar{l} \ll 1$, retaining one term. Denoting this result by $\hat{u}^{(1,1)}$, we have in this case, since the leading-order term in the expansion (A.11) is unity, the trivial result $\hat{u}^{(1,1)} = 1$. Consequently, imposing the matching condition $u^{(1,1)} = \hat{u}^{(1,1)}$ implies $\mu_0 = \bar{q}_1$, which is the correct result according to Eq. (A.6). In addition, a composite solution $u_c^{(1,1)}$, obtained by adding the one-term outer solution to the one-term inner solution and subtracting their common limit $u^{(1,1)}$, is given by

$$u_c^{(1,1)} = (-\ln \bar{l})^{-1} \ln \left(\frac{1 - \alpha_s}{1 - \chi} \right). \quad (\text{A.14})$$

This is the same as the leading-order outer solution, which satisfies the boundary condition at $\chi = \alpha_s$ but has an $O[(-\ln \bar{l})^{-1}]$ singularity as $\chi \rightarrow 1$.

Considering next a $(2, 2)$ match, we take the two-term outer solution $(-\ln \bar{l})^{-1} u_{0,0}(\chi) + (-\ln \bar{l})^{-2} u_{0,1}$, rewrite it in terms of the inner variable γ and expand the result for $\bar{l} \ll 1$, retaining two terms. Denoting the result by $u^{(2,2)}$, we obtain

$$u^{(2,2)} = \frac{\bar{q}_1}{\mu_0} \left[1 + \frac{\ln(1 - \alpha_s) - \mu_1/\mu_0 - \ln \gamma}{(-\ln \bar{l})} \right] = \frac{\bar{q}_1}{\mu_0 (-\ln \bar{l})} \left[\ln(1 - \alpha_s) - \frac{\mu_1}{\mu_0} - \ln(1 - \chi) \right], \quad (\text{A.15})$$

where, in the last equality, we have rewritten the result in terms of the outer variable χ . Similarly, we take the two-term inner solution, rewrite it in terms of the outer variable and expand the result for $\bar{l} \ll 1$, retaining two terms. Denoting this result by $\hat{u}^{(2,2)}$, we calculate

$$\hat{u}^{(2,2)} = 1 - \frac{\bar{q}_1}{\mu_0} \left[1 + \frac{\ln(1 - \chi)}{(-\ln \bar{l})} \right]. \quad (\text{A.16})$$

Equating $u^{(2,2)}$ and $\hat{u}^{(2,2)}$ and collecting coefficients of like orders with respect to $\bar{l}$, we obtain at $O(1)$ and $O[(-\ln \bar{l})^{-1}]$ the respective results

$$\mu_0 = \bar{q}_1, \quad \mu_1 = \mu_0 \ln(1 - \alpha_s), \quad (\text{A.17})$$

which agrees with Eqs. (A.5) – (A.6) to the calculated order. The composite solution $u_c^{(2,2)}$, obtained by adding the two-term outer and two-term inner solutions and subtracting their common limit given by either (A.15) or (A.16), is given by

$$u_c^{(2,2)} = (-\ln \bar{l})^{-1} \ln \left(\frac{1 - \alpha_s}{1 + \bar{l} - \chi} \right) + \frac{\ln(1 - \alpha_s)}{(-\ln \bar{l})^2} \ln \left(\frac{1 - \alpha_s}{1 - \chi} \right). \quad (\text{A.18})$$

This approximation has an $O[\bar{l}(-\ln \bar{l})^{-1}]$ error at $\chi = \alpha_s$, but the singularity at $\chi = 1$ is now only $O[(-\ln \bar{l})^{-2}]$, as compared with the $O[(-\ln \bar{l})^{-1}]$ singularity for the lower-order composite approximation $u_c^{(1,1)}$.

