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Exact and Variational Solutions of 3D Eigenmodes in High Gain FELs

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Abstract

Exact solution and variational approximation of eigenmodes in high gain FELs are presented. These eigenmodes specify transverse profiles and exponential growth rates of the laser field before saturation. They are self-consistent solutions of coupled Maxwell-Vlasov equations describing FEL interaction taking into account the effects due to energy spread, emittance and betatron oscillations of the electron beam, as well as diffraction and optical guiding of the laser field. A new formalism of scaling is introduced and based on which solutions in various limiting cases are discussed. In addition, a fitting formula is obtained from interpolating the variational solution for quick calculation of exponential growth rate of the fundamental mode.

Key words: 3D, High Gain FEL theory; Optical Guiding; SASE

1 Introduction

The main objective of this article is the determination of exponential growing modes (eigenmodes) in high gain FELs, taking into account the effects due to energy spread, emittance and betatron oscillations of the electron beam, as well as diffraction and optical guiding of the laser field. To deal with all these effects simultaneously, the most effective approach for analytical investigation is through coupled Maxwell-Vlasov equations. An equation satisfied by the modes of laser field was first derived by Kim [1], but without providing a solution. The first approximate solution of the equation was obtained by Yu et al. [2] for the fundamental mode, using a variational technique first

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introduced by Xie et al. [3]. The solution by Yu et al. assumes a waterbag model for unperturbed electron distribution in transverse phase space. Later, a special 2D case (sheet beam) of Gaussian model was considered by Hafizi et al. [4] for the fundamental mode, using also the variational technique. Taking a different approach from that of Kim [1] in handling the coupled Maxwell-Vlasov equations, Chin et al. [5] derived an equation satisfied by the mode of perturbed distribution function and obtained another approximate solution for the fundamental mode. However, this solution is known [5] to have a significant systematic error when approaching to the 1D limit.

In this article, we present the first exact solutions of 3D FEL eigenmodes, both fundamental and higher order. The unperturbed electron distribution is assumed to be of Gaussian shape in four dimensional transverse phase space and in energy variable, but uniform in longitudinal coordinate. For the fundamental mode, a variational solution is derived and from this solution a fitting formula is generated for the growth rate. A new formalism of scaling is introduced and based on which solutions are presented and discussed in various limiting cases.

2 Eigenmode Equation

The eigenmodes of laser field independent of initial condition can be determined by the following equation [1]

$$\left(\frac{\partial^2}{\partial \mathbf{x}^2} + \frac{ik_r}{L_{1d}} q\right) E(\mathbf{x}) = \frac{hk_r}{4L_{1d}^3} \int_{-\infty}^{\infty} d^2\mathbf{p} \int_{-\infty}^{\infty} d\eta \int_{-\infty}^{0} ds s e^{\Phi}$$
$$f_{\|}(\eta) f_{\perp}(\mathbf{x}^2 + \mathbf{p}^2/k_\beta^2) E[\mathbf{x}\cos(k_\beta s) + (\mathbf{p}/k_\beta)\sin(k_\beta s)], \tag{1}$$

where $\Phi = [q/2L_{1d} + i2k_w\eta - ik_w\Delta\nu - ik_r k_\beta^2(\mathbf{x}^2 + \mathbf{p}^2/k_\beta^2)/2]s$, $\eta = (\gamma - \gamma_0)/\gamma_0$, $\Delta\nu = (\omega - \omega_r)/\omega_r$, $k_r = 2\gamma_0^2 k_w/(1 + a_w^2)$, and $h = (2/\sqrt{3})^3$. The transverse profile of the slowly varying laser field, $E(\mathbf{x})$, is defined by $\mathcal{E}(\mathbf{x}, z, t) = E(\mathbf{x}, z)\exp[i(kz - \omega t)]$ and $E(\mathbf{x}, z) = E(\mathbf{x})\exp(qz/2L_{1d})$. To comply with a new scaling to be introduced later, here the complex exponential growth rate (eigenvalue), q, is scaled by the 1D power gain length, $L_{1d} = 1/2\sqrt{3}k_w\rho$, where $\rho = \sqrt[3]{\pi r_e n_0 A_w^2/4k_w^2\gamma_0^3}$ is the Pierce parameter, $A_w = a_w$ for helical wiggler, $A_w = a_w[J_0(a_w^2/2(1 + a_w^2)) - J_1(a_w^2/2(1 + a_w^2))]$ for planar wiggler, $a_w = 0.934\lambda_w[\text{cm}]B_{rms}[\text{T}]$, n_0 is the peak electron density on the axis, and r_e is the classical radius of electron. It is noted that the term proportional to $\mathbf{x}^2$ was absent from the phase factor Φ in the original equation derived by Kim [1], apparently due to a typo, and corrected later by Yu et al. [2].

