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ARRAYS

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OPERATIONAL EQUATIONS FOR DATA IN COMMON ARRAYS

by

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ABSTRACT

A new method for interpolating experimental data by means of the shifting operator was introduced in 1985. This report illustrates new interpolating equations for data in the five-point rectangle and diamond configurations, new measures of central tendency, and new equations for data at the vertices of a cube.

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INTRODUCTION

By means of the shifting operator, $\exp(x)F(x) = F(x+h)$, a variety of new equations for data in the four- and five-point rectangular array can be prepared [1,2]. Analogous equations for data in the eight- and nine-point cubical array are also possible [3]. The equations are divided into four categories:

- (1) polynomial equations that contain linear and quadratic terms
- (2) equations that use circular and hyperbolic sines and cosines
- (3) equations that use exponential functions like p^x
- (4) equations that use powers of linear expressions

Operational equations that do not generate ridges or troughs within the interpolated space ordinarily yield better approximations to known surfaces than the linear equations that have traditionally been used for interpolations. The criterion for "better approximations" is a lower sum of squares of deviations between typical test surfaces and the modeling surfaces. The equations can be adapted to other configurations composed of a like number of points. Most of the equations in this report are new. The letter R on the left-hand sides of the equations represent interpolated values, often denoted "responses." The -1...1 coordinate system used is for all equations.

THE 8-POINT CUBE INTERPOLATED BY A SUM OF EXPONENTIALS

A diagram of the cubical arrangement of data is shown in Fig. 1 of Ref. 3.

Measurements, denoted by letters in alphabetical order from near lower left to far upper right, are made at the vertices of the cube. Data in cubical array have traditionally been represented by the standard, trilinear equation [3]. An operational equation for the cube that is based on a sum of exponential terms is Eq. 1:

$$R = T + MJ^{(x+1)} + NK^{(y+1)} + PL^{(z+1)} + QJ^{(x+1)}K^{(y+1)} + SJ^{(x+1)}L^{(z+1)} + UK^{(y+1)}L^{(z+1)} + VJ^{(x+1)}K^{(y+1)}L^{(z+1)} \quad (1)$$

In Eq. 1:

$$J = [(B-I)(G-D)/((A-H)(F-C))]^{(1/4)} \quad (2)$$

$$K = [(C-I)(H-D)/((A-G)(F-B))]^{(1/4)} \quad (3)$$

$$L = [(F-I)(H-G)/((A-D)(C-B))]^{(1/4)} \quad (4)$$

$$D1 = (J^2-1)(K^2-1)(L^2-1) \quad (5)$$

$$T = (L^2(K^2(AJ^2-B)-CJ^2+D)+K^2(G-FJ^2)+HJ^2-I)/D1 \quad (6)$$

$$M = (L^2(K^2(A-B)-C+D)+K^2(G-F)+H-I)/(-D1) \quad (7)$$

$$N = (L^2(J^2(A-C)-B+D)+J^2(H-F)+G-I)/(-D1) \quad (8)$$

$$P = (K^2(J^2(A-F)-B+G)+J^2(H-C)+D-I)/(-D1) \quad (9)$$

$$Q = (L^2(A-B-C+D)-F+G+H-I)/D1 \quad (10)$$

$$S = (K^2(A-B-F+G)-C+D+H-I)/D1 \quad (11)$$

$$U = (J^2(A-C-F+H)-B+D+G-I)/D1 \quad (12)$$

$$V = (A-B-C+D-F+G+H-I)/(-D1) \quad (13)$$

Example 1

Let Eq. 1 be substituted with Eqs. (2-13), and let the responses at the cube vertices A-D and F-I be generated by the function $2^{(5+x/2+y+5z/2)}$. In other words, the data are the geometrically progressing numbers 2, 4, 8, 16, and 64, 128, 256, 512, as A, B, C, D, F, G, H, I, respectively. After the numeric substitutions, Eq. 1 reduces to Eq. 14.

$$R = 2[\text{SQRT}(2)]^{(x+1)}(2)^{(y+1)}[\text{SQRT}(32)]^{(z+1)} \quad (14)$$

Eq. 14 replicates the data and represents them as an equation containing three exponential terms. It predicts the center point response (E) of the cube to be 32. This is the correct response using the generating function. By contrast, the standard trilinear equation predicts the center point response to be 123.75.

In this case, as well as in many other cases, operational Eq. 1 yields better predictions than the standard, trilinear equation. The difference between the two equations can be stated differently: the trilinear equation is exact on data in arithmetic progression, whereas Eq. 1 is exact on data in geometric progression.

Example 2

The following data are assigned to the vertices of a cube: 1, $1/2$, $1/3$, $1/4$, and $1/6$, $1/7$, $1/8$, and $1/9$ as A-D, and F-I, respectively. These numbers represent the reciprocals of the first nine integers, omitting the center point $E=1/5$. The operational, polynomial equation for the eight-point cube, Eq. 10 in Ref. 3, yields 0.2028 as the response at center point E, whereas Eq. 1 yields 0.2006 as the same response. The standard trilinear equation (Eq. 1 in Ref. 3) yields 0.3286, whereas the true value is $E = 1/5 = 0.2000$. Eq. 1 gives the best estimate for the true response at E. Both operational equations are superior to the trilinear equation in this case.

Although Eq. 1 is tedious to put into a computer program, the effort is often rewarded by increased accuracy compared to the standard trilinear equation. This is particularly true for data that are monotonic in the directions of the coordinate axes. A program for Eq. 1 can be checked by applying it to 1,4,9,16 as A,B,C,D, and 36,49,64,81

as F,G,H,I, respectively. These data yield $J=1.1055$, $K=1.2247$, and $L=1.7321$, approximately. If the equation is substituted with the cited data and then expanded, Eq. 15 results.

$$R = 32.5 - 49.75J^x - 48.99K^y - 43.30L^z + 48.74J^xK^y + 43.08J^xL^z + 42.43K^yL^z \quad (15)$$

The foregoing data represent the squares of the integer sequences 1...4, and 6...9, respectively. Eq. 15 predicts $E=24.71$, while the true value for the series of squared integers is $E=25$. If complex numbers are generated by Eq. 1, interpolate with their real parts or use another method.

CUBES REPRESENTED BY POWERS OF LINEAR EXPRESSIONS

The method of roots for the four-point rectangle has been illustrated elsewhere [4]. The method can be extended to cubes by means of Eq. 16.

$$A^{(1/n)} + D^{(1/n)} + G^{(1/n)} + H^{(1/n)} - B^{(1/n)} - C^{(1/n)} - F^{(1/n)} - I^{(1/n)} = 0 \quad (16)$$

If Eq. 16 has a root, use it to find the n th roots of the data. Represent these n th roots by the standard, trilinear equation. The interpolating equation for the cube is the n th power

of the substituted trilinear equation. For example, if the data are the squares of the first nine integers, the value of the exponent n is found to be 1 or 2. For $n=2$ the interpolating equation is:

$$R = (5 + 0.5x + y + 2.5z)^2 \quad (17)$$

This method for the cube parallels the technique illustrated in Ref. 4 for the four-point rectangle, but its applications are limited because Eq. 16 does not always have a real root or the case is ambiguous. Eq. 16 can be applied with a translating term and an assumed value of the exponent n , as illustrated in Ref. 4 for the four-point rectangle.

CURVATURE AND NET CURVATURE

Traditionally, curvature in the four-point rectangle has been detected by the difference between the datum at the center point and the average of the four corner points. The estimator is the value of the difference $E - (A+C+G+I)/4$. For the cube, curvature can be detected similarly: the value of the difference between and the average of the eight vertex data and the measurement at the center point:

$$E - (A+B+C+D+F+G+H+I)/8.$$

This scale of curvature is easy to use with operational, nine-point polynomial

equations for the cube. The sum of the "corrected" quadratic coefficients [5] is also $E - (A+B+C+D+F+G+H+I)/8$. If this difference is zero, the data can be regarded as displaying no curvature. However, zero can be formed in many ways. Each way represents a different interpretation of the data. An example of this observation is provided in the data given by Barrentine [6].

Example 3

Barrentine [6] assures us that Eq. 18, containing no quadratic coefficients, represents the numbers 74, 102, 102, 110, 80, 112, 104, 116 at vertices A...I of the eight-point cube, respectively. The center point, E, is 100.

$$R = 100 + 10x + 8y + 3z - 5xy + xz - yz \quad (18)$$

Is it possible that Barrentine's data could have been generated by a function with nonzero quadratic coefficients?

