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Time Scaling of The Rates of Produced Fluids in Laboratory Displacements

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Abstract

In a recent study¹ it was shown that the ratio of the flow rates of the produced fluids in an immiscible displacement can be used to identify geometrical and petrophysical characteristics of a reservoir. In this paper we develop an extended version of this approach and apply it to displacements in 1-D laboratory cores. Because of the possible importance of capillary effects, we derive asymptotic theories for the time dependence of the produced fluids ratio in the limits where capillarity is negligible and where it dominates, respectively. Attention is paid to the capillary end effect at the outlet and to late times.

It is shown that the exponent in the power-law dependence of the relative permeability to saturation, near its residual value, can be determined from a segment of the curve of the outlet fractional flow sometime after breakthrough. The asymptotic predictions are verified by obtaining an analytical solution to the full problem, including end-effects, for a model case corresponding to Burgers' equation. Solutions are obtained for both drainage and imbibition. Experimental results for the high-rate displacement of some gas-liquids pairs are then reported. In agreement with the theoretical predictions, a power-law segment in the outlet fractional flow curve can be identified, before capillary effects set in. The exponents obtained are discussed in terms of pore-scale models for gas-liquid displacement.

INTRODUCTION

Relative permeabilities are important functions for the modeling and prediction of the recovery of fluids from porous media. They express integrated information on the pore structure topology and geometry and on the pore occupancy by the corresponding fluids. Of particular interest is their dependence near the end-point saturations (corresponding to residual or trapped fluids), where a power-law behavior is expected. Sensitivity studies using numerical simulation have shown that this behavior has the most important effect on recovery rates. Reflecting the same fact, most of the empirical models used (e.g. Corey, see Bear²) contain a power law in the residual saturation region. From theoretical considerations, this power-law dependence reflects two different mechanisms, the trapping of a disconnected non-wetting phase, where film flow is absent, in the case of imbibition, and the continuous drainage of a wetting phase, through film flow, in the case of drainage. In the latter case, the residual saturation is theoretically equal to zero.

The trapping of a non-wetting phase by capillarity can be described by percolation theory and its extensions, such as invasion percolation³ and gradient percolation⁴. Percolation theory predicts that the approach of various functions to the trapped saturation satisfies a power law with universal exponents, independent of the local pore structure. For example, Wilkinson⁵ showed that the relative permeability of the displaced phase, k_{rd} , near the trapping threshold in a random medium obeys the scaling

$$k_{rd} \sim (S - S_r)^b \tag{1}$$

where S is the displaced phase saturation and subscript r denotes trapped. The important parameter in (1) is exponent b , which depends only on the dimensionality of the pore network through the relation

$$b = \frac{\mu}{1 + \beta} \tag{2}$$

Here μ is the conductivity exponent and β the percolation exponent, both of which are universal. For example, in 3-D we have $\mu \approx 2$, $\beta = 0.45$, hence $b \approx 1.38$.

This analysis applies under the assumption that the displaced phase is non-wetting, thus it becomes trapped by capillary forces. However, if the displaced phase is wetting, it can escape through film flow. Now drainage of the wetting phase continues, albeit at low rates. Film flow occurs either in the form of thick films and corner (or pocket) flows or through continuous thin films. In such cases, the residual saturation will ultimately vanish, while the exponent of the relative permeability curve is different from that for drainage. Theoretical results for the dependence of the relative permeability of the wetting phase under conditions of film flow were obtained by de Gennes⁶, Novy et al.⁷ and Toledo et al.⁸ In the thick-film flow regime, de Gennes' theory predicts an exponent of the form

$$b = \frac{5 - D}{3 - D} \quad (3)$$

where D is the fractal dimension of the pore surface. For values of D in the order of 2-2.5, one obtains estimates for b in the range 3-5. The theory was based on the assumption of parallel pores. If the flow is controlled by thin films, the power-law exponent is additionally related to the power m of the dependence of the disjoining pressure Π on the film thickness h (namely $\Pi(h) \sim h^{-m}$), through the relation

$$b = \frac{3}{m(3 - D)} \quad (4)$$

(see Toledo et al.⁸). For values of D in the order of 2.3-2.6 and $m = 1/2$, one obtains estimates for b in the range 8.4 to 16.8. These values are significantly higher and different than for drainage.

The experimental determination of the relative permeability functions, including their behavior near the trapped saturation, is usually done using steady- or unsteady-state displacements. The well-known JBN technique⁹ provides information on the relative permeabilities in the saturation region after breakthrough of the injected fluid. It is based on an analysis of fluid rate production data and of the pressure drop across the sample, using the classical Buckley-Leverett equation¹⁰. In general, however, the method is most accurate away from trapped saturations, where relative permeabilities are not too small. An alternative for determining the exponent of the power law dependence near residual saturations is to consider the asymptotic behavior at late times. In a recent study¹, we showed that for

displacements in porous media and various injection patterns, the variation of the ratio of the produced flow rates as a function of time can be used to identify certain geometrical and petrophysical characteristics of the medium. In particular, for an 1-D displacement this ratio can be used to estimate the relative permeability exponents. Motivated by this finding, the objective of this paper is to evaluate the potential of such a method as a diagnostic tool for displacements in laboratory cores. For this purpose we will assess the conditions for which the approach of Ref. 1 is applicable, provide a suitable extension and subsequently apply to experimental data in gas-liquid displacements.

Deriving an expression for the time dependence of the ratio of the produced fluids is straightforward in the absence of capillary effects. Capillarity will affect saturation profiles, and possibly flow rates, in places where permeability varies with position. A most extreme, but routine, example of capillary heterogeneity occurs at the ends of the core sample, where the permeability contrast is theoretically infinite and gives rise to well known end effects. Effects of capillary heterogeneity were analyzed in detail under steady-state flow conditions¹¹. In unsteady-state displacements, most of our understanding comes from numerical investigations¹². Analytical results do exist for a number of problems involving counter-current imbibition^{13,14}. In the latter, however, the process is controlled by capillary dispersion, rather than by viscous forces, as is the case in the Buckley-Leverett problem. To assess the effect of capillarity we will consider the influence of end effects, other effects of capillary heterogeneity, and late-time regimes, where capillarity may be important, even at high rates of displacement. For this purpose, boundary-layer analyses and late-time asymptotics will be provided. These will be verified using analytical solutions of the full problem based on a simple model that follows Burgers' equation¹⁵. The latter is one among a few nonlinear equations of the second order, which are exactly solvable¹⁶. The results will then be applied to obtain relative permeability exponents from a set of gas-liquid displacements.