The focusing system for the confinement of electron beam in wiggler is assumed to have a transverse gradient invariant along the beam axis. It is characterized by a constant betafunction β in both transverse planes. Thus betatron motion is governed by $\mathbf{p} = d\mathbf{x}/dz, d\mathbf{p}/dz = -k_\beta^2 \mathbf{x}$. In particular for natural wiggler focusing, $k_\beta = k_w a_w/\sqrt{2}\gamma_0$. Respectively, the unperturbed longitudinal and transverse distribution functions are normalized according to $\int_{-\infty}^{\infty} d\eta f_\parallel(\eta) = 1$ and $u(\mathbf{x} = 0) = 1$, where $u(\mathbf{x}) = \int_{-\infty}^{\infty} d^2\mathbf{p} f_\perp(\mathbf{x}^2 + \mathbf{p}^2/k_\beta^2)$.

Equation (1) can be reduced to a more convenient form [2]

$$\left(\frac{\partial^2}{\partial \mathbf{x}^2} + \frac{ik_r}{L_{1d}} q\right) E(\mathbf{x}) = \int_{-\infty}^{\infty} d^2\mathbf{x}' \Gamma(\mathbf{x}, \mathbf{x}') E(\mathbf{x}'), \tag{2}$$

where

$$\Gamma(\mathbf{x}, \mathbf{x}') = \int_{-\infty}^{0} ds \int_{-\infty}^{\infty} d\eta \frac{h k_r k_\beta^2 s}{4L_{1d}^3 \sin^2(k_\beta s)} e^{\Phi} f_\parallel(\eta) f_\perp(\chi),$$

$$\chi \equiv \mathbf{x}^2 + \mathbf{p}^2/k_\beta^2 = \frac{\mathbf{x}^2 + \mathbf{x}'^2 - 2\mathbf{x}\cdot\mathbf{x}' \cos(k_\beta s)}{\sin^2(k_\beta s)}.$$

3 Scaling and Limiting Cases

We now specify the unperturbed electron distribution as:

$$f_\parallel(\eta) = \frac{1}{\sqrt{2\pi}\sigma_\eta} e^{-\eta^2/2\sigma_\eta^2}, \tag{3}$$

$$f_\perp(\mathbf{x}^2 + \mathbf{p}^2/k_\beta^2) = \frac{1}{2\pi\sigma_x^2 k_\beta^2} e^{-(\mathbf{x}^2 + \mathbf{p}^2/k_\beta^2)/2\sigma_x^2}. \tag{4}$$

Here the transverse distribution is matched to the betatron focusing channel, giving rise to a constant beam size, σ_x. With Eqs.(3,4) and $\mathbf{X} = \mathbf{x}/\sigma_x, \tau = s/2L_{1d}$, Eq.(2) can be expressed in a scaled form

$$\left(2\eta_d \frac{\partial^2}{\partial \mathbf{X}^2} + iq\right) E(\mathbf{X}) = \int_{-\infty}^{\infty} d^2\mathbf{X}' \Pi(\mathbf{X}, \mathbf{X}') E(\mathbf{X}'), \tag{5}$$

where

$$\Pi(\mathbf{X},\mathbf{X}') = \int_{-\infty}^{0} \frac{\tau d\tau h}{2\pi \sin^2(2\sqrt{\eta_d\eta_\epsilon}\tau)} e^{\Psi},$$

$$\Psi = (q - i\eta_\omega)\tau - 2\eta_\gamma^2\tau^2 - \frac{(1+i\eta_\epsilon\tau)[\mathbf{X}^2 + \mathbf{X}'^2 - 2\mathbf{X}\cdot\mathbf{X}'\cos(2\sqrt{\eta_d\eta_\epsilon}\tau)]}{2\sin^2(2\sqrt{\eta_d\eta_\epsilon}\tau)}.$$