A set of "corrected" quadratic coefficients suitable for a nine-point polynomial for the cube is given as x^2 , y^2 , and z^2 in Eqs. 19-21. These three expressions are the coefficients of x^2 , y^2 , and z^2 , respectively, in a polynomial equation that represents the

cube of measurements. They are added to equations like Eq. 18 to render the polynomial equation exact at the cube center and vertex points while including quadratic terms [5]. The sum $(x_{2n}+y_{2n}+z_{2n})$ is $(A+B+C+D+F+G+H+I)/8-E$.

$$x_{2n} = (1/3)[2(x^2) - E - x_c - y_c - z_c - xyc - xzc - yzc - xyzc - y^2 - z^2 + I] \quad (19)$$

$$y_{2n} = (1/3)[2(y^2) - E - x_c - y_c - z_c - xyc - xzc - yzc - xyzc - x^2 - z^2 + I] \quad (20)$$

$$z_{2n} = (1/3)[2(z^2) - E - x_c - y_c - z_c - xyc - xzc - yzc - xyzc - x^2 - y^2 + I] \quad (21)$$

The terms x_c , y_c , z_c , xyc , xzc , yzc , $xyzc$, x^2 , y^2 , and z^2 are defined in Ref. 3 by Eqs. 2-13 therein. When Barrentine's data are substituted into Eqs. 19-21, the coefficients of x^2 , y^2 , and z^2 are 0.8, 0.8, and -1.6, respectively. The sum of these coefficients is zero, so $(A+B+C+D+F+G+H+I)/8-E$ is also zero. This implies that there is no net curvature in the experimental space defined by Barrentine's data. However, the nonzero quadratic coefficients generate curvature along the edges of the cube. In other words, the absence of net curvature does not guarantee the absence of all curvature.

EQUATIONS FOR THE FIVE-POINT RECTANGLE

The five-point rectangle is represented by data measured at rectangle points A,C,E,G,I, as illustrated in Fig. 1.

G	H	I
D	E	F
A	B	C

Fig. 1. Arrangement of measurements for the
five-point rectangle and diamond configurations.

Three operational equations for the five-point rectangle have been illustrated elsewhere [7]. The equations are exact on bilinear data in rectangular array as well as on the squares of the bilinear data. Other operational equations for the five-point rectangle are known. All of the equations have the same coefficients for the constant, linear, and cross-product terms.

$$R = E + (XC)x + (YC)y + (XYC)xy + (X2)x^2 + (Y2)y^2 \quad (22)$$

$$XC = (I-A+C-G)/4 \quad (23)$$

$$YC = (I-A-C+G)/4 \quad (24)$$

$$XYC = (A-C-G+I)/4 \quad (25)$$

$$X2 = (I-A+C-G)(EI-EG-EA+EC+AG-IC) / [4(I-A)(G-C)] \quad (26a)$$

$$Y2 = (-1)(I-A-C+G)(EI+EG-EA-EC+AC-IG) / [4(I-A)(G-C)] \quad (26b)$$

$$X2 = (A+3G+I-C-4E)/8 + (IC-CG-IG+G^2) / [4(I-A+C-G)] \\ + (C^2-IC-CG+IG) / [4(A-I+C-G)] \quad (27a)$$

$$Y2 = (A+3C+I-G-4E)/8 - (IC-CG-IG+G^2) / [4(I-A+C-G)] \\ - (C^2-IC-CG+IG) / [4(A-I+C-G)] \quad (27b)$$

Coefficients (26a, 26b) and (27a, 27b) are used as pairs. The latter pair is preferred to the same but more complicated versions given as Eqs. 10 and 11 in Ref. 7. The standard equation for the five-point rectangle has equal quadratic coefficients: $(A+C+G+I-4E)/8$.

Example 4

The bilinear data (1,3,5,7,9) substituted as (A,C,E,G,I) into Eq. 22 yield $R = (5+x+3y)$ for both sets of the quadratic coefficients. The data when changed to the squares of the cited integers yield $R = (5+x+3y)^2$ using the same sets of quadratic coefficients. The two operational equations are exact for these cases. If the data are taken as the sines of (10°, 30°, 50°, 70°, 90°) as (A,C,E,G,I), respectively, the two equations yield:

$$R = 0.7660 + 0.0966x + 0.3165y - 0.0665xy - 0.0108x^2 - 0.1019y^2 \quad (28)$$

$$R = 0.7660 + 0.0966x + 0.3165y - 0.0665xy - 0.0070x^2 - 0.1057y^2 \quad (29)$$

The sums of the squares of the deviations of Eqs. 28 and 29 from the surface defined by the function $\sin((50+10x+30y)^0)$ are 0.000305 and 0.000315, respectively. The like sum for the standard equation for the 5-point rectangle is 0.00179.

All three modeling surfaces are "close" to the test surface, but the two surfaces defined by the operational equations are closer than the surface defined by the standard equation. Using symbols, the sum of the quadratic coefficients for all three equations is $(I+A+C+G-4E)/4$, but the quadratic coefficients in the operational equations are not equal like they are in the standard equation for the five-point rectangle.

Equation 30 is another operational equation for the five-point rectangle.

$$R = T + PJ^{(x+1)} + QK^{(y+1)} + SJ^{(x+1)}K^{(y+1)} \quad (30)$$

$$D1 = (1-J^2)(1-K^2) \quad (31)$$

$$T = (AJ^2K^2 + I - K^2C - GJ^2) / D1 \quad (32)$$

$$P = (-1)(I - G + AK^2 - CK^2) / D1 \quad (33)$$

$$Q = (-1)(I + AJ^2 - GJ^2 - C) / D1 \quad (34)$$

$$S = (I + A - C - G) / D1 \quad (35)$$

$$J^2 = [(I-E)(C-E)] / [(E-A)(E-G)] \quad (36)$$

$$K^2 = [(G-E)(I-E)] / [(E-A)(E-C)] \quad (37)$$

Eq. 30 is restricted to data in which center point E lies near the median of the corner point data. If $A < C < G < I$, then E should lie between C and G. This is the usual case for monotonic surfaces such as appear in tabular data and in many experimental designs. The method does not apply if J^2 or K^2 is negative.

Example 5

Assume the data are (1,9,25,49,81) as (A,C,E,G,I), respectively. Eq. 30 yields

$$R = -15.32 - (2.88)(1.247)^{(x+1)} + (1.92)(1.871)^{(y+1)} + (17.28)(1.247)^{(x+1)}(1.871)^{(y+1)} \quad (38)$$

Eq. 38 reproduces all of the original data. Now suppose that the center point measurement is changed to 48. The equation representing the new data is Eq. 39. It reproduces all of the new numbers.

$$R = 48.650 + (1.230)(5.233)^{(x+1)} - (47.95)(0.134)^{(y+1)} - (0.926)(5.233)^{(x+1)}(0.134)^{(y+1)} \quad (39)$$

If the center point measurement lies outside the range of applicability of Eq. 30, use the method of quadrants or select another five-point equation [4]. Eq. 30 applies

primarily to data of one sign. Appendix I contains more discussion of this subject.

Another method for representing five data in rectangular array is found by solving Eq. 40 for the exponent n .

$$(A)^{(1/n)} + (C)^{(1/n)} - 4(E)^{(1/n)} + (G)^{(1/n)} + (I)^{(1/n)} = 0 \quad (40)$$

If Eq. 40 has a solution for the exponent n , use it to find the n th roots of all of the original data. Substitute these n th roots into the standard, bilinear equation for the four-point rectangle, and raise the expression to the n th power. The resulting expression is the interpolating equation for the five-point rectangle.

