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By

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Simple numerical schemes for the Korteweg-deVries equation

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Abstract

Two numerical schemes, which simulate the propagation of dispersive nonlinear waves, are described. The first is a split-step Fourier scheme for the Korteweg-de Vries (KdV) equation. The second is a finite-difference scheme for the modified KdV equation. The stability and accuracy of both schemes are discussed. These simple schemes can be used to study a wide variety of physical processes that involve dispersive nonlinear waves.

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I. INTRODUCTION

The Korteweg-de Vries (KdV) equation [1]

$$\partial_t u + (1 + u)\partial_x u + \partial_{xxx}^3 u = 0 \quad (1)$$

is the simplest wave equation that incorporates the effects of dispersion and nonlinear convection. Consequently, it is used to model the propagation of a wide variety of waves. Gravity waves on the surface of shallow water [2], magnetosonic waves in conducting fluid [3] and ion-acoustic waves in plasma [4] are well-known examples.

We were motivated by the need to model stimulated Brillouin scattering (SBS) in the context of inertial confinement fusion [5]. SBS is the decay of an incident light wave into a scattered light wave and an ion-acoustic wave [6]. The density perturbation associated with the ion-acoustic wave scatters the incident light wave. In turn, the ponderomotive (PM) force associated with the incident and scattered light waves drives the ion-acoustic wave. To model the evolution of SBS realistically, one must include nonlinear mechanisms that stop the growth of the ion-acoustic wave [7]. Two such effects are harmonic generation and nonlinear detuning. If dispersion is weak energy is transferred efficiently from the fundamental ion-acoustic wave to its harmonics. This nonlinear damping counteracts the PM force and stops the growth of the ion-acoustic wave. If dispersion is strong the energy of the fundamental is only shared with a few harmonics. However, the beating of the second harmonic with the fundamental shifts the natural frequency of the fundamental. This nonlinear frequency shift detunes the ion-acoustic wave from the PM force and stops its growth. Because the KdV equation is the simplest wave equation that incorporates both saturation mechanisms, it plays an important role in the modeling of SBS.

This report is organized as follows: In Sec. II some analytical solutions of the linear dispersion, nonlinear convection and KdV equations are described. These solutions illustrate the wave physics associated with the KdV equation and can be used to test putative numerical schemes. A split-step Fourier scheme is described in Sec. III. This scheme is based on

the split-step scheme for the nonlinear Schroedinger (NS) equation [8], but is modified, as required, for the KdV equation. A finite-difference scheme for the modified KdV equation is described in Sec. IV. This scheme allows the use of periodic and aperiodic (transmitting) boundary conditions. Finally, in Sec. VI the main results of this report are summarized.

II. ANALYTICAL RESULTS

In certain cases the KdV equation can be solved analytically. These analytical solutions illustrate the wave physics encompassed by the KdV equation and provide useful checks of codes that purport to solve the KdV equation numerically.

If the nonlinear convection term is omitted, the KdV equation reduces to the linear dispersion equation

$$\partial_t u + \partial_x u + \partial_{xxx}^3 u = 0. \quad (2)$$

It is shown in App. A that, subject to the initial condition

$$u(0, x) = a \exp(-x^2/2l^2), \quad (3)$$

the solution of Eq. (2) is

$$u(t, x) = \frac{a(2\pi)^{1/2}}{\tau^{1/3}} \text{Ai}\left(\frac{1 + 4\tau\xi}{4\tau^{4/3}}\right) \exp\left(\frac{1 + 6\tau\xi}{12\tau^2}\right), \quad (4)$$

where Ai is an Airy function [9], the time variable $\tau = 3t/l^3$ and the distance variable $\xi = (x - t)/l$. Solution (4) shows that as a wave packet convects, it leaves behind it a dispersive wake. No disturbance appears in front of the wave packet because the group velocities associated with Eq. (2) do not exceed unity.

If the linear dispersion terms are omitted, the KdV equation reduces to the nonlinear convection equation

$$\partial_t u + u\partial_x u = 0. \quad (5)$$

Equation (5) shows that the larger-amplitude parts of the wave convect more quickly than the smaller-amplitude parts. The wave profile steepens and, ultimately, a shock forms. In App. B it is shown that the shock-formation time associated with initial condition (3) is

$$t_s = le^{1/2}/a. \quad (6)$$

Shocks form quickly if the amplitude is large or the scale length is short.

