Title: EXPONENTIAL MONTE CARLO CONVERGENCE ON A HOMOGENEOUS RIGHT PARALLELEPIDED USING THE REDUCED SOURCE METHOD WITH LEGENDRE EXPANSION

Author(s): J. A. FAVORITE

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Exponential Monte Carlo Convergence on a Homogeneous Right Parallelepiped Using the Reduced Source Method with Legendre Expansion

Jeffrey A. Favorite
Applied Theoretical and Computational Physics Division
Los Alamos National Laboratory
Los Alamos, New Mexico USA
fave@lanl.gov

Abstract

In previous work, exponential convergence of Monte Carlo solutions using the reduced source method with Legendre expansion has been achieved only in one-dimensional rod and slab geometries. In this paper, the method is applied to three-dimensional (right parallelepiped) problems, with resulting evidence suggesting success. As implemented in this paper, the method approximates an angular integral of the flux with a discrete-ordinates numerical quadrature. It is possible that this approximation introduces an inconsistency that must be addressed.

1 Introduction

Exponential convergence has been obtained for a relatively few simple transport problems using various combinations of adaptive Monte Carlo learning algorithms and solution techniques. This paper has as its concern the correlated sampling or reduced source method, first suggested by Halton [1962, 1994] and first used, apparently, by Fraley et al. [1962]. McKinney [1996] and Kong and McKinney [1996] used the reduced source method with Monte Carlo estimation of the coefficients of a Legendre expansion to solve a spatially continuous one-dimensional transport problem with no scattering. These results have since been extended to the rod problem with scattering [Favorite, 1998] and to general bi-directional slab problems with scattering as well as to continuous-angle slab problems [Kong and Spanier, 1998; Spanier, 1998]. In these works, the generalized track length estimator was used to obtain Monte Carlo estimates of the flux-weighted integral definitions of the Legendre expansion coefficients.

This paper documents the first attempt to generalize the procedures that have proved successful in obtaining exponential convergence in slab geometry to full three-dimensional geometry. Although the problem here considered, a homogeneous right parallelepiped, is still a rather simple one, there are many computational difficulties encountered in extending the method to the three-dimensional case. This paper discusses these difficulties, their present solution, and provides numerical evidence of exponential convergence in some three-dimensional problems.

The next section of this paper briefly describes the reduced source method. Section 3 describes the Legendre expansion (in the spatial variables) of the scalar flux and the implications of using that procedure. Section 4 describes the present application of the Monte Carlo method (i.e., scoring and
Section 5 presents some of the computational difficulties of the three-dimensional problem. Numerical results are presented in Sec. 6.

2 The reduced source method

The monoenergetic neutral particle transport equation for a homogeneous medium with isotropic scattering and no internal source, rendered in standard notation, is

\[ \Omega \cdot \nabla \psi(r, \Omega) + \Sigma \psi(r, \Omega) - \Sigma_s \int d\Omega' \psi(r, \Omega') = 0 \]

(1)

with boundary condition (on surface points \( r \) with outward normal \( n_s \))

\[ |\Omega \cdot n_s|\psi(r_s, \Omega) = S(r_s, \Omega), \quad \Omega \cdot n_s < 0 \]

(2)

Given some estimate \( \tilde{\psi} \) of the true angular flux \( \psi \), the difference \( \psi - \tilde{\psi} \) is defined as the angular flux residual, \( \Delta \psi \). Using \( \psi = \tilde{\psi} + \Delta \psi \) in Eqs. (1) and (2) yields an equation for the angular flux residual:

\[ \Omega \cdot \nabla \Delta \psi(r, \Omega) + \Sigma \Delta \psi(r, \Omega) - \Sigma_s \int d\Omega' \Delta \psi(r, \Omega') = -\Omega \cdot \nabla \tilde{\psi}(r, \Omega) - \Sigma_s \tilde{\psi}(r, \Omega) + \Sigma_s \int d\Omega' \tilde{\psi}(r, \Omega') \]

(3)

with boundary condition

\[ |\Omega \cdot n_s| \Delta \psi(r_s, \Omega) = S(r_s, \Omega) - |\Omega \cdot n_s| \tilde{\psi}(r_s, \Omega), \quad \Omega \cdot n_s < 0 \]

(4)

In the reduced source method, Eq. (1) is solved for an initial estimate \( \tilde{\psi}^0 \), which is then used on the right-hand side of Eq. (3) for the first order residual \( \Delta \psi^0 \). Using the resulting estimate \( \tilde{\psi} = \tilde{\psi}^0 + \Delta \psi^0 \) on the right-hand side of Eq. (3) provides an equation for the second order residual \( \Delta \psi^1 \) and a prescription for an iterative strategy. The angular flux estimate after \( n \) such iterations is

\[ \tilde{\psi}(r, \Omega) = \tilde{\psi}^0(r, \Omega) + \sum_{m=1}^n \Delta \psi^m(r, \Omega) \]

(5)

Note that the reduced source, i.e., the right-hand side of Eq. (3), is a measure of the amount by which \( \tilde{\psi} \) fails to satisfy Eq. (1).