Example 6

Let the data be (1,9,26,49,81) for (A,C,E,G,I), respectively. The interpolating equation is

$$R = (5.899 + 1.234x + 3.744y + 0.0785xy)^{1.8357} \quad (41)$$

The measurements can also be represented by arbitrarily assigning a reasonable value (like $n=3$) to the exponent in Eq. 42 and solving Eq. 42 for the translating term, t .

$$(A+t)^{(1/n)} + (C+t)^{(1/n)} - 4(E+t)^{(1/n)} + (G+t)^{(1/n)} + (I+t)^{(1/n)} = 0 \quad (42)$$

The terms $(A+t)^{(1/n)}$, $(C+t)^{(1/n)}$, $(G+t)^{(1/n)}$, and $(I+t)^{(1/n)}$ are then substituted into the bilinear equation. The resulting expression is raised to the n th power, and t is subtracted from it. Using $n=3$, the interpolating equation for the data becomes Eq. 43.

$$R = (3.235 + 0.278x + 0.919y + 0.0304xy)^3 - 7.846 \quad (43)$$

Analogous processes apply to the nine-point cube, where the measurement at the center point of the cube counts as one of the data. In this case, Eq. 44 replaces Eq. 40, but the treatment of the data is similar, and the trilinear equation is used instead of the bilinear equation.

$$(A)^{(1/n)} + (B)^{(1/n)} + (C)^{(1/n)} + D^{(1/n)} - 8(E)^{(1/n)} + (F)^{(1/n)} + (G)^{(1/n)} + (H)^{(1/n)} + (I)^{(1/n)} = 0 \quad (44)$$

Eqs. 40 and 44 apply where the measurement at center point E is near the median of the corner point measurements. Eq. 44 can also be used with a translating term, t .

The eight-point cube represented by sines and cosines has been illustrated elsewhere [3]. The expression for the response at the faces of the cube, Expression 23 in Ref. 3, contains the letter E that represents the response at the center point of the cube. However, this eight-point algorithm is independent of the center point response. The reason is that E cancels from the numerator and denominator of the ratios used to find p , q , and r , the arguments of the sine and cosine functions. Once the algorithm has been reduced to a computer program, the letter E can be removed from the program. It is easy to change this technique into a method for the nine-point cube so that it reproduces the center point measurement as well as all of the corner point measurements [8].

Two different methods for representing the five-point rectangle by sines and cosines have been illustrated in Ref. 7. The rectangle is rotated about its center point E and formulas for side points B or D (Eqs. 17 and 18 in Ref. 7) are repeatedly applied to determine the responses at the remaining side points H and F. The absolute values of B,D,F,H are used in this method for the five-point rectangle, so the algorithm is limited to positive data. The calculated values for B,D,F,H are used as illustrated in Ref. 3 for the four-point rectangle. Eqs. 15 and 16 in Ref. 7 can also be used for the five-point rectangle, but they contain an ambiguity of signs, so these equations yield two interpolating equations for the five-point rectangle. The equation with coefficients that are similar to the coefficients generated by the cited Eqs. 17 and 18 is usually preferred.

As an illustration of an expression for the five-point rectangle, let $A=1$, $C=9$, $E=25$, $G=49$, $I=81$. The expression $R = 25\cosh(Px)\cosh(Qy) + 31.52\cosh(Px)\sinh(Qy) + 31.52\sinh(Px)\cosh(Qy) + 27.82\sinh(Px)\sinh(Qy)$ represents these data. In this expression, $P=0.2306$ and $Q=0.8828$, nearly.

DIAMOND CONFIGURATIONS

Interpolating equations for the four- and five-point diamond configurations can be prepared from the equations for four- and five-point rectangles. The four- and five-point diamond configurations are defined by points B,D,F,H and points B,D,E,F,H in Fig. 1, respectively. One way to obtain equations for the diamond is to assign the numerical values of B,D,E,F,H to A,C,E,G,I in the equations for the five-point rectangle. Then change x into $(y-x)$ and y into $(y+x)$ in the substituted equations. These substitutions preserve the numeric order of data $A<C<E<G<I$ so that the diamond configuration has $B<D<E<F<H$. To preserve the alphabetic order, change A to B , C to F , I to H , G to D , x to $(y+x)$, and y to $(y-x)$, and then make the numeric substitutions. The substitutions necessary to transform an equation for a rectangle into an equation for a diamond are easier to understand with the aid of a diagram. Rotate the rectangle $ACEGI$ 45 degrees counterclockwise, rename the vertices, and then make the changes to x and y . Other substitution routines can also be used the effect the change.

Example 7

The foregoing process applied to the operational, polynomial equation for the four-point rectangle yields an equation like Eq. 22. The coefficients in the equation are defined by Eqs. 45-50. This equation is appropriate for diamond BDFH [9].

$$E = (D+F)/2 + (BD-FD-BF+F^2) / [2(B+D-H-F)] - (BD+FD-BF-D^2) / [2(B-D-H+F)] \quad (45)$$

$$XC = (F-D) / 2 \quad (46)$$

$$YC = (H-B) / 2 \quad (47)$$

$$XYC = [(D-F)(B-H)(H-D+B-F)] / [(B+D-H-F)(B-D-H+F)] \quad (48)$$

$$X2 = (D-F)^2(H-D+B-F) / [2(B+D-H-F)(B-D-H+F)] \quad (49)$$

$$Y2 = (B-H)^2(H-D+B-F) / [2(B+D-H-F)(B-D-H+F)] \quad (50)$$

Sometimes it is easier to make the transformation from the four- or five-point rectangle to the four- or five-diamond by the coordinate substitutions described above. This avoids the need to generate literal expressions such as Eqs. 45-50. If the equation for the four-point rectangle is substituted with (1,3,7,9) as (A,C,G,I), while the equation for the four-point diamond is substituted with (2,4,6,8) as (B,D,F,H), the same interpolating equation results: $R=(5+x+3y)$. Identical expressions also result if the squares of the cited data are entered into the two equations: $R=(5+x+3y)^2$.

The equation for the diamond configuration that is based on Eqs. 45-50 permits the estimation of the interaction and quadratic coefficients even though the data comprise a figure that has only two points in each of the vertical and horizontal directions. This remark applies to most data that are monotonic in both the horizontal and vertical directions. If the data are (2,25,27,8) as (B,D,F,H), respectively, a ridge is suggested along segment DF. Such data do not present a strictly monotonic surface, so four-point operational equations do not work as well for them.

The method of substitutions can be applied to the five-point polynomial equation with quadratic coefficients given by Eqs. 27a and 27b. This changes the application of the polynomial from rectangle ACEGI to diamond BDEFH. The new equation is

$$R = E + (F-D)x/2 + (H-B)y/2 + (H-B)(F-D)(F-H-B+D)xy / [(F-B-D+H)(B-H+F-D)] \\ + (F-2E+D)x^2/2 + (H-2E+B)y^2/2 \quad (51)$$

Eq. 51 is exact on bilinear data and on the squares of bilinear data. It estimates the interaction and quadratic terms even though the data fall on two perpendicular straight lines.

Sometimes measurements follow exponential laws. In this case, an operational

equation can be obtained for the diamond configuration by appeal to the shifting operator.

Start with Eqs. 52-55, and then convert the sines and cosines to their Euler forms.

$$2\cos(x) + 2\cos(y) = J + 1/J + K + 1/K \quad (52)$$

$$2\cos(x) - 2\cos(y) = J + 1/J - K - 1/K \quad (53)$$

$$(2I)\sin(x) + (2I)\sin(y) = J - 1/J + K - 1/K \quad (54)$$

$$(2I)\sin(x) - (2I)\sin(y) = J - 1/J - K + 1/K \quad (55)$$

In Eqs. 52-55, $\cos(x)$ is rewritten as $[\exp(Ix) + \exp(-Ix)]/2$ in the customary manner, and $\sin(x)$ is also given its Euler interpretation. I represents $\text{SQRT}(-1)$. Multiply the left-hand sides of Eqs. 52-55 by unity as expressed by the ratio $F(x,y)/F(x,y) = E/E$, using the convention $\exp(Ix)F(x,y) = F(x+h,y)$. The notation $F(x+h,y)$ represents the response at point F, while $F(x-h,y)$ represents the response at point D. Addition of a translating term, T , modifies Eqs. 52-55 so that they become Eqs. 56-59.

$$(F + D + B + H + 4T)/(E + T) = J + 1/J + K + 1/K \quad (56)$$

$$(F + D - B - H)/(E + T) = J + 1/J - K - 1/K \quad (57)$$

$$(F - D + H - B)/(E + T) = J - 1/J + K - 1/K \quad (58)$$

$$(F - D - H + B)/(E + T) = J - 1/J - K + 1/K \quad (59)$$

Eqs. 56-59 can be solved for J, K, E, T, and from these expressions an equation for the diamond configuration is constructed. It is Eq. 60. This equation is exact on simple exponential functions like (4,16,64,256) as (B,D,F,H), respectively.

$$z = J^K(E+T) - T \quad (60)$$

$$E = [JK(J+K-2)(B+D) + (J(2K-1)-K)(F+H)] / [2(J^2K + J(K^2-1)-K)] \quad (61)$$

$$T = [JK(B+D+F+H) - E(J^2K + J(K^2+1)+K)] / [J^2K + J(K^2-4K+1)+K] \quad (62)$$

$$K = [(H-F)(D-H)]^{(1/2)} / [B(D+F) - DF - B^2]^{(1/2)} \quad (63)$$

$$J = (B-F)K / (D-H) \quad (64)$$

Example 8

If the measurements at (B,D,F,H) are (4,16,64,256), respectively, the interpolating equation for the diamond is $R = 32(2^x)(8^y)$. If the measurements are (14,26,74,266), respectively, the interpolating equation is $R = 32(2^x)(8^y)+10$.