There are four scaling parameters [6] in Eq.(5): $\eta_d = 1/F_d$ is a diffraction parameter, where $F_d = 2k_r\sigma_x^2/L_{1d}$ is the Fresnel number of electron beam corresponding to a length scale of L_{1d}; $\eta_\epsilon = 4\pi\,(L_{1d}/\lambda_\beta)\,k_r\epsilon$ and $\eta_\gamma = 4\pi\,(L_{1d}/\lambda_w)\,\sigma_\eta$ characterize the effective spread in longitudinal velocity due to emittance and betatron focusing and due to energy spread, respectively, where $\lambda_\beta = 2\pi\beta$, $\epsilon = k_\beta\sigma_x^2$ is rms beam emittance and σ_η is relative rms energy spread; finally, $\eta_\omega = 4\pi\,(L_{1d}/\lambda_w)\,\Delta\nu$ is a frequency detuning parameter. The Pierce parameter can now be expressed in a more convenient form $\rho = \sqrt[3]{(I/I_A)(\lambda_w A_w/2\pi\sigma_x)^2(1/2\gamma_0)^3}$, where I is the peak beam current and $I_A = 17.05$ kA is the Alfv̀en current.

It has been shown [2,7] that FEL equations can be scaled with a minimum number of scaling parameters and in different ways. The scaling formalism introduced here, termed L_{1d} scaling, differs from the previous ones in the following aspects. First, parameter η_ϵ is chosen to emphasize the combined effect of emittance and betatron focusing, rather than using $k_r\epsilon$ or $k_\beta/k_w\rho$ separately. Second, by employing the scaling with L_{1d}, rather than with ρ or D, the formulation is made more transparent and elegant, and its presentation and elucidation more convenient, as it will become evident later on.

In the 1D limit without diffraction, $\eta_d = 0$, all modes are degenerate with the same eigenvalue and Eq.(5) bcomes

$$q + ih \int_{-\infty}^{0} \frac{\tau d\tau e^{(q-i\eta_\omega)\tau - 2\eta_\gamma^2\tau^2}}{1 + i\eta_\epsilon\tau} = 0. \qquad (6)$$

Further, if $\eta_\epsilon = 0$ and $\eta_\gamma = 0$, then, $q(q - i\eta_\omega)^2 - ih = 0$. This is the well-known 1D cubic equation [8] which admits root with the highest growth rate, $q = 1 + i/\sqrt{3}$, at $\eta_\omega = 0$. From now on we will refer the origin, $\{0,0,0,0\}$, in the scaled parameter space $\{\eta_d, \eta_\epsilon, \eta_\gamma, \eta_\omega\}$ as the 1D, ideal beam limit. In this limit, the scaled growth rate, $q_r \equiv L_{1d}/L_g$, reaches the absolute maximum of unity, where L_g is the power gain length by definition.

Another limiting case is known as the parallel beam limit, which can be obtained from Eq.(5) with $\eta_\epsilon = 0$

$$\left[2\eta_d \frac{\partial^2}{\partial \mathbf{X}^2} + iq - he^{-\mathbf{X}^2/2} \int_{-\infty}^{0} \tau d\tau e^{(q-i\eta_\omega)\tau - 2\eta_\gamma^2 \tau^2}\right] E(\mathbf{X}) = 0. \tag{7}$$

Further, if $\eta_\gamma = 0$, then for axially symmetric modes

$$\left[2\eta_d \frac{d}{RdR}\left(R\frac{d}{dR}\right) + iq + \frac{h}{(q-i\eta_\omega)^2} e^{-R^2/2}\right] E(R) = 0. \tag{8}$$

This is the same equation systematically studied earlier by Xie et al. [3,9], where both exact solutions for all modes and variational approximation for the fundamental mode were obtained.

4 Exact Solutions

In polar coordinate $\mathbf{X} = \{R, \phi\}$, Eq.(5) assumes the following form

$$\begin{aligned} &\left[2\eta_d \left[\frac{d}{RdR}\left(R\frac{d}{dR}\right) - \frac{m^2}{R^2}\right] + iq\right] E_m(R) \\ &\quad = \int_0^\infty R' dR' G_m(R, R') E_m(R'), \end{aligned} \tag{9}$$

where m is the azimuthal mode index and

$$G_m(R, R') = \int_{-\infty}^{0} \frac{\tau d\tau i^{-m} h}{\sin^2(2\sqrt{\eta_d \eta_\epsilon}\tau)} e^U J_m(V),$$

$$U = (q - i\eta_\omega)\tau - 2\eta_\gamma^2 \tau^2 - \frac{(1 + i\eta_\epsilon \tau)(R^2 + R'^2)}{2\sin^2(2\sqrt{\eta_d \eta_\epsilon}\tau)},$$

$$V = \frac{i(1 + i\eta_\epsilon \tau)\cos(2\sqrt{\eta_d \eta_\epsilon}\tau) R R'}{\sin^2(2\sqrt{\eta_d \eta_\epsilon}\tau)}.$$