3 The reduced source method with Legendre expansion of the scalar flux

To obtain a Monte Carlo estimate of the solutions of Eqs. (1) and (3), the spatial dependence of the scalar flux \( \phi(r) = \int d\Omega \psi(r, \Omega) \) is approximated with a Legendre polynomial expansion of the form

\[ \phi(x, y, z) = \tilde{\phi}(x, y, z) = \sum_{i=0}^{I} \sum_{j=0}^{J} \sum_{k=0}^{K} a_{ijk} P_i(\frac{x}{X}-1)P_j(\frac{y}{Y}-1)P_k(\frac{z}{Z}-1) \]

(6)

where the coefficients \( a_{ijk} \) are defined by

\[ a_{ijk} = \frac{(2i+1)(2j+1)(2k+1)}{X Y Z} \int_0^X dx \int_0^Y dy \int_0^Z dz \phi(x, y, z) P_i(\frac{x}{X}-1)P_j(\frac{y}{Y}-1)P_k(\frac{z}{Z}-1) \]

(7)

[In Eqs. (6) and (7), \( X, Y, \) and \( Z \) are the dimensions of the volume, a right parallelepiped.] An estimate \( \tilde{\psi}(r, \Omega) \) of the angular flux at a given point in the phase space is recovered by solving the following equation, an approximation to Eq. (1):

\[ \Omega \cdot \nabla \tilde{\psi}(r, \Omega) + \Sigma \tilde{\psi}(r, \Omega) = \Sigma_s \tilde{\phi}(r) \]

(8)

with boundary condition

\[ |\Omega \cdot n_s| \tilde{\psi}(r_s, \Omega) = S(r_s, \Omega), \quad \Omega \cdot n_s < 0 \]

(9)
This idea of estimating the scalar flux and using it to compute the angular flux was proposed by Booth [1998] and implemented by Kong and Spanier [1998]. At the time, the latter investigators were solving the slab problem (\(x, \mu\) geometry) using separate Legendre polynomial expansions of the spatial (\(x\)) and angular (\(\mu\)) dependencies of the angular flux, and finding that the approximation failed for small values of \(\mu\). Booth’s suggestion obviated the expansion in \(\mu\). The application of this procedure to the three-dimensional case is detailed in Sec. 5 below.

The solution to Eq. (3) is obtained by evaluating the internal (volumetric) and surface sources using the approximation of Eqs. (8) and (9). Using Eqs. (8) and (9) on the right-hand sides of Eqs. (3) and (4), respectively, yields

\[
\mathbf{\Omega} \cdot \nabla \Delta \psi(r, \mathbf{\Omega}) + \mathbf{\Sigma}_s \Delta \psi(r, \mathbf{\Omega}) - \mathbf{\Sigma}_s \int d\Omega' \Delta \psi(r, \mathbf{\Omega}') = \mathbf{\Sigma}_s \left[ \int d\Omega' \bar{\psi}(r, \mathbf{\Omega}') - \bar{\phi}(r) \right]
\]

with boundary condition

\[\left| \mathbf{\Omega} \cdot \mathbf{n} \right| \Delta \psi(r, \mathbf{\Omega}) = 0 , \quad \mathbf{\Omega} \cdot \mathbf{n} < 0 .\]

It is Eq. (10), with the boundary condition of Eq. (11), that is solved for the angular flux residual in the present method.