Reasonable care should be exercised when using operational equations. It should be verified that the equations reproduce the original data. The four rotations of the data around their center point should yield the same response at the center point. If the equations generate complex numbers, interpolate with their real parts. Appendix I

contains an alternative method that can be applied when Eq. 60 contains I , the square root of (-1) .

It is not customary to represent measurements as powers of linear expressions, but there seems to be no law of nature denying the potential advantage of this representation. A third method for developing operational equations is by appeal to the algebra of operations. Let p and q represent increments in the horizontal and vertical directions, respectively. Then Eqs. 65-67 follow from Fig. 1.

$$(B^{(1/n)} - p + q)^n - D = 0 \quad (65)$$

$$(B^{(1/n)} + 2q)^n - H = 0 \quad (66)$$

$$(B^{(1/n)} + p + q)^n - F = 0 \quad (67)$$

Eqs. 65 and 66 can be solved for p and q , and the results substituted into Eq. 67. The result is Eq. 68.

$$(B^{(1/n)} + H^{(1/n)} - D^{(1/n)})^n - F \quad (68)$$

If Eq. 68 has a solution for the exponent n , take the n th roots of the original data. The data so reduced form a bilinear set. Substitute the reduced data into the operational

polynomial equation for the four-point diamond configuration, Eq. 22 substituted with Eqs. 46-50. The interaction and quadratic terms disappear, or represent round-off errors that can be discarded. The interpolating equation is the remaining bilinear expression raised to the n th power. Where a, b, k are constants, the interpolating equation takes the form of Eq. 69:

$$R = (k + ax + by)^n \quad (69)$$

Eq. 69 is limited to positive data, but this is not a severe limitation for most problems in engineering and science. The versatility of Eq. 69 can be extended if it is rewritten with a translating factor as in Eq. 70.

$$((B+t)^{(1/n)} + (H+t)^{(1/n)} - (D+t)^{(1/n)})^n - (F+t) \quad (70)$$

A reasonable number for the exponent n is chosen, and Eq. 70 is solved (if possible) for the translating term, t . This method for the diamond configuration closely parallels the similar method for the four-point rectangle [4]. Illustrations of that approach that also apply to the diamond configuration are found in Ref. 4.

Example 9

Suppose the measurements at (B,D,F,H) are (8,64,216,512), respectively. Eq. 68 yields the interpolating equation representing them as $R = (5 + x + 3y)^3$. Suppose the exponent n is changed to 4. Solve Eq. 70 for t . Then add t to each datum, and take the fourth root of each sum. These numbers are entered into the polynomial for the four-point diamond. Raise the expression to the fourth power, and subtract t . The new interpolating equation is $R = (3.393 + 0.477x + 1.383y)^4 - 8.335$. This equation is a false representation of the data, because the exponent was arbitrarily assigned, but the false representation and the true representation give results that are acceptably close for many practical purposes.

Experimental data arranged in the diamond configuration can be represented by functions that depend on the circular or hyperbolic sine and cosine as shown in Eq. 71.

$$z = M\sinh(Px+Qy) + N\cosh(Px+Qy) \quad (71)$$

This method can be applied to the four-point diamond configuration by using Eq. 72 for the corner point A. Point A is at lower left-hand corner of the square, as shown in Fig. 1.

$$A^2 = (D^2 - FD - B^2 + HB)^2 (BD - FH) / [(B - H + D - F)(D - F + H - B)(DH - BF)] \quad (72)$$

In other words, find the square of the response at corner point A in terms of the data at B,D,F,H. Rotation of the data about the center point of the diamond design yields the corresponding equations for C^2 , G^2 , and I^2 . These three equations are found by renaming the letters in Eq. 72. Each equation produces a positive and a negative number for the response at a vertex of the rectangle. For most positive data, all corner point responses are also positive, or three are positive and the numerically smallest is negative, or two are positive and the two numerically smallest are negative.

The derived responses at A,C,G,I are then used in the method that generates Eq. 71 for the four-point rectangle [3]. The combination of values for A,C,G,I that reproduces the original data at B,D,F,H is proper choice for interpolating diamond configuration BDFH. At most, sixteen combinations (the plus-or-minus values of each of A,C,G,I) must be examined to determine which combination reproduces the original data at points B,D,F,H. The equation for the diamond can also be obtained by the method of substitutions, as described in the paragraph that precedes Example 7.

Eq. 71 can also be applied to the five-point diamond configuration. The simplest approach appears to be the method of substitutions, as described in the paragraph

preceding Example 7. Two distinct methods for generating forms of Eq. 71 for the five-point rectangle are described in Ref. 7. They are the equation pairs (15,16) and (17,18) therein. These two pairs of equations represent two distinct approaches to the five-point diamond as represented by Eq. 71. The method based on the cited Eqs. 15 and 16 contains an ambiguity of signs, so this method generates two equations for the five-point rectangle and diamond. The proper selection is usually the one with coefficients that more nearly agree with the coefficients obtained from the cited Eqs. 17 and 18.

Example 10

Suppose the data at points (B,D,E,F,H) are (4,16,25,36,64), respectively. The five-point diamond can be represented by Eqs. 73, 74, or 75.

$$R = 25\cosh(Q)\cosh(P) + 32.8297\cosh(Q)\sinh(P) + 32.8297\sinh(Q)\cosh(P) \\ + 25.8869\sinh(Q)\sinh(P) \quad (73)$$

where $P = 0.5596y + 0.5596x$, and $Q = 0.2595y - 0.2595x$.

$$R = 25\cosh(Q)\cosh(P) + 33.1292\cosh(Q)\sinh(P) + 31.4426\sinh(Q)\cosh(P) \\ + 25\sinh(Q)\sinh(P) \quad (74)$$

where $P = 0.5534y + 0.5534x$, and $Q = 0.2715y - 0.2715x$

$$\begin{aligned} R = & 25\cosh(Q)\cosh(P) + 62.8851\cosh(Q)\sinh(P) + 16.5646\sinh(Q)\cosh(P) \\ & + 25\sinh(Q)\sinh(P) \end{aligned} \quad (75)$$

where $P = 0.2715y + 0.2715x$, and $Q = 0.5534y - 0.5534x$

DISCUSSION OF OPERATIONAL EQUATIONS

In the 16th century, Viète developed many of the identities found in textbooks of trigonometry. Application of the shifting operator $[\exp(x)F(x) = F(x+h)]$ to the Euler forms of trigonometric identities generates finite difference equations. These finite difference equations are used to derive equations that estimate interaction and quadratic coefficients for four or five data in various configurations, including the rectangle and diamond arrays. As a method for developing interpolating equations for geometric arrays of measurements, this approach apparently escaped notice for 400 years. Operational equations raise interesting questions: Should the shifting operator be a part of analytic geometry? Is analytic geometry complete?

Four-point operational equations are useful for many data sets that derive from monotonic, differentiable surfaces. A nine-point Latin square is suggested if an extremum, a ridge, or a trough is present in the data, or if the opposite sides of the design have slopes of contrary sign. The method of quadrants can be applied in those cases [4]. When an operational equation suggests a ridge or a trough in data that are known to represent a monotonic surface, choose another method.

If the L_2 norm test is the criterion of merit, operational equations are often closer to known surfaces than the standard bilinear equation, or the standard equation for the five-point rectangle. This remark also applies to data in cubical array, and to numbers arranged in increasing or decreasing sequences. For data ordered by magnitude, operational center points are usually closer to the true values than the arithmetic average, the most commonly used measure of central tendency. This remark applies primarily (but not exclusively) to data of one sign. See Appendix II and Ref. 5.