Applying Hankel transform pair

$$E_m(Q) = \int_0^\infty RdR J_m(QR) E_m(R),$$

$$E_m(R) = \int_0^\infty QdQ J_m(QR) E_m(Q),$$

Eq.(9) can be converted into an integral equation

$$E_m(Q) = \int_0^\infty Q'dQ' T_m(Q,Q') E_m(Q'), \tag{10}$$

where

$$\begin{aligned} T_m(Q,Q') &= \frac{1}{iq - 2\eta_d Q^2} \int_0^\infty RdR \int_0^\infty R'dR' J_m(QR) J_m(Q'R') G_m(R,R') \\ &= \frac{i^{-m}h}{iq - 2\eta_d Q^2} \int_{-\infty}^0 \frac{\tau d\tau}{(1+i\eta_\epsilon \tau)^2} e^{(q-i\eta_\omega)\tau - 2\eta_\gamma^2 \tau^2 - (Q^2+Q'^2)/2(1+i\eta_\epsilon \tau)} \\ & J_m \left[\frac{i\cos(2\sqrt{\eta_d \eta_\epsilon}\tau) QQ'}{1+i\eta_\epsilon \tau} \right]. \end{aligned}$$

Upon discretization in Q space, the integral equation, Eq.(10), can be casted to a matrix form, $[\mathbf{T}_m(q) - \mathbf{I}]\mathbf{E}_m = 0$, where $\mathbf{I}$ is a unit matrix. Then all the eigenvalues, q_{nm}, can be determined by solving equation $|\mathbf{T}_m(q) - \mathbf{I}| = 0$, and the corresponding eigenmode, $\mathbf{E}_{nm}$, can be calculated subsequently given the matrix $\mathbf{T}_m(q_{nm})$, where n is the radial mode index.

Consider LCLS nominal case as an example with the following parameters [10]: $\lambda_r = 1.5$Å, $\gamma_0 = 28009$, $I = 3.4$kA, $\gamma_0\epsilon = 1.5$mm-mrad, $\sigma_\eta = 2\times10^{-4}$, $\beta =$ 18m, a planar wiggler with $\lambda_w = 3$cm and $\sqrt{2}a_w = 3.7$. The scaled parameters take the values: $\eta_d = 0.0367$, $\eta_\epsilon = 0.739$ and $\eta_\gamma = 0.248$. The intensity profiles of the three lowest order modes, E_{00}, E_{10} and E_{01} are shown in Fig.(1-3). Respectively, their eigenvalues and the corresponding optimal detunings are: $q_{00} = 0.4901 + i0.227$ ($\eta_\omega = -1.161$), $q_{10} = 0.125 - i0.0245$ ($\eta_\omega = -1.52$) and $q_{01} = 0.297 + i0.0662$ ($\eta_\omega = -1.40$).

5 Variational Approximation of Fundamental Mode

An approximate solution for the fundamental mode, E_{00}, is presented in this section. Similar solutions for the higher order modes, E_{10} and E_{01}, are given in another paper [11] in connection with the study of transverse coherence of SASE. The approximate solution derived here is, first of all, more efficient in calculation than the exact one. Secondly, it provides more physical insights, in particular on the mode profile, in a simpler manner. The solution is based on an approximation technique introduced by Xie et al. [3]. There, standard variational method [12] was first extended to treat the eigenvalue problem in which the eigenmode equation has a nonlinear dependence on the eigenvalue and the eigenvalue is complex. The generalized variational technique, acclaimed as one of the most flexible and general approximation method [13], has later been proven effective for a variety of 2D or 3D FEL eigenvalue problems [2,4,14]. According to the recipe [3], a variational functional may be constructed from Eq.(9) as follows:

$$\int_0^\infty RdRE(R)\left[2\eta_d\frac{d}{RdR}\left(R\frac{d}{dR}\right)+iq\right]E(R) = \int_0^\infty RdR\int_0^\infty R'dR'E(R)G_0(R,R')E(R'). \tag{11}$$