Applying the same procedure to obtain a (3,3) match, we calculate

$$u^{(3,3)} = \frac{\bar{q}_1}{\mu_0(-\ln \bar{l})} \left\{ \ln(1 - \alpha_s) - \ln(1 - \chi) - \frac{\mu_1}{\mu_0(-\ln \bar{l})} \left[\ln(1 - \alpha_s) - \frac{\mu_1}{\mu_0} - \ln(1 - \chi) \right] \right\}, \quad (\text{A.19})$$

$$\hat{u}^{(3,3)} = 1 - \frac{\bar{q}_1}{\mu_0} \left\{ 1 + \frac{\ln(1 - \chi)}{(-\ln \bar{l})} + \frac{\mu_1}{\mu_0(-\ln \bar{l})} \left[1 + \frac{\ln(1 - \chi)}{(-\ln \bar{l})} \right] \right\}. \quad (\text{A.20})$$

Equating $u^{(3,3)}$ and $\hat{u}^{(3,3)}$, we again obtain, at $O(1)$ and $O[(-\ln \bar{l})^{-1}]$, respectively, the expressions for μ_0 and μ_1 given in Eqs. (A.17), with the coefficients at $O[(-\ln \bar{l})^{-2}]$ then summing to zero. Thus, no further information is obtained regarding the coefficients in the eigenvalue expansion, but the composite expansion $u_c^{(3,3)}$, given by

$$u_c^{(3,3)} = (-\ln \bar{l})^{-1} \left[1 - \frac{\ln(1 - \alpha_s)}{(-\ln \bar{l})} \right] \ln \left(\frac{1 - \alpha_s}{1 + \bar{l} - \chi} \right) + \frac{\ln^2(1 - \alpha_s)}{(-\ln \bar{l})^3} \ln \left(\frac{1 - \alpha_s}{1 - \chi} \right), \quad (\text{A.21})$$

moves the singularity at $\chi = 1$ to $O[(-\ln \bar{l})^{-3}]$ while leaving the error at $\chi = \alpha_s$ unchanged at $O[\bar{l}(-\ln \bar{l})^{-1}]$.

Continuing with the present series of higher-order approximations, it is now clear that no further eigenvalue information will be obtained beyond the coefficients μ_0 and μ_1 , and that the m th order composite solution $u_c^{(m,m)}$ will have an $O[(-\ln \bar{l})^{-m}]$ singularity at $\chi = 1$. It is also clear that in order to calculate higher-order coefficients in the eigenvalue expansion, one must consider the next-higher-order group of terms, namely those proportional to $\bar{l}$ times various powers of $(-\ln \bar{l})$, in the expansions (A.5) and (A.7). This of course requires the retention of *all* lower-order terms, and thus we now consider the limit $m \rightarrow \infty$ in the above matching procedure. Because of the need to include all terms that differ by no more than an algebraic power of $(-\ln \bar{l})$ in the formulation of an appropriate matching procedure, it is convenient to introduce a generalized

ordering notation by using quotation marks to denote all orders that differ by no more than an algebraic power of $(-\ln \bar{l})$. Thus, for example, any term of order $\bar{l}^n(-\ln \bar{l})^{-m}$ is defined to be $O(\bar{l}^n)$, and hence only terms that differ by an algebraic power of $\bar{l}$ itself are regarded as being of different orders in applying the following generalization of Van Dyke's matching formalism.

Using the results (A.9) and (A.13), the "1"-term outer and inner solutions u_0 and $\hat{u}_0$, consisting of the sum of all terms in Eqs. (A.7) and (A.11) of $O(\bar{l})$, are given by

$$u_0 = (-\ln \bar{l})^{-1} \sum_{m=0}^{\infty} \frac{u_{0,m}}{(-\ln \bar{l})^m} = \frac{\bar{q}_1}{\mu_0(-\ln \bar{l})} \left[1 + \frac{\mu_1/\mu_0}{(-\ln \bar{l})} \right]^{-1} \ln \left(\frac{1-\alpha_s}{1-\chi} \right), \quad (\text{A.22})$$

$$\hat{u}_0 = 1 + \sum_{m=1}^{\infty} \frac{u_{0,m}}{(-\ln \bar{l})^m} = 1 - \frac{\bar{q}_1}{\mu_0(-\ln \bar{l})} \left[1 + \frac{\mu_1/\mu_0}{(-\ln \bar{l})} \right]^{-1} \ln(\gamma + 1). \quad (\text{A.23})$$