A curious application of operational equations is extrapolation over short intervals by means of echo functions [1]. Operational equations permit a somewhat analogous process as applied to the cube. See Appendix III.

In addition to the literature citations in the references, other papers about operational equations by the author have appeared in Quality Engineering 4 (1991) 57-61, Quality Engineering 6 (1993-94) 307-310, Analytica Chimica Acta 197 (1987) 129-135, and Analytica Chimica Acta 217 (1989) 395-399.

There are also other U.S. Department of Energy Research and Development reports, prepared by the author, describing applications of the shifting operator:

"Operational Methods for Data Interpolation" MLM-3277 (August 1985)

"Extensions of the Operational Method" MLM-3382 (August 1986)

"Remarks on the Operational Method of Data Treatment" MLM-3444 (November 1987)

"Operational Measures of Central Tendency" MLM-3598 (August 1989)

"Operational Center Point Formulas" MLM-3638 (June 1990)

"Three-Dimensional Operational Interpolation" MLM-3722 (December 1991)

"Operational Surfaces for the Rectangle and Triangle" MLM-3748 (June 1992)

These reports, and the references cited therein, contain most of what was known about operational interpolating equations by the end of 1999.

APPENDIX I. ALTERNATIVE METHODS.

There are two other methods for using Eq. 60 on four data. If one or both of J and K are imaginary, change them to real numbers by removing the symbol I , where I represents $\text{SQRT}(-1)$. If one or both of J or K are negative, change their signs so that after the changes only real, positive numbers are found in the first three terms of Eq. 60. These terms are multiplied by a factor that is chosen from Table 1. For example, if (B,D,F,H) are (4,16,256,64), then substituted Eq. 60 changes from Eq. 76 to Eq. 77.

$$z = (25.88I)(-9.17I)^x(-1.75I)^y + 18.82 \quad (76)$$

$$z = (x+y)(25.88)(9.17)^x(1.75)^y + 18.82 \quad (77)$$

The advantage of Eq. 77 is that it often provides a simpler surface through the data, one that minimizes ridges and troughs that can arise by the exponentiation of negative or imaginary numbers. Note that

$$K = [(F-H)^{(1/2)}(D-H)^{(1/2)}] / [(B-D)^{(1/2)}(B-F)^{(1/2)}] \quad (78)$$

$$J = [(F-H)^{(1/2)}(B-F)^{(1/2)}] / [(B-D)^{(1/2)}(D-H)^{(1/2)}] \quad (79)$$

can also be used as Eqs. 63 and 64, respectively, so Eq. 60 can be written in more than one way.

Table 1. Coefficients for the Real Form of Eq. 60.

<u>Leading Term</u>	<u>I</u>	<u>K</u>	<u>Multiplier</u>
-Im	-Im	-Im	$(-x-y)$
-Im	-Im	+Im	$(y-x)$
-Im	+Im	-Im	$(x-y)$
-Im	+Im	+Im	$(x+y)$
+Im	+Im	+Im	$(-x-y)$
+Im	+Im	-Im	$(y-x)$
+Im	-Im	+Im	$(x-y)$
+Im	-Im	-Im	$(x+y)$
-Re	-Re	+Re	(x^2-y^2)
-Re	+Re	-Re	(y^2-x^2)
+Re	-Re	+Re	(y^2-x^2)
+Re	+Re	-Re	(x^2-y^2)

Im represents an imaginary number. Re represents a real number.

Another method for the four-point rectangle and diamond configurations proceeds as follows. The equation to be modified is Eq. 80.

$$R = [J^{(x+1)}K^{(y+1)}(A-G)(A-C) + AI - CG] / [A-C-G+I] \quad (80)$$

Eq. 80 applies to the four-point rectangle and is described in Refs. 2 and 4.

Eq. 80 is an operational equation for rectangle ACIG. J is the square root of the ratio $(C-I)/(A-G)$. If the ratio is negative, take the absolute value and find its square root. This root is the new value of J. Now multiply the first term in the numerator of Eq. 80 (the four-member product) by $(-x)$. K is the square root of the ratio $(G-I)/(A-C)$. If the ratio is negative, take the absolute value and find its square root. This root is the new value of K. Now multiply the first term in the numerator of Eq. 80 by $(-y)$. If both ratios for J and K are negative, make both substitutions. Modified Eq. 80 should reproduce the original data [10].

Example 11

Measurements at corner points (A,C,G,I) are (1,9,49,81) in Fig. 1. The equation for this table of perfect squares $(1,3^2,7^2,9^2)$, based on Eq. 80, is Eq. 81.

$$R = (16)(1.225)^{(x+1)}(2)^{(y+1)} - 15 \quad (81)$$

The advantage of Eq. 81 is that it yields more accurate interpolations than the standard, bilinear equation on the cited table of squares, and on many other tables, as well. If the table is misconstrued so that the measurements are thought to be (9,1,49,81) as (A,C,G,I), the new interpolating equation is Eq. 82.

$$R = (8y)(1.414)^{(x+1)}(2)^{(y+1)} + 17 \quad (82)$$

Another misconception has the data (1,81,9,49) so that the interpolating equation becomes Eq. 83.

$$R = (16x)(2)^{(x+1)}(0.7071)^{(y+1)} + 17 \quad (83)$$

The third misconception is (1,49,81,9). In this case, the interpolating equation is Eq. 84.

$$R = (-32xy)(0.7071)^{(x+1)}(1.225)^{(y+1)} + 33 \quad (84)$$

In all of the misconstrued cases, substituting and evaluating Eq. 80 yields terms

containing I , the square root of -1 . The real parts of the responses calculated by this equation can also be used as the interpolated values, if that is desired. The advantage of modifying Eq. 80 is that it usually ameliorates the curvature that can result from powers of imaginary numbers. The modified form of Eq. 80 usually gives better interpolations than the bilinear equation for the four-point rectangle, too. If the method of substitutions is used, modified Eq. 80 yields an equation for the four-point diamond configuration that is easier to prepare than the equation that relies on choosing a coefficient from Table 1.

Eq. 30 is an operational equation for the five-point rectangle. Another operational equation for the same rectangle is Eq. 85.

$$P = R(J^x)(K^y)(M^z) + Q \quad (85)$$

The response in Eq. 85 is designated by the letter P to conform to the notation given in Ref. 7. The remaining letters J, K, Q, M , and R in Eq. 85 are defined in Ref. 7. The exponent z is taken as $1-(x^2+y^2)/2$, although other forms of the exponent are possible. The applicability of Eq. 85 is not limited by the magnitude of the response at the center point of the rectangle. Eq. 85 yields acceptable interpolating equations for the five-point rectangle, as judged by the L_2 norm test. Eq. 85 can also be adapted to the five-point diamond configuration by the method of coordinate substitutions already described.

Example 12

Data are sampled at (A,C,E,G,I) in a five-point rectangle. The true experimental surface is Eq. 86.

$$R = (5+x+3y)^{(5+x+3y)} / (5+x+3y)! \quad (86)$$

Three equations for the modeling the data are selected: (1) the standard, five-point polynomial, (2) Eq. 30, and (3) Eq. 85. The sum of the squares of the deviations of the true surface from the surface predicted by the first selection is 77244. The same sum, using Eq. 30 is 7.3. Using Eq. 85, the sum is 4.9. The equations based on exponential functions are better in this experiment because the true surface behaves more like an exponential function than a polynomial function. If the true surface behaved like a polynomial function of low order, the results would be different.

The three equations that are cited in Example 12 are adaptable to the five-point diamond configuration by the method of substitutions as described in the text preceding Example 7.

Operational equations can be used to estimate interaction and quadratic term coefficients on four or five points in the diamond array. Tables 2-5 illustrate the estimation of the coefficients of the interaction (xy) term and the quadratic terms on

four simple functions. These functions are $(5+x+3y)^2$, $(5+x+3y)^3$, $\ln(6+x+3y)$, and $1/(5+x+3y)$. Taylor expansion of each of the four generating functions listed in the left-hand column of Table 2 generated the coefficients in the middle column of the table. The right-hand column of Table 2 lists the same coefficients as they are approximated when Eq. 22 is substituted with Eqs. 45-50. Substituted Eq. 22 is then expanded by a Taylor series. In other words, the coefficients in the right-hand column of Table 2 are obtained only from the numbers produced by the generating functions at points B,D,F,H of the four-point diamond configuration. It can be seen that the coefficients are reasonable approximations to the true values as listed in the middle column of Table 2.