Substituting into Eq.(11) a trial function of the form, $E(R) = \exp(-aR^2)$, where a is a complex variational parameter to be determined, and applying the variational condition, $\delta q/\delta a = 0$, to the resulting equation, we obtain the following two equations from which the eigenvalue q and mode parameter a can be determined,

$$F_1(q,a) \equiv \frac{iq}{4a} - \eta_d - \int_{-\infty}^0 \tau d\tau h\frac{e^{f_1}}{f_2} = 0, \tag{12}$$

$$F_2(q,a) \equiv \frac{\partial F_1}{\partial a} = -\frac{iq}{4a^2} + \int_{-\infty}^0 \tau d\tau h\frac{f_3 e^{f_1}}{f_2^2} = 0, \tag{13}$$

where

$$\begin{aligned} f_1 &= (q - i\eta_\omega)\tau - 2\eta_\gamma^2\tau^2, \\ f_2 &= (1+i\eta_\epsilon\tau)^2 + 4a(1+i\eta_\epsilon\tau) + 4a^2\sin^2(2\sqrt{\eta_d\eta_\epsilon}\tau), \\ f_3 &= 4(1+i\eta_\epsilon\tau) + 8a\sin^2(2\sqrt{\eta_d\eta_\epsilon}\tau). \end{aligned}$$

To take the 1D limit appropriately, a singularity is removed by introducing $a = a_s(\eta_d, \eta_\epsilon, \eta_\gamma, \eta_\omega)/\sqrt{\eta_d}$, where a_s is a well-behaved, smooth function of its arguments. Then by taking $\eta_d = 0$, Eqs.(12,13) lead to Eq.(6) for the eigenvalue and for the mode parameter

$$a_s = \frac{1}{4}\sqrt{\frac{h\int_{-\infty}^0 \tau d\tau e^{f_1}}{\eta_\epsilon h\int_{-\infty}^0 \tau^3 d\tau e^{f_1}/(1+i\eta_\epsilon\tau)^2 - 1}}. \tag{14}$$

Therefore in the 1D limit, variational solutions are the same as the exact ones. On the other hand, in the parallel beam limit with $\eta_\epsilon = 0$, and furthermore with $\eta_\gamma = 0$, Eqs.(12,13) become

$$iq\bar{q}^2 + \eta_d\bar{q}^2 - 2\sqrt{h\eta_d}\bar{q} + h = 0, \tag{15}$$

$$a = \frac{1}{4}\left(\sqrt{\frac{h}{\eta_d\bar{q}^2}} - 1\right), \tag{16}$$

where $\bar{q} = q - i\eta_\omega$. These are the same equations derived earlier by Xie et al. [3]. Equation (15) is a 3D extension of the usual 1D cubic equation.

Given parameter a, mode properties can be determined completely by comparing $E(R \equiv r/\sigma_x) = \exp(-aR^2)$ with the usual Gaussian mode $E(r) = \exp(-r^2/w^2 + ik_r r^2/2R_c)$. Thus $w/2\sigma_x = \sqrt{1/4a_r}$ and $R_c/L_{1d} = -F_d/4a_i$, where w is the mode size and R_c the radius of phasefront curvature. Due to optical guiding, w remains constant along the wiggler and R_c is always positive for the growing mode [3]. If such a mode is allowed to propagate in free space from a location such as the end of the wiggler, the mode will diverge with a Rayleigh length L_r, and have an apparent waist w_0, located within the wiggler at a distance z_0 from the end of the wiggler, specifically: $L_r/L_{1d} = F_d/4a_r[1 + (a_i/a_r)^2]$, $w_0/2\sigma_x = 1/\sqrt{4a_r[1 + (a_i/a_r)^2]}$ and $z_0/L_{1d} = -F_d/4a_i[1 + (a_r/a_i)^2]$. In addition, the far field divergence angle is $\theta_d \equiv w_0/L_r = \sqrt{a_r[1 + (a_i/a_r)^2]}(\lambda_r/\pi\sigma_x)$. For the LCLS example, the variational method yields $q = 0.4902 + i0.2271$ and $a_s = 0.099 - i0.11$, optimized at $\eta_\omega = -1.161$. The comparison of mode profiles is shown in Fig.(1). Detuning curves are given in Fig.(4).

Another approximate solution for Gaussian beam distribution was derived earlier by Chin et al. [5] using a truncated orthogonal expansion method. There, the resulting zeroth order dispersion relation for the eigenvalue, when expressed in terms the scaling introduced here, can be simply written as:

$$\int_{-\infty}^{\infty}\int_{0}^{\infty} \frac{dxdy\ y^3 e^{-x^2-y^2}}{(iq + \eta_\omega - 2\sqrt{2}\eta_\gamma x + \eta_\epsilon y^2)^2} \int_{0}^{\infty} \frac{dz\ ze^{-z^2}}{(iq - 2\eta_d z^2)} = \frac{\sqrt{\pi}}{2h}. \tag{17}$$

The solution of Eq.(17) is known to be more accurate for larger value of η_d. However, for the LCLS example, η_d is quite small and Eq.(17) yields $q = 0.313 + i0.138$ optimized at $\eta_\omega = -1.865$, showing a significant difference from the exact solution. In the 1D, ideal beam limit, Eq.(17) gives $q = (1/\sqrt[3]{2})q_{exact}$.