Rewriting the "1"-term outer solution u_0 in terms of the inner variable, we expand with respect to $\bar{l}$, retaining all terms that are $O(\bar{l})$. In this case, no terms in the "1"-term inner expansion $u_0^{("1,1")}$ of the "1"-term outer solution are dropped according to our generalized ordering rule, and thus the final result, rewritten in terms of the outer variable χ , is identical to the original expression (A.22) for u_0 . Similarly, we rewrite the "1"-term inner solution $\hat{u}_0$ in terms of the outer variable, expand with respect to $\bar{l}$ and retain all terms that are $O(\bar{l})$. This determines the "1" term outer expansion $\hat{u}^{("1,1")}$ of the "1"-term inner solution as

$$\hat{u}^{("1,1")} = 1 - \frac{\bar{q}_1}{\mu_0} \left[1 + \frac{\mu_1/\mu_0}{(-\ln \bar{l})} \right]^{-1} \left[1 + \frac{\ln(1-\chi)}{(-\ln \bar{l})} \right]. \quad (\text{A.24})$$

Imposing the matching condition $u^{("1,1")} = \hat{u}^{("1,1")}$ and multiplying each side of the equation by $1 + (\mu_1/\mu_0)(-\ln \bar{l})^{-1}$ to clear that factor from the denominator in Eqs. (A.23) and (A.24), we find, by collecting coefficients of like orders (now interpreted in the classical sense) with respect to $\bar{l}$, that the terms proportional to $\ln(1-\chi)$ cancel, and the remaining terms of $O(1)$ and $(-\ln \bar{l})^{-1}$ lead to the previous results (A.17) for μ_0 and μ_1 . In addition, the generalized composite solution $u_c^{("1,1")} = u_0 + \hat{u}_0 - u^{("1,1")}$, which in this case is equal to the inner solution $\hat{u}_0$ expressed in terms of χ , is given by

$$u_c^{("1,1")} = \frac{\ln(1-\alpha_s) - \ln(1+\bar{l}-\chi)}{\ln(1-\alpha_s) - \ln \bar{l}}. \quad (\text{A.25})$$

As anticipated by the removal of the previous singularity at $\chi = 1$ to higher and higher orders as the number of logarithmic terms included in the expansions increased, the limit in which all such terms are included results in the elimination of that singularity altogether. Indeed, since Eq. (A.25) is identical to the inner solution $\hat{u}_0$, the boundary condition at $\chi = 1$ is now satisfied exactly, while the error at $\chi = \alpha_s$ remains $O(\bar{l})$ or, more precisely, $O[\bar{l}(-\ln \bar{l})^{-1}]$. By way of comparison, the different composite expansions (A.14), (A.18), (A.21) and (A.25) are shown in Figure A.2.

The above generalized approach to asymptotic matching is consistent with the sometimes-expressed notion that it is best not to break an expansion between terms that differ by at most logarithmic factors (cf. [8]), and in this case, it clearly leads to an improved composite result. In the present problem, however, such an approach is also needed because it is necessary to calculate all $O(1)$ logarithmic terms if one desires to obtain $O(\bar{l})$ corrections. In order to further illustrate the formal correctness of this approach, we proceed to determine the $O(\bar{l})$ coefficients μ_2 and μ_3 in the eigenvalue expansion (A.5).