Tables 3-5 also list the generating functions in the left-hand column of each table. Another equation for the four-point diamond configuration, substituted with data from points B,D,F,H, is shown in the middle column of each table. The coefficients of the cross-product and quadratic terms, as derived by Taylor expansion of each substituted equation, are shown in the right-hand column of each table. As can be seen from the right hand columns of all of Tables 2-5, the coefficients obtained from the four-point, operational equations for the diamond configuration are reasonable approximations to the Taylor series coefficients as given in the middle column of Table 2. The right-hand column coefficients in Tables 2-5 are reasonable estimations in spite of sampling the function at only four points that define two perpendicular straight lines.

Tables similar to Tables 2-5 can also be constructed from operational equations for the five-point diamond array. They are shown as Tables 6-9. Although the numerical entries are different, equations for the five-point diamond array also generate acceptable estimates of the interaction and quadratic term coefficients for most differentiable, monotonic surfaces that do not contain steep gradients, interior extrema, ridges or troughs.

Table 2. Interaction plus Quadratic Terms of Test

Functions using Eq. 22 substituted with Eqs. 45-50.

Four-point diamond array sampled at points B, D, F, H.

<u>Function</u>	<u>$xy+x^2+y^2$</u> <u>from Taylor series of Function</u>	<u>$xy+x^2+y^2$</u> <u>from Taylor series of Eq. 22</u>
$(5+x+3y)^2$	$6xy+x^2+9y^2$	$6xy+x^2+9y^2$
$(5+x+3y)^3$	$90xy+15x^2+135y^2$	$80xy+12x^2+132y^2$
$\ln(6+x+3y)$	$-0.083xy-0.014x^2-0.13y^2$	$-0.088xy-0.013x^2-0.14y^2$
$1/(5+x+3y)$	$0.048xy+0.008x^2+0.072y^2$	$0.049xy+0.005x^2+0.11y^2$

Table 3. Substituted Eq. 60 and Interaction and Quadratic Terms from its Taylor series. Four-point diamond array sampled at points B, D, F, H.

<u>Function</u>	<u>Eq. 60</u>	<u>$xy+x^2+y^2$</u> <u>From Taylor series of Eq. 60</u>
$(5+x+3y)^2$	$44.9(1.247)^x(1.871)^y-20$	$6.2xy+1.1x^2+8.8y^2$
$(5+x+3y)^3$	$163.8(1.567)^x(3.374)^y-40.5$	$89xy+16x^2+121y^2$
$\ln(6+x+3y)$	$-0.97(0.842)^x(0.584)^y+2.76$	$-0.090xy-0.014x^2-0.14y^2$
$1/(5+x+3y)$	$0.1(0.667)^x(0.25)^y+0.1$	$0.056xy+0.0082x^2+0.096y^2$

Table 4. Substituted Eq. 69 and Interaction and Quadratic
Terms from its Taylor series. Four-point diamond array
sampled at points B, D, F, H.

<u>Function</u>	<u>Eq. 69</u>	<u>$xy+x^2+y^2$</u>
		<u>From Taylor series of Eq. 69</u>
$(5+x+3y)^2$	$(5+x+3y)^2$	$6xy+x^2+9y^2$
$(5+x+3y)^3$	$(5+x+3y)^3$	$90xy+15x^2+135y^2$
$\ln(6+x+3y)$	$(4.77+1.19x+3.49y)^{0.373}$	$-0.076xy-0.013x^2-0.11y^2$
$1/(5+x+3y)$	$1/(5+x+3y)$	$0.048xy+0.008x^2+0.072y^2$

Table 5. Substituted Eq. 71 and Interaction and Quadratic
Terms from its Taylor series. Four-point diamond array
sampled at points B, D, F, H.

<u>Function</u>	<u>Eq. 71</u>	<u>$xy+x^2+y^2$</u> <u>From Taylor series of Eq. 71</u>
$(5+x+3y)^2$	$24.8\cosh(0.307x+0.836y) +$ $32.0\sinh(0.307x+0.836y)$	$6.4xy+1.2x^2+8.7y^2$
$(5+x+3y)^3$	$122\cosh(0.543x+1.39y)+$ $133\sinh(0.543x+1.39y)$	$92xy+18x^2+118y^2$
$\ln(6+x+3y)$	$1.79\cos(0.120x+0.401y)+$ $1.40\sin(0.120x+0.401y)$	$-0.086xy-0.013x^2-0.14y^2$
$1/(5+x+3y)$	$0.201\cosh(0.261x+1.01y)-$ $0.158\sinh(0.261x+1.01y)$	$0.053xy+0.007x^2+0.10y^2$

Table 6. Substituted Eq. 51 (with alternative interaction term*)

and Interaction and Quadratic Terms from its Taylor series.

Five-point diamond array sampled at points B, D, E, F, H.

<u>Function</u>	<u>$xy+x^2+y^2$</u> <u>from Taylor series of Function</u>	<u>$xy+x^2+y^2$</u> <u>from Taylor series of Eq. 51*</u>
$(5+x+3y)^2$	$6xy+x^2+9y^2$	$6xy+x^2+9y^2$
$(5+x+3y)^3$	$90xy+15x^2+135y^2$	$83xy+15x^2+135y^2$
$\ln(6+x+3y)$	$-0.083xy-0.014x^2-0.13y^2$	$-0.088xy-0.014x^2-0.14y^2$
$1/(5+x+3y)$	$0.048xy+0.008x^2+0.072y^2$	$0.051xy+0.008x^2+0.11y^2$

*Another polynomial equation for the five-point diamond array has the interaction term as

$XYC = (H - B)(F - D)(H + B - 4E + D + F) / [(H - B)^2 + (F - D)^2]$. It is this interaction term that is presently used in Eq. 51, not the interaction term as listed in that equation.

Otherwise, the five-point equations are the same.

Table 7. Interaction and Quadratic Terms from
rotated and substituted Eq. 30 and Eq. 85. Five-
point diamond array sampled at points B, D, E, F, H.

<u>Function</u>	<u>Eq. 30*</u>	<u>Eq. 85*</u>
$(5+x+3y)^2$	$5.9xy+x^2+8.7y^2$	$6.2xy+x^2+8.7y^2$
$(5+x+3y)^3$	$84xy+15x^2+120y^2$	$90xy+15x^2+121y^2$
$\ln(6+x+3y)$	$-0.089xy-0.014x^2-0.14y^2$	$-0.090xy-0.014x^2-0.14y^2$
$1/(5+x+3y)$	$0.056xy+0.008x^2+0.096y^2$	$0.056xy+0.008x^2+0.096y^2$

*Eqs. 30 and 85 rotated to apply to the five-point diamond array and substituted with data at points B, D, E, F, H.

Table 8. Interaction and Quadratic Terms derived from
the interpolating equation obtained from rotated Eq. 40.
Five-point diamond array sampled at points B, D, E, F, H.

<u>Function</u>	<u>Eq. 69</u>	<u>$xy+x^2+y^2$</u> <u>From Taylor series of Eq. 69</u>
$(5+x+3y)^2$	$(5+x+3y)^2$	$6xy+x^2+9y^2$
$(5+x+3y)^3$	$(5+x+3y)^3$	$90xy+15x^2+135y^2$
$\ln(6+x+3y)$	$(4.83+1.21x+3.55y+$ $0.0046(y-x)(y+x))^{0.370}$	$-0.077xy-0.014x^2-0.11y^2$
$1/(5+x+3y)$	$1/(5+x+3y)$	$0.048xy+0.008x^2+0.072y^2$

Table 9. Rotated and substituted Eq. 71 and Interaction and Quadratic Terms from its Taylor series. Five-point diamond array sampled at points B, D, E, F, H.