The preceding examples show that linear dispersion tends to retard the wave, whereas nonlinear convection tends to advance the wave. For certain wave profiles these effects cancel. It is shown in App. C that the KdV equation has a two-parameter family of traveling-wave solutions

$$u(t, x) = 3acn^2(k\xi|m), \quad (7)$$

where cn is the cnoidal elliptic function [10,11] of modulus $m = (a/b)^{1/2}$, $k = (b/4)^{1/2}$, $\xi = x - vt$ and $v = (1 + 2a - b)t$. One can think of a as the amplitude parameter and $b^{1/2}$ as the wavenumber parameter. Larger-amplitude waves convect more quickly than smaller-amplitude waves.

For the case in which $a = b$, solutions (7) reduce to the solitary-wave solutions

$$u(t, x) = 3asech^2(k\xi), \quad (8)$$

where $k = (a/4)^{1/2}$ and $v = (1 + a)$. Larger-amplitude waves convect more quickly and have shorter scale lengths than smaller-amplitude waves.

III. SPLIT-STEP SCHEME

In this section a split-step scheme for the KdV equation is described. It is a variation of the scheme for the NS equation, which models optical pulse propagation in fibers [8].

The KdV equation can be rewritten in the form

$$\partial_t u = (L + N)u, \quad (9)$$

where the operators

$$L = -\partial_x - \partial_{xxx}^3, \quad (10)$$

$$N = -u\partial_x u. \quad (11)$$

During each time step dispersion and convection act simultaneously. The idea of operator splitting is to replace each integral step by a sequence of partial steps during which dispersion and convection act independently. One can then use standard (individually optimized) schemes for solving the equations

$$\partial_t u = Lu, \quad (12)$$

$$\partial_t u = Nu. \quad (13)$$

The simplest splitting scheme, which consists of a dispersive step ($\delta t L$) followed by a convective step ($\delta t N$), has a local truncation error of order δt^2 and a global truncation error of order δt . It is not accurate enough for practical use. It is shown in App. D that the symmetric splitting scheme, which consists of a dispersive half-step ($\delta t L/2$) followed by a nonlinear step ($\delta t N$) and a dispersive half-step ($\delta t L/2$), has a local truncation error of order δt^3 and a global truncation error of order δt^2 . Our code is based on the symmetric scheme.

A. Dispersion step

Since Eq. (12) is linear, one can use the Fourier-transformation method to solve it. The transform of the initial wave amplitude

$$\bar{u}(0, k) = F[u(0, x)], \quad (14)$$

where F denotes the transformation operator. The transformed version of Eq. (12) has the solution

$$\bar{u}(\delta t, k) = \bar{u}(0, k) \exp[-i\omega(k)\delta t], \quad (15)$$

where the dispersion function

$$w(k) = k - k^3. \quad (16)$$

The final wave amplitude

$$u(\delta t, x) = F^{-1}[\bar{u}(\delta t, k)]. \quad (17)$$

In the numerical implementation of this method F represents a discrete Fourier transformation (DFT). The theory of the DFT and a subroutine that does it efficiently are described by Press *et al.* [12].

We tested this dispersion scheme by using it to solve Eq. (12), subject to initial condition (3), for $l = 10$ and $t = 256$. Snapshots of the initial and final wave amplitudes are displayed in Fig. 1. Solid lines denote the numerical solutions (obtained with 128 data points) and dashed lines denote the exact analytical solutions (4). The agreement between the analytical and numerical solutions is excellent.

B. Convection step

Second-order Runge-Kutta (RK) schemes are often used to solve the NS version of Eq. (13). However, second-order RK schemes are unstable when applied to Eq. (13), as shown in App. E. To solve Eq. (13), we used the method of characteristics together with the three-point Lagrange formula [12,13] that interpolates the values of u from the characteristic curves, on which they are known, to the Eulerian grid points, at which they are needed. This scheme is referred to as the characteristic-interpolation (CI) scheme.

One can appraise the CI scheme by applying it to the linear convection equation

$$\partial_t u + \partial_x u = 0. \quad (18)$$

Suppose that the values of u are known at time $i\delta t$, at the Eulerian positions $x_j^E = j\delta x$. Then the values of u are known at time $(i+1)\delta t$, at the Lagrangian positions $x_j^L = x_j^E + \delta t$. Let u_{j-1}^L , u_j^L and u_{j+1}^L denote the values of u at the Lagrangian positions x_{j-1}^L , x_j^L and x_{j+1}^L , respectively. In the interval $x_{j-1}^L \leq x \leq x_{j+1}^L$ one can approximate u by the three-point interpolating polynomial

$$u^{(3)}(x) = u_j^L + \left(\frac{u_{j+1}^L - u_{j-1}^L}{2\delta x} \right) (x - x_j^L) + \left(\frac{u_{j+1}^L - 2u_j^L + u_{j-1}^L}{\delta x^2} \right) \frac{(x - x_j^L)^2}{2}. \quad (19)$$