Thus, continuing with the sequence of problems obtained from the substitution of the outer expansion (A.7) into the problem (A.1), we obtain at $O[\bar{l}(-\ln \bar{l})^{-n-1}]$, for $n = 0, 1, 2, \dots$, the problems

$$(1 - \chi) \frac{du_{1,n}}{d\chi} + \chi \frac{du_{0,n}}{d\chi} = (-1)^{n+1} \frac{\bar{q}_1}{\mu_0} \left(\frac{\mu_1}{\mu_0} \right)^{n-1} \left[(n+1) \frac{\mu_1 \mu_2}{\mu_0^2} - n \frac{\mu_3}{\mu_0} \right]; \quad u_{1,n}(\alpha_s) = 0, \quad (\text{A.26})$$

where the $u_{0,n}$ were given by Eq. (A.9). Equations (A.26) have the solution

$$u_{1,n} = (-1)^n \frac{\bar{q}_1}{\mu_0} \left(\frac{\mu_1}{\mu_0} \right)^n \left\{ \frac{\chi - \alpha_s}{(1 - \chi)(1 - \alpha_s)} - \left[1 - (n+1) \frac{\mu_2}{\mu_0} + n \frac{\mu_3}{\mu_1} \right] \ln \left(\frac{1 - \alpha_s}{1 - \chi} \right) \right\}, \quad (\text{A.27})$$

from which we calculate the $O(\bar{l})$ outer solution u_1 as

$$u_1 = \sum_{n=0}^{\infty} \frac{u_{1,n}}{(-\ln \bar{l})^{n+1}} = \frac{\bar{q}_1}{\mu_0(-\ln \bar{l})} \left\{ \left[1 + \frac{\mu_1/\mu_0}{(-\ln \bar{l})} \right]^{-1} \left[\ln \left(\frac{1 - \alpha_s}{1 - \chi} \right) - \frac{\chi - \alpha_s}{(1 - \chi)(1 - \alpha_s)} \right] - \left[1 + \frac{\mu_1/\mu_0}{(-\ln \bar{l})} \right]^{-2} \left[\frac{\mu_2}{\mu_0} + \frac{\mu_3}{\mu_0(-\ln \bar{l})} \right] \ln \left(\frac{1 - \alpha_s}{1 - \chi} \right) \right\}. \quad (\text{A.28})$$

Similarly, continuing with the sequence of problems obtained from the substitution of the inner expansion (A.11) into the problem (A.10), we obtain at $O[\bar{l}(-\ln \bar{l})^{-n-1}]$, for $n = 0, 1, 2, \dots$, the problems

$$(\gamma + 1) \frac{d\hat{u}_{1,n}}{d\gamma} - \gamma \frac{d\hat{u}_{0,n+1}}{d\gamma} = (-1)^n \frac{\bar{q}_1}{\mu_0} \left(\frac{\mu_1}{\mu_0} \right)^{n-1} \left[(n+1) \frac{\mu_1 \mu_2}{\mu_0^2} - n \frac{\mu_3}{\mu_0} \right]; \quad \hat{u}_{1,n}(0) = 0, \quad (\text{A.29})$$

where the $\hat{u}_{0,n+1}$ are determined from Eq. (A.13) as $\hat{u}_{0,n+1} = (-1)^{n+1} (\bar{q}_1/\mu_0) (\mu_1/\mu_0)^n \ln(\gamma + 1)$. Equations (A.29) have the solution

$$\hat{u}_{1,n} = (-1)^n \frac{\bar{q}_1}{\mu_0} \left(\frac{\mu_1}{\mu_0} \right)^n \left\{ \frac{\gamma}{\gamma + 1} - \left[1 - (n+1) \frac{\mu_2}{\mu_0} + n \frac{\mu_3}{\mu_1} \right] \ln(\gamma + 1) \right\}, \quad (\text{A.30})$$

from which we calculate the $O(\bar{l})$ inner solution $\hat{u}_1$ as

$$\hat{u}_1 = \sum_{n=0}^{\infty} \frac{\hat{u}_{1,n}}{(-\ln \bar{l})^{n+1}} = \frac{\bar{q}_1}{\mu_0(-\ln \bar{l})} \left\{ \left[1 + \frac{\mu_1/\mu_0}{(-\ln \bar{l})} \right]^{-1} \left[\frac{\gamma}{\gamma + 1} - \ln(\gamma + 1) \right] + \left[1 + \frac{\mu_1/\mu_0}{(-\ln \bar{l})} \right]^{-2} \left[\frac{\mu_2}{\mu_0} + \frac{\mu_3}{\mu_0(-\ln \bar{l})} \right] \ln(\gamma + 1) \right\}. \quad (\text{A.31})$$