<u>Function</u>	<u>Eq. 71</u>	<u>$xy+x^2+y^2$</u> <u>From Taylor series of Eq. 71</u>
$(5+x+3y)^2$	$-0.4335\cosh(p) + 25.433\cosh(q)$ $+32.83\sinh(q)$ $p=0.3001y+0.8191x,$ $q=0.3001x+0.8191y$	$6.14xy+1.0x^2+8.51y^2$
$(5+x+3y)^3$	$127.5\cosh(p) - 2.474\cosh(q)$ $139.8\sinh(p)$ $p=1.352y+0.520x$ $q=0.520y+1.352x$	$87.9xy+15x^2+116y^2$
$\ln(6+x+3y)$	$0.0136\cos(p) + 1.778\cos(q)$ $1.395\sin(q)$ $p=0.121y+0.405x$ $q=0.405y+0.121x$	$-0.088xy-0.014x^2-0.146y^2$

$$1/(5+x+3y) \quad -0.155\sinh(p) + 0.002\sinh(q)$$

$$+ 0.198\cosh(p)$$

$$0.054xy+0.008x^2+0.10y^2$$

$$p=1.022y+0.265x$$

$$q=0.265y+1.022x$$

Another operational equation for five data in rectangular array has quadratic coefficients given Eqs. 87 and 88.

$$X2 = (1/8)(A - C - G + I)^2 / (A + C + G + I - 4E) \\ (-/+) (2)SQRT[(C-2E+G)(A-2E+I)] \quad (87)$$

$$Y2 = (1/8)(A - C - G + I)^2 / (A + C + G + I - 4E) \\ (+/-) (2)SQRT[(C-2E+G)(A-2E+I)] \quad (88)$$

These coefficients differ by alternative plus or minus signs in their denominators. The proper choice of sign can usually be made by comparison of the numerical values of the coefficients to the numerical values of the like coefficients in another operational equation for the five-point rectangle. In this respect, Eqs. 87 and 88 behave like the quadratic coefficients given as Eqs. 6 and 7 in Ref. 7.

APPENDIX II. CENTRAL TENDENCY

Two series of operational center point formulas for ordered data have already been illustrated [5]. They are based on the polynomial and compound exponential methods for representing geometric figures such as squares and cubes. (These methods were formerly denoted the algebraic and first exponential methods, respectively.) Center point estimators based on the method of representing geometric figures by simple exponential functions (like 2^x) can also be described. Like the second series of central tendency measures illustrated in Ref. 5, the new series of center point estimators is useful primarily for data of one sign: positive data, ordinarily.

The estimators do not apply universally. In the first one to be described, excluded are cases that lead to division by zero or exponentiation of zero or unity. The method is exact on simple functions like 2^x , $1/x$, $\tanh(x)$, $\coth(x)$, $\cot(x)$, and $\tan(x)$, but in the last two cases the domain of permissible data does not include divergence of the functions. The method is not recommended if it generates complex numbers.

The measure of central tendency for four ordered data is the center point of the four-point rectangle as it is represented by Eq. 9 in Ref. 2 or Eq. 80 above. Order the four data as $A < C < G < I$ for use in this equation. The center point estimate approaches the

limiting expression

$$\text{SQRT}(C-A)\text{SQRT}(G-A) + A$$

as the value of I increases indefinitely.

On five ordered data, remove the median of the set and estimate the center point on the remaining four ordered data using Eq. 80. Take the square root of the product of the removed median and the estimated center point on the four remaining data. This root is the estimate of the central tendency of the five ordered data.

Let six ordered data be denoted A, B, C, E, F, G . Define six quantities $J_1, J_2, K_1, K_2, L_1, L_2$ as the square roots of the following six ratios, respectively: $(F-E)(E-B)/((C-B)(F-C))$, $(G-E)(E-A)/((G-C)(C-A))$, $(F-E)(F-C)/((E-B)(C-B))$, $(G-F)(F-A)/((G-B)(B-A))$, $(G-C)(G-E)/((E-A)(C-A))$, $(G-B)(G-F)/((B-A)(F-A))$. Define three new quantities J, K, L as the square roots of the products J_1J_2, K_1K_2 , and L_1L_2 , respectively.

Use a convenient method, like least squares, to solve Eqs. 89-94 for P and T .

$$PJ + T - E = 0 \quad (89)$$

$$P/J + T - C = 0 \quad (90)$$

$$PK + T - F = 0 \quad (91)$$

$$P/K + T - B = 0 \quad (92)$$

$$PL + T - G = 0 \quad (93)$$

$$P/L + T - A = 0 \quad (94)$$

The estimate of central tendency on the six ordered data is the sum (P+T).

The center point of seven ordered data is found by removing the median of the set and finding the center point of the six remaining data, as above. The square root of the product of the removed median and the determined center point on the remaining six ordered data estimates the center point of the seven ordered data.

The foregoing description of the estimation of the central tendencies on 4-7 data is satisfactory for expository purposes, but the calculations could have been described in other ways. The estimation of the center point of eight ordered data illustrates this remark. Instead of lengthy formulas and least-squares procedures, partition the ordered data into four groups of five ordered numbers: $[A < B < C < D < E]$, $[B < C < D < E < F]$, $[C < D < E < F < G]$, $[D < E < F < G < H]$. Estimate the central tendency of each group of five data. Take these four ordered estimates and use them as A,C,G,I in Eq. 80 or Eq. 9 in Ref. 2. In other words, each estimate of the center point of each of the four groups is

taken as one vertex of rectangle ACIG as described above. The data are arranged so that $A < C < G < I$ in Eq. 80 or Eq. 9 of Ref. 2. The estimated response at the center point of this rectangle is the estimate of the center point of the eight original data.

For example, let selected functions be applied to the eight numbers 1,2,3,4,5,6,7,8 as trial data. The center points (cp) of all eight data, the last seven data, and the last six data are shown in Table 10. The true center point is the selected function applied to the median of the indicated sequence of integers. It can be seen from this table that the estimates and true center points agree very well.

The method of roots for squares and cubes can be used to estimate center points on ordered data. Use four ordered data as the vertices of the rectangle as illustrated in Ref. 4. The center point of the rectangle is also the center point of the ordered measurements. For five ordered data, remove the median of the set, and treat the remaining four ordered data as the vertices of the rectangle, using the method of roots, as illustrated in Ref. 4. The arithmetic average of the n th roots of both the removed median and the determined center point, raised to the n th power, is the estimate of the center point on the group of five ordered numbers.

For six ordered data A,B,C,E,F,G use Eq. 95 to find the degree of the data, n .

$$(A^{(1/n)} + B^{(1/n)} + F^{(1/n)} + G^{(1/n)})^n - (2C^{(1/n)} + 2E^{(1/n)})^n = 0 \quad (95)$$

The center point of the six ordered data is the arithmetic average of the n th roots of the data when this average is raised to the n th power. For seven ordered data, remove the median of the set and proceed as for six ordered data. Take the arithmetic average of the n th roots of the determined center point and the removed median and raise this average to the n th power. For eight ordered data, partition the data into four groups of five data each. Find the center points on these four groups, and then find the center point of the four determined center points as described above. In all cases, the method can be tested for its exactness on powers of the first four to eight integers.

The method of roots as applied to center point determinations has a number of defects. The method is restricted to positive numbers that are more or less evenly separated. Real data do not always meet this requirement. The technique requires the determination of the root of an equation. Although this seems easy, it is not necessarily easy in practice. The necessary equation might not have a root. If it has a root, commercial software might not be able to find it. Even if a root is found, its uniqueness has not been demonstrated. When the data occur in bunches, the center point is not necessarily a good representation of the measurements. These considerations do not recommend the method over any of the other operational measures of central tendency.

Table 11 is similar to Table 10 in that it gives representative results for this technique using four common functions. Center points obtained by this method are usually closer to the true values than the estimates obtained by the arithmetic average, but the cost is much more extensive calculations. The method works on many sets of ordered data, but an answer cannot be guaranteed as it can by means of the arithmetic average.

Another series of center point estimators is obtained as the difference between the arithmetic average of a set of points and the quadratic coefficients for the corresponding geometric figure. This series is illustrated for four and eight ordered data by the constant terms in the polynomial equations for the rectangle and the cube.

TABLE 10. CENTER POINTS (cp) ON TRIAL DATA
BY METHOD OF SIMPLE EXPONENTIALS*

<u>Data used</u>	<u>Applied Function</u>			
	<u>x^2</u>	<u>$1/x$</u>	<u>$\tan(10x^\circ)$</u>	<u>$\ln(x)$</u>
All 8 data	20.22 (20.25)	0.2222 (0.2222)	1.000 (1.000)	1.504 (1.504)
Last 7 data	24.91 (25.00)	0.2000 (0.2000)	1.192 (1.192)	1.610 (1.609)
Last 6 data	30.20 (30.25)	0.1818 (0.1818)	1.428 (1.428)	1.705 (1.705)

*Numbers in parentheses are true values found by evaluating $u(4.5)$, $u(5)$, and $u(5.5)$, respectively, where "u" is the function applied to the cited sequence of integers.