6 Fitting Formula for Gain Length

One of the most important FEL performance parameter is the gain length of the fundamental mode, L_g. To facilitate quick calculation of this quantity, a fitting formula is generated in a scaled form

$$\frac{L_{1d}}{L_g} \equiv q_r = \mathcal{F}(\eta_d, \eta_\epsilon, \eta_\gamma)|_{\eta_\omega^*}, \tag{18}$$

where at each point in the three dimensional parameter space $\{\eta_d, \eta_\epsilon, \eta_\gamma\}$ the scaled quantity L_{1d}/L_g is maximized at the optimal detuning η_ω^*.

The function $\mathcal{F}$ is determined by interpolating the variational solutions with the following functional form

$$\frac{L_{1d}}{L_g} = \frac{1}{1+\Lambda}, \tag{19}$$

where

$$\begin{aligned}\Lambda = & a_1 {\eta_d}^{a_2} + a_3 {\eta_\epsilon}^{a_4} + a_5 {\eta_\gamma}^{a_6} \\ & + a_7 {\eta_\epsilon}^{a_8} {\eta_\gamma}^{a_9} + a_{10} {\eta_d}^{a_{11}} {\eta_\gamma}^{a_{12}} + a_{13} {\eta_d}^{a_{14}} {\eta_\epsilon}^{a_{15}} \\ & + a_{16} {\eta_d}^{a_{17}} {\eta_\epsilon}^{a_{18}} {\eta_\gamma}^{a_{19}},\end{aligned}$$

and the 19 fitting parameters are given in Table 1.

Table 1. Fitting parameters for gain length.

$a_1 = 0.45$	$a_2 = 0.57$	$a_3 = 0.55$	$a_4 = 1.6$
$a_5 = 3$	$a_6 = 2$	$a_7 = 0.35$	$a_8 = 2.9$
$a_9 = 2.4$	$a_{10} = 51$	$a_{11} = 0.95$	$a_{12} = 3$
$a_{13} = 5.4$	$a_{14} = 0.7$	$a_{15} = 1.9$	$a_{16} = 1140$
$a_{17} = 2.2$	$a_{18} = 2.9$	$a_{19} = 3.2$	

This is the same fitting formula published before without giving the derivation [6]. The accuracy of the fitting formula is shown in Fig.(5) for a typical case. In the special case with $\eta_d = 0$, the formula reproduces the exact solution from Eq.(6) practically with no discrepancy, as seen in Fig.(6).

7 Conclusions

A systematic approach is developed in three steps for the determination of 3D eigenmodes from Eq.(1). First and foremost, the exact solutions of both fundamental and higher order modes are obtained for the first time. Based on these solutions, complete information on the eigenmodes including eigenvalues and mode profiles can be extracted, examined, and used as a benchmark as well as an inspiration for approximate solutions. Secondly, a variational approximate solution of the fundamental mode is derived for the first time for Gaussian model. The solution is shown to be highly accurate in the parameter regime of interest to short wavelength FELs. It is also very efficient and robust in calculation, and as a result, the solution has been mapped out in the entire scaled parameter space. Finally, based on the wealth of information obtained with the variational solution, a transparent and elegant fitting formula for the gain length is generated. Apart from being compared with the variational solution, the formula has been found to be in good agreement with full-blown simulations [15]. Because of its convenience and accuracy, the formula has been widely used for design and optimization of high gain FEL systems [10]. A Chinese philosopher, Mao Tse-Tung, once said: let philosophy be liberated from the classrooms and books of philosophers, and turned into weapons in the hands of the masses. The three steps taken here is indeed a journey in that direction. The L_{1d} scaling introduced here has made that journey a pleasant trip in style.