We now apply Van Dyke's matching algorithm using our generalized concept of ordering. In particular, we perform a ("2,2") matching as follows. First, we take the "2" term inner expansion $\hat{u}_0(\gamma; \bar{l}) + \bar{l}\hat{u}_1(\gamma; \bar{l})$, rewrite it in terms of the outer variable χ and expand the result for small $\bar{l}$, retaining "2" terms. Denoting this "2"-term outer expansion (of the "2"-term inner solution) by $\hat{u}^{("2,2")}$, we obtain

$$\begin{aligned} \left[1 + \frac{\mu_1/\mu_0}{(-\ln \bar{l})}\right]^2 \hat{u}^{("2,2")} = & \\ & 1 - \frac{\bar{q}_1}{\mu_0} + \frac{1}{(-\ln \bar{l})} \left[\frac{\mu_1}{\mu_0} \left(2 - \frac{\bar{q}_1}{\mu_0}\right) - \frac{\bar{q}_1}{\mu_0} \ln(1 - \chi) \right] + \frac{\mu_1/\mu_0}{(-\ln \bar{l})^2} \left[\frac{\mu_1}{\mu_0} - \frac{\bar{q}_1}{\mu_0} \ln(1 - \chi) \right] \\ & + \bar{l} \left(\frac{\bar{q}_1}{\mu_0} \right) \left(\frac{\mu_2}{\mu_0} - 1 \right) + \frac{\bar{l}}{(-\ln \bar{l})} \left(\frac{\bar{q}_1}{\mu_0} \right) \left[1 - \frac{\mu_1}{\mu_0} + \frac{\mu_3}{\mu_0} - \frac{1}{1 - \chi} + \left(\frac{\mu_2}{\mu_0} - 1 \right) \ln(1 - \chi) \right] \\ & + \frac{\bar{l}}{(-\ln \bar{l})^2} \left(\frac{\bar{q}_1}{\mu_0} \right) \left[\frac{\mu_1}{\mu_0} \left(1 - \frac{1}{1 - \chi}\right) + \left(\frac{\mu_3}{\mu_0} - \frac{\mu_1}{\mu_0} \right) \ln(1 - \chi) \right]. \end{aligned} \quad (\text{A.32})$$

Similarly, we take the "2" term outer expansion $u_0(\chi; \bar{l}) + \bar{l}u_1(\chi; \bar{l})$, rewrite it in terms of the inner variable γ and expand the result for small $\bar{l}$, retaining "2" terms. Denoting this "2"-term inner expansion (of the "2"-term outer solution) by $u^{("2,2")}$, we obtain