It follows from Eq. (19), and the relation $x_j^E = x_j^L - \delta t$, that

$$u_j^{i+1} = u_j^i - c(u_{j+1}^i - u_{j-1}^i)/2 + c^2(u_{j+1}^i - 2u_j^i + u_{j-1}^i)/2, \quad (20)$$

where $c = \delta t/\delta x$ is the convection number. Although our method of solution differs from that of Lax and Wendroff, Eq. (20) is the Lax-Wendroff formula [12,13].

To make a stability analysis of Eq. (20), one assumes that

$$u_j^i \propto r^i \exp(ikj\delta x), \quad (21)$$

where k is a wavenumber. By making substitutions of the form (21) in Eq. (20), one can show that

$$r = 1 - c^2[1 - \cos(k\delta x)] - ic \sin(k\delta x). \quad (22)$$

It follows from Eq. (22) that

$$|r|^2 = 1 - 4c^2(1 - c^2) \sin^4(k\delta x/2). \quad (23)$$

Thus, for $c \leq 1$ the CI scheme is stable. For the special case in which $c = 1$, formula (20) reduces to the exact formula $u_j^{i+1} = u_{j-1}^i$. It follows from Eq. (23) that the magnitude of r differs from 1 by an amount that is of order $(k\delta x)^4$. For well-resolved wavenumbers ($k\delta x \ll 1$) one can neglect this difference. By making the substitution $r = \exp(-i\omega\delta t)$ in Eq. (22) one finds that

$$\sin(\omega\delta t) \approx c \sin(k\delta x) \quad (24)$$

and, hence, that

$$\omega \approx k - [(1 - c^2)\delta x^2/6]k^3. \quad (25)$$

Thus, for $c < 1$ the CI scheme is weakly dispersive. For the special case in which $c = 1$, the CI scheme is not dispersive. (As noted above, it is exact.)

The preceding analysis does not apply directly to the nonlinear convection equation. However, because of the similarity between the linear and nonlinear equations, one might expect the CI method to be stable when $|u|c \ll 1$, where $|u|$ is the maximal wave amplitude. This expectation was verified numerically.

We tested our convection scheme by using it to solve Eq. (13), subject to initial condition (3), for $l = 10.0$ and $t = 15.7$, which is 95% of the shock-formation time (6). Snapshots of the initial and final wave amplitudes are displayed in Fig. 2. Solid lines denote the numerical solutions (obtained with 256 data points) and dashed lines denote the exact analytical solutions. The agreement between the analytical and numerical solutions is excellent.

C. Test of the split-step scheme

To check the complete split-step scheme we solved the KdV equation numerically, subject to the initial condition associated with the solitary-wave solution (8), for $a = 0.1$ and $t = 1320$. Snapshots of the initial and final wave amplitudes are displayed in Fig. 3. Solid lines denote the numerical solutions (obtained with 256 data points) and dashed lines denote the exact analytical solutions. Although the numerical wave speed associated with the split-step scheme is slightly lower than the actual wave speed $(1+a)$, the wave profile is preserved very well.

IV. FINITE-DIFFERENCE SCHEME

The results of Sec. III show that our split-step scheme works very well. The main disadvantage of this scheme is the requirement for periodic boundary conditions, which is inherent in the use of Fourier transforms. In this section a finite-difference scheme is described, which allows the use of periodic and aperiodic boundary conditions.

A. Crank-Nicholson scheme

Since the KdV equation (1) models dispersive wave propagation, one might use the Crank-Nicholson (CN) scheme [12,13] to solve it. One can appraise the CN scheme by applying it to the linear KdV equation (2). The finite-difference version of this equation is

$$\begin{aligned} & u_j^{i+1} + c(u_{j+1}^{i+1} - u_{j-1}^{i+1})/4 + d(u_{j+2}^{i+1} - 2u_{j-1}^{i+1} + 2u_{j-1}^{i+1} - u_{j-2}^{i+1})/4 \\ & = u_j^i - c(u_{j+1}^i - u_{j-1}^i)/4 - d(u_{j+2}^i - 2u_{j-1}^i + 2u_{j-1}^i - u_{j-2}^i)/4, \end{aligned} \quad (26)$$

where the convection number $c = \delta t / \delta x$ and the dispersion number $d = \delta t / \delta x^3$. By substituting Ansatz (21) in Eq. (26), one can show that

$$r = \frac{1 - ic \sin(k\delta x)/2 - id[\sin(2k\delta x) - 2\sin(k\delta x)]/2}{1 + ic \sin(k\delta x)/2 + id[\sin(2k\delta x) - 2\sin(k\delta x)]/2}. \quad (27)$$