TABLE 11. CENTER POINTS (cp) ON TRIAL DATA
BY METHOD OF ROOTS*

<u>Data used</u>	<u>Applied Function</u>			
	<u>x^2</u>	<u>$1/x$</u>	<u>$\tan(7x^\circ)^{**}$</u>	<u>$\ln(x)$</u>
All 8 data	20.25 (20.25)	0.2222 (0.2222)	0.6130 (0.6128)	1.503 (1.504)
Last 7 data	25.00 (25.00)	0.2000 (0.2000)	0.7004 (0.7002)	1.609(1.609)
Last 6 data	30.25 (30.25)	0.1818 (0.1818)	0.7955 (0.7954)	1.705(1.705)

*Numbers in parentheses are true values found by evaluating $u(4.5)$, $u(5)$, and $u(5.5)$, respectively, where "u" is the function applied to the cited sequence of integers.

**The method fails on $\tan(10x^\circ)$ so $\tan(7x^\circ)$ was substituted.

Appendix III. Completing the Cube.

A design sometimes recommended for economy is the partial cube described by vertices A, D, G, H in Fig. 1 of Ref. 3. If the measurement at cube center point E is added, it is sometimes possible to estimate the responses at the remaining vertices of the cube. The proposed method does not apply if the data enclose a ridge, a trough, an extremum, a hole, or some other unusual feature.

The first step in the process is to estimate the degree of the data. If possible, solve Eq. 96 for exponent n. The degree of the data N is (1/n).

$$[E^n - (H^n + A^n)/2] / [(G^n + D^n)/2 - E^n] - 1 = 0 \quad (96)$$

Reduce the data by finding their Nth roots. These roots are entered into the coefficients of the linear terms x, y, and z as indicated in Eqs. 97-99.

$$x_c = (G' + D' - A' - H') / 4 \quad (97)$$

$$y_c = (H' + D' - A' - G') / 4 \quad (98)$$

$$z_c = (H' + G' - A' - D') / 4 \quad (99)$$

where the apostrophe following each letter indicates its Nth root.

The expression that represents the cube, based on the limited available data, is Eq. 100.

$$R = (E + (xc)x + (yc)y + (zc)z)^{(N)} \quad (100)$$

For example, suppose the measured data are A=1, D=16, E=25, G=49, H=64.

These data represent a cube of the squares of the first nine integers. The value of n determined from Eq. 96 is 0.5, so the degree of the data, N, is 2. The responses at vertices B and I are 4 and 81, respectively. These are also the predicted values, because Eq. 100 generates Eq. 101.

$$R = (5 + x/2 + y + 5z/2)^2 \quad (101)$$

The method is exact on powers of the integers. This is often better than a linear extrapolation, but it is not entirely satisfactory, as is illustrated if the data are sin(10x degrees), where x represents one of original integers. In this case:

$$R = (0.562 + 0.067x + 0.114y + 0.358z)^{(0.463)} \quad (102)$$

The predicted datum at vertex B is 0.425, but the real value at B is 0.342. The reason for

the poor prediction is the failure of a simple function like p^N to represent the sine function satisfactorily. The predicted response at I is 1.05, compared to the real value 1.00.

The method can be modified by the introduction of a translating factor, t . Add t to every letter in Eq. 96 and assume some reasonable number for the degree of the data. Solve modified Eq. 96 for t . Add the determined value of t to each datum, find the N th roots of each sum, and proceed as above. For example, in the cited case involving the sine function, assume that the degree of the data is 0.5. The function representing the data under this assumption is Eq. 103.

$$R = (0.472 + 0.057x + 0.097y + 0.308z)^{(0.5)} + 0.079 \quad (103)$$

The equation for the complete cube of the cited sines, based on Eq. 44, is Eq. 104:

$$R = (0.513 + 0.051x + 0.104y + 0.366z - 0.003xy - 0.002xz - 0.002yz - 0.027xyz)^{(0.399)} \quad (104)$$

Quadratic polynomial coefficients for data in cubical array divide into three categories: those that use and replicate the corner points only, those that use the center point and the corner points but replicate the center point only, and those that use and

replicate the center point and the corner points. Quadratic coefficients in all three classes are exact on trilinear data and the squares of trilinear data. Another method for generating a set of coefficients in the third category is to apply the polynomial equation that uses the values at the six faces of the cube. Treat each face of the cube as a four-point rectangle and estimate its center point by means of the operational polynomial equation for the four-point rectangle. These center points are used as the numerical entries for the four letter abbreviations such as ABCD and BDGI that appear in Eqs. 9-11 in Ref. 5.

When applied with Barrentine's data, five sets of quadratic coefficients belonging to the third category yield the following approximate numbers for the x^2 -, y^2 -, and z^2 -coefficients, respectively: [0.279,-3.634,3.355], [0,0,0], [0.8,0.8,-1.6], [-2.167,-0.324,2.491], and [0.294,-3.699,3.405]. Four of these sets contain nonzero members. These four sets suggest the forms of local curvature that could be present within the space defined by Barrentine's data. Nothing in the responses themselves rules out the possibility of local curvature that the standard, trilinear equation cannot detect. Even so, the sum of the numbers within each set is zero, indicating that no net curvature is still a possibility within the experimental space. The nonzero operational coefficients do not contradict Eq. 18, but they suggest that the standard, trilinear equation is not the only perspective on data that otherwise appear to be strictly determined by that equation.

Echo functions were introduced many years ago [1]. They express the equation of a curve as $y=f(y)$ instead of $y=f(x)$. They typically apply to five or more curvilinear data A,B,C,D,E. Point C lies midway between points B and D, but points A and E are permitted to float [1]. Echo functions can be used for interpolation, extrapolation, and arc length. An example of an echo function is Eq. 105.

$$P(A^2 - AB - DA + BD) + AE - BD + C(B + D - E - A) = 0 \quad (105)$$

$$P = [\text{SQRT}((B-E)/(A-D))][\text{SQRT}((D-E)/(A-B))] \quad (106)$$

Example 13

Eq. 105 can be solved for E. Suppose three equidistant points on a curve have responses $B=2^1=2$, $C=2^2=4$, $D=2^3=8$. If $A=4$, then $E=4$. The stimulus is $A=4$ and its echo is $E=4$. Both stimulus and echo coincide with the center point datum $C=4$. If $A=2$, then $E=8$, and if $A=1$, then $E=16$. E and A depart from midpoint C in geometric progression. If "deda" represents the derivative of E with respect to A, then the length of the arc $16/A$ is found from the definite integral of $\text{SQRT}(1+\text{deda}^2)$. The echo function is exact on 2^x , as illustrated, and it is also exact on x , $1/x$, $\tan(x^\circ)$, and $\tanh(x)$.

Another echo function is implicit in Eq. 10 in Ref. 4. Where A,B,D,E are four curvilinear data, it can be written as Eq. 107.

$$E = (B^{(1/N)} + D^{(1/N)} - A^{(1/N)})^N \quad (107)$$

Example 14

The exponent N is 2 for the case of numbers that are known to be the squares of integers in sequence. Let $B=4^2$ and $D=6^2$. If $A=5^2$, then $E=5^2$. The coincidence of E and A represents the midpoint C between B and D. If $A=4^2$, then $E=6^2$, and if $A=3^2$, then $E=7^2$. Likewise, if $A=1^2$, then $E=9^2$. Floating points A and E move away from midpoint C as the squares of numbers in an arithmetic sequence. If data to the left of the midpoint are measured, values to the right of the midpoint are interpolated.

The degree of the data N is appropriate if the calculated value of E is close to the measured value at midpoint C. For example, let $B=\ln(5)$ and $D=\ln(7)$. Midpoint C is $\ln(6)$, so this is the value used as A. When $N=0.3595$, E is calculated to be about 1.79175. This is close to 1.79176, the true value of $\ln(6)$, so $N=0.3595$ is sufficiently accurate to interpolate logarithms between 5 and 7. The echo function should be checked to make sure that it also reproduces the end point values B and D.

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