$$\begin{aligned} \left[1 + \frac{\mu_1/\mu_0}{(-\ln \bar{l})}\right]^2 u^{("2,2")} = & \\ & \frac{\bar{q}_1}{\mu_0} \left\{ 1 + \frac{1}{(-\ln \bar{l})} \left[\frac{\mu_1}{\mu_0} + \ln(1 - \alpha_s) - \ln \gamma - \frac{1}{\gamma} \right] + \frac{\mu_1/\mu_0}{(-\ln \bar{l})^2} \left[\ln(1 - \alpha_s) - \ln \gamma - \frac{1}{\gamma} \right] \right. \\ & + \bar{l} \left(1 - \frac{\mu_2}{\mu_0} \right) + \frac{\bar{l}}{(-\ln \bar{l})} \left[\frac{\mu_1}{\mu_0} - \frac{\mu_3}{\mu_0} + \frac{1}{1 - \alpha_s} - \left(\frac{\mu_2}{\mu_0} - 1 \right) \left[\ln(1 - \alpha_s) - \ln \gamma \right] \right] \\ & \left. + \frac{\bar{l}}{(-\ln \bar{l})^2} \left[\frac{\mu_1/\mu_0}{1 - \alpha_s} + \left(\frac{\mu_1}{\mu_0} - \frac{\mu_3}{\mu_0} \right) \left[\ln(1 - \alpha_s) - \ln \gamma \right] \right] \right\} \\ = & \frac{\bar{q}_1}{\mu_0} \left\{ \frac{1}{(-\ln \bar{l})} \ln \left(\frac{1 - \alpha_s}{1 - \chi} \right) + \frac{\mu_1/\mu_0}{(-\ln \bar{l})^2} \ln \left(\frac{1 - \alpha_s}{1 - \chi} \right) \right. \\ & + \frac{\bar{l}}{(-\ln \bar{l})} \left[\frac{1}{1 - \alpha_s} - \frac{1}{1 - \chi} - \left(\frac{\mu_2}{\mu_0} - 1 \right) \ln \left(\frac{1 - \alpha_s}{1 - \chi} \right) \right] \\ & \left. + \frac{\bar{l}}{(-\ln \bar{l})^2} \left[\frac{\mu_1}{\mu_0} \left(\frac{1}{1 - \alpha_s} - \frac{1}{1 - \chi} \right) + \left(\frac{\mu_1}{\mu_0} - \frac{\mu_3}{\mu_0} \right) \ln \left(\frac{1 - \alpha_s}{1 - \chi} \right) \right] \right\}, \end{aligned} \quad (\text{A.33})$$

where, in the last equality, the result has been rewritten in terms of the outer coordinate.

We now impose the matching condition $\hat{u}^{("2,2")} = u^{("2,2")}$, which implies that the right-hand sides of Eqs. (A.32) and (A.33) be equal. In order for this to be valid for arbitrarily

small $\bar{l}$, this equality must hold at each order with respect to $\bar{l}$, including logarithmic orders. In particular, equality must hold separately for terms of order 1, $(-\ln \bar{l})^{-1}$, $(-\ln \bar{l})^{-2}$, $\bar{l}$, $\bar{l}(-\ln \bar{l})^{-1}$ and $\bar{l}(-\ln \bar{l})^{-2}$. The first two of these orders reproduce the result (A.6) for μ_0 and μ_1 , in which case the third simply gives an identity. The fourth and fifth orders then give the correct result (A.6) for μ_2 and μ_3 , with the last order again collapsing to an identity. Using these results, the composite solution $u_c^{(2,2)} = u_0 + \bar{l}u_1 + \hat{u}_0 + \bar{l}\hat{u}_1 - u^{(2,2)}$, with all terms expressed in terms of the outer variable, is given by

$$u_c^{(2,2)} = \frac{\ln(1 - \alpha_s) - \ln(1 + \bar{l} - \chi) + \bar{l}(1 - \chi)(1 + \bar{l} - \chi)^{-1}}{\ln(1 - \alpha_s) - \ln \bar{l}} + \bar{l} \frac{\ln(1 + \bar{l} - \chi) - \ln \bar{l}}{[\ln(1 - \alpha_s) - \ln \bar{l}]^2}. \quad (\text{A.34})$$

Comparing Eq. (A.34) with the lower-order approximation (A.25), we observe that the boundary condition at $\chi = 1$ remains identically satisfied, while the condition at $\chi = \alpha_s$ is now satisfied to within $O[\bar{l}^2(-\ln \bar{l})^{-1}]$. A comparison of $u_c^{(1,1)}$, $u_c^{(2,2)}$ and the exact solution u for a characteristically small value of $\bar{l}$ is shown in Figure A.3. Finally, Figure A.4 shows a comparison of the eigenvalue approximations $\Lambda_0 \approx \lambda_0 = (-\ln \bar{l})\mu_0 + \mu_1$ and $\Lambda_0 \approx \lambda_0 + \bar{l}\lambda_1$, where $\lambda_1 = [(-\ln \bar{l})\mu_2 + \mu_3]$, to the exact value Λ_0 given by Eq. (A.3). Both approximations provide good agreement even as $\bar{l}$ approaches $O(1)$ values.