It follows from Eq. (27) that the CN scheme is unconditionally stable. This feature is an advantage of the CN scheme. However, it is also important to determine the numerical dispersion associated with the CN scheme. By making the substitution $r = \exp(-i\omega\delta t)$ in Eq. (27), one can show that

$$\sin(\omega\delta t) = \frac{c \sin(k\delta x) + d \sin(2k\delta x) - 2d \sin(k\delta x)}{1 + [c \sin(k\delta x) + d \sin(2k\delta x) - 2d \sin(k\delta x)]^2/4}. \quad (28)$$

For well-resolved wavenumbers ($k\delta x \ll 1$) Eq. (28) reduces to

$$\omega \approx k - k^3 - [(2 + c^2)\delta x^2/12]k^3. \quad (29)$$

It follows from Eq. (29) that weak numerical dispersion is associated with the CN scheme for all values of c . The origin of this spurious dispersion is the convection terms in Eq. (26). Although one can limit the magnitude of the dispersion coefficient by requiring that $c \sim 1$, even weak dispersion will distort the wave on the long time scales of interest to us. This feature is a disadvantage of the CN scheme. Furthermore, since the CN scheme is implicit, its application to the KdV equation, which contains a nonlinear convection term, requires further work.

B. Leap-frog scheme

Alternately, one might use the leap-frog (LF) scheme. The finite difference version of Eq. (2) is

$$u_j^{i+1} = u_j^{i-1} - c(u_{j+1}^i - u_{j-1}^i) - d(u_{j+2}^i - 2u_{j-1}^i + 2u_{j-1}^i - u_{j-2}^i). \quad (30)$$

By substituting Ansatz (21) in Eq. (32) and using the fact that $r - 1/r = -2i \sin(\omega \delta t)$, one can show that

$$\sin(\omega \delta t) = c \sin(k \delta x) + d[\sin(2k \delta x) - 2 \sin(k \delta x)]. \quad (31)$$

In the absence of dispersion ($d = 0$), it follows from Eq. (31) that the LF scheme is stable for $c \leq 1$. For well-resolved wavenumbers the convection terms in Eq. (31) reduce to

$$\omega \approx k - [(1 - c^2) \delta x^2 / 6] k^3. \quad (32)$$

For the special case in which $c = 1$, numerical dispersion is absent and Eq. (30) reduces to $u_j^{i+1} - u_j^{i-1} = -(u_{j+1}^i - u_{j-1}^i)$, which is consistent with the exact formula. In the absence of convection ($c = 0$), analysis of Eq. (31) shows that the LF scheme is stable for $d \leq 2/(3\sqrt{3})$. For well-resolved wavenumbers the dispersion terms in Eq. (31) reduce to

$$\omega \approx -k^3. \quad (33)$$

It follows from Eqs. (32) and (33) that one can eliminate the (third-order) numerical dispersion associated with the LF scheme. Furthermore, since the LF scheme is explicit, its application to the nonlinear convection equation is trivial. These features are advantages of the LF scheme. However, the constraint on δt associated with the second stability condition renders the LF scheme impractical.

C. Hybrid scheme

Recall that one derives the KdV equation from the exact fluid equations by making the assumptions that the amplitude of the wave is small and the scale length of the wave is long.

It follows from these assumptions that $\partial_t u \approx -\partial_x u$ and, hence, that the KdV equation is physically equivalent to the equation

$$(1 - \partial_{xx}^2)\partial_t u + \partial_x(u + u^2/2) = 0. \quad (34)$$

We refer to Eq. (34) as the modified KdV equation. As discussed in Sec. IIIA, one can apply the LF scheme to the convection terms in Eq. (34). When one considers the extension of this scheme to the whole equation, the advantages of the modification become clear. First, the term ∂_{xxt}^3 makes the LF scheme implicit, like the CN scheme. Second, it converts the pentadiagonal system associated with Eq. (26) to a tridiagonal system. The finite-difference version of the modified KdV equation is

$$\begin{aligned} -eu_{j-1}^{i+1} + (1 + 2e)u_j^{i+1} - eu_{j+1}^{i+1} = & -eu_{j-1}^{i-1} + (1 + 2e)u_j^{i-1} - eu_{j+1}^{i-1} \\ & -c[f(u_{j+1}^i) - f(u_{j-1}^i)], \end{aligned} \quad (35)$$

where $e = 1/\delta x^2$ and $f(u) = u + u^2/2$. By substituting Ansatz (21) in the linear version of Eq. (35), one can show that

$$\sin(\omega\delta t) = \frac{c \sin(k\delta x)}{1 + 4e \sin^2(k\delta x/2)}. \quad (36)$$