The paper is organized as follows: We first derive the basic theory for the dependence of the ratio of the produced fluids as a function of time in homogeneous media and under viscous control. Then, the end-effect at the production end is analyzed using a boundary layer description, when capillarity is small. Under the same conditions, we derive the dependence for late-time regimes, where capillarity may be significant. A discussion of more

general effects of heterogeneity is also provided. Subsequently, we verify the asymptotic results analytically, by constructing solutions of a model problem based on Burgers' equation. Experiments are then reported involving the displacement of various liquids (brine, methanol and toluene) with methane at high rates. The exponents obtained are interpreted using pore scale models, which allow inferring possible flow mechanisms near the residual saturation.

THEORY

The equation describing 1-D immiscible 2-phase flow in porous media, where capillary forces are accounted for, is well known. For homogeneous media, it reads in dimensionless form

$$u_t + (F(u))_x - \epsilon(\mathcal{D}(u)u_x)_x = 0 \quad (5)$$

where u is the reduced saturation of the displacing phase (namely, $u = \frac{S-S_r}{S_m-S_r}$, where S_m is the maximum saturation), $F(u)$ is its fractional flow function and $\mathcal{D}(u)$ is a dimensionless capillary diffusivity. In the general case, the fractional flow function has the form

$$F(u) = \frac{1 - N_{gx}k_{r1}(u)}{\left(1 + \frac{\kappa(u)}{M}\right)} \quad (6)$$

where $\kappa(u)$ is the ratio of the relative permeabilities of displaced to displacing phases, $M = \frac{\mu_2}{\mu_1}$ is the ratio of the corresponding viscosities, and $N_{gx} = \frac{(\rho_2 - \rho_1)kg_x}{q_T\mu_1}$ is the gravity number, where g_x is the acceleration of gravity component in the direction of displacement, subscripts 1 and 2 denote initial and injected phases, respectively, and ρ denotes density. However, most of our interest is on the approach near the trapped saturation of the displaced phase. There, gravity effects are secondary and can be neglected. Near that region we can assume, in view of the theoretical expectations cited above, that the ratio of the relative permeabilities obeys a power law

$$\kappa(u) \approx c(1 - u)^b \quad (7)$$

where $b > 1$ and c is a constant prefactor. We must also note that at intermediate saturations, an exponential dependence (the so-called X-plot) was found to successfully describe relative permeability field data¹⁷.

Parameter ϵ is the inverse of a modified capillary number,

$$\epsilon = \frac{\sqrt{k\phi}\gamma}{\mu_1 q_T L} \quad (8)$$

where k denotes permeability, ϕ is porosity, γ is interfacial tension, μ is viscosity, q_T is the injection rate and L is the system length. Small or large ϵ denote, respectively, the limits of viscous or capillary control. The dimensionless capillary dispersivity is

$$\mathcal{D}(u) = (\text{const}) \frac{k_{r1}(u)}{\left(1 + \frac{\kappa(u)}{M}\right)} \left| \frac{dJ}{du} \right| \quad (9)$$

where the constant prefactor accounts for the fact that u is a normalized saturation, J is the conventional J-function representing capillary pressure and the absolute value is taken to allow for a common description for both drainage and imbibition. The dependence of $\mathcal{D}(u)$ near the residual saturation has been discussed by various authors. In the corner (pocket) flow regime, de Gennes⁶ predicted hypodiffusion, namely the vanishing of the capillary dispersivity, following the power law

$$\mathcal{D}(u) \sim (1 - u)^{\frac{1}{3-D}} \quad (10)$$

When thin films dominate, the behavior is more complicated. de Gennes' theory predicts hyperdispersion, where

$$\mathcal{D}(u) \sim (1 - u)^{-1} \quad (11)$$

but Novy et al.⁷ provide arguments for both hypodiffusion and hyperdispersion, depending on the values of the fractal dimension of the pore surface and the exponent of the disjoining pressure.

The problem is completed with the initial condition

$$u = 0 \quad \text{at} \quad t = 0 \quad (12)$$

and the following boundary conditions. At the inlet, assuming injection at a constant rate, we have

$$F(u) - \epsilon \mathcal{D}(u)u_x = 1 \quad \text{at} \quad x = 0 \quad (13)$$

At the outlet, the end effect dictates

$$u = 0 \quad \text{at} \quad x = 1 \quad (14)$$

for the case of drainage, and

$$u = 0 \quad \text{when} \quad F(u) - \epsilon \mathcal{D}(u)u_x = 0, \quad \text{otherwise} \quad u = 1, \quad \text{at} \quad x = 1 \quad (15)$$

for the case of imbibition. Because of its nonlinearity, the above problem must generally be solved numerically. However, analytical solutions are possible in some limiting cases, where the functions $F(u)$ and $D(u)$ take a specific form^{15,16}. One such model will be used below to verify the asymptotic predictions. In the next section we consider, first, the solution in the limit when viscous forces are dominant.

Viscous Control, $\epsilon \ll 1$.

In the case of viscous control, $\epsilon \ll 1$, capillarity is negligible, and equation (5) becomes the well-known Buckley-Leverett equation. Its solution will be referred to as the outer solution and denoted by superscript (0). This solution is singular in the following places: around the shock front, if one develops, around the inlet $x = 0$ at early times of order $O(\epsilon)$, around the outlet $x = 1$ near and after breakthrough, and possibly at late times of order $O(\epsilon^{-1})$. Corrections to the Buckley-Leverett solutions in these limits will be discussed in later sections.

1. The outer solution

In the small ϵ limit, we obtain

$$u_t^{(0)} + F(u^{(0)})_x = 0 \quad (16)$$

the solution of which is obtained by the method of characteristics,

$$F'(u^{(0)}) = \frac{x}{t} \quad (17)$$

This solution is also valid for variable injection rates, where t expresses the fraction of pore volumes injected. Of interest to this paper is the reduced saturation at the outlet $u^{(0)}(1) = u_L^{(0)}$. From (17), this is readily shown to satisfy

$$\frac{\kappa'(u_L^{(0)})}{\left(1 + \frac{\kappa(u_L^{(0)})}{M}\right)^2} = -\frac{M}{t} \quad (18)$$

Then, the fractional flow of the displaced phase at the outlet, $F_{1,L}^{(0)}$, can be calculated

$$F_{1,L}^{(0)} = 1 - F(u_L^{(0)}) = 1 - \frac{1}{1 + \frac{\kappa(u_L^{(0)})}{M}} \quad (19)$$

The variation of the produced flow rate ratio with time has been used originally by Welge¹⁸, in laboratory displacements, by Omoregie and Ershaghi¹⁷ (X-plot method) for field analysis, and by Yortsos et al.¹, as a diagnostic tool in waterfloods.