4 Present application of the Monte Carlo method

Monte Carlo estimates of the Legendre expansion coefficients for the scalar flux are obtained using the generalized track length estimator. For simplicity, the following expansion, which is a subset of Eq. (6), is used:

\[
\bar{\phi}(x, y, z) = a_{000} + \sum_{i=1}^{I} a_i P_i(\frac{x}{L}) - 1 + \sum_{j=1}^{J} b_j P_j(\frac{y}{W}) - 1 + \sum_{k=1}^{K} c_k P_k(\frac{z}{H}) - 1 ,
\]

where \(a_i, b_j,\) and \(c_k\) correspond to \(a_{i00}, a_{0j0},\) and \(a_{00k}\) of Eq. (6), respectively. Thus, the definition of Eq. (7) applies. Using the generalized track length estimator of Eq. (7), in which \(w \int_C ds P_i P_j P_k\) is scored rather than the usual \(w \int_C ds\) (\(C\) represents each track segment and \(w\) is the particle’s weight), each particle track segment contributes a score for all of the coefficients as follows [the notation \(s(a_i)\) denotes the score for coefficient \(a_i; h\) is the track length; the beginning and end points of the track are \((x_1, y_1, z_1)\) and \((x_2, y_2, z_2)\), respectively; and the usual direction cosines of the track are \(\mu, \eta,\) and \(\xi\)]:

\[
s(a_{000}) = \frac{1}{X/H} w \int_0^h ds = \frac{1}{X/H} wh ,
\]

\[
s(a_i) = \frac{(2i+1)}{X/H} w \int_0^h ds P_i(\frac{2x(s)}{X} - 1) = \frac{(2i+1)}{X/H} \frac{w}{\mu} \int_{x_1}^{x_2} dx P_i(\frac{2s}{X} - 1) ,
\]

\[
s(b_j) = \frac{(2j+1)}{Y/W} w \int_0^h ds P_j(\frac{2y(s)}{Y} - 1) = \frac{(2j+1)}{Y/W} \frac{w}{\eta} \int_{y_1}^{y_2} dy P_j(\frac{2s}{Y} - 1) ,
\]

and

\[
s(c_k) = \frac{(2k+1)}{Z/H} w \int_0^h ds P_k(\frac{2z(s)}{Z} - 1) = \frac{(2k+1)}{Z/H} \frac{w}{\xi} \int_{z_1}^{z_2} dz P_k(\frac{2s}{Z} - 1) .
\]
Note that the vector representation of a point along the track is given by \( r(s) = r_0 + s\Omega \). (In the unlikely event that a direction cosine is identically zero, a zero is scored for the appropriate coefficient.)

The "zero'th" stage of the Monte Carlo reduced source solution procedure, corresponding to an approximate solution of Eq. (1), is treated as a conventional (non-adaptive) Monte Carlo problem. Particles are started from a physical (in this case, surface) source location with appropriate weights. The zero'th stage results in an estimate \( \tilde{\phi}(r) \) of the scalar flux everywhere, which may be used to obtain an estimate \( \tilde{\psi}(r, \Omega) \) of the angular flux at any point in the phase space using Eqs. (8) and (9).

In all subsequent stages, it is the angular flux residual, \( \Delta \psi(r, \Omega) \) of Eq. (10), that is desired. This requires that "residual particles" be started in the phase space in accordance with the internal source condition on the right-hand side of Eq. (10) [the Monte Carlo procedure automatically accounts for the boundary source condition of Eq. (11) by ignoring it]. This internal source, while not dependent on angle, may be positive or negative anywhere and would require some expensive computational work to convert to a probability density function from which to sample particle starting points. An alternative procedure, used here, is to sample a starting point uniformly in the volume and use as the particle's weight the value of the source at that point. In other words, randomly sample a starting point \( r_{\text{start}} \) in the volume and use as the particle's weight the quantity \( \Sigma_s \left[ d\Omega \tilde{\psi}(r_{\text{start}}, \Omega) - \tilde{\phi}(r_{\text{start}}) \right] \). When this quantity is multiplied by the volume of the source region, \( XYZ \), the total weight is conserved. Thus, each residual particle is started at random spatial point \( r_{\text{start}} \) in random (isotropically distributed) direction \( \Omega_{\text{start}} \) with weight

\[
w = XYZ \Sigma_s \left[ d\Omega \tilde{\psi}(r_{\text{start}}, \Omega) - \tilde{\phi}(r_{\text{start}}) \right].
\]

Details of the evaluation of \( \int d\Omega \tilde{\psi}(r_{\text{start}}, \Omega) \) are given in the next section.

5 Some calculational details of the present application

The next section of this paper presents some preliminary numerical results. In this section, some details are provided for two aspects of the calculational procedure.