For well-resolved wavenumbers Eq. (36) reduces to

$$\omega \approx k - k^3 - [(1 - c^2)\delta x^2/6]k^3. \quad (37)$$

Thus, one can eliminate the (third-order) numerical dispersion associated with formula (35) and maintain a practical time step simultaneously. This property is the third advantage of the hybrid scheme.

Since formula (35) describes a two-step scheme, a separate formula is required for the first step. We used the explicit formula

$$\begin{aligned} u_j^1 = & u_j^0 - c[f(u_{j+1}^0) - f(u_{j-1}^0)]/2 \\ & -d(u_{j+2}^0 - 2u_{j+1}^0 + 2u_{j-1}^0 - u_{j-2}^0)/2, \end{aligned} \quad (38)$$

which is a finite-difference version of the original KdV equation. The local truncation error associated with Eq. (38) is of order δt^2 . One can attain a local truncation error of order δt^3 by adding a correction step. However, we found this addition to be unnecessary.

D. Tests of the finite-difference scheme

To test our hybrid finite-difference scheme we repeated the tests described in Sec. III, which were made using periodic boundary conditions. Although these conditions make the tridiagonal system associated with Eq. (35) periodic, it can be solved by a simple modification of the Thomas algorithm [13,14]. In the first two tests we used the same numbers of data points and the same initial conditions. In the third test we used the same number of data points, but modified the third initial condition because the solitary-wave solution of the modified KdV equation is

$$u(t, x) = 3a \operatorname{sech}^2(k\xi), \quad (39)$$

where $k = (a/4v)^{1/2}$, $\xi = x - vt$ and $v = (1 + a)$. If the modified solitary wave has the same amplitude as the unmodified solitary wave, it propagates at the same speed, but is wider. Snapshots of the final wave amplitudes are displayed in Fig. 4. Solid lines denote the numerical solutions and dashed lines denote the exact analytical solutions. (a) Test of linear convection and dispersion. Although there is a perceptible difference between the analytical and numerical wakes, the finite-difference scheme models linear convection and dispersion very well. (b) Test of nonlinear convection. Although the overall agreement between the analytical and numerical profiles is very good, there is a nascent numerical instability near the wave crest. The onset of this instability can be delayed by reducing the time step, at the expense of introducing numerical dispersion, or by reducing the time and space steps simultaneously. This instability will not complicate our studies of ion-acoustic waves because the dispersion term in the KdV equation prevents these waves from forming shocks. (c) Test of solitary-wave propagation. Although the numerical wave speed associated with the finite-

difference scheme is slightly higher than the actual wave speed, the wave profile is preserved very well. Overall, the finite-difference scheme works well.

E. Aperiodic boundary conditions

The main advantage of our finite-difference scheme is its ability to be used with aperiodic boundary conditions. Physical waves modeled by the KdV equation propagate from left to right. [See the comment preceding Eq. (34).] Consequently, at the left boundary we used the condition

$$u_0^{i+1} = u_b[(i+1)\delta t], \quad (40)$$

where u_b is an arbitrary function of its argument. At the right boundary we replaced the KdV equation by the linear convection equation $(\partial_t + \partial_x)u = 0$, and used a forward difference in time and a backward difference in space to obtain the finite-difference equation

$$u_n^{i+1} = u_n^i - c(u_n^i - u_{n-1}^i). \quad (41)$$

When $c = 1$, Eq. (41) models linear convection exactly. With these modifications, the system associated with Eq. (35) is tridiagonal.

Several types of boundary conditions were discussed by Durrant [14]. Although the physical waves associated with the KdV equation propagate to the right, there are computational waves associated with Eq. (35) that propagate to the left. [See Eq. (36).] Because Eqs. (35) and (41) have different dispersion relations associated with them, these two types of wave are coupled at the right boundary. When physical waves arrive, some of their energy is transferred to computational waves that propagate to the left. At the left boundary the energy associated with the computational waves is transferred to physical waves by condition (40), and the reflection process continues. If the boundary conditions are chosen and implemented properly, the trapped energy should be a small fraction of the incident energy, and a nonincreasing function of time.