Consider, now, relatively large times (to be more precisely defined below), where $u_L^{(0)}$ is sufficiently large for equation (7) to apply. Substitution of the latter in (18) gives after some manipulations the result

$$\left(\frac{F_{1,L}^{(0)}}{1 - F_{1,L}^{(0)}}\right)^{\frac{1}{b}} \approx bc^{\frac{1}{b}} M^{-\frac{1}{b}} t \quad (20)$$

This can be further re-arranged to read

$$\log\left(\frac{F_{1,L}^{(0)}}{1 - F_{1,L}^{(0)}}\right) = b \log(t F_{1,L}^{(0)} (1 - F_{1,L}^{(0)})) + \text{const} \quad (21)$$

Thus, under conditions of viscous control and at sufficiently large times, a log-log plot of $\frac{F_{1,L}^{(0)}}{1 - F_{1,L}^{(0)}}$ vs $t F_{1,L}^{(0)} (1 - F_{1,L}^{(0)})$ will result into a straight line with slope b . By definition ($b > 1$), the latter slope must be larger than 1. Equation (21) is a more general and, as it will turn out, a better diagnostic tool for relative permeability exponents, than what was presented in Ref. [1] (see also discussion below). In the latter reference, we considered the further limit of small $F_{1,L}^{(0)}$, to obtain the power law dependence

$$F_{1,L}^{(0)} \approx \frac{\kappa(u_L^{(0)})}{M} \approx \frac{c}{M} \left(\frac{M}{c b t} \right)^{\frac{b}{b-1}} \quad (22)$$

from which one can infer exponent b . For reasons that will become clear below, in this paper we elected to consider the more general expression (21). An important reason is that the use of such a plot allows to differentiate sharply the power-law regime from that of the X-plot. Indeed, if X-plot conditions were applicable, in which case the following relation $\kappa(u) \sim \exp(-du)$ is assumed to apply, the expression equivalent to (21) reads

$$1 = (\text{const}) \, t F_{1,L}^{(0)} (1 - F_{1,L}^{(0)}) \quad (23)$$

suggesting that in the appropriate plot, the combination $t F_{1,L}^{(0)} (1 - F_{1,L}^{(0)})$ is a constant. In simulations and experiments to be described below, this property will be used to differentiate between the various regimes.

To verify the above results, we computed the solution of problem (18) and plotted the results in the logarithmic plot suggested by (21). Results for $b = 2$ and $b = 5$ and for four different values of M are shown in Figures 1a and 1b, respectively. In these examples, we have taken the relative permeability dependence, $\kappa(u) = (1 - u)^b / u^2$. The arrows in the curves of Figure 1 point in the direction of increasing time. As expected, for sufficiently small $F_{1,L}^{(0)}$, at sufficiently large times, the plots show the existence of a straight segment with slope b equal to what is theoretically expected, regardless of the value of M . The latter only affects the results by shifting downwards the region of applicability of the power-law regime as M increases. The straight line segment is preceded by an earlier-time regime in which X-plot conditions apply, and in which a constant ordinate is expected (region near the vertical slope). At least for the simulations of Figure 1 it is apparent that this region does not last for an extended period in the logarithmic coordinates. The examples in Figure 1 confirm the validity of the above analysis and its scaling predictions in the absence of capillarity. The results will be contrasted to the experiments, to be analyzed later in this paper.

Equation (21) was derived assuming t sufficiently larger than 1 and the absence of capillary effects. As pointed out above, however, even under the condition $\epsilon \ll 1$, the solution (17) becomes singular at various places: at the inlet, at the shock front of the propagating Buckley-Leverett front (if any) and at the outlet, where boundary layers must be inserted

(see Figure 2). These layers have a thickness of order ϵ , in space or time, or around the shock. In addition, capillarity cannot be neglected at large times of the order of $\frac{1}{\epsilon}$. How these affect the above scaling of the effluent flow ratio will be analyzed below.

2. Outlet-end effect

Let t_f denote the time of breakthrough of the injected fluid. In the neighborhood of the outlet and for $t < t_f$, the solution of the full problem is identically equal to $u = 0$ (if $\mathcal{D}(0) = 0$) or it is exponentially small (if $\mathcal{D}(0) \neq 0$), and the viscous solution satisfies both the differential equation and the boundary condition, for either drainage or imbibition. After breakthrough, however, the outer solution at the outlet, $u_L^{(0)}$, does not satisfy boundary conditions (14) or (15), in the respective cases. A boundary layer in terms of the stretched coordinate $\xi = \frac{1-x}{\epsilon}$ must be inserted. Rescaling equation (5) in this coordinate gives to leading-order the equation

$$-(F(u))_\xi = (\mathcal{D}(u)u_\xi)_\xi \quad (24)$$

the integration of which is

$$\int_0^u \frac{\mathcal{D}(u)du}{F(u) - A(t)} = -\xi \quad (25)$$

for the case of drainage, and

$$\int_1^u \frac{\mathcal{D}(u)du}{F(u) - A(t)} = -\xi \quad (26)$$

for the case of imbibition. Parameter $A(t)$ is independent of ξ and is determined by matching the outer limit ($\xi \rightarrow \infty$) of (25) and (26) with the outer solution $u_1^{(0)}$. Then, we find, to leading-order

$$A(t) = F(u_{1,L}^{(0)}) \quad (27)$$

Inserted back in (25) and (26), this expression gives closed-form expressions for the behavior of the saturation near the outlet.

For illustration, we consider the case $F(u) = 1 - (1 - u)^2$ (shown in the schematic of Figure 3), which implies $u_L^{(0)} = 1 - \frac{1}{2t}$ and $F(u_{1,L}^{(0)}) = 1 - \frac{1}{4t^2}$, for $t > t_f = \frac{1}{2}$. We also take for simplicity, $\mathcal{D}(u) = 1$. Direct substitution gives the following expression near the outlet,

$$u = 1 - \frac{1}{2t} \left[\frac{1 + \left(\frac{2t-1}{2t+1} \right) \exp \left(-\frac{1-x}{\epsilon t} \right)}{1 - \left(\frac{2t-1}{2t+1} \right) \exp \left(-\frac{1-x}{\epsilon t} \right)} \right] \quad \text{for } t > \frac{1}{2} \quad (28)$$

for the case of drainage, and

$$u = 1 - \frac{1}{2t} \left[\frac{1 - \exp \left(-\frac{1-x}{\epsilon t} \right)}{1 + \exp \left(-\frac{1-x}{\epsilon t} \right)} \right] \quad \text{for } t > \frac{1}{2} \quad (29)$$

for the case of imbibition.

The variation of u as a function of time and position for the above two cases is shown in Figures 4a and 4b. The well-known end effect is apparent in both drainage and imbibition. However, and even though the saturation profile undergoes a significant change near the outlet, the fluid rates remain *continuous* across the thin boundary layer. Hence, in this limit, expressions (21) and/or (22), which are based on $u_L^{(0)}$, still give at leading-order the correct expressions for the flow ratio at the outlet. This will be demonstrated by a specific example in a later section. Therefore, we conclude that even though they affect the saturation profile, end-effects do not change to leading order the time scaling of the outlet fractional flow rates.