5.1 Extracting the Angular Flux from the Scalar Flux

The Legendre expansion of the scalar flux, \( \tilde{\phi}(r) \) of Eq. (12), is used as a known source in Eq. (8) to solve for the angular flux at any point in the phase space. For a particular phase space point \((r_p, \Omega_p)\), the points \( r(s) \) along the trajectory from the surface point \( r_s \) to the spatial point \( r_p \) along the direction \( \Omega_p \), when the points \( r_p \) and \( r_s \) are specified as vectors, satisfy \( r(s) = r_s + s\Omega_p \), and \( r_p = r_s + h\Omega_p \), where, as before, \( h \) is the path length. Equations (8) and (9) can now be written

\[
\frac{d\tilde{\psi}(r(s), \Omega_p)}{ds} + \Sigma_s \tilde{\psi}(r(s), \Omega_p) = \Sigma_s \tilde{\phi}(r_s),
\]

(18)
with boundary condition
\[ \Omega \cdot n_s |\psi(r_s, \Omega_p) = S(r_s, \Omega_p) , \Omega_p \cdot n_s < 0 . \quad (19) \]

Equation (18) is just a first-order differential equation, with an inhomogeneous source, and Eq. (19) is its boundary condition. In the present method, whenever an estimate of the angular flux is desired at some specified point, Eq. (18) is solved numerically, using a fifth-order Runge-Kutta method with adaptive step-size control [Press et al., 1992].

5.2 Estimating the Reduced Source

In the adaptive stages of the reduced source problem (i.e., all stages after the zero'th), the reduced source is evaluated at each particle’s starting point, and the particle’s weight is assigned as in Eq. (17). The estimate of the scalar flux, \( \Phi(r_{\text{start}}) \), is available at any time from Eq. (12), using the latest value of the coefficients. Unfortunately, there appears to be no readily available estimate of \( dK^2 \mathcal{P}^{-1} \). That quantity is estimated in this work using a discrete-ordinates type of angular quadrature:
\[ \int d\Omega \Psi(r_{\text{start}}, \Omega) = \sum_{m=1}^{M} t_m \Psi(r_{\text{start}}, \Omega_m) , \quad (20) \]
where the set of ordinates \( \Omega_m \) and corresponding weights \( t_m \) are preselected. Each \( \Psi(r_{\text{start}}, \Omega_m) \) is calculated using the method of Sec. 5.1.

It should also be noted that Eq. (17) for the value of the reduced source at a particular point is not simply proportional to the difference of two arbitrary estimates of the scalar flux; rather, it is the simplification of the right-hand side of Eq. (3).

6 Numerical results

Some preliminary results are given in this section for two problems: a three-dimensional “slab” and a cube. It must be noted that in conventional (i.e., non-adaptive) Monte Carlo, the statistical uncertainty is governed by and calculated using the Central Limit Theorem, which requires that the samples be independent and identically distributed (e.g., all source particles must be sampled from the same distribution). In the reduced source method, the probability distributions change from stage to stage. At present, there is no mathematical theorem that allows estimates of statistical uncertainty in adaptive Monte Carlo methods. Therefore, the results of this section do not include statistical uncertainties.

6.1 The Three-Dimensional “Slab”

The theory and algorithms of this paper were first tested on a three-dimensional version of a homogeneous slab problem with isotropic scattering that had been analyzed in previous work [Lichtenstein, 1999]. There, the slab was 1 cm thick with \( \Sigma_t = 1 \) and \( \Sigma_c = 0.5 \), and the source \( S(r_x, \Omega) \) of Eq. (2) was \( \mu \) for \( x = 0 \) and \( \mu > 0 \):
\[ \mu \psi(0, \mu) = \mu , \mu > 0 , \quad (21) \]
or
\[ \psi(0, \mu) = 1 , \mu > 0 . \quad (22) \]
Lichtenstein [1999], using the reduced source method to compute the coefficients of Cases's solution rather than the coefficients of a Legendre expansion of the scalar flux, reported values of the angular flux in the center of the slab for $\mu = 0.4995$ and $\mu = 0.5005$. His results agreed with those of Kong [1998], who used the reduced source method to compute the coefficients of a Legendre expansion of the scalar flux, to about one part in $10^5$. Lichtenstein's [1999] results are shown in Table 1.