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Figure Captions

Fig. 1. Physical sketch of the model. The two-phase-flow nature of the problem is enhanced by the effects of confinement, which leads to a pressure-driven permeation of the burned gas into the pores of the unburned solid.

Fig. 2. Profile of the normalized temperature-difference approximation $w_0^* = w_0 \bar{k} / [\bar{q}_1 (\bar{q}_2 - \bar{q}_1)]$ for $\alpha_s = 0.2$.

Fig. 3. "Two"-term approximation of the normalized eigenvalue $\Lambda_0 / \bar{q}_1$ for $\alpha_s = 0.2$ and $\bar{q}_2 / \bar{q}_1 = 1.75$. The curves correspond to $\bar{k} / \bar{q}_1^2 = \infty$ (—), 0.2 (— · —), 0.1 (---), 0.075 (— · · —) and 0.05 (— · · · —).

Fig. 4. "Two"-term composite approximation $u_c^{(2,2)}$ of the temperature profile $u(\chi)$ for $\alpha_s = 0.2$, $q_2/q_1 = 1.75$ and $\bar{l} = 0.1$. The curves correspond to $\bar{k} / \bar{q}_1^2 = \infty$ (—), 0.75 (---), 0.5 (— · —) and 0.25 (— · · —). The exact solution for $\bar{k} = \infty$ is represented by the solid curve. As described in the Appendix, the approximations at this order of matching satisfy the boundary condition at $\chi = 1$ exactly, and possess an $O[\bar{l}^2(-\ln \bar{l})^{-1}]$ error at $\chi = \alpha_s$.

Fig. 5. Plot of the burning-rate factor U_n as a function of overpressure for $\bar{k} / \bar{q}_1^2 = \infty$ (—), 0.5 (— · —), 0.2 (---), 0.1 (— · —) and 0.005 (— · · —). Parameter values used were $\alpha_s = 0.3$, $\gamma = 1.4$, $b = \hat{b} = 1.0$, $\hat{r} = 0.008$, $Q = 6.0$, $\gamma_s = -0.5$, $N_u = 175$ and $\bar{l} = 0.1$.

Fig. A.1. Exact solution $u(\chi)$ for $\bar{l} = 0.00001$ (—), 0.001 (— · —), 0.01 (---), 0.1 (— · · —) and 0.5 (— · · · —). The profiles have a boundary-layer structure for $\bar{l} \ll 1$.

Fig. A.2. The composite approximations $u_c^{(1,1)}$ (— · —), $u_c^{(2,2)}$ (---), $u_c^{(3,3)}$ (— · · —), $u_c^{(1,1)}$ (— · · · —) and the exact solution $u(\chi)$ (—) for $\bar{l} = 0.1$. (a) The approximations $u_c^{(1,1)}$, $u_c^{(2,2)}$ and $u_c^{(3,3)}$ are singular, and hence nonuniform, as $\chi \rightarrow 1$. (b) The approximation $u_c^{(1,1)}$ satisfies the boundary condition at $\chi = 1$ and represents a uniform approximation to the exact solution.

Fig. A.3. Comparison of the uniform approximations $u_c^{(1,1)}$ (— · —), $u_c^{(2,2)}$ (---) and the exact solution $u(\chi)$ (—) for $\bar{l} = 0.1$.

Fig. A.4. Comparison of the eigenvalue approximations $\Lambda_0 \approx \lambda_0$ (— · —) and $\Lambda_0 \approx \lambda_0 + \bar{l}\lambda_1$ (---) with the exact expression (—) given by Eq. (A.3). The second approximation offers excellent agreement even for only modestly small values of $\bar{l}$.

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Figure A.2b

Figure A.3. Comparison of the uniform approximations $u_c^{(1,1'')}$ (— —), $u_c^{(2,2'')}$ (- - -) and the exact solution $u(\chi)$ (—) for $\bar{l} = 0.1$.

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