3. Capillary heterogeneity

This also applied to the more general heterogeneous media. There, the permeability variation will induce a corresponding capillary heterogeneity. The effect will be manifested primarily as an additional term to the fractional flow, the strength of which is proportional to ϵ and the gradient of the dimensionless permeability, k_D , as shown in the following equation

$$u_t + (F(u))_x + \epsilon \left(\mathcal{D}(u) \frac{\frac{d\sqrt{k_D}}{dx}}{\frac{J}{J'}} \right)_x - \epsilon (\mathcal{D}(u) \sqrt{k_D} u_x)_x = 0 \quad (30)$$

which is the corresponding version of (5) for heterogeneous media^{11–13}. Steady- and unsteady-state saturation profiles in the presence of capillary heterogeneity have been discussed before^{11–14}.

From (30) it is apparent that the effect of heterogeneity is strongest when the permeability is discontinuous, in which case it acts as an equivalent end-effect. As in the end effect in homogeneous media, however, there exists a sufficiently small ϵ (sufficiently high rate) such that the heterogeneity effect is negligible on the flow rate ratio, although not necessarily on the saturation. Therefore, in the theoretical limit $\epsilon \ll 1$, we expect that equation (21) would still be valid for describing the ratio of the flow rates at the outlet. The more heterogeneous the medium, the smallest the value of ϵ required for the power-law scaling behavior previously derived to apply. For example, Figure 5 reprinted from Ref. [11] shows that even for $\epsilon = 0.1$, heterogeneity affects non-trivially the saturation profile. In general, its effect would be to lower the average value of the saturation profile. (In an unpublished numerical study we have also found that at small values of ϵ , the effective relative permeabilities, hence the effective fractional flow rates, are the same with the point relative permeabilities evaluated at the average saturation.) Therefore, the main effect of heterogeneity would be to delay the onset of the power-law behavior (7) over the sample. Precise estimates on the value of ϵ for which this delay is negligible require knowledge of the detailed permeability field and flow parameters, however.

We conclude that if ϵ is sufficiently small, capillary heterogeneity or end effects, although affecting saturation profiles, would not substantially affect the flow rate ratio, which should approach the scaling given by (21) and/or (22), sometime after breakthrough. Capillarity is expected to affect the eventual approach to a steady-state, at a later time, however. This is addressed next.

4. Late-time behavior

In the large-time limit, equations (25) and (26) show that u approaches a steady state $\bar{u}$. In the case of imbibition, it is the flat profile $\bar{u} = 1$. In the case of drainage, the steady state is a function of ξ , obtained from the solution of

$$\epsilon \int_0^{\bar{u}} \frac{\mathcal{D}(u) du}{1 - F(u)} = 1 - x \quad (31)$$

Our interest in this section is on the rate of approach to this asymptotic state. Consider, first, the case of constant capillary dispersivity.

a. Case of constant dispersivity, $\mathcal{D}_\infty = \mathcal{D}(1) \neq 0$.

In the viscous control region outside the boundary layer at the outlet end, $\bar{u} \rightarrow 1$ as $\epsilon \rightarrow 0$. Defining $v = 1 - u$, and taking into account that in this limit equation (7) is valid, we find that v satisfies the equation

$$v_t + \frac{cb}{M} v^{b-1} v_x \approx \epsilon \mathcal{D}_\infty v_{xx} \quad (32)$$

where we assumed $\mathcal{D}_\infty = \mathcal{D}(1) \neq 0$. This is to be solved subject to the boundary conditions

$$-\frac{1}{M} v^b + \epsilon \mathcal{D}_\infty v_x = 0 \quad \text{at } x = 0 \quad \text{and} \quad v_x = 0 \quad \text{at } x = 1 \quad (33)$$

Because of the assumed constant capillary diffusivity and the fact that $b > 1$, equation (32) admits a solution which decays exponentially in time,

$$v = \exp(-\lambda t) g(x, t) \quad (34)$$

The positive constant λ is determined by substituting in (32) and taking the large-time limit, $\lambda t \gg 1$. Then, the viscous term contributes only a higher-order correction, and g satisfies the eigenvalue problem

$$\epsilon \mathcal{D}_\infty g_{xx} + \lambda g = 0 \quad (35)$$

with

$$g_x = 0 \quad \text{at } x = 0 \quad \text{and at } x = 1 \quad (36)$$

The solution of this problem is straightforward. We find $g \sim \cos \sqrt{\frac{\lambda}{\epsilon \mathcal{D}_\infty}} x$, where the eigenvalue is obtained by applying the boundary condition at $x = 1$, hence

$$\lambda = \epsilon \mathcal{D}_\infty \pi^2 \quad (37)$$

Thus, the solution in the outer region outside the boundary layer, approaches at late times, $t \gg \frac{1}{\epsilon}$, the expression

$$u = 1 - \exp(-\pi^2 \epsilon \mathcal{D}_\infty t) g(x, t) + \dots \quad (38)$$

indicating an exponential decay in time. Regarding the flow rate ratio, we note that the dominant component in the flux of the displaced phase, $1 - F(u) - \epsilon \mathcal{D}_\infty v_x \approx \frac{cv^b}{M} - \epsilon \mathcal{D}_\infty v_x$, is the capillary term $\epsilon \mathcal{D}_\infty v_x$, as the nonlinear term contributes only a higher-order correction. Using (38) we then find in this limit the scaling behavior

$$F_{1,L} \sim \exp(-\pi^2 \epsilon \mathcal{D}_\infty t) \quad (39)$$

Equation (39) shows that at large times, where capillarity dominates, the log-log plot of $\frac{F_{1,L}}{1-F_{1,L}}$ vs $t F_{1,L}(1-F_{1,L})$ is a straight line, but now with slope equal to unity, subject to some logarithmic corrections. To examine the behavior when the capillary dispersivity vanishes, we consider the following case.

b. Hypodiffusive case, $\mathcal{D} \sim (1-u)^{b+n-1}$.