To test boundary conditions (40) and (41), and our implementation of them, we launched the solitary wave (39) from the left boundary at $t = -96$. The other simulation parameters were the same as those used for Figs. 3 and 4. At $t = 0$ the wave was in the middle of the box with the correct shape. By $t = 96$ the wave had exited the box through the right boundary. At this time the (energy) reflectivity was 0.7×10^{-5} . Subsequently, the reflectivity stayed below 10^{-5} . Despite the simplicity of our boundary conditions, wave generation and transmission works well enough for our purposes.

V. SUMMARY

In this report two numerical schemes, which simulate the propagation of dispersive nonlinear waves, were described. The first is a split-step Fourier scheme for the KdV equation (1). The second is a finite-difference scheme for the modified KdV equation (35). The stability of each scheme was discussed and the accuracy with which each scheme models linear convection and dispersion, nonlinear convection and solitary-wave propagation was studied. Overall, both schemes work well. The split-step scheme models dispersion and convection more accurately than the finite-difference scheme, but can only be used with periodic boundary conditions. In contrast, the finite-difference scheme can be used with periodic or aperiodic boundary conditions. The implementation of transmitting boundary conditions for the finite-difference scheme was described briefly. These simple schemes can be used to study a wide variety of physical processes that involve dispersive nonlinear waves.

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APPENDIX A: LINEAR CONVECTION AND DISPERSION

In this appendix the solution of Eq. (2), subject to initial condition (3) is described. Since the equation is linear, one can use the Fourier transform method to write its solution as

$$u(t, x) = \int_{-\infty}^{\infty} \bar{u}(0, k) \exp[ikx - iw(k)t] dk, \quad (\text{A1})$$

where the transformed initial amplitude

$$\bar{u}(0, k) = [l/(2\pi)^{1/2}] \exp(-l^2 k^2/2) \quad (\text{A2})$$

and the dispersion function

$$w(k) = k - k^3. \quad (\text{A3})$$

For generality, we rewrite the dispersion function as

$$w(k) = w_1 k + w_3 k^3/6. \quad (\text{A4})$$

By combining Eqs. (A2) and (A4), one finds that the integrand in Eq. (A1) is an exponential function with argument

$$\phi(k) = -iw_3 t k^3/6 - l^2 k^2/2 + i(x - w_1 t)k. \quad (\text{A5})$$

One can do the inversion integral analytically by completing the cube and using the identity [9]

$$\text{Ai}(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(ik^3/3 + izk) dk. \quad (\text{A6})$$

The algebra is tedious, but straightforward. The final result is

$$u(t, x) = \frac{(2\pi)^{1/2}}{\tau^{1/3}} \text{Ai}\left(\frac{1 - 4\tau\xi}{4\tau^{4/3}}\right) \exp\left(\frac{1 - 6\tau\xi}{12\tau^2}\right), \quad (\text{A7})$$

where the time variable $\tau = |w_3|t/2l^3$, the distance variable $\xi = s_3(x - t)/l$ and the sign parameter $s_3 = w_3/|w_3|$. One can check that solution (A7) reproduces the initial condition by using the fact that $\text{Ai}(z) \sim \exp(-2z^{3/2}/3)/(2\pi^{1/2}z^{1/4})$ as $z \rightarrow \infty$ [9].

APPENDIX B: NONLINEAR CONVECTION

The partial differential equation (5) is equivalent to the coupled ordinary differential equations

$$d_t u = 0, \quad (\text{B1})$$

$$d_t x = u. \quad (\text{B2})$$

According to Eqs. (B1) and (B2), the wave amplitude is constant on each characteristic curve, which is a straight line from the initial position $(\xi, 0)$ to the current position (x, t) . The equation that describes the characteristics is $\xi(x, t) = x - u(x, t)t$ or, equivalently, $x(\xi, t) = \xi + u(\xi)t$. Neighboring characteristics intersect when $d_\xi x(\xi, t) = 0$. For initial condition (3), this condition is

$$1 - (at\xi/l^2) \exp(-\xi^2/2l^2) = 0. \quad (\text{B3})$$

The first shock originates from the initial position that corresponds to the maximal coefficient of the t term. It is not difficult to verify that $\xi_s = l$ and, hence, that

$$t_s = le^{1/2}/a, \quad (\text{B4})$$

$$x_s = 2l. \quad (\text{B5})$$

APPENDIX C: TRAVELING WAVES

By making the substitution $u = 3v$, one can rewrite KdV equation as

$$\partial_t v + (1 + 3v)\partial_x v + \partial_{xxx}^3 v = 0. \quad (\text{C1})$$