To apply the methods of this paper, the same homogeneous slab was used but given dimensions $1 \text{ cm} \times 10^6 \text{ cm} \times 10^6 \text{ cm}$. This three-dimensional slab had the same cross sections as the one-dimensional slab. The source $S(r, \Omega)$ of Eq. (2) was

$$S(r, \Omega) = S(0, y, z, \mu, \omega) = \mu, \quad \mu > 0 .$$  \hspace{1cm} (23)

In the one-dimensional slab, the surface source is assumed constant in the "y" and "z" directions. In three dimensions, the same rule applies, and the total source strength [using the source of Eq. (23)] is

$$S = \int_0^1 dy \int_0^1 dz \int_4\Omega S(r, \Omega)$$

$$= \int_0^1 dy \int_0^1 dz \int_0^{2\pi} \frac{d\phi}{2\pi} \int_0^\phi \frac{d\theta}{2} \mu$$

$$= \frac{\pi}{4} .$$  \hspace{1cm} (24)

The estimate of $\int d\Omega \psi(r, \Omega)$, as discussed in Sec. 5.2 [c.f. Eq. (20)], used the same standard $S_{16}$ quadrature set used in the discrete-ordinates code THREEDANT [Alcouffe et al., 1997]. In addition, the Monte Carlo calculation used 1000 particles per stage and Legendre polynomials of order 10 in each coordinate direction [i.e., $I = J = K = 10$ in Eq. (12)].

The angular fluxes computed at Lichtenstein's [1999] values of $\mu$ are compared with his results in Table 1. The values would not agree exactly, since the three-dimensional problem explicitly models leakage effects not present in the one-dimensional slab problem. Nevertheless, the close agreement obtained is reassuring.

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>Lichtenstein [1999]</th>
<th>Present study$^a$</th>
<th>Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.4995</td>
<td>0.4795008</td>
<td>0.4795146</td>
<td>0.00288%</td>
</tr>
<tr>
<td>0.0555</td>
<td>0.4801217</td>
<td>0.4801349</td>
<td>0.00296%</td>
</tr>
</tbody>
</table>

$^a$ Values after 13 adaptive stages. Values used for $\eta$ and $\xi$ were 0 and $\sqrt{1-\mu^2}$, respectively.

The average scalar flux in the entire slab and the scalar flux at the midpoint of the slab (denoted $r_{mid}$) were compared with the same values from THREEDANT, whose calculation used a $51 \times 51 \times 51$ mesh grid, a standard $S_{16}$ quadrature set, and a standard diamond difference scheme. For the Monte Carlo calculation, two values of the scalar flux were considered: the Legendre expansion $\tilde{\phi}(r_{mid})$ of Eq. (12), and the angular quadrature $\sum_{m=1}^N t_m \tilde{\psi}(r_{mid}, \Omega_m)$ of Eq. (20).

The errors (with respect to the THREEDANT values) in the quadrature estimate of the scalar flux and the Legendre expansion of the scalar flux at the midpoint, and the average scalar flux in the slab, are plotted together on Figure 1. By any of these measures, it is clear that exponential convergence is being achieved, at least for the first 11 to 13 stages. In subsequent stages the errors fluctuate,
indicating noise in the solution. In general, the method will converge only until the limiting error is reached, suggesting that the minimum error achievable using the approximations of the present application of the method is about 0.01% – that is, with respect to THREEDANT.

This problem was also simulated using the Monte Carlo code MCNP [Briesmeister, 1997] with $10^8$ particles and no variance reduction. A point detector (an F5 tally) for the scalar flux in the center of the slab failed because the slab was too big. The average scalar flux in the slab (an F4 tally), however, was 0.58% higher than that calculated by THREEDANT (the MCNP statistical error was reduced to 0.01%). While an analysis of the differences between general-purpose discrete ordinates and Monte Carlo codes is outside the scope of this work, it is worth noting that the reduced source Monte Carlo method of this paper more accurately represents the results of the latter for the slab problem.

More insight into the nature of the reduced source solutions was gained by analyzing a more difficult problem, a cube.

6.2 The Cube

A true three-dimensional object, a 2-cm $\times$ 2-cm $\times$ 2-cm cube, has also been studied. The small cube is a much more difficult problem than the slab of Sec. 6.2, and detailed analysis of this problem, beyond the preliminary results shown here, is continuing. The cube had the same physical parameters and the same source as the slab: $\Sigma_t = 1$, $\Sigma_r = 0.5$, and $S(0, y, z, \Omega) = \mu$ for $\mu > 0$. In addition, Legendre
polynomials of order 10 were used in each coordinate direction, and 10,000 particles were used per stage. Three standard quadrature sets for the angular quadrature estimate of the scalar flux [Eq. (20)] were studied: $S_8$, $S_{12}$, and $S_{16}$.