Consider the asymptotic behavior in the large-time limit for the hypo-diffusive case, where $\mathcal{D}(1) = 0$. For this, we need to assume the behavior $|\frac{dJ}{du}| \sim (1-u)^{n-1}$, where $1-b < n < 1$. For example, according to de Gennes' theory⁶ for corner flow, $n = 1 + \frac{1}{3-D} - b$. Now, the equivalent to (32) equation is

$$v_t + \frac{cb}{M} v^{b-1} v_x \approx \epsilon_D (v^{(b+n-1)} v_x)_x \quad (40)$$

where we have incorporated various constants in the modified capillary dispersivity, ϵ_D . By further defining $w = v^{b+n}$, this equation reads

$$w_t + \frac{cb}{M} w^\zeta w_x \approx \epsilon_D w^\gamma w_{xx} \quad (41)$$

where $\zeta = \frac{b-1}{b+n}$ and $\gamma = \frac{b+n-1}{b+n}$. At large times, we take a solution of (41) in the asymptotic form

$$w \sim t^{-\nu} \quad (42)$$

where the exponent ν is to be determined. Substituting in (41) shows that balancing the dominant terms is possible if the transient term is balanced by the capillary term, namely if

$$\nu = \frac{1}{\gamma} = \frac{b+n}{b+n-1} \quad (43)$$

This implies that in the capillary limit, the viscous term makes a negligible contribution to leading order. The contribution to the flux comes only from capillary effects, hence

$$F_{1,L} \sim t^{-\frac{b+n}{b+n-1}} \quad (44)$$

It follows that in this case, the log-log plot of $\frac{F_{1,L}}{1-F_{1,L}}$ vs $tF_{1,L}(1-F_{1,L})$ is a straight line with a slope $b+n$, greater than 1. For example, using de Gennes' theory, we find a slope equal to $1 + \frac{1}{3-D}$, which for $D = 1$, which is the non-fractal limit in his theory, gives the value 1.5. The scaling predicted by (44) was verified with numerical simulations of the full problem, which will not be reported here for the sake of brevity.

In theory, therefore, the capillary control segment, with slope 1 or $1 + \frac{1}{3-D}$ follows the straight-line segment corresponding to viscous control, which has the slope $b > 1$. For the two segments to be clearly distinguished requires sufficiently small ϵ , so that the late-time regime where capillarity is important, $\epsilon t \gg 1$, does not overlap with the viscous control regime after breakthrough, where t is sufficiently large, but not too large, for equation (7) to apply. An analytical example illustrating these regimes will be provided below.

Exact Solution for a Model Case

In the above we showed that the flow rate ratio has a different behavior depending on whether the problem is viscous- or capillary- controlled. To verify these limits, we will use an exact result for a model case in a homogeneous medium, including end effects, based on Burgers equation. This involves the quadratic fractional flow function

$$F(u) = 1 - (1 - u)^2 \quad (45)$$

which corresponds to a power-law near the trapping threshold ($u \sim 1$) with exponent $b = 2$. Function $F(u)$ was plotted in Figure 3. Based on (45), the corresponding Buckley-Leverett

problem ($\epsilon = 0$) admits the spreading wave (no shock) solution, $u^{(0)} = 1 - \frac{x}{2t}$. For a constant dispersivity, $\mathcal{D} = 1$, the general displacement is described by the non-linear equation

$$u_t + 2(1 - u)u_x = \epsilon u_{xx} \quad (46)$$

which through the further substitution $v = 1 - u$, can be transformed to Burgers' equation¹³

$$v_t + 2vv_x = \epsilon v_{xx} \quad (47)$$

This is to be solved subject to the initial condition $v = 1$ at $t = 0$, and the boundary conditions: $v^2 - \epsilon v_x = 0$ at $x = 0$, and

$$v = 1 \quad \text{at} \quad x = 1 \quad (48)$$

for the case of drainage, or

$$1 - v^2 + \epsilon v_x = 0 \quad \text{if} \quad v \neq 0, \quad \text{and} \quad 1 - v^2 + \epsilon v_x \neq 0 \quad \text{if} \quad v = 0, \quad \text{at} \quad x = 1 \quad (49)$$

for the case of imbibition. Now, it is well-known that using the Cole-Hopf transformation,

$$v = -\epsilon \frac{\phi_x}{\phi} \quad (50)$$

Burgers' equation is mapped to the linear heat equation

$$\phi_\tau = \phi_{xx} \quad (51)$$

where we introduced the rescaled time $\tau = \epsilon t$. Then, the initial and boundary conditions become $\phi = \exp\left(-\frac{x}{\epsilon}\right)$ at $\tau = 0$, $\phi = 1$ at $x = 0$, and

$$\frac{\phi_x}{\phi} = -\frac{1}{\epsilon} \quad \text{at} \quad x = 1 \quad (52)$$

for the case of drainage, or

$$\phi = \exp\left[\frac{\tau}{\epsilon^2} - \frac{1}{\epsilon}\right] \quad \text{if} \quad \phi_x \neq 0, \quad \text{and} \quad \phi_x = 0 \quad \text{otherwise, at} \quad x = 1 \quad (53)$$

for the case of imbibition. In this notation, the fractional flow of the displaced phase at the outlet is

$$F_{1,L} = \epsilon^2 \frac{\phi_\tau}{\phi} \Big|_{x=1} \quad (54)$$

for either drainage or imbibition.

Consider, now the solution of this problem in the case of drainage. The corresponding solution for imbibition is described in the Appendix. By subtracting the steady-state solution, $\bar{\phi} = 1 - \frac{x}{1+\epsilon}$, from ϕ and defining the variable $\omega = \phi - \bar{\phi}$, we obtain an initial-boundary value problem which can be solved using classical methods¹⁹. After several manipulations we find the result

$$\omega = \sum_{n=1}^{\infty} Z_n(x) \exp(-\beta_n^2 \tau) \int_0^1 Z_n(\xi) G(\xi) d\xi \quad (55)$$

where Z_n ($n = 1, 2, 3, \dots$) is the eigenfunction

$$Z_n(x) = \frac{\sqrt{2(\epsilon^2 \beta_n^2 + 1)} \sin \beta_n x}{\sqrt{\epsilon^2 \beta_n^2 + 1 + \epsilon}} \quad (56)$$

β_n ($n = 1, 2, 3, \dots$) is the eigenvalue, obtained as the n^{th} root of the algebraic equation

$$\tan \beta_n = -\beta_n \epsilon \quad (57)$$

and $G(x) = \exp\left(-\frac{x}{\epsilon}\right) + \frac{x}{1+\epsilon} - 1$. The solution for ϕ reads

$$\phi(x, \tau) = 1 - \frac{x}{1+\epsilon} - 2 \sum_{n=1}^{\infty} \frac{\sin(\beta_n x) \exp(-\beta_n^2 \tau)}{\beta_n (\epsilon^2 \beta_n^2 + 1 + \epsilon)} \quad (58)$$

from which the saturation $u = 1 + \epsilon \frac{\phi_x}{\phi}$ and other quantities can be explicitly expressed. In particular, we find the following expression for the fractional flow at the outlet,

$$F_{1,L}(\tau) = - \frac{\epsilon^2 (1 + \epsilon) \sum_{n=1}^{\infty} \frac{\beta_n^2 \cos \beta_n \exp(-\beta_n^2 \tau)}{\epsilon^2 \beta_n^2 + 1 + \epsilon}}{\left[\frac{1}{2} + (1 + \epsilon) \sum_{n=1}^{\infty} \frac{\cos \beta_n \exp(-\beta_n^2 \tau)}{\epsilon^2 \beta_n^2 + 1 + \epsilon} \right]} \quad (59)$$

The analytical saturation profiles are plotted in Figure 6 for two different values of the dimensionless time τ (small τ in Figure 6a and large τ in Figure 6b) and for various values

of ϵ . At small ϵ (Figure 6a), the profile is close to the outer viscous solution (Buckley-Leverett) profile, $u^{(0)} = 1 - \frac{x}{2t}$, as expected (curves with $\epsilon = 0.001$ and 0.01 in Figure 6a, where we should also note that for constant τ , different values of ϵ correspond to different times). In both cases, inlet and outlet effects are apparent, the inlet effect delaying the approach of the inlet saturation to its maximum value of 1. At larger times (Figure 6b), the corresponding profiles are almost flat, except near the boundary layer at the outlet. As ϵ increases, the saturation profiles are being controlled by capillarity and lose the linear relationship predicted by the viscous solution.