Traveling-wave solutions of Eq. (C1) are assumed to depend on the single variable $\xi = x - (1+c)t$, where the velocity increment c is to be determined self-consistently. By applying such an assumption to Eq. (C1) and integrating twice, one finds that

$$(d_\xi v)^2 = \alpha + \beta v + cv^2 - v^3, \quad (\text{C2})$$

where α and β are constants of integration. Equation (C2) is equivalent to the equation governing the motion of a particle in a potential well. The terms on the right side of Eq. (C2) constitute the negative of the potential function.

Traveling-wave solutions of Eq. (C1) exist when the potential has three real roots. Since the coefficient of the v^3 term is negative, the wave amplitude must be bounded by the two larger roots. For a wave governed by the KdV equation, changing the wave amplitude by a constant is equivalent to changing the reference frame from which the wave is observed. Consequently, one has the freedom to choose an arbitrary reference value for the wave amplitude. We define the minimum of v , at which $d_\xi v = 0$, to be 0 and denote the three roots by $a - b$, 0, and a , where $b \geq a$. It follows from the preceding discussion that Eq. (C2) can be rewritten as

$$(d_\xi v)^2 = v(a - v)(b - a + v). \quad (\text{C3})$$

By comparing the v^2 terms in Eqs. (C2) and (C3), one finds that $c = 2a - b$.

Equation (C3) is equivalent to the differential form

$$\frac{dv}{[v(a - v)(b - a + v)]^{1/2}} = -d\xi. \quad (\text{C4})$$

By making the substitution $v = aw^2$, one can rewrite Eq. (C4) as

$$\frac{dw}{[(1 - w^2)(1 - m^2 + m^2 w^2)]^{1/2}} = -kd\xi, \quad (\text{C5})$$

where $k = (b/4)^{1/2}$ and $m = (a/b)^{1/2}$. It follows immediately that

$$w(\xi) = \text{cn}^2(k\xi|m), \quad (\text{C6})$$

where cn is the cnoidal elliptic function of modulus m [10,11]. By making the substitutions described before Eqs. (C1) and (C5), one reproduces Eq. (7). The traveling-wave solutions have period

$$\lambda = 4K[(a/b)^{1/2}]/b^{1/2}, \quad (\text{C7})$$

where K is the complete elliptic integral of the first kind [10,11]. Equation (C7) relates b to the amplitude and length of the wave, which are physical quantities.

In the small-amplitude limit ($a \ll b$), solutions (7) reduce to

$$u(\xi) = (3a/2)[1 + \cos(b^{1/2}\xi)], \quad (\text{C8})$$

where $\xi = x - (1-b)t$. In this limit $b^{1/2}$ is the wavenumber and phase velocity is wavenumber dependent.

APPENDIX D: SPLIT-STEP SCHEME

The solution of Eq. (9) can be written as

$$u(\delta t, x) = \exp\{\delta t(L + N)\}u(0, x), \quad (\text{D1})$$

where the exponential of an operator is defined by a series expansion (just as the exponential of a complex number is defined by a series expansion). If L and N were numbers, one could write $\exp\{\delta t(L + N)\} = \exp(\delta tL)\exp(\delta tN)$. However, because L and N are noncommuting operators, the product of the series expansions for $\exp(\delta tL)$ and $\exp(\delta tN)$ does not equal the series expansion for $\exp\{\delta t(L + N)\}$. This inequality means that operator splitting adds errors to the solution scheme.

One can analyze these errors explicitly by using the formula [8]

$$\exp(A)\exp(B) = \exp\{A + B + [A, B]/2 + \dots\}, \quad (\text{D2})$$

where $[A, B] = AB - BA$ and the dots represent a series of higher-order commutators. In the following analysis A and B represent simple combinations of δtL and δtN , so the higher-order commutators are of order δt^3 and can be neglected.