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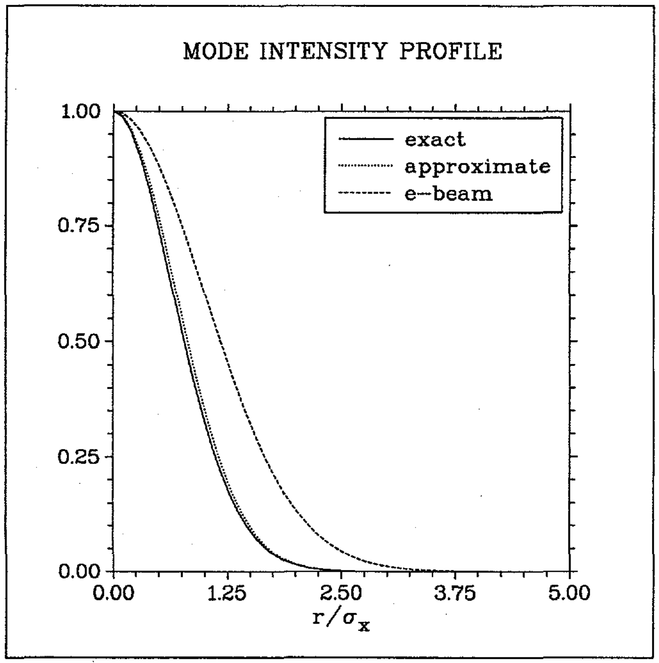

Fig. 1. Intensity profile of E_{00} mode from both the exact solution and variational approximation, superimposed with the electron density profile.

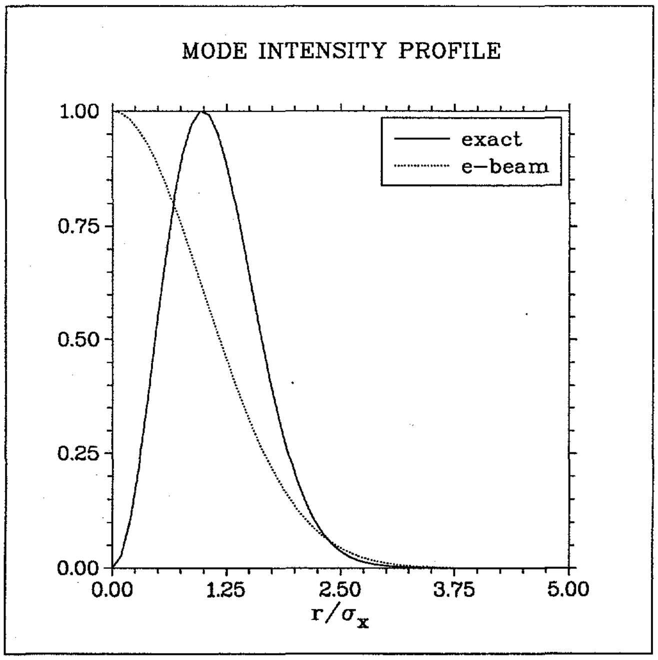

Fig. 2. Intensity profile of E_{10} mode from the exact solution, with the electron density profile.

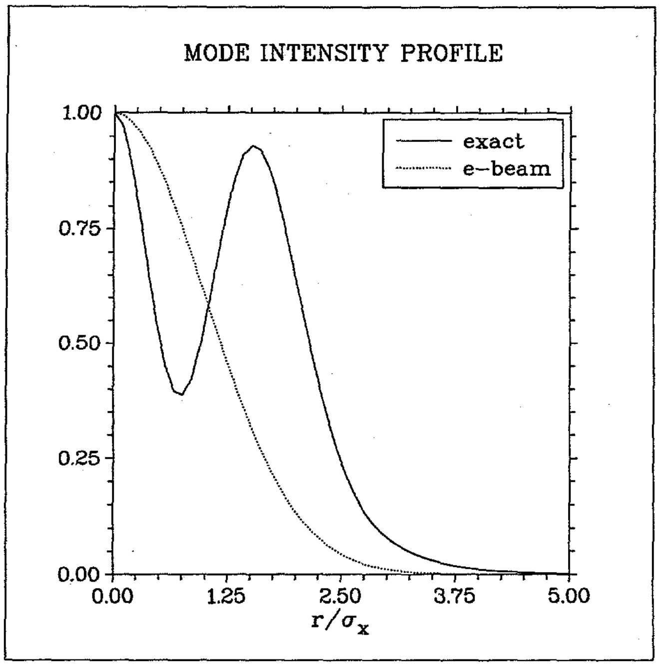

Fig. 3. Intensity profile of E_{01} mode from the exact solution, with the electron density profile.