The variation of $\frac{F_{1,L}}{1-F_{1,L}}$ with $tF_{1,L}(1-F_{1,L})$ for different values of ϵ is shown in Figure 7. At small values of ϵ the plot has the main features of the viscous solution (compare with Figure 1), provided that time is not too large. At early times, before breakthrough, the curve has a negative slope. For sufficiently small times after breakthrough the X-plot regime sets in, and the curve approaches a vertical slope. The regime of the power-law approach to the trapped saturation is the straight line segment that follows with slope $b = 2$. This is as expected from the outer viscous solution, where $b = 2$. This regime is quite apparent for sufficiently small ϵ ($\epsilon < 0.1$ in Figure 7), although its extent decreases with increasing ϵ . Also shown in Figure 7 is the viscous solution (denoted in the figure as the B-L curve), which for the particular fractional flow taken for Burgers' equation satisfies the implicit equation

$$Y = -\log(4t^2 - 1) \quad ; \quad X = \log \frac{1}{4t} \left(1 - \frac{1}{4t^2} \right) \quad (60)$$

in the logarithmic coordinates X, Y of Figure 7. As expected the analytical solution approaches this limiting curve, as ϵ becomes smaller.

At larger times, the curve changes slope, reflecting the onset of the exponential decay regime. According to the asymptotic theory, in this region the plot should be a straight line with a slope equal to unity. We first remark that this is also the limit of the exact solution (59). Indeed, in the limit of large times, the leading-order in (59) is the first term in the series, namely

$$F_{1,L} \approx -2\epsilon^2 \beta_1^2 \cos \beta_1 \exp(-\beta_1^2 \tau) \quad (61)$$

where β_1 is the first root of the algebraic equation. In the small ϵ limit, we further have

$\beta_1 \approx \pi$, thus

$$F_{1,L} \approx 2\pi^2 \epsilon^2 \exp(-\pi^2 \tau) \quad (62)$$

The exponential decay is as predicted theoretically and coincides with (39), where $\mathcal{D}_\infty = 1$. Figure 7 shows that this capillary regime with unit slope is indeed approached at sufficiently large times.

As ϵ increases, the curves in Figure 7 retain the linear character of the viscous solution, but in progressively smaller intervals. In fact, above $\epsilon \approx 0.1$, the capillary regime overlaps the viscous regime, and it is difficult to discern the existence of the latter. For $\epsilon > 10$, all curves practically coincide to a single, capillary-controlled curve. This curve can be obtained by considering equation (59) in the limit $\epsilon \gg 1$. Then,

$$F_{1,L} \approx 2 \frac{\sin \beta_1}{\beta_1} \exp(-\beta_1^2 \tau) \quad (63)$$

where β_1 is the first root of the algebraic equation and, in the large ϵ limit, $\beta_1 \approx \frac{\pi}{2}$. Thus,

$$F_{1,L} \approx \frac{4}{\pi} \exp\left(-\frac{\pi^2}{4} \tau\right) \quad (64)$$

In the X, Y plot of Figure 7, this curve has the implicit dependence

$$Y = -\log \left[\frac{\pi}{4} \exp\left(\frac{\pi^2}{4} \epsilon t\right) - 1 \right] \quad ; \quad X = \log \left[\frac{te^Y}{(1 + e^Y)^2} \right] \quad (65)$$

Similar results were obtained for the case of imbibition, as described separately in the Appendix.

In summary, the above theory suggests that for sufficiently small values of the capillary parameter ϵ , corresponding to sufficiently high rates, a plot of the effluent flow rate ratio in the form $\frac{F_{1,L}}{1-F_{1,L}}$ vs. $tF_{1,L}(1-F_{1,L})$ will give a straight-line segment with a slope greater than one. This is equal to the exponent of the power-law approach of the relative permeability of the displaced phase to its trapped saturation. This segment is followed by a subsequent one with slope equal to one (or with a slope greater than one, in the hypodiffusive case), where capillarity dominates. The above behavior is valid regardless of whether the displacement is drainage or imbibition (see Appendix).

EXPERIMENTS AND DISCUSSION

The theoretical predictions were subsequently used to analyze laboratory experiments. The experiments consisted of a series of room-temperature methane floods of a core sample containing various liquids (brine, brine/methanol mixture and toluene). Humid methane was injected from the top, while maintaining a constant pressure drop of 10 psi. The core sample consisted of a preserved, composite (3 plugs) sandstone with porosity 0.16, permeability 14 md, length 16 cm and pore volume of 30 cc. Since the injection rate was variable, the value of ϵ was estimated at the later times of the displacement, where gas is the main flowing phase. For $\gamma = 50$ mN/m, and a viscosity ratio $M \approx 100$, equation (8) shows that $\epsilon \approx 0.02$ for these experiments.

The measured data consisted of the gas flow rate, the volume of gas injected, the flow rate of the liquid expelled, and the change in the weight of the core at the end of each flood. Figure 8 shows log-log plots of $\frac{F_{1,L}}{1-F_{1,L}}$ vs. $tF_{1,L}(1 - F_{1,L})$ for different experiments, where methane displaces respectively, methanol (MeOH), toluene, brine (KCl), a 1:1 mixture of brine and methanol, and a 3:1 mixture of brine and methanol. All these plots should be compared to the small ϵ curves in the corresponding figure for the Burgers' model, Figure 7. We note similar features in all the experimental results. The curves consist of segment with a negative slope, at early times, followed by a segment with an almost vertical slope, which then becomes a straight line with a slope greater than one, and eventually approaches at large times a segment with slope close to unity. Guided by the previous theory, we identify these regimes as corresponding to a regime before breakthrough, the X-plot regime, the regime of the power-law dependence near the trapped saturation and the capillary regime, respectively.