It follows directly from Eq. (D2) that

$$\exp(\delta tL)\exp(\delta tN) = \exp\{\delta t(L + N) + O(\delta t^2)\}, \quad (\text{D3})$$

as stated in the text. By using Eq. (D2) twice, one can show that

$$\exp(B) \exp(A/2) = \exp\{C + O(\delta t^3)\}, \quad (\text{D4})$$

where

$$C = B + A/2 + [B, A]/4, \quad (\text{D5})$$

and

$$\begin{aligned} \exp(A/2) \exp(C) &= \exp\{A/2 + B + A/2 + [A, B]/4 \\ &\quad + [A, A]/8 + [B, A]/4 + O(\delta t^3)\}. \end{aligned} \quad (\text{D6})$$

It follows from Eq. (D6), and the facts that $[A, A] = 0$ and $[B, A] = -[A, B]$, that

$$\exp(\delta t L/2) \exp(\delta t N) \exp(\delta t L/2) = \exp\{\delta t(L + N) + O(\delta t^3)\}, \quad (\text{D7})$$

as stated in the text.

APPENDIX E: STABILITY ANALYSIS

In this section a stability analysis is made of second-order RK schemes applied to the centered-difference-in-space (CS) version of the linear convection equation

$$\partial_t u + \partial_x u = 0. \quad (\text{E1})$$

As a preliminary, consider the ordinary differential equation

$$d_t u = f(u), \quad (\text{E2})$$

where f is a prescribed function. Let the superscript i denote the time $i\delta t$. The RK formula is

$$u^{i+1} = u^i + [(1 - \alpha)s^i + \alpha s^p]\delta t, \quad (\text{E3})$$

where $\alpha > 0$ and the initial slope $s^i = f(u^i)$. The predicted intermediate slope $s^p = f(u^p)$, where the predicted intermediate solution $u^p = u^i + s^i\beta\delta t$. The consistency condition

$$\alpha\beta = 1/2 \quad (\text{E4})$$

ensures that formula (E3) has a local truncation error of order δt^3 and a global truncation error of order δt^2 .

Now consider Eq. (E1). Let the subscript j denote the position $j\delta x$. The CS version of Eq. (E1) is

$$d_t u_j = (u_{j-1} - u_{j+1})/(2\delta x). \quad (\text{E5})$$

According to formula (E3),

$$u_j^{i+1} = u_j^i + (1 - \alpha)c(u_{j-1}^i - u_{j+1}^i)/2 + \alpha c(u_{j-1}^p - u_{j+1}^p)/2, \quad (\text{E6})$$

where $c = \delta t/\delta x$ is the convection number and the predicted intermediate solution

$$u_j^p = u_j^i + \beta c(u_{j-1}^i - u_{j+1}^i)/2. \quad (\text{E7})$$

By substituting Ansatz (21) in Eqs. (E6) and (E7), one can show that

$$r^p = 1 - i\beta c \sin(k\delta x) \quad (\text{E8})$$

and, hence, that

$$r = 1 - ic \sin(k\delta x) - \alpha\beta c^2 \sin^2(k\delta x). \quad (\text{E9})$$

It follows from Eqs. (E4) and (E9) that

$$|r|^2 = 1 + c^4 \sin^4(k\delta x)/4. \quad (\text{E10})$$

Thus, all second-order RK schemes are unconditionally unstable when applied to the CS version of the linear convection equation.

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FIGURES

FIG. 1. Test of linear convection and dispersion. The wave amplitude is plotted as a function of distance for (a) $t = 0$ and (b) $t = 256$. In each figure the solid line denotes the numerical solution and the dashed line denotes the exact analytical solution (4). The Fourier-transformation scheme models linear convection and dispersion very well.

FIG. 2. Test of nonlinear convection. The wave amplitude is plotted as a function of distance for (a) $t = 0.0$ and (b) $t = 15.7$, which is 95% of the shock-formation time (6). In each figure the solid line denotes the numerical solution and the dashed line denotes the exact analytical solution. The characteristic-interpolation scheme models nonlinear convection very well.

FIG. 3. Test of solitary-wave propagation. The wave amplitude is plotted as a function of distance for (a) $t = 0$ and (b) $t = 1320$. In each figure the solid line denotes the numerical solution and the dashed line denotes the exact analytical solution (8). Although the numerical wave speed associated with the split-step scheme is slightly less than the actual wave speed, the wave profile is preserved very well.

FIG. 4. Tests of (a) linear convection and dispersion, (b) nonlinear convection and (c) solitary-wave propagation. The initial conditions and simulation times were described in Figs. 1, 2 and 3, respectively. In this figure the final wave amplitudes are plotted as functions of distance. The solid lines denote the numerical solutions and the dashed lines denote the exact analytical solutions. Overall, the finite-difference scheme works well.