Results were compared with a THREEDANT calculation using a standard $S_{16}$ quadrature set, a $51 \times 51 \times 51$ mesh grid, and a standard diamond difference scheme. Results were also compared with an MCNP calculation using $10^8$ particles, no variance reduction, and a track length (F4) tally for the average scalar flux in the cube.

Results for the average scalar flux in the cube are presented in Figure 2. These results are not as sensitive to the order of the quadrature set as was initially anticipated. For all three values of the quadrature order, the reduced source solution is seen to converge exponentially to the DANT solution for the first five stages; subsequently, the noise becomes great. (The statistical uncertainty of the MCNP calculation was 0.01%.)

Results for the scalar flux along the cube centerline ($x$ varying from $0$ to $X$, $y = Y/2$, and $z = Z/2$) are presented in Figure 3, in which the reduced source results are for a 10-stage calculation. It is clear that the angular quadrature of the reduced source method using the $S_{16}$ quadrature set does a much better job of reproducing the THREEDANT results than the angular quadrature of the reduced source method using the $S_8$ quadrature set, even matching the irregularity in the smoothness for $0.2 < x < 0.7$ that is an artifact of the discrete ordinates approximation. (Results from the MCNP simulation of this problem, using 20 point detectors along the centerline, followed nearly exactly the DANT solution but were smooth in this range. These MCNP results are not shown for improved clarity.) Also in Figure 3, it is

![Figure 2: Cube problem results. Volume average scalar flux from DANT, MCNP, and the reduced source method using $S_8$, $S_{12}$, and $S_{16}$ quadrature sets to estimate the scalar flux [Eq. (20)]. The $S_{16}$ calculation was stopped before reaching 40 stages.](image-url)
Figure 3. Cube problem results. Scalar flux along the cube centerline ($y = Y/2$, $z = Z/2$) from DANT and the reduced source method using $S_8$ and $S_{16}$ quadrature sets to estimate the scalar flux [Eq. (20)]. Reduced source results are for a 10-stage calculation.

seen that the Legendre expansion estimate of the scalar flux is insensitive to the order of the angular quadrature, and that it deviates from the DANT results. Not shown in Figure 3 are the results of the reduced source method using the $S_{12}$ quadrature set; these results were generally between the $S_{16}$ and the $S_8$ results.

Figure 3 sheds light on the cause of the large amount of noise observed in the reduced source solutions for the slab (Figure 1) as well as for the cube (Figure 2). The difficulty is not that the quadrature estimate does not accurately represent the “exact” scalar flux; indeed, the quadrature estimate is remarkably accurate, considering the relatively poor value of the scalar flux scattering source that is used to compute it via Eq. (18).

One possible explanation is that the Legendre expansion of the scalar flux that uses ten terms in each coordinate direction and no cross terms [cf. Eq. (12)] may not be sufficient to accurately represent the flux in the cube. On the other hand, consider a thought experiment in which an infinite expansion with all cross terms could be computed [cf. Eq. (6)]. To what solution would such a “perfect” expansion converge? It is by no means clear that the expansion would converge to the DANT solution, the MCNP solution, or even the angular quadrature solution of the reduced source method [Eq. (20) with a finite quadrature order $M$]. In this latter case, the reduced source of Eq. (3) and particularly Eq. (10) will not go to zero, and the solution will exhibit the noise seen on Figures 1 and 2. In short, this is an unresolved issue pointing to a possible inconsistency in the use of the angular quadrature to estimate the angular flux integral. This issue is currently under investigation.
7 Conclusions

Evidence of exponential convergence for solutions of three-dimensional Monte Carlo problems using the reduced source method has been observed. The method implemented as described in this paper, however, is computationally expensive, and several details remain to be improved, chief among which is the evaluation of the residual source. It is possible that the source evaluation procedure used in this work has introduced an internal inconsistency that inhibits the ability of the method to converge. This issue is under investigation.

Another issue that has not been addressed in this work is whether there is a useful metric for gauging the convergence rate. (The metric used here, the error relative to another answer, is not, of course, useful in general.) It is expected that the (absolute) value of the residual scalar flux and the residual angular flux should decrease everywhere with each adaptive stage, providing an estimate of the convergence rate. This issue is also under investigation.

Although major challenges remain, the Monte Carlo reduced source method has the potential to increase significantly the efficiency of Monte Carlo calculations, while simultaneously providing global solutions to three-dimensional transport problems.

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