The segment with the power-law behavior can be identified well in most of the experiments but not as well in the displacement of brine (Figure 8c). The methanol, toluene and KCL-MeOH data (Figures 8a, 8b, 8d and 8e) show well-defined power-law regions in the viscous control regime. However, the region of applicability of the power-law decreases in this sequence, due to the increase of ϵ , since the interfacial tension increases in the sequence, and the increasing overlap with the capillary regime (compare also with Figure 7). The experiment involving the displacement of brine (Figure 8c) has the least well-defined

power-law region, which is also consistent with the high interfacial tension (72 dyn/cm) for this system. From the theory, we can extract the appropriate value of b . We estimate $b = 3.15, 3.28, 3.54, 4.5$ and 4.8 for Figures 8a, 8b, 8c, 8d and 8e, respectively.

The capillary regime has a slope close to unity, with an exponent varying in the range 1.18-1.20 for MeOH and toluene and in the range 1.02-1.05 for the brine solution. It is suggesting either a constant dispersivity (for the latter case) or a weakly-hypodiffusive case (for the former). The first case predicts a slope equal to one, but subject to logarithmic corrections, which are not negligible in this time regime. The second predicts a slope greater than one (and equal to 1.5 in the case of a Euclidean (not fractal) surface), according to de Gennes' theory.

Comparison with the percolation results of Wilkinson⁴, equations (1) and (2), which predict an exponent of the order of 1.38 for 3-D, clearly shows that trapping in the absence of film flow is not the operating mechanism in these experiments. Instead, it is apparent that drainage continues, albeit at slow rates, through the action of film flow. The values of the exponents obtained are consistent with the theory of de Gennes⁶ for pocket flow over a weakly fractal pore-surface (for example, where $D = 2.2$ for $b = 3.5$ and $D = 2.42$ for $b = 4.5$). We note that our results cannot be compared with Novy et al.⁷ or Toledo et al.⁸, whose pore-network simulations do not agree with de Gennes, as their saturations take much smaller values (in the range 10^{-3} - 10^{-1}) compared to those in our experiments (of the order of 10^{-1} and higher). In either of the latter theories, the phase distribution at the pore-network scale is controlled by capillary forces. It is possible, although this was not confirmed, that the trapped saturation in our experiments reflects macroscopic trapping due to permeability heterogeneity (similar to that shown in Figure 5) or viscous fingering at various scales. In such cases, the exponents obtained would reflect drainage through film flow along the displacement patterns in a fashion different than the one predicted by de Gennes. As far as we know, an analysis of this problem does not exist at present.

In summary, we are led to the conclusion that the displacement in the experiments reported is a drainage process, where following the trapping of the displaced phase, drainage continues, albeit at slow rates, through corner and pocket flow. The exponent of the relative permeability obtained, of the order of 3-4, is consistent with de Gennes' theory, with flow

occurring over a weakly fractal pore surface. At the same time, we cannot exclude the possibility of trapping due to heterogeneity or viscous fingering, the slow drainage around which may contribute to analogous exponents. What appears to be certain is that thin film flow did not contribute to the observed behavior in this regime. This is consistent with two facts: that the exponents are much smaller than what would be expected if thin films dominated (order of 6 and higher), and that they appear to be weakly dependent on the chemistry of the displaced fluid, which rules out disjoining pressure effects.

Finally, we must add that in a preliminary report²⁰ of this analysis, we were led to a different conclusion and reported considerably larger exponents, which supported flow dominated by thin films. That analysis was based on analyzing plots of $\log F_{1,L}$ vs. $\log t$, which according to this paper and Ref. [1], should asymptotically approach a straight line with slope equal to $-b/(b-1)$ (for example, compare with (22)). The slopes obtained were close to one, which is possible only if b is large indeed. It was realized in retrospect, however, that in fact we were matching data, which did not pertain to the power-law regime of the curve, but rather to the X-plot regime. In a $\log F_{1,L}$ - $\log t$ plot, this regime is also a straight line with slope 1, but more importantly, it cannot be clearly distinguished from the power-law segment. Using the new plot $\frac{F_{1,L}}{1-F_{1,L}}$ vs $t F_{1,L}(1-F_{1,L})$, on the other hand, allows for a much sharper distinction, as the X-plot regime theoretically corresponds to a vertical slope, which can be clearly separated from the power-law segment, which has a finite slope. To demonstrate the usefulness of this new plot, as a diagnostic tool, we have plotted all the curves for the various experiments in one Figure (Figure 9). The curves have very similar characteristics with the theoretically expected (e.g. compare with Figure 7) in the limit of small capillarity, and allow for the identification of the viscous control regime.

CONCLUSIONS

In this paper, we proposed the use of an asymptotic method, based on the time scaling of the ratio of produced fluids, to infer the relative permeability exponent of the displaced phase near its residual saturation, for immiscible displacements in laboratory cores. We showed that at sufficiently large injection rates, the existence of a power law can be detected, and its exponent inferred, by plotting in an appropriate plot the ratio of the flow rates of the

two fluids at the effluent for some time after breakthrough. In this time interval, the plot is a straight line with a slope equal to the exponent in the relative permeability power law. Capillary end effects do not affect the results to leading order at high rates. However, at sufficiently large times, capillarity becomes important. In such a plot, it is manifested by a straight line with a slope equal to one, if the capillary dispersivity is constant, or greater than one, if the process is hypodiffusive. The theoretical findings were verified by comparison with an exact solution. Application of the results to laboratory displacements of various liquids by methane resulted in exponents for the relative permeability of the wetting phase which were consistent with pore-scale models of corner (or pocket) film flow.

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APPENDIX

In this Appendix we provide the derivation of the exact solution in the case of imbibition. As noted in the main text, for this we will seek a solution of the heat equation (51) subject to the boundary condition (53). The first part of the boundary condition, namely before the breakthrough of the displacing phase, applies first. Hence, for $\tau < \tau_f$, where the breakthrough time τ_f is determined from the condition $\phi_x = 0$, we take

$$\phi = \exp\left[\frac{\tau}{\epsilon^2} - \frac{1}{\epsilon}\right] \quad \text{at } x = 1 \quad (1)$$

The solution of this problem is obtained by standard methods. We find

$$\phi = 1 - x + \sum_1^{\infty} b_n \sin n\pi x \exp(-n^2 \pi^2 \tau) + \sum_1^{\infty} c_n \sin n\pi x \exp\left(\frac{\tau}{\epsilon^2}\right) \quad (2)$$

where the unknown coefficients are

$$b_n = -\frac{2}{n\pi(1 + \epsilon^2 n^2 \pi^2)} \quad (3)$$

and

$$c_n = (-1)^{n+1} \frac{2\pi \epsilon^2 n}{1 + \epsilon^2 n^2 \pi^2} \exp\left(-\frac{1}{\epsilon}\right) \quad (4)$$