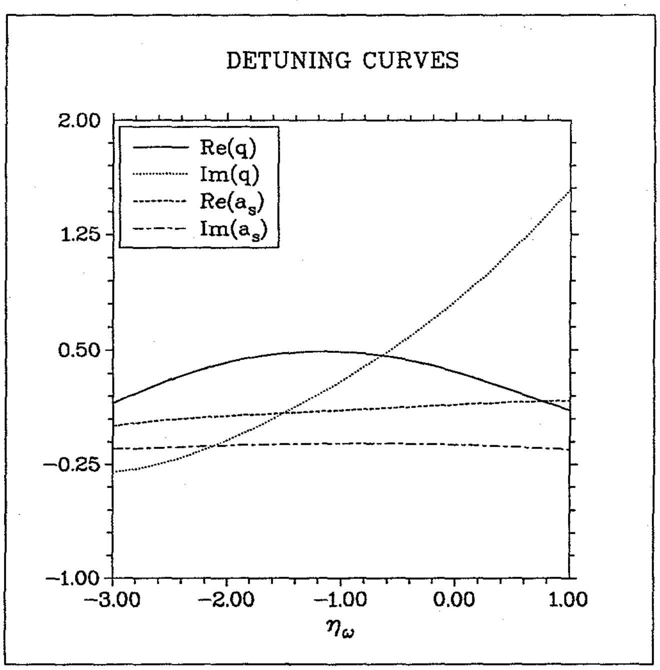

Fig. 4. Detuning curves from the variational solution.

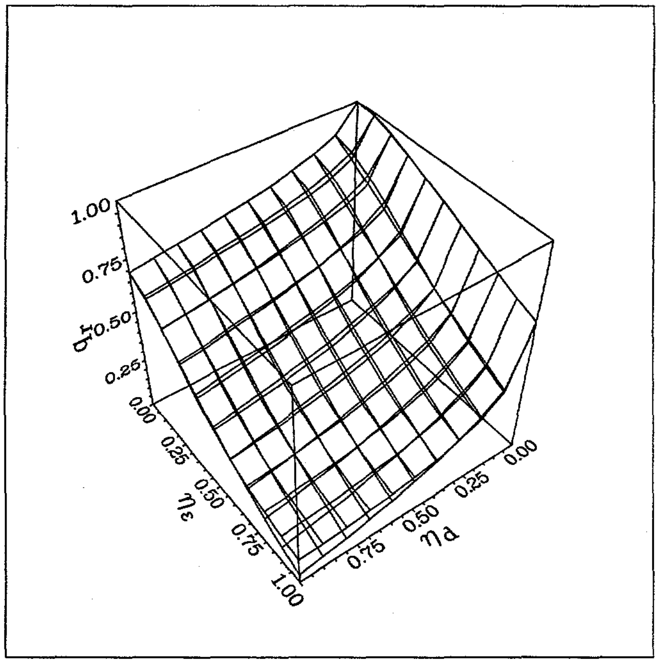

Fig. 5. Two surface plots showing $q_r = \mathcal{F}(\eta_d, \eta_\epsilon, \eta_\gamma = 0)$ are superimposed, one from the variational solution and another from the fitting formula.

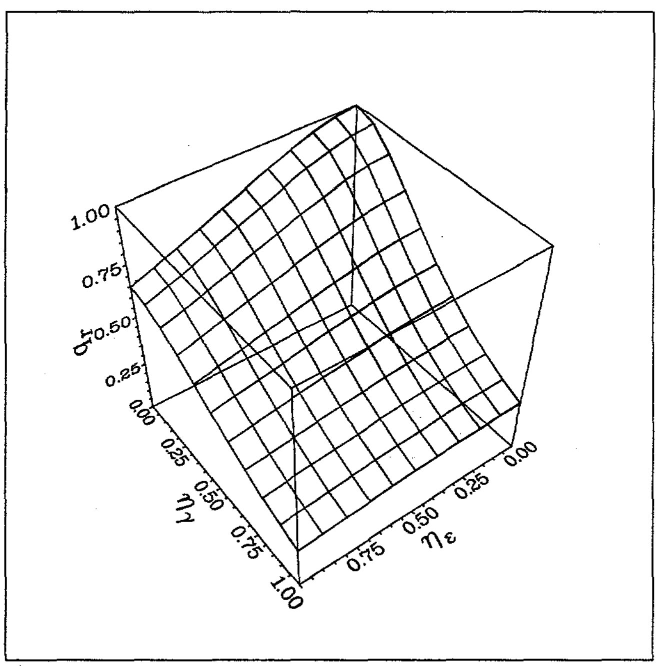

Fig. 6. Two surface plots showing $q_r = \mathcal{F}(\eta_d = 0, \eta_\epsilon, \eta_\gamma)$ are superimposed, one from the variational solution (same as the exact solution in this case) and another from the fitting formula. The difference between the two can hardly be seen in this case.