The breakthrough time is the solution of the equation

$$\frac{1}{\pi} = \sum_1^{\infty} (-1)^n n b_n \exp(-n^2 \pi^2 \tau_f) + \sum_1^{\infty} (-1)^n n c_n \exp\left(\frac{\tau_f}{\epsilon^2}\right) \quad (5)$$

Equivalently, one may use a different expansion for the solution of this problem, by considering a Laplace transform. Then, it is not difficult to show that τ_f solves the following equation

$$\begin{aligned} 2 = & \sum_{n=-\infty}^{\infty} \left[\exp\left(\frac{2}{\epsilon}(n+1)\right) \left(\operatorname{erfc}\left(\frac{\sqrt{\tau_f}}{\epsilon} + \frac{(n+\frac{1}{2})}{\sqrt{\tau_f}}\right) - s_n \right) \right. \\ & \left. + \exp\left(-\frac{2n}{\epsilon}\right) \left(\operatorname{erfc}\left(\frac{\sqrt{\tau_f}}{\epsilon} - \frac{(n+\frac{1}{2})}{\sqrt{\tau_f}}\right) + s_n \right) \right] \end{aligned} \quad (6)$$

where $s_n = \text{sign}(n + 1/2)$. This equation is better suited both for numerical evaluation and for analytical manipulations. For example, one can show after some manipulations, that the solution of (6) in the limit $\epsilon \ll 1$ is $\tau_f = \frac{\epsilon}{2}$, as expected from the viscous solution case, where $t_f = 1/2$. For completeness, we show in Figure A1 the solution of (6) for different values of ϵ . As expected, the breakthrough time is given by the Buckley-Leverett solution, $t_f = 1/2$ at small ϵ and approaches $t_f = 1$ as the problem is controlled by capillarity.

For times $\tau > \tau_f$, we need to solve again the heat equation, but now using the boundary condition $\phi_x = 0$ at $x = 1$. Based on the initial condition $\phi = \phi(x, \tau_f)$, obtained from the above, we obtain the result

$$\phi = 1 - 2 \sum_1^{\infty} \sin(\lambda_n x) \exp(-\lambda_n^2 (\tau - \tau_f)) \left[\frac{\sin \lambda_n}{\lambda_n^2} - \frac{1}{2} \sum_0^{\infty} f_m(\tau_f) \left(\frac{\sin(\lambda_n - m\pi)}{\lambda_n - m\pi} - \frac{\sin(\lambda_n + m\pi)}{\lambda_n + m\pi} \right) \right] \quad (7)$$

where $\lambda_n = (2n + 1)\frac{\pi}{2}$ and

$$f_m(\tau_f) = b_m \exp(-m^2 \pi^2 \tau_f) + c_m \exp\left(\frac{\tau_f}{\epsilon^2}\right) \quad (8)$$

Although in an explicit form, the solution of the problem requires the numerical evaluation of τ_f . The results for the fractional flow at the outlet, $F_{1,L}$, are plotted in the log-log plot of $\frac{F_{1,L}}{1-F_{1,L}}$ vs $tF_{1,L}(1 - F_{1,L})$ in Figure A2 for various values of ϵ . The curves obtained have similar characteristics with the drainage case discussed in the main text.

FIGURE CAPTIONS

- Figure 1. Log-log plot of $\frac{F_{1,L}^{(0)}}{1-F_{1,L}^{(0)}}$ vs $tF_{1,L}^{(0)}(1-F_{1,L}^{(0)})$ for the outlet fractional flow of the liquid under viscous control, for different values of b and M . (a) $b = 2$, (b) $b = 5$. Arrows indicate the direction of increasing time. The straight-line segment at large times has slope equal to b . The X-plot regime corresponds to the region around the vertical slope.
- Figure 2. Location of boundary layers in the limit of small ϵ .
- Figure 3. The model fractional flow function $F(u) = 1 - (1 - u)^2$.
- Figure 4. The saturation profile in boundary layer coordinates, near the outlet end for the fractional flow of Figure 3: (a) drainage, (b) imbibition.
- Figure 5. Drainage saturation profiles in a heterogeneous medium for $\epsilon = 0.1$, $M = 10$ and for different values of time (0.1, 0.25, 0.50, 0.75, 1.0, 2.0 and ∞): (a) Saturation profiles, (b) the imposed permeability variation. (From Reference [12]).
- Figure 6. Saturation profiles for drainage obtained from the exact solution based on Burgers' equation, for different values of ϵ and two different values of τ : (a) $\tau = 0.01$, (b) $\tau = 10$.
- Figure 7. Log-log plot of $\frac{F_{1,L}}{1-F_{1,L}}$ vs $tF_{1,L}(1-F_{1,L})$ for the outlet fractional flow of the liquid as obtained from the exact solution of the Burgers' equation for drainage, for different values of ϵ . Plotted also is the solution corresponding to the viscous solution. For $\epsilon < 0.1$, a straight line segment with slope 2, corresponding to the viscous solution can be identified. The late-time segment with slope 1 corresponds to capillary control. The two different slopes are separately indicated in the figure.
- Figure 8. Log-log plot of $\frac{F_{1,L}}{1-F_{1,L}}$ vs $tF_{1,L}(1-F_{1,L})$ for the outlet fractional flow in the following experiments: (a) methanol (MeOH) displaced by methane, (b) toluene displaced by methane, (c) brine (KCl) displaced by methane, (d) an 1:1 mixture of brine and methanol displaced by methane, and (e) a 3:1 mixture of brine and methanol

displaced by methane. The slopes of the viscous control and capillary control straight-line segments are 3.15 and 1.21, 3.28 and 1.18, 3.54 and 1.02, 4.5 and 1.04, and 4.8 and 1.05, respectively.

- Figure 9. Log-log plot of $\frac{F_{1,L}}{1-F_{1,L}}$ vs $tF_{1,L}(1 - F_{1,L})$ for the outlet fractional flow of all the curves in Figures 8-12. Comparison with Figure 7 at small ϵ shows that the experimental curves are consistent with the theoretical expectations.
- Figure A1. The breakthrough time t_f as a function of the parameter ϵ for imbibition, obtained from the exact solution corresponding to Burgers' equation. At small ϵ , the breakthrough time approaches the viscous limit $\frac{1}{2}$, while at large ϵ it approaches the capillary limit 1.
- Figure A2. The counterpart of Figure 7 for the case of imbibition. At small ϵ , the two curves are qualitatively similar.

Figure 1(a)

Figure 1(b)

Figure 2

Figure 3

Figure 4a

Figure 4b

Figure 5a

Figure 5b

Figure 6a

Figure 6b

Figure 7

8a. Displacement of MeOH with methane

8b. Displacement of Toluene with methane

8c. Displacement of brine (KCl) with methane

8d. Displacement of KCl-MeOH1:1 with methane

8e. Displacement of KCl-MeOH3:1 with methane

Figure 9

Figure A1

